DIGITAL NOTES
ON
PROBABILITY & STATISTICS

(R22A0026)
B.TECH

II YEAR – I SEM

(2023-2024)

PREPARED BY
RADHA PYAR

DEPARTMENT OF COMPUTER SCIENCE & INFORMATION TECHNOLOGY

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution– UGC, Govt. of India)

(Affiliated to JNTUH, Hyderabad, Approved by AICTE – Accredited by NBA&NAAC –'A' Grade- ISO 9001:2015 Certified)
Maisammaguda, Dhulapally (Post Via. Hakimpet), Secunderabad–500100,TelanganaState,India

SYLLABUS

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

B. TECH- II- YEAR- I-SEM **L T/P/ C**
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PROBABILITY & STATISTICS

Course Objectives:

- To understand a random variable that describes randomness or an uncertainty in certain realistic situation. It can be either discrete or continuous type.
- To learn important probability distributions like: in the discrete case, study of the Binomial and the Poisson Distributions and in the continuous case the Normal Distributions.
- To Understand linear relationship between two variables and also to predict how a dependent variable changes based on adjustments to an independent variable.
- To learn the types of sampling, sampling distribution of means and variance, Estimations of statistical parameters.
- Use of probability theory to make inferences about a population from large and small samples.
- To understand different queuing models.

UNIT – I: Basic Probability and Random Variables

Basic Probability: Definition,The axioms of probability and basic problems.

Single Random Variables:Discrete and Continuous. Probability distribution function,Probability mass and density functions, mathematical expectation.
Multiple Random variables: Discrete and Continuous, Joint probability distributions-Joint probability mass and density functions, Marginal probability mass and density functions.

UNIT-II: Probability Distributions

Binomial distribution – properties, mean, variance and recurrence formula for Binomial distribution, Poisson distribution – Poisson distribution as Limiting case of Binomial distribution, properties, mean variance and recurrence formula for Poisson distribution,Normal distribution – mean, variance, median,mode and characteristics of Normal distribution.

UNIT -III:Correlation and Regression

Correlation -Coefficient of correlation, Rank correlation, Regression- Regression coefficients, Lines of regression.
Multiple correlation and regression- Coefficient of multiple Correlation, multiple regression, Multiple linear regression equations.

UNIT –IV: Testing of Hypothesis

Sampling: Definitions ,Standard error . Estimation - Point estimation and Interval estimation.

Testing of hypothesis: Null and Alternative hypothesis - Type I and Type II errors, Critical region - confidence interval - Level of significance, One tailed and Two tailed test.

Large sample Tests: Test of significance - Large sample test for single mean, difference of means, single proportion, difference of proportions.

Small samples: Test for single mean, difference of means, paired t-test, test for ratio of variances (F-test) ,Chi- square test for goodness of fit and independence of attributes.

UNIT V: Queuing Theory

Queuing theory –Structure of a queuing system and its characteristics-Arrival pattern and service pattern-Pure birth and Death process.
Terminology of Queuing systems-queuing models and its types - M/M/1 Model of infinite queue(without proofs) and M/M/1 Model of finite queue(without proofs).

Suggested Text Books:

i) Fundamental of Statistics by S.C. Gupta,7thEdition,2016.
ii) Fundamentals of Mathematical Statistics by SC Gupta and V.K.Kapoor
iii) Higher Engineering Mathematics by B.S. Grewal, Khanna Publishers, 35th Edition,2000.
iv) R. A. Johnson, Miller and Freund's "Probability and Statistics for Engineers", Pearson Publishers, 9th Edition, 2017.

References :

i) Introduction to Probability and Statistics for Engineers and Scientists by SheldonM.Ross.
ii) Probability and Statistics for Engineers by Dr. J. Ravichandran.

Course Outcomes: After learning the contents of this paper the student must be able to

1. Describe randomness in certain realistic situation which can be either discrete or continuous type and compute statistical constants of these random variables.
2. Provide very good insight which is essential for industrial applications by learning probability distributions.
3. Make objective, data-driven decisions by using correlation and regression.
4. *Draw statistical inference* using samples of a given size which is taken from a population.
5. To design balanced systems that serve customers quickly and efficiently but it is not cost effective.

CONTENTS


A CITADEL OF LEARNING

UNIT – I

Basic Probability and Random Variables

Random Experiment

If an experiment is conducted any number of times under identical conditions, there will be a set of outcomes associated with it. If the result is not certain and is any one of the several possible outcomes, the experiment is called a random experiment.

Each outcome is known as an elementary event.

Sample Space

The set of all possible elementary events in a trail is called a sample space (denoted by S) and each element of a sample space is called a sample point. Any subset of a sample space is an event(denoted by E)

Equally Likely Events

Events are said to be equally likely when there is no reason to expect any one of them rather than any one of the others.

Eg. When a card is drawn from a pack of cards, any card may be obtained. ie, all the 52 elementary events are equally likely.

Exhaustive Events

All possible events in a trail are called exhaustive events.

Eg. In tossing a coin, there are two exhaustive elementary events, head and tail.

Mutually Exclusive Events

Events are said to be mutually exclusive, if the happening of any one of the event in a trail excludes the happening of any one of the others.

Classical definition of Probability

In a random experiment let there be ***n*** mutually exclusive and equally likely elementary events. Let E be an event of the experiment. If ***m*** elementary events are in E(favourable to the event E), then probability of E is defined as

$$P(E)=\frac{m}{n}=\frac{Number\ of\ elementary\ evevnts\ in\ E}{Total\ number\ of\ elementary\ evevnts\ in\ the\ random\ experiment}$$

If $\bar{E}$(Complementary event of E) denotes the event of non-occurrence of E, then the number of elementary events in $\bar{E}$

Is n-m and hence the probability of $\bar{E}$is defined as

$P(\bar{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E)$

ie $P(E) + P(\bar{E}) = 1$

Since $\boldsymbol{m}$ is a non negative integer, $\boldsymbol{n}$ is a natural number and $\boldsymbol{m \leq n}$, we have $0 \leq \frac{m}{n} \leq 1$

Hence $0 \leq P(E) \leq 1$

Example 1: What is the probability for a leap year to have 52 Mondays and 53 Sundays?

Solution: A leap year has 366 days i.e., 52 weeks and 2 days. These two days can be any one of the following 7 ways-

(i) Mon & Tue
(ii) Tue & Wed
(iii) Wed & Thurs
(iv) Thurs & Fri
(v) Fri & Sat
(vi) Sat & Sun
(vii) Sun & Mon

Let E be the event of having 52 Mondays and 53 Sundays in the year.

Total number of possible cases is $n = 7$

Number of favorable cases to E is $m = 1$

$$\therefore \quad P(E) = \frac{m}{n} = \frac{1}{7}$$

Example 2: A class consists of 6 girls and 10 boys. If a committee of 3 is chosen at random from the class, find the probability that (i) 3 boys are selected (ii) exactly 2 girls are selected.

Solution: Total number of students = 16

$n(S) =$ no.of ways of choosing 3 from 16 $= {}^{16}C_3$

(i) Suppose 3 boys are selected. This can be done in ${}^{10}C_3$ ways

Here, $n(E) = {}^{10}C_7$

$\therefore P(E) =$ The probability that 3 boys are selected $= \frac{n(E)}{n(S)}$

$$= \frac{{}^{10}C_7}{{}^{10}C_3} = 0.2143$$

(ii) Suppose exactly 2 girls are selected. Then-

$$n(E) = {}^{6}C_2 \times {}^{10}C_1$$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{{}^{6}C_2 \times {}^{10}C_1}{{}^{16}C_3} = 0.2678$$

PROBABILITY-AXIOMATIC APPROACH

Let S be a finite sample space. A real valued function P from the power set of S into R is called a probability function on if the following axioms are satisfied.

Axioms of probability:

(i) Axiom of positivity : $P(E) \geq 0$

(ii) Axiom of certainty : $P(S) = 1$

(iii) Axiom of union : If E_1 and E_2 are disjoint subsets of S, then
$P(E_1 \cup E_2) = P(E_1) + P(E_2)$

Addition Theorem on Probability

If S is a sample space, and E_1, E_2 are any events in S then-

$$P(E_1 \text{ or } E_2) = P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

Multiplication Theorem of Probability

In a random experiment, if E_1, E_2 are two events such that $P(E_1) \neq 0$ and $P(E_2) \neq 0$, then-

$$P(E_1 \cap E_2) = P(E_1).P(E_2/E_1)$$

$$P(E_2 \cap E_1) = P(E_2).P(E_1/E_2)$$

Conditional Probability

If E_1, E_2 are two events in a sample space and $P(E_1) \neq 0$, then the probability of E_2, after the event E_1 has occurred, is called the conditional probability of the event of E_2 given E_1 and is denoted by $P\left(\frac{E_2}{E_1}\right)$ or $P(E_2/E_1)$ and we define $P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}$

Similarly, $P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}$

Random Variable

A Random Variable X is a real valued function from sample space S to a real number R.

(or)

A Random Variable X is a real number which is determined by the outcomes of the random experiment.

Eg:1. Tossing 2 coins simultaneously

Sample space ={HH,HT,TH,TT}

Let the random variable be getting number of heads then

X(S)={0,1,2}.

2.Sum of the two numbers on throwing 2 dice

X(S)={2,3,4,5,6,7,8,9,10,11,12}.

Types of Random Variables:

1.**Discrete Random Variables :** A Random Variable X is said to be discrete if it takes only the values of the set {0,1,2…..n}.

Eg:1.Tosssing a coin, throwing a dice,number of defective items in a bag.

2.Continuous Random Variables: A Random Variable X which takes all possible values in a given interval of domain.

Eg: Heights, weights of students in a class.

Discrete Probability Distribution:

Let x is a Discrete Random Variable with possible outcomes $x_1, x_2, x_3 \ldots . x_n$ having probabilities $p(x_i) for\ i = 1,2 \ldots n$ .If $p(x_i) > 0\ and\ \sum_{i=1}^{n} p(x_i) = 1$ then the function $p(x_i)$ is called **Probability mass function** of a random variable X and $\{ x_i, p(x_i)\}\ for\ i = 1,2 \ldots n$ is called **Discrete Probability Distribution.**

Eg: Tossing 2 coins simultaneously

Sample space ={HH,HT,TH,TT}

Let the random variable be getting number of heads then

X(S)={0,1,2}.

Probability of getting no heads = $\frac{1}{4}$, Probability of getting 1 head = $\frac{1}{2}$, Probability of getting 2 heads = $\frac{1}{4}$

∴ Discrete Probability Distribution is

x_i	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Cumulative Distribution function is given by $F(x) = p[X \leq x] = \sum_{i=0}^{x} p(x_i)$.

Properties of Cumulative Distribution function:

1. $P[a < x < b] = F(b) - F(a) - P[X = b]$

2. $P[a \leq x \leq b] = F(b) - F(a) - P[X = a]$

3. $P[a < x \leq b] = F(b) - F(a)$

4. $P[a \leq x < b] = F(b) - F(a) - P[X = b] + P[X = a]$

Mean: The mean of the discrete Probability Distribution is defined as

$\mu = \frac{\sum_{i=1}^{n} x_i p(x_i)}{\sum_{i=1}^{n} p(x_i)} = \sum_{i=1}^{n} x_i p(x_i)$ since $\sum_{i=1}^{n} p(x_i) = 1$

Expectation: The Expectation of the discrete Probability Distribution is defined as

E(X) = $\sum_{i=1}^{n} x_i p(x_i)$

In general, $E(g(x)) = \sum_{i=1}^{n} g(x_i) p(x_i)$

Properties:

1) $E(X) = \mu$

2) $E(X) = k\,E(X)$

3) $E(X + k) = E(X) + k$

4)) $E(aX \pm b) = aE(X) \pm b$

Variance: The variance of the discrete Probability Distribution is defined as

$Var(X) = V(X) = E[X - E(X)]^2$

$$\therefore V(X) = E[X]^2 - [E(X)]^2$$

$$= \sum x_i^2 p_i - \mu^2$$

Properties:

1) $V(c) = 0$ where c is a constant

2) $V(kX) = k^2 V(X)$

3) $V(X + k) = V(X)$

4) $V(aX \pm b) = a^2 V(X)$

Problems

1. If 3 cars are selected randomly from 6 cars having 2 defective cars.

a) Find the Probability distribution of defective cars.

b) Find the Expected number of defective cars.

Sol: Number of ways to select 3 cars from 6 cars $=6_{c_3}$

Let random variable X(S) = Number of defective cars = {0,1,2}

Probability of non defective cars = $\frac{^4w_3 \, ^2w_0}{6_{w_3}} = \frac{1}{5}$

Probability of one defective cars = $\frac{^4c_2 \, ^2c_1}{6_{c_3}} = \frac{3}{5}$

Probability of two defective cars = $\frac{^4w_1 \, ^2w_2}{6_{w_3}} = \frac{1}{5}$

Clearly , $p(x_i) > 0 \; and \; \sum_{i=1}^{n} p(x_i) = 1$

Probability distribution of defective cars is

x_i	0	1	2
$p(x_i)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$

Expected number of defective cars = $\sum_{i=1}^{n} x_i p(x_i) = 0\left(\frac{1}{5}\right) + 1\left(\frac{3}{5}\right) + 2\left(\frac{1}{5}\right) = 1$

2. Let X be a random variable of sum of two numbers in throwing two fair dice. Find the probability distribution of X, mean ,variance.

Sol: Sample space of throwing two dices is

S ={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

$\therefore n(S) = 36.$

Let X = Sum of two numbers in throwing two dice = {2,3,4,5,6,7,8,9,10,11,12}

X	Favorable cases	No of Favorable cases	$p(x)$
2	(1,1)	1	$\frac{1}{36}$
3	(2,1),)(1,2)	2	$\frac{2}{36}$
4	(3,1),(2,2),(1,3)	3	$\frac{3}{36}$
5	(4,1),(3,2),(2,3),(1,4)	4	$\frac{4}{36}$
6	(5,1),(4,2),(3,3),(2,4),(1,5)	5	$\frac{5}{36}$
7	(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)	6	$\frac{6}{36}$
8	(6,2),(5,3),(4,4),(3,5),(2,6)	5	$\frac{5}{36}$
9	(6,3),(5,4),(4,5),(3,6)	4	$\frac{4}{36}$
10	(6,4),(5,5),(4,6)	3	$\frac{3}{36}$
11	(6,5),(5,6)	2	$\frac{2}{36}$
12	(6,6)	1	$\frac{1}{36}$

Clearly , $p(x_i) > 0$ and $\sum_{i=1}^{n} p(x_i) = 1$

Probability distribution is given by

x_i	2	3	4	5	6	7	8	9	10	11	12
$p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$\text{Mean} = \mu = \sum_{i=1}^{n} x_i p(x_i)$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$= 7.$

$$\text{Variance} = V(X) = \sum x_i^2 p_i - \mu^2$$

$$= 4\left(\frac{1}{36}\right) + 9\left(\frac{2}{36}\right) + 16\left(\frac{3}{36}\right) + 25\left(\frac{4}{36}\right) + 36\left(\frac{5}{36}\right) + 49\left(\frac{6}{36}\right) + 64\left(\frac{5}{36}\right) + 81\left(\frac{4}{36}\right) + 100\left(\frac{3}{36}\right) + 121\left(\frac{2}{36}\right) + 144\left(\frac{1}{36}\right) - 49$$

$\therefore \text{Variance} = 5.83$

3. Let X be a random variable of maximum of two numbers in throwing two fair dice simultaneously. Find the

a) probability distribution of X

b)mean

c)variance

d)P(1<x<4)

e)P(2≤ $x \leq 4$).

Sol: Sample space of throwing two dices = S ={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)

(2,1),(2,2),(2,3),(2,4),(2,5),(2,6)

(3,1),(3,2),(3,3),(3,4),(3,5),(3,6)

(4,1),(4,2),(4,3),(4,4),(4,5),(4,6)

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

$\therefore n(S) = 36.$

Let X = Maximum of two numbers in throwing two dice = {1,2,3,4,5,6,}

X	Favorable cases	No of Favorable cases	$p(x)$
1	(1,1)	1	$\frac{1}{36}$
2	(2,1),)(1,2),(2,2)	3	$\frac{3}{36}$
3	(3,1),(1,3),(2,3)(3,3),(3,2)	5	$\frac{5}{36}$
4	(1,4),(4,1),(4,2),(2,4)(4,3),(3,4),(4,4)	7	$\frac{7}{36}$
5	(1,5),(5,1),(2,5),(5,2)(3,5),(5,3),(5,4),(4,5),(5,5)	9	$\frac{9}{36}$
6	(1,6)(6,1),(6,2),(2,6),(6,3),(3,6),(4,6),(6,4),(6,5)(5,6),(6,6)	11	$\frac{11}{36}$

Clearly , $p(x_i) > 0$ and $\sum_{i=1}^{n} p(x_i) = 1$

Probability distribution is given by

x_i	1	2	3	4	5	6
$p(x_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$\text{Mean} = \mu = \sum_{i=1}^{n} x_i p(x_i) = 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right)$$

$$= 4.4\ 7.$$

$$\text{Variance} = V(X) = \sum x_i^2 p_i - \mu^2$$

$$= 1\left(\frac{1}{36}\right) + 4\left(\frac{3}{36}\right) + 9\left(\frac{5}{36}\right) + 16\left(\frac{7}{36}\right) + 25\left(\frac{9}{36}\right) + 36\left(\frac{11}{36}\right)$$

$$\therefore \text{Variance} = 1.99.$$

4. A random variable X has the following probability function

x_i	-3	-2	-1	0	1	2	3
$p(x_i)$	k	0.1	k	0.2	2k	0.4	2k

Find k ,mean, variance.

Sol: We know that $\sum_{i=1}^{n} p(x_i) = 1$

i.e k+0.1+k+0.2+2k+0.4+2k = 1

i.e 6k+0.7 = 1 $\therefore k = 0.05$

$$\text{Mean} = \mu = \sum_{i=1}^{n} x_i p(x_i) = k(-3) + 0.1(-2) + k(-1) + 2k(1) + 2(0.4) + 3(2k)$$

$$= 0.8.$$

$$\text{Variance} = V(X) = \sum x_i^2 p_i - \mu^2$$

$$= k(-3)^2 + 0.1^2(-2) + k(-1)^2 + 2k(1) + 4(0.4) + 9(2k)$$

$$\therefore \text{Variance} = 2.86.$$

Continuous Probability distribution:

Let X be a continuous random variable taking values on the interval (a,b). A function f(x) is said to be the Probability density function of x if

i) $f(x) > 0\ \forall\ x \in (a, b)$

ii) Total area under the probability curve is 1 i. e, $\int_a^b f(x)dx = 1.$

iii) For two distinct numbers 'c' and 'd' in (a, b) is given by $P(c < x < d) =$ Area under the probability curve between ordinates $x = c$ and $x = d$ i. e $\int_c^d f(x)dx.$

Note: $P(c < x < d) = P(c \le x \le d) = P(c \le x < d) = P(c < x \le d)$

Cumulative distribution function of $f(x)$ is given by

$\int_{-\infty}^{x} f(x)dx$ i.e, $f(x) = \frac{d}{dx}F(x)$

Mean: The mean of the continuous Probability Distribution is defined as

$$\mu = \int_{-\infty}^{\infty} x\, f(x)dx.$$

Expectation: The Expectation of the continuous Probability Distribution is defined as

$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx.$

In general, $E(g(x)) = \int_{-\infty}^{\infty} g(x)\, f(x)dx.$

Properties:

1) $E(X) = \mu$

2) $E(X) = k\,E(X)$

3) $E(X + k) = E(X) + k$

4)) $E(aX \pm b) = aE(X) \pm b$

Variance: The variance of the Continuous Probability Distribution is defined as

$Var(X) = V(X) = \int_{-\infty}^{\infty} x^2\, f(x)dx - \mu^2.$

Properties:

1) $V(c) = 0$ where c is a constant

2) $V(kX) = k^2\, V(X)$

3) $V(X + k) = V(X)$

4) $V(aX \pm b) = a^2\, V(X)$

Mean Deviation: Mean deviation of continuous probability distribution function is defined as

$$\int_{-\infty}^{\infty} |x - \mu|\, f(x)dx.$$

Median: Median is the point which divides the entire distribution in to two equal parts. In case of continuous distribution,median is the point which divides the total area in to two equal parts i.e, $\int_a^M f(x)dx = \int_M^b f(x)dx = \frac{1}{2}\ \forall\ x \in (a,b)$.

Mode: Mode is the value of x for which f(x) is maximum.

i.e $f'(x) = 0$ and $f''(x) < 0$ for $x \in (a,b)$

Problems

1. If the probability density function $f(x) = \frac{k}{1+x^2}$ $-\infty < x < \infty$. Find the value of 'k' and probability distribution function of f(x).

Sol: Since total area under the probability curve is 1 i.e, $\int_a^b f(x)dx = 1$.

$$\int_{-\infty}^{\infty} \frac{\mathbf{k}}{\mathbf{1+x^2}}dx = 1.$$

$$2k(\tan^{-1}x)_0^{\infty} = 1$$

$$2k(\tan^{-1}\infty - \tan^{-1}0) = 1$$

$$\therefore k = \frac{1}{\pi}$$

Cumulative distribution function of f(x) is given by

$$\int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{\mathbf{k}}{\mathbf{1+x^2}}dx = \frac{1}{\pi}(\tan^{-1}x)_{-\infty}^{x} = \frac{1}{\pi}\left[\frac{\pi}{2} + (\tan^{-1}x)\right].$$

2. If the probability density function $f(x) = ce^{-|x|}$ $-\infty < x < \infty$.

Find the value of 'c', mean and variance.

Sol: Since total area under the probability curve is 1 i.e, $\int_a^b f(x)dx = 1$.

$$\int_{-\infty}^{\infty} \mathbf{ce^{-|x|}}\,dx = 1$$

$$2\int_0^{\infty} \mathbf{ce^{-x}}\,dx = 1$$

$$2c\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1$$

$$\therefore c = \frac{1}{2}$$

Mean=$\mu = \int_{-\infty}^{\infty} x\, f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} x\mathbf{e}^{-|x|}\, dx = 0$ since $x\mathbf{e}^{-|x|}$ is an odd function .

variance $= V(X)$

$$= \int_{-\infty}^{\infty} x^2\, f(x)dx - \mu^2$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2\mathbf{e}^{-|x|}\, dx$$

$$= \frac{1}{2}\int_{0}^{\infty} 2x^2\mathbf{e}^{-x}\, dx = [x^2(-\mathbf{e}^{-x}) - 2x(\mathbf{e}^{-x}) + 2(-\mathbf{e}^{-x})]_0^{\infty} = 2\,.$$

3. If the probability density function $\mathbf{f(x)} = \{$ $\frac{\mathbf{sinx}}{\mathbf{2}}$ **if $\mathbf{0 \le x \le \pi}$; **0 otherwise**

Find mean,median,mode and P($\mathbf{0 < x < \frac{\pi}{2}}$).

Sol: Mean = $\mu = \int_{-\infty}^{\infty} x\, f(x)dx = \frac{1}{2}\int_0^{\pi} x\frac{\mathbf{sinx}}{\mathbf{2}}\, dx = \frac{1}{2}[-x\cos x + \sin x]_0^{\pi} = \frac{\pi}{2}$.

Let M be the Median then

$$\int_0^M f(x)dx = \int_M^{\pi} f(x)dx = \frac{1}{2}\ \forall\, x \in (-\infty, \infty)$$

$$\int_0^M \frac{\mathbf{sinx}}{\mathbf{2}}\, dx = \int_M^{\pi} \frac{\mathbf{sinx}}{\mathbf{2}}\, dx = \frac{1}{2}\ \forall\, x \in (-\infty, \infty)$$

consider $\int_M^{\pi} \frac{\mathbf{sinx}}{\mathbf{2}}dx = \frac{1}{2}$ then $(-\cos x)_M^{\pi} = 1$

$$\therefore M = \frac{\pi}{2}$$

Since $f(x) = \{$ $\frac{\sin x}{2}$ if $0 \le x \le \pi$; 0 otherwise

To find maximum, we have $f'(x) = 0$

i.e, $\cos x = 0$ implies that $x = \frac{\pi}{2}$

and $f''(x) = -\frac{\sin x}{2}$ which is less than 0 at $x = \frac{\pi}{2}$

$\therefore$ Mode $= \frac{\pi}{2}$

4. If the distributed function is given by

$$\mathbf{F(x)} = \begin{cases} \mathbf{0 \text{ if } x \le 1} \\ \mathbf{k(x-1)^4 \text{ if } 1 \le x \le 3} \\ \mathbf{1 \text{ if } x > 3} \end{cases}$$

Find $\mathbf{k, f(x)}$, mean.

Sol: Cumulative distribution function of f(x) is given by

$\int_{-\infty}^{x} f(x)dx$ i.e, $f(x) = \frac{d}{dx}F(x)$

$$\textbf{i.e, } f(x) = \begin{cases} 0 \text{ if } x \le 1 \\ 4k(x-1)^3 \text{ if } 1 \le x \le 3 \\ 0 \text{ if } x > 3 \end{cases}$$

Since total area under the probability curve is 1 i. e, $\int_a^b f(x)dx = 1$

$$\int_1^3 4k(x-1)^3 dx = 1$$

$$[k(x-1)^4]_1^3 = 1$$

$$\therefore k = \frac{1}{16}$$

$$\therefore f(x) = \begin{cases} 0 \text{ if } x \le 1 \\ \frac{1}{4}(x-1)^3 \text{ if } 1 \le x \le 3 \\ 0 \text{ if } x > 3 \end{cases}$$

$$\text{Mean}=\mu = \int_{-\infty}^{\infty} x\, f(x)dx = \frac{1}{4}\int_1^3 x(x-1)^3 dx = 19.6.$$

Multiple Random variables

Discrete two dimensional random variable:

Joint probability mass function is defined as $f(x, y) = P(X = x_i, Y = y_i)$

Joint probability distribution function is defined as

$\mathbf{F_{XY}(x, y)} = P(X < x_i, Y < y_i) = \sum_{<x} \sum_{<y} p(x_i, y_i)$

Marginal probability mass functions of X and Y are defined as

$$P(X = x_i) = p(x_i) = \sum_j p(x_i, y_j)$$

$$P(Y = y_j) = p(y_j) = \sum_i p(x_i, y_j)$$

Continuous two dimensional random variable:

Joint probability density function is defined as

$f_{XY}(x, y) = P(x \le X \le x + dx, y \le Y \le y + dy)$

and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dxdy = 1$

Joint probability distribution function is defined as

$F_{XY}(x, y) = P(X < x_i, Y < y_i) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x, y)\, dxdy$

and $f_{XY}(x, y) = \frac{\partial^2}{\partial x\, \partial y} [F_{XY}(x, y)]$

Marginal probability density functions of X is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

Marginal probability density functions of Y is defined as

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx$$

Problems

1. For the following 2-d probability distribution of X and Y

X\Y	**1**	**2**	**3**	**4**
1	**0.1**	**0**	**0.2**	**0.1**
2	**0.05**	**0.12**	**0.08**	**0.01**
3	**0.1**	**0.05**	**0.1**	**0.09**

Find i) $P(X \le 2, Y = 2)$ ii) $F_X(2)$ iii)P(Y=3) iv) $P(X < 3, Y \le 4)$ v) $F_y(3)$.

Sol: Given

X\Y	1	2	3	4
1	0.1	0	0.2	0.1
2	0.05	0.12	0.08	0.01

3	0.1	0.05	0.1	0.09

i)$P(X \leq 2, Y = 2) = P(X = 1, Y = 2) + P(X = 2, Y = 2)$

$= 0+0.12$

$= 0.12$

ii) $F_X(2) = P(X \leq 2) = P(X = 1) + P(X = 2)$

$= \sum_j p(x_i, y_j) + \sum_j p(x_i, y_j)$

$=(0.1+0+0.2+0.1)+(0.05+0.2+0.08+0.1)$

$=0.66$

iii) $P(Y=3) = \sum_i p(x_i, y_j)$

$= 0.2+0.08+0.1$

$= 0.38.$

iv) $P(X < 3, Y \leq 4) = P(X < 3, Y = 1) + P(X < 3, Y = 2) + P(X < 3, Y = 3) + P(X < 3, Y = 4)$

$= P(X = 1, Y = 1) + P(X = 2, Y = 1) + P(X = 1, Y = 2) + P(X = 2, Y = 2) + P(X = 1, Y = 3) + P(X = 2, Y = 3) + P(X = 1, Y = 4) + P(X = 2, Y = 4)$

$=(0.1+0+0.2+0.1)+(0.05+0.2+0.08+0.1)$

$=0.66$

v) $F_y(3) = P(Y \leq 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$

$=(0.1+0.05+0.1)+(0+0.12+0.05)+(0.2+0.08+0.1)$

$=0.8$

2. Suppose the random variables X and Y have the joint density function defined by

$$\mathbf{f(x,y)} = \begin{cases} \mathbf{c(2x+y)\ if\ 2 < x < 6, 0 < y < 5} \\ \mathbf{0 \quad otherwise} \end{cases}$$

Find i)c ii)P(X>3,Y>2) iii) P(X>3)

Sol: Since $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1$

$$\int_2^6 \int_0^5 c(\mathbf{2x+y})\,dydx = 1$$

$$\int_2^6 c(2xy + \frac{y^2}{2})_0^5 dx = 1$$

$$\int_2^6 c(10x + \frac{25}{2}) dx = 1$$

$$c(10\frac{x^2}{2} + \frac{25x}{2})_2^6 = 1$$

$$\therefore c = \frac{1}{210}$$

ii) $P(X > 3, Y > 2) = \int_3^6 \int_2^5 f(x,y)\, dydx$

$$= \int_3^6 \int_2^5 \frac{1}{210}(2x + y)dydx$$

$$= \frac{1}{210}\int_3^6 (2xy + \frac{y^2}{2})_2^5 dx = \frac{15}{28}.$$

iii)) $P(X > 3) = \frac{1}{210}\int_3^6 \int_0^5 f(x,y)\, dydx$

$$= \frac{1}{210}\int_3^6 \int_0^5 (\mathbf{2x + y})dydx$$

$$= \frac{1}{210}\int_3^6 (2xy + \frac{y^2}{2})_0^5 dx$$

$$= \frac{1}{210}\int_3^6 (10x + \frac{25}{2}) dx$$

$$= \frac{1}{210}[10x^2 + (10x + \frac{25x}{2})]_3^6 = \frac{23}{28}.$$

3. The joint density function defined by

$$\mathbf{f(x,y)} = \begin{cases} \mathbf{c(xy)\ if\ 1 < x < 3, 2 < y < 4} \\ \mathbf{0\quad otherwise} \end{cases}$$

Find i) c

ii) Marginal probability density functions of X and Y

iii) Show that X and Y are independent.

Sol: Since $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dxdy = 1$

$$\int_2^4 \int_1^3 \mathbf{c(xy)} dxdy = 1$$

$$\int_2^4 cy\left(\frac{x^2}{2}\right)_1^3 dy = 1$$

$$\frac{8c}{2}\left(\frac{y^2}{2}\right)_2^4 = 1 \quad \therefore c = \frac{1}{24}.$$

ii) Marginal probability density functions of X and Y

Marginal probability density functions of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \frac{1}{24}\int_2^4 xy\,dy = \frac{x}{4}$$

Marginal probability density functions of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \frac{1}{24}\int_1^4 xy\,dx = \frac{y}{6}$$

iii) Clearly $f_{XY}(x,y) = \frac{xy}{24} = \frac{x}{4}\frac{y}{6} = f_X(x)\,f_Y(y)$

Therefore ,X and Y are independent.

Conditional probability density function :

Conditional probability density function of X on Y is

$$f_{XY}(X/Y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Conditional probability density function of Y on X is

$$f_{XY}(Y/X) = \frac{f_{XY}(x,y)}{f_X(X)}$$

4. The joint density function defined by

$$\mathbf{f(x,y)} = \begin{cases} \left(\mathbf{x^2 + \frac{xy}{3}}\right) \textbf{ if } \mathbf{0 < x < 1, 0 < y < 2} \\ \mathbf{0} \quad \textbf{otherwise} \end{cases}$$

Find

i) Conditional probability density functions.
ii) Marginal probability density functions
iii) Check whether the functions X and Y are independent or not

Sol: Given $f(x,y) = \begin{cases} \left(x^2 + \frac{xy}{3}\right) \text{ if } 0 < x < 1, 0 < y < 2 \\ 0 \quad \text{otherwise} \end{cases}$

Marginal probability density functions of X is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^2 \left(\mathbf{x^2} + \frac{\mathbf{xy}}{\mathbf{3}}\right) dy = 2x\left(x + \frac{1}{3}\right)$$

Marginal probability density functions of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^1 \left(\mathbf{x^2} + \frac{\mathbf{xy}}{\mathbf{3}}\right) dx = \frac{1}{3} + \frac{y}{6}$$

Here $f_Y(y)\ f_X(x) = 2x\left(x + \frac{1}{3}\right)\left(\frac{1}{3} + \frac{y}{6}\right)$

Therefore, $f_{XY}(x,y) \neq f_X(x)\ f_Y(y)$

Hence X and Y are not Independent.

Conditional probability density function of X on Y is

$$f_{XY}(X/Y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\left(\mathbf{x^2} + \frac{\mathbf{xy}}{\mathbf{3}}\right)}{\left(\frac{1}{3} + \frac{y}{6}\right)}$$

Conditional probability density function of Y on X is

$$f_{XY}(Y/X) = \frac{f_{XY}(x,y)}{f_X(X)} = \frac{\left(\mathbf{x^2} + \frac{\mathbf{xy}}{\mathbf{3}}\right)}{2x\left(x + \frac{1}{3}\right)}$$

UNIT- II

PROBABILITY DISTRIBUTIONS

Binomial Distribution: A Random variable 'X' has binomial distribution if it assumes only non-negative values with probability mass function given by

$$p(x=r)=\begin{cases} {}^{n}c_{r}\, p^{r}q^{n-r} & r=0,1,2,----n \\ 0 & otherwise \end{cases}$$

$= b(r;n,p)$

Conditions For Applicability Of Binomial Distributions:

1. Number of trials must be finite (n is finite)
2. The trails are independent
3. There are only two possible outcomes in any event i.e., success and failure.
4. Probability of success in each trail remains constant.

Examples:

1. Tossing a coin n times
2. Throwing a die
3. No. of defective items in the box

Mean Of The Binomial Distribution

$\mu = \sum_{r=0}^{n} r.P(r)$

$= \sum_{r=0}^{n} r.{}^{n}c_{r}\, P^{r}q^{n-r}$

$= {}^{n}c_{1}p^{1}q^{n-1} + 2{}^{n}c_{2}P^{r}q^{n-2} + 3{}^{n}c_{3}p^{3}q^{n-3} + \cdots \ldots {}^{n}c_{n}p^{n}q^{n-n}$

$= np^{1}q^{n-1} + 2.\frac{n(n-1)}{2!}p^{2}q^{n-2} + 3.\frac{n(n-1)(n-2)}{3!}p^{3}q^{n-3} + -- + np^{n}$

$= np[\, q^{(n-1)} + (n-1){}^{c}_{1}p^{1}q^{(n-1)-1} + ---+ p^{n-1}$

$= np[p+q]^{n-1}$

$= np$ since $[p+q=1]$

Mean = np.

Variance Of The Binomial Distribution

$$\sigma^{2} = \sum_{r=0}^{n} r^{2}\, p(r) - \mu^{2}$$

$$= \sum_{r=0}^{n}[r(r-1)+r]P(r) - \mu^2$$

$$= \sum_{r=0}^{n} r(r-1)P(r) + \sum_{r=0}^{n} r.P(r) - n^2p^2$$

$$= \sum_{r=0}^{n} r(r-1)n_{c_r}P^r q^{n-r} + np - n^2P^2$$

$$\text{let } \sum_{r=0}^{n} r(r-1)P(r) = \sum_{r=0}^{n} r(r-1)n_{c_r}P^r q^{n-r} = 2\,n_{c_{2r}}P^2q^2n^{n-2} + 6n_{c_3}P^3q^{n-3} + 12n_{c_r}P^4q^{n-4} + \cdots + n(n-1)\,P^n$$

$$= n(n-1)P^2[q^{n-2} + +(n-2)_{c_1}p^1q^{(n-2)-1} + - \ - \ - + p^2]$$

$$= n(n-1)P^2(p+q)^{n-2}$$

$$= n^2P^2 - nP^2$$

$$\sigma^2 = n^2P^2 - nP^2 + np - n^2P^2$$

$$= np(1-p)$$

$$= npq.$$

Problems

1. In tossing a coin 10 times simultaneously. Find the probability of getting

i) at least 7 heads ii) almost 3 heads iii) exactly 6 heads.

Sol: Given $n = 10$

Probability of getting a head in tossing a coin = $\frac{1}{2} = p$.

Probability of getting no head = $q = 1 - \frac{1}{2} = \frac{1}{2}$

The probability of getting r heads in a throw of 10 coins is

$$P(X = r) = p(r) = 10_{C_r}\left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{10-r}; r = 0,1,2,\ldots\ldots\ldots,10$$

(i) Probability of getting at least seven heads is given by

$$P(X \geq 7) = P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$$

$$= 10_{C_7}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^{10-7} + 10_{C_8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^{10-8} + 10_{C_9}\left(\frac{1}{2}\right)^9\left(\frac{1}{2}\right)^{10-9} + 10_{C_{10}}\left(\frac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}}[10_{C_7} + 10_{C_8} + 10_{C_9} + 10_{C_{10}}]$$

$$= \frac{1}{2^{10}}[120 + 45 + 10 + 1]$$

$$= \frac{176}{1024}$$

$$= 0.1719$$

ii) Probability of getting at most 3 heads is given by

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$

$$= {}^{10}c_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{10-1} + {}^{10}c_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{10-2} + {}^{10}c_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{10-3} + {}^{10}c_0 \left(\frac{1}{2}\right)^{10}$$

$$= \frac{1}{2^{10}}[{}^{10}c_0 + {}^{10}c_1 + {}^{10}c_2 + {}^{10}c_3]$$

$$= \frac{1}{2^{10}}[120 + 45 + 10 + 1]$$

$$= \frac{176}{1024}$$

$$= 0.1719$$

iii) Probability of getting exactly six heads is given by

$$P(X = 6) = {}^{10}c_6 \left(\frac{1}{2}\right)^6 \left(\frac{1}{2}\right)^{10-6}$$

=0.205.

2.In 256 sets of 12 tosses of a coin ,in how many cases one can expect 8 Heads and 4 Tails.

Sol: The probability of getting a head, $p = \frac{1}{2}$

The probability of getting a tail, $q = \frac{1}{2}$

Here $n = 12$

The probability of getting 8heads and 4Tails in 12trials $= P(X = 8) = {}^{12}c_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^4$

$$= \frac{12!}{8!\,4!}\left(\frac{1}{2}\right)^{12} = \frac{495}{2^{12}}$$

The expected number of getting 8 heads and 4 Tails in 12 trials of such cases in256 sets

$$= 256 \times P(X = 8) = 2^8 \times \frac{495}{2^{12}} = \frac{495}{16} = 30.9375 \sim 31$$

3.Find the probability of getting an even number 3 or 4 or 5 times in throwing a die 10 times

Sol: Probability of getting an even number in throwing a die $= \frac{3}{6} = \frac{1}{2} = p.$

Probability of getting an odd number in throwing a die $= q = \frac{1}{2}$

∴Probability of getting an even number 3 or 4 or 5 times in throwing a die 10 times is

$$P(X = 3) + P(X = 4) + P(X = 5)$$

$$= 10_{c_3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{10-3} + 10_{c_4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{10-4} + 10_{C_5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{10-5}$$

$$= \frac{1}{2^{10}}[10_{C_3} + 10_{C_4} + 10_{c_5}]$$

$$= \frac{1}{2^{10}}[120 + 252 + 210]$$

=0.568.

4. Out of 800 families with 4 children each ,how many could you expect to have

a) three boys b)five girls c) 2 or 3 boys d)at least 1 boy.

Sol: : Given $n = 5, N = 800$

Let having boys be success

Probability of having a boy $= \frac{1}{2} = p.$

Probability of having girl $= q = 1 - \frac{1}{2} = \frac{1}{2}$

The probability of having r boyss in 5 children is

$$P(X = r) = p(r) = 5_{C_r}\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{5-r}; r = 0,1,2 \ldots \ldots 5$$

a)Probability of having 3 boys is given by

$$P(X = 3) = 5_{C_r}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{5-3} = \frac{5}{16}$$

Expected number of families having 3 boys = $N\,p(3)$ =800($\frac{5}{16}$) =250 families.

b) Probability of having 5 girls = Probability of having no boys is given by

$$P(X = 0) = 5_{C_0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{5-0} = \frac{1}{32}$$

Expected number of families having 5 girls = $N\,p(0)$ =800($\frac{1}{32}$) =25 families.

c) Probability of having either 2 or 3 boys is given by

$$P(X=2)+P(X=3)=5_{C_2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right)^{5-2}+5_{C_3}\left(\frac{1}{2}\right)^{3}\left(\frac{1}{2}\right)^{5-3}=\frac{5}{18}$$

Expected number of families having 3 boys = $N\,p(3)$ =800($\frac{5}{8}$) =500 families.

d) Probability of having at least 1 boy is given by

$$P(X\geq 1)=1-P(X=0)$$

$$=1-5_{C_0}\left(\frac{1}{2}\right)^{0}\left(\frac{1}{2}\right)^{5-0}=\frac{31}{32}$$

Expected number of families having at least 1 boy =800($\frac{31}{32}$) =775 families.

5. Fit a Binomial distribution for the following data.

x	**0**	**1**	**2**	**3**	**4**	**5**
f	**2**	**14**	**20**	**34**	**22**	**8**

Sol: Given n= 5,$\sum f = 2+14+20+34+22+8=100$

$\sum x_i f_i$ == $0(2)+1(14)+2(20)+3(34)+4(22)+5(8)$ = 284

$\therefore$ Mean of the distribution $=\frac{\sum x_i f_i}{\sum f_i}=\frac{284}{100}=2.84$

We have Mean of the binomial distribution = np = 2.84

$$\therefore p=\frac{2.84}{5}=0.568;\ q=1-0.568=0.432.$$

Table To Fit Binomial Distribution

X	P(x_i)	E(x_i)
0	$5_{C_0}(0.568)^0\,(0.432)^{5-0}$=0.02	N p(0) =100(0.02)=2
1	$5_{C_1}(0.568)^1\,(0.432)^{5-1}$=0.09	9
2	$5_{C_2}(0.568)^2\,(0.432)^{5-2}$=0.26	26
3	$5_{C_3}(0.568)^3\,(0.432)^{5-3}$=0.34	34
4	$5_{C_4}(0.568)^4\,(0.432)^{5-4}$=0.22	22

5	$5_{C_5}(0.568)^5\ (0.432)^{5-5}$=0.059	5.9

Fitted Binomial distribution is

x	0	1	2	3	4	5
f	2	10	26	34	22	6

Recurrence Relation

$p(r+1) = n_{C_{r+1}}\ (p)^{r+1}\ (q)^{n-r-1}$ ..... (1)

$p(r) = n_{C_r}(p)^r\ (q)^{n-r}$ ..... (2)

$\frac{(1)}{(2)} = \frac{p(r+1)}{p(r)} = \frac{n_{C_{r+1}}(p)^{r+1}\ (q)^{n-r-1}}{n_{C_r}\ (p)^r\ (q)^{n-r}}$

$\therefore \frac{p(r+1)}{p(r)} = \frac{n_{C_{r+1}}}{n_{C_r}}(\frac{p}{q})$

$p(r+1) = \frac{n_{C_{r+1}}}{n_{C_r}}(\frac{p}{q})\ p(r).$

Poisson Distribution

A random variable 'X' follows Poisson distribution if it assumes only non-negative values with probability mass function is given by

$$P(x=r) = P(r,\lambda) = \begin{cases} \frac{e^{-\lambda}\lambda^r}{r!}\ for & y = 0,1,--(\lambda > 0) \\ 0 & otherwise \end{cases}$$

Conditions For Poisson Distribution

1. The number of trials are very large (infinite)
2. The probability of occurrence of an event is very small ($\lambda = np$)
3. $\lambda = np = finite$

Examples:

1. The number of printing mistakes per page in a large text
2. The number of telephone calls per minute at a switch board
3. The number of defective items manufactured by a company.

Recurrence Relation

$P(r+1) = \frac{e^{-\lambda}\lambda^{r+1}}{(r+1)!}$ ------(1)

$P(r) = \frac{e^{-\lambda}\lambda^{r}}{(r)!}$ ------(2)

$$\frac{1}{2} = \frac{P(r+1)}{P(r)} = \frac{e^{-\lambda}\lambda^2.\ \lambda}{(r+1)r!} X \frac{r!}{e^{-\lambda}\lambda^2}$$

$$P(r+1) = (\frac{\lambda}{r+1})\, P(r)\ for\ r = 0,1,2 - - - -$$

Problems

1. Using Recurrence relation find probability when x=0,1,2,3,4,5, if mean of P.D is 3.

Sol: We have

$P(r+1) = (\frac{\lambda}{r+1})\, P(r)\ for\ r = 0,1,2 - - - -$(1)

Given λ= 3

$P(0) = \frac{e^{-3}\lambda^{0}}{(0)!} = e^{-3}$ [by definition of Poisson distribution]

From (1),

For $r = 0$, $P(1) = (\frac{3}{0+1})\, P(0)$ =3 e^{-3}

For $r = 1$, $P(2) = (\frac{3}{1+1})\, P(0)$ =$\frac{3}{2}$ e^{-3}

For $r = 2$, $P(3) = (\frac{3}{2+1})\, P(0)$ = e^{-3}

F For $r = 3$, $P(4) = (\frac{3}{3+1})\, P(0)$ = $\frac{3}{4} e^{-3}$

or $r = 4$, $P(5) = (\frac{3}{4+1})\, P(0) = \frac{3}{5}\, e^{-3}$.

2. If X is a random variable such that $3P(X = 4) = \frac{P(X=2)}{2} + P(X = 0)$.

Find mean, $P(X \leq 2)$.

Sol: Given $3P(X = 4) = \frac{P(X=2)}{2} + P(X = 0)$…..(1)

Since X is a Poisson variable,

$$P(x = r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

$$\therefore 3\frac{e^{-\lambda}\lambda^2\,\lambda^4}{4!} = \frac{e^{-\lambda}\lambda^2}{(2)2!} + \frac{e^{-\lambda}\lambda^0}{0!}$$

Solving it we get $\lambda^4 - 2\lambda^2 - 4 = 0$

Taking $\lambda^2 = k, we\ get\ k^2 - 2k - 4 = 0$

$$\therefore k = 4, -2$$

$$\therefore \lambda^2 = 4\ implies\ that\ \lambda = 2$$

Therefore, Mean of the Poisson distribution = 2

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)$$

$$= \frac{e^{-2}\lambda^0}{0!} + \frac{e^{-2}2^1}{1!} + \frac{e^{-\lambda}2^2}{2!} = 0.54.$$

3. A car hire firm has 2 cars which it hires out day by day. The number of demands for a car on each day is distributed as poisson with mean 1.5 Calculate the proportion of days

i) on which there is no demand

ii) on which demand is refused.

Sol: Let number of demands for cars be the success.

Given mean = 1.5= λ

Using Poisson distribution,

$$P(x = r) = \frac{e^{-\lambda}\lambda^r}{r!}$$

i) Probability that there is no demand for car is

$$P(x = 0) = \frac{e^{-1.5}(1.5)^0}{0!} = 0.223$$

Expected number of days that there is no demand =N$P(0) = 365(0.223) = 81.39 \sim 81\ days$

ii) Probability that demand refused for car is

$$P(x > 2) = 1 - P(x = 0) - P(x = 1) - P(x = 2)$$

$$= 1 - \frac{e^{-1.5}(1.5)^0}{0!} - \frac{e^{-1.5}(1.5)^1}{1!} - \frac{e^{-1.5}(1.5)^2}{2!} = 0.191$$

Expected number of days that demand refused for car = N$P(x > 2)$

$$= 365(0.191) = 69.7 \sim 70\ days.$$

4. The distribution of typing mistakes committed by typist is given below.

Fit a Poisson distribution for it.

Mistakes per page	**0**	**1**	**2**	**3**	**4**	**5**
Number of pages	**142**	**156**	**69**	**27**	**5**	**1**

Sol: Given n= 5,$\sum f = 142 + 156 + 69 + 27 + 5 + 1 = 400$

$\sum x_i f_i == 0(142) + 1(156) + 2(69) + 3(27) + 4(5) + 5(1) = 400$

$$\therefore \text{Mean of the distribution} = \frac{\sum x_i f_i}{\sum f_i}$$

$$= \frac{400}{400} = 1.$$

We have Mean of the Poisson distribution = $\lambda = 1$

Table To Fit Poisson Distribution

X	P(x_i)	E(x_i)
0	$\frac{e^{-1}(1)^0}{0!}$=0.368	N p(0) =400(0.368)=147.2~147
1	$\frac{e^{-1}(1)^1}{1!}$=0.368	147
2	$\frac{e^{-1}(1)^2}{2!}$=0.184	74
3	$\frac{e^{-1}(1)^3}{3!}$=0.061	24
4	$\frac{e^{-1}(1)^4}{4!} = 0.015$	6

5	$\frac{e^{-1}(1)^5}{5!} = 0.003$	1

Fitted Poisson distribution is

Mistakes per page	0	1	2	3	4	5
Number of pages	147	147	74	24	6	1

Normal Distribution (Gaussian Distribution)

Let X be a continuous random variable, then it is said to follow normal distribution if its pdf is given by

$$f(x,\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty \le x \le \infty, \mu, \sigma > 0$$

Here , σ are the mean & S.D of X.

Properties Of Normal Distribution

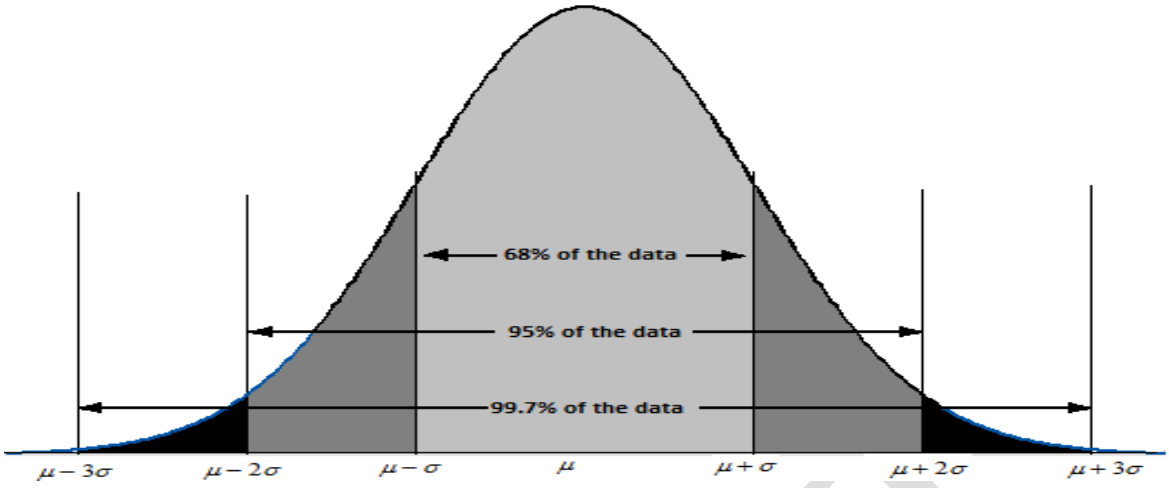
1. Normal curve is always centered at mean
2. Mean, median and mode coincide (i.e., equal)
3. It is unimodal
4. It is a symmetrical curve and bell shaped curve
5. X-axis is an asymptote to the normal curve
6. The total area under the normal curve from $-\infty\ to \infty$ is "1"
7. The points of inflection of the normal curve are $\mu \pm \sigma, \mu \pm 3\sigma$
8. The area of the normal curve between

$$\mu - \sigma \text{ to } \mu + \sigma = 68.27\%$$

$$\mu - 2\sigma\ to\ \mu + 2\sigma = 95.44\%$$

$$\mu - 3\sigma\ to\ \mu + 3\sigma = 99.73\%$$

9.The curve for normal distribution is given below



Standard Normal Variable

Let $Z = \frac{x-\mu}{\sigma}$ with mean '0' and variance is '1' then the normal variable is said to be standard normal variable.

Standard Normal Distribution

The normal distribution with man '0' and variance '1' is said to be standard normal distribution of its probability density function is defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-u}{\sigma}\right)^2} \qquad -\infty < x \leq \infty$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \qquad -\infty \leq x \leq \infty \ (\mu = 0, \sigma = 1)$$

Mean Of The Normal Distribution

Consider Normal distribution with b,σ as parameters Then

$$f(x; b, \sigma)\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}}$$

Mean of the continuous distribution is given by

$$\mu = \int_{-\infty}^{\infty} x\, f(x)dx = \int_{-\infty}^{\infty} x\, \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-b)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + b)\, e^{-\frac{(z)^2}{2}}\, dz \quad [\text{Putting } z = \frac{x-b}{\sigma} \text{ so that } dx = \sigma\, dz]$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{(z)^2}{2}}\, dz + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(z)^2}{2}} dz$$

$$= \frac{2b}{\sqrt{2\pi}} \int_{-0}^{\infty} e^{-\frac{(z)^2}{2}}\, dz$$

[since $z\, e^{-\frac{(z)^2}{2}}$ is an odd function and $e^{-\frac{(z)^2}{2}}$ is an even function]

$$\mu = \frac{2b}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = b$$

$$\therefore \text{Mean} = b$$

$$\sigma^2 = \int_{-\infty}^{\infty} x^2\, f(x)dx - \mu^2.$$

Variance Of The Normal Distribution

$$\text{Variance} = \int_{-\infty}^{\infty} x^2\, f(x)dx - \mu^2$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx - \mu^2$$

$$\text{Let } z = \frac{x-\mu}{\sigma} \Longrightarrow dx = \sigma dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu^2 + \sigma^2 z^2 + 2\mu\sigma z)\, e^{-\frac{z^2}{2}} \sigma dz - \mu^2$$

$$= \frac{\mu^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\, e^{-\frac{3^2}{2}} dz + \frac{2\mu\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2\, e^{-\frac{3^2}{2}}\, dz - \mu^2$$

$$= \frac{2\mu^2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{z^2}{2}} dz + \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} z^2\, e^{-\frac{3^2}{2}}\, dz - \mu^2$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} z^2 e^{-\frac{z^2}{2}} dz$$

$$\because \frac{z^2}{2} = + \Rightarrow \frac{2zdz}{2} = dt \qquad dz = \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^{\infty} (2 +)^2 e^t \frac{dt}{\sqrt{2t}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} + ^{2-1}.dt$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma(\tfrac{3}{2})$$

$$== \frac{2\sigma^2}{\sqrt{\pi}} \frac{1}{2} \Gamma(\tfrac{1}{2})$$

$$= \frac{\sigma^2}{\sqrt{\pi}} \sqrt{\pi} = \sigma^2$$

Median Of The Normal Distribution

Let x=M be the median, then

$$\int_{-\infty}^{M} f(x)dx = \int_{M}^{\infty} f(x)dx = \frac{1}{2}$$

Let $\mu \epsilon (-\infty, M)$

Let $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\mu} f(x)dx + \int_{\mu}^{M} f(x)dx = \frac{1}{2}$

Consider $\int_{-\infty}^{\mu} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\mu} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} dx$

Let $z = \frac{x-\mu}{\sigma} \Rightarrow dx = \sigma dz$ $\qquad [\because \text{Limits of } z - \infty \longrightarrow 0]$

$$\int_{-\infty}^{\mu} f(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{0} e^{-\frac{z^2}{2}} \sigma dz$$

$= \frac{1}{\sqrt{2\pi}} \int_{\infty}^{0} e^{-\frac{t^2}{2}}$ (dt) (by taking z=-t again)

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\pi}{2}} = \frac{1}{2}$$

From (1)

$$\int_{\mu}^{\mu} f(x)dx = 0 \Rightarrow \mu = M$$

Mode Of The Normal Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} - \left(\frac{x-\mu}{\sigma}\right)^2$$

$$f'(x) = 0 \Rightarrow \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left(\frac{-1}{2}\right) 2 \left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sigma} = 0$$

$$\Rightarrow x - \mu = 0 \Rightarrow x = \mu$$

$$\Rightarrow x = \mu$$

$$f^{11}(x) = \frac{-1}{\sigma^3\sqrt{2\pi}} [e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . 1 + (x-\mu) e^{-\frac{1}{2}} \left(\frac{x-\mu}{\sigma}\right)^2 \left(\frac{-1}{2}\right) 2 \left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sigma}]$$

$$= \frac{-1}{\sigma^3\sqrt{2\pi}} [e^0 + 0]$$

$$= \frac{-1}{\sigma^3\sqrt{2\pi}} < 0$$

$\because x = \mu$ is the mode of normal distribution.

Problems :

1. If X is a normal variate, find the area A

i) to the left of $z = 1.78$

ii) to the right of $z = -1.45$

iii) Corresponding to $-0.8 \leq z \leq 1.53$

iv) **to the left of $z = -2.52$ and to the right of $z = 1.83$.**

Sol: i) $P(z < -1.78) = 0.5 - P(-1.78 < z < 0)$

$$= 0.5 - P(0 < z < 1.78)$$

= 0.5-0.4625 =0.0375.

ii) $P(z > -1.45) = 0.5 + P(-1.45 < z < 0)$

$= 0.5 + P(0 < z < 1.45)$

= 0.5+0.4625 =0.9265.

iii) $P(-0.8 \leq z \leq 1.53) = P(-0.8 \leq z \leq 0) + P(0 \leq z \leq 1.53)$

= 0.2881+0.4370=0.7251.

iv **)** $P(z < -2.52) = 0.5 - P(0 < z < 2.52)$=0.0059

$P(z > 1.83)$ $= 0.5 - P(0 < z < 1.83)$

=0.036

2.If the masses of 300 students are normally distributed with mean 68 kgs and standard deviation 3kgs.How many students have masses

i) greater than 72kgs.

ii) less than or equal to 64 kgs

iii)between 65 and 71 kgs inclusive.

Sol: Given N=300,$\mu = 68, \sigma = 3$.Let X be the masses of the students.

i) Standard normal variate for X=72 is

$$z = \frac{x - \mu}{\sigma} = \frac{72 - 68}{3} = 1.33$$

$P(X > 72) = P(z > 1.33)$

$= 0.5 - P(0 < z < 1.33)$

=0.5-0.4082

=0.092

Expected number of students greater than 72 = E(X>72)

=300(0.092)

=27.54~28 students

ii) Standard normal variate for X=64 is

$$z = \frac{x - \mu}{\sigma} = \frac{64 - 68}{3} = -1.33$$

$P(X \leq 64) = P(z \leq -1.33)$

$= 0.5 - P(0 < z < 1.33)$ (Using symmetry)

=0.5-0.4082

=0.092

Expected number of students less than or equal to 64 = E(X less than or equal to 64)

=300(0.092)

=27.54~28 students .

iii) Standard normal variate for X=65 is

$$z_1 = \frac{x - \mu}{\sigma} = \frac{65 - 68}{3} = -1$$

Standard normal variate for X=71 is

$$z_2 = \frac{x - \mu}{\sigma} = \frac{71 - 68}{3} = 1$$

$$P(65 \leq X \leq 71) = P(-1 \leq z \leq 1)$$
$$= P(-1 \leq z \leq 0) + P(-0 \leq z \leq 1)$$
$$= 2\,P(-0 \leq z \leq 1)$$

=2(0.341)= 0.6826

$$E(65 \leq X \leq 71) = 300(0.6826) = 205\ Students.$$

∴Expected number of students between 65 and 71 kgs inclusive = 205 students.

3. In a normal distribution 31% of the items are under 45 and 8% of the items are over 64. Find mean and variance of the distribution.

Sol: Given $P(X < 45) = 31\% = 0.31$

And $P(X > 64) = 8\% = 0.08$

Let Mean and variances of the normal distributions are μ, σ^2.

Standard normal variate for X is

$$z = \frac{x - \mu}{\sigma}$$

Standard normal variate for X_1=45 is

$$z_1 = \frac{X_1 - \mu}{\sigma} = \frac{45 - \mu}{\sigma}$$
$$\Rightarrow \mu + \sigma z_1 = 45 \ldots\ldots\ldots (1)$$

Standard normal variate for X_2=64 is

$$z_2 = \frac{X_2 - \mu}{\sigma} = \frac{64 - \mu}{\sigma}$$
$$\Rightarrow \mu + \sigma z_2 = 64 \ldots\ldots\ldots (2)$$

From normal curve ,we $have\ P(-z_1 \le z \le 0) = 0.19$

$$\Rightarrow z_1 = -0.5$$

$P(0 \le z \le z_2)$=0.42

$$\Rightarrow z_2 = 1.41$$

$substituting\ the\ values\ of\ z_1, z_2$ in (1) and (2),we get

$\mu = 50, \sigma^2 = 98.$

4. In a normal distribution 7% of the items are under 35 and 89% of the items are under 63. Find mean and variance of the distribution.

Sol: Given $P(X < 35)$= $7\% = 0.07$

And $P(X < 63)$= $89\% = 0.89$

Let Mean and variances of the normal distributions are μ, σ^2.

Standard normal variate for X is

$$z = \frac{x - \mu}{\sigma}$$

Standard normal variate for X_1=35 is

$$z_1 = \frac{X_1 - \mu}{\sigma} = \frac{35 - \mu}{\sigma}$$
$$\Rightarrow \mu + \sigma z_1 = 35 \ldots\ldots\ldots (1)$$

Standard normal variate for X_2=63 is

$$z_2 = \frac{X_2 - \mu}{\sigma} = \frac{63 - \mu}{\sigma}$$
$$\Rightarrow \mu + \sigma z_2 = 63 \ldots\ldots\ldots (2)$$

Given $P(X < 35) = P(z < z_1)$

$0.07 = 0.5 - P(-z_1 \le z \le 0)$

$P(0 \le z \le z_1) = 0.43$

From normal curve ,we $have$

$$\Rightarrow z_1 = 1.48$$

We have$P(X < 63) = P(z < z_2)$

$0.89 = 0.5 + P(0 \leq z \leq z_2)$

$P(0 \leq z \leq z_2) = 0.39$

From normal curve ,we *have*

$$\Rightarrow z_2 = 1.23$$

substituting the values of z_1, z_2 in (1) and (2),we get

$\mu = 50, \sigma^2 = 100.$

UNIT-III

CORRELATION AND REGRESSION

CORRELATION

Introduction

In a bivariate distribution and multivariate distribution we may be interested to find if there is any relationship between the two variables under study. Correlation refers to the relationship between two or more variables. The correlation expresses the relationship or interdependence of two sets of variables upon each other.

Definition Correlation is a statistical tool which studies the relationship b/w 2 variables & correlation analysis involves various methods & techniques used for studying & measuring the extent of the relationship b/w them.

Two variables are said to be correlated if the change in one variable results in a corresponding change in the other.

The Types of Correlation

1) Positive and Negative Correlation: If the values of the 2 variables deviate in the same direction

i.e., if the increase in the values of one variable results in a corresponding increase in the values of other variable (or) if the decrease in the values of one variable results in a corresponding decrease in the values of other variable is called Positive Correlation.

e.g. Heights & weights of the individuals If the increase (decrease) in the values of one variable results in a corresponding decrease (increase) in the values of other variable is called Negative Correlation.

e.g, Price and demand of a commodity.

2) Linear and Non-linear Correlation: The correlation between two variables is said to be Linear if the corresponding to a unit change in one variable there is a constant change in the other variable over the entire range of the values (or) two variables x, y are said to be linearly related if there exists a relationship of the form $y = a + bx$.

e.g when the amount of output in a factory is doubled by doubling the number of workers.

Two variables are said to be Non linear or curvilinear if corresponding to a unit change in one variable the other variable does not change at a constant rate but at fluctuating rate.

i.e Correlation is said to be non linear if the ratio of change is not constant. In other words, when all the points on the scatter diagram tend to lie near a smooth curve, the correlation is said to be non linear (curvilinear).

3) Partial and Total correlation: The study of two variables excluding some other variables is called Partial correlation .

e.g. We study price and demand eliminating the supply.

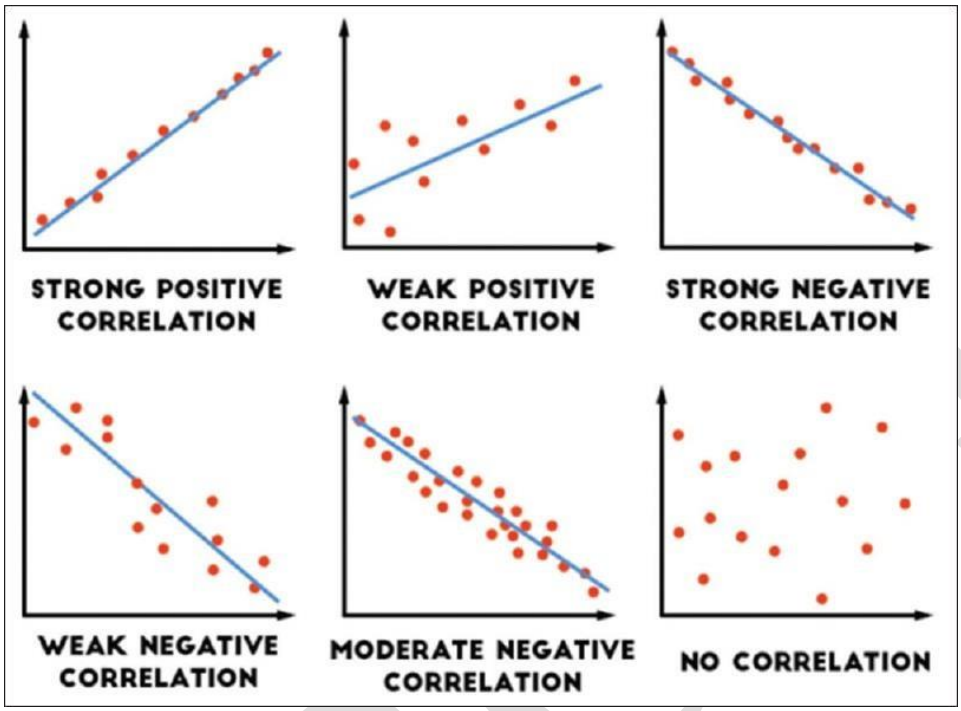
In Total correlation all the facts are taken into account.

e.g Price, demand & supply ,all are taken into account.

4) Simple and Multiple correlation: When we study only two variables, the relationship is described as Simple correlation.

E.g quantity of money and price level, demand and price.

The following are scatter diagrams of Correlation.



Karl Pearson's Coefficient of Correlation

Karl Pearson suggested a mathematical method for measuring the magnitude of linear relationship between 2 variables. This is known as Pearsonian Coefficient of correlation. It is denoted by '*r*'. This method is also known as Product-Moment correlation coefficient

$$r = \frac{Cov(xy)}{\sigma_x \sigma_y}$$

$$= \frac{\sum xy}{N\sigma_x \sigma_y}$$

$$= \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}}$$

$X = (x - \bar{X})$, $Y = (y - \bar{Y})$ where, $\bar{X}, \bar{Y}$ are means of the series x & y.

σ_x = standard deviation of series x

$\sigma_y =$ standard deviation of series y

Properties

1. The Coefficient of correlation lies b/w -1 & $+1$.
2. The Coefficient of correlation is independent of change of origin & scale of measurements.
3. If X, Y are random variables and a, b, c, d are any numbers such that $a \neq 0, c \neq 0$ then

$$r(aX + b, cY + d) = \frac{ac}{|ac|} r(X, Y)$$

4. Two independent variables are uncorrelated. That is if X and Y are independent variables then $r(X, Y) = 0$

Rank Correlation Coefficient

Charles Edward Spearman found out the method of finding the Coefficient of correlation by ranks. This method is based on rank & is useful in dealing with qualitative characteristics such as morality, character, intelligence and beauty. Rank correlation is applicable to only to the individual observations.

formula: $\rho = 6 \frac{\sum D^2}{N(N^2-1)}$

where : ρ - Rank Coefficient of correlation

D^2- Sum of the squares of the differences of two ranks

N- Number of paired observations.

Properties

1. The value of ρ lies between $+1$ and -1.
2. If $\rho = 1$, then there is complete agreement in the order of the ranks & the direction of the rank is same.

3. If $\rho = -1$, then there is complete disagreement in the order of the ranks & they are in opposite directions.

Equal or Repeated ranks

If any 2 or more items are with same value the in that case common ranks are given to repeated items. The common rank is the average of the ranks which these items would have assumed, if they were different from each other and the next item will get the rank next to ranks already assumed.

Formula: $\rho = 1 - 6\{\frac{\sum D^2 + \frac{1}{12}(m^3 - m) + \frac{1}{12}(m^3 - m)....}{N^3 - N}\}$

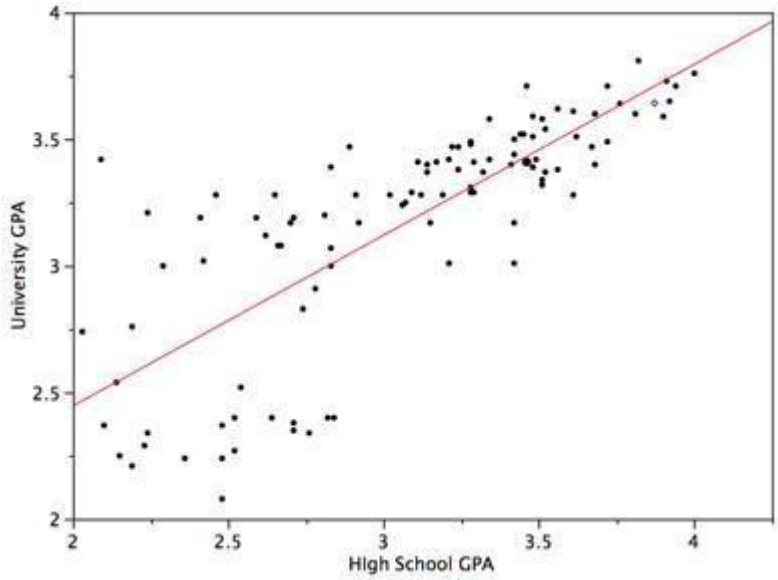
where m = the number of items whose ranks are common.

N- Number of paired observations.

D^2- Sum of the squares of the differences of two ranks

REGRESSION

In regression we can estimate value of one variable with the value of the other variable which is known. The statistical method which helps us to estimate the unknown value of one variable from the known value of the related variable is called 'Regression'. The line described in the average relationship b/w 2 variables is known as Line of Regression.



Regression Equation:

The standard form of the Regression equation is $Y = a + b X$ where a, b are called constants. 'a' indicates value of Y when $X = 0$. It is called Y-intercept. 'b' indicates the value of slope of the regression line & gives a measure of change of y for a unit change in X . it is also called as regression coefficient of Y on X. The values of a, b are found with the help of following Normal Equations.

Regression Equation of Y on X: $\sum Y = Na + b\sum X$

$$\sum XY = a\sum X + b\sum X^2$$

Regression Equation of X on Y : $\sum X = Na + b\sum Y$

$$\sum XY = a\sum Y + b\sum Y^2$$

Regression equations when deviations taken from the arithmetic mean :

Regression equation of Y on X : $Y - \bar{Y} = b_{yx}(X - \bar{X})$ where $b_{yx} = \frac{\sum XY}{\sum X^2}$

Regression equation of X on Y : $X - \bar{X} = b_{xy}(Y - \bar{Y})$ where $b_{xy} = \frac{\sum XY}{\sum Y^2}$

Angle b/w Two Regression lines : $\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$

Note:

1. If θ is acute then $\tan\theta = \frac{\sigma_x \sigma_y}{\sigma^2_x + \sigma^2_y}\left(\frac{1 - r^2}{r}\right)$
2. If θ is obtuse then $\tan\theta = \frac{\sigma_x \sigma_y}{\sigma^2_x + \sigma^2_y}\left(\frac{r^2 - 1}{r}\right)$
3. If $r = 0$ then $\tan\theta = \infty$ then $\theta = \frac{\pi}{2}$. Thus if there is no relationship between the 2 variables (i.e, they are independent) then $\theta = \frac{\pi}{2}$.
4. If $r = \pm 1$ then $\tan\theta = 0$ then $\theta = 0$ or π. Hence the 2 regression lines are parallel or coincident. The correlation between 2 variables is perfect.

Problems

1. Find Karl Pearson's coefficient of correlation from the following data.

Ht. in inches	**57**	**59**	**62**	**63**	**64**	**65**	**55**	**58**	**57**
Weight in lbs	**113**	**117**	**126**	**126**	**130**	**129**	**111**	**116**	**112**

Solution:

Ht. in inches X	Deviation from mean X = x-$\bar{x}$	X^2	Wt. in lbs Y	Deviation from mean Y = y-$\bar{y}$	Y^2	Product of deviations of X and Y series (XY)
57	-3	9	113	-7	49	21
59	-1	1	117	-3	9	3
62	2	4	126	6	36	12
63	3	9	126	6	36	18
64	4	16	130	10	100	40
65	5	25	129	9	81	45
55	-5	25	111	-9	81	45
58	-2	4	116	-4	16	8
57	-3	9	112	-8	64	24
540	0	102	1080	0	472	216

Coefficient of correlation $r = \frac{\sum XY}{\sqrt{\sum X^2 \sum Y^2}} = \frac{216}{\sqrt{(102)(471)}} = 0.98$

2. Calculate Coefficient of correlation for the following data.

X	**12**	**9**	**8**	**10**	**11**	**13**	**7**
Y	**14**	**8**	**6**	**9**	**11**	**12**	**3**

Solution: In both series items are in small number.

So there is no need to take deviations.

Formula used: $r = \frac{Cov(XY)}{\sigma_x \sigma_y}$

X	Y	X^2	Y^2	XY
12	14	144	196	168
9	8	81	64	72
8	6	64	36	48
10	9	100	81	90
11	11	121	121	121
13	12	169	144	156
7	3	49	9	21
$\sum X = 70$	$\sum Y = 63$	$\sum X^2 = 728$	$\sum Y^2 = 651$	$\sum XY = 676$

$$r = \frac{\sum XY - (\sum X \sum Y)/N}{\sqrt{(\sum X^2) - (\sum X)^2/N)(\sum Y^2 - (\sum Y)^2)/N}}$$

Here $N = 7$.

$$r = \frac{4732 - 4410}{\sqrt{5096 - 4900}\sqrt{4557 - 3969}} = \frac{322}{\sqrt{(196)(588)}} = \frac{322}{339.48} = +0.95$$

3. A sample of 12 fathers and their elder sons gave the following data about their elder sons. Calculate the rank correlation coefficient.

Fathers	**65**	**63**	**67**	**64**	**68**	**62**	**70**	**66**	**68**	**67**	**69**	**71**
Sons	**68**	**66**	**68**	**65**	**69**	**66**	**68**	**65**	**71**	**67**	**68**	**70**

Solution:

Fathers(x)	Sons(y)	Rank(x)	Rank(y)	$d_i = x_i - y_i$	d_i^2
65	68	9	5.5	3.5	12.25
63	66	11	9.5	1.5	2.25
67	68	6.5	5.5	1.0	1
64	65	10	11.5	-1.5	2.25
68	69	4.5	3	1.5	2.25
62	66	12	9.5	2.5	6.25
70	68	2	5.5	=3.5	12.25
66	65	8	11.5	3.5	12.25
68	71	4.5	1	=3.5	12.25
67	67	6.5	8	-1.5	2.25
69	68	3	5.5	-2.5	6.25
71	70	1	2	-1	1
					$\sum d_i^2 = 72.5$

Repeated values are given common rank, which is the mean of the ranks .In X: 68 & 67 appear twice.

In Y : 68 appears 4 times , 66 appears twice & 65 appears twice. Here N = 12.

$$\rho = 1 - 6\left\{\frac{\sum D^2 + \frac{1}{12}(m^3 - m) + \frac{1}{12}(m^3 - m)}{N^3 - N}\right\} = 1 - \frac{6(72.5 + 7)}{12(12^2 - 1)} = 0.722$$

4. Given n = 10 , σ_x= 5.4, σ_y = 6.2 and sum of product of deviation from the mean of X & Y is 66. Find the correlation coefficient.

Solution: $n = 10, \sigma_x = 5.4, \sigma_y = 6.2$

$$\sigma_x^2 = \frac{\sum(x - \bar{x})^2}{n}$$

$$\sigma_y^2 = \frac{\sum(y - \bar{y})^2}{n}$$

$$\sum(x - \bar{x})(y - \bar{y}) = 66$$

$$r = \frac{\sum(x - \bar{x})(y - \bar{y})}{\sqrt{\sum(x - \bar{x})^2 \sum(y - \bar{y})^2}} = \frac{66}{(5.)(6.2)} = 0.1971$$

5. The heights of mothers & daughters are given in the following table. From the 2 tables of regression estimate the expected average height of daughter when the height of the mother is 64.5 inches.

Ht. of Mother(inches)	**62**	**63**	**64**	**64**	**65**	**66**	**68**	**70**
Ht. of the daughter(inches)	**64**	**65**	**61**	**69**	**67**	**68**	**71**	**65**

Solution:

Let X = heights of the mother

Y = heights of the daughter

Let $dx = X - 65, dy = Y - 67, \sum x = 522, \sum dx = 2, \sum dx^2 = 50, \sum y = 530,$

$$\sum dy = -6 \sum dy^2 = 74, \sum dxdy = 20$$

$$\bar{X} = \frac{\sum X}{N} = \frac{522}{8} = 66.25$$

$$\bar{Y} = \frac{\sum Y}{N} = \frac{530}{8} = 65.25$$

$$b_{yx} = \frac{\sum dxdy - \frac{\sum dx \sum dy}{N}}{\sum dx^2 - \frac{(\sum dx)^2}{N}} = \frac{20 - \frac{2(-6)}{8}}{50 - \frac{2}{8}} = 0.434$$

Regression equation of Y on : $Y - \bar{Y} = b_{yx}(X - \bar{X})$

$$Y = 37.93 + 0.434X$$

when $X = 64.5$ then $Y = 69.923$

6. The equations of two regression lines are $7x - 16y + 9 = 0$ and $5y - 4x - 3 = 0$.

Find the coefficient of correlation and the means of x& *y* .

Solution: Given equations are $7x - 16y + 9 = 0$.....(1)

$$5y - 4x - 3 = 0 \text{.....(2)}$$

(1) × 4 gives $28x - 64y + 36 = 0$

(2) × 7 gives $-28x + 35y - 21 = 0$

On adding we get $-29y + 15 = 0$

$y = 0.5172$

from(1) $7x = 16y - 9$ which gives $x = 0.1034$

since regression line passes through $(\bar{x}, \bar{y})$ we have $\bar{x} = 0.1034$

$\bar{y} = 0.5172$

From(1) $x = \frac{16}{7}y - \frac{9}{7}$

From (2) $y = \frac{4}{5}x + \frac{3}{5}$,

$r\frac{\sigma_x}{\sigma_y} = \frac{16}{7}$ and $r\frac{\sigma_y}{\sigma_x} = \frac{4}{5}$

Multiplying these 2 equations , we get $r^2 = \frac{16}{7}\frac{4}{5} = \frac{64}{35}$

$$r = \frac{8}{\sqrt{35}}.$$

7. If $\sigma_x = \sigma_y = \sigma$ and the angle between the regression lines is $\text{Tan}^{-1}\left(\frac{4}{3}\right)$. Find r.

Solution: $\tan\theta = \frac{\sigma_x\sigma_y}{\sigma^2_x+\sigma^2_y}\left(\frac{1-r^2}{r}\right)$

$$= \frac{\sigma^2}{2\sigma^2}\left(\frac{1-r^2}{r}\right)$$

By data, $\theta = \text{Tan}^{-1}\left(\frac{4}{3}\right)$.

$$\frac{1-r^2}{2r} = \frac{4}{3}$$

$$3 - 3r^2 - 8r = 0$$

$$(3r - 1)(r + 3) = 0$$

$$r = \frac{1}{3} \text{ or } -3$$

Since we cannot have $r = -3$

Thus $r = \frac{1}{3}$

8. Given the following information regarding a distribution $N = 5$,

$\bar{X} = 10, \bar{Y} = 20, \sum(X - Y)^2 = 100, \sum(Y - 10)^2 = 160$. Find the regression coefficients and hence coefficient of correlation.

Solution: Here $dx = X - 4, dy = Y - 10$

$$\bar{X} = A + \frac{\sum dx}{N} \Rightarrow 10 = Y + \frac{\sum dx}{5} \Rightarrow \sum dx = 30 (\text{here } A = 4)$$

$$\bar{Y} = B + \frac{\sum dy}{N} \Rightarrow 20 = 10 + \frac{\sum dy}{5} \Rightarrow \sum dy = 50 (\text{here } B = 10)$$

$$b_{yx} = \frac{\sum dxdy - \frac{\sum dx \sum dy}{N}}{\sum dx^2 - \frac{(\sum dx)^2}{N}} = \frac{-220}{-80} = 2.75$$

$$b_{xy} = \frac{\sum dxdy - \frac{\sum dx \sum dy}{N}}{\sum dy^2 - \frac{(\sum dy)^2}{N}} = \frac{-220}{-340} = 0.65$$

$$\text{Coefficient of correlation } r = \pm\sqrt{b_{xy} \times b_{yx}} = \sqrt{(0.65)(2.75)} = \sqrt{1.7875} = 1.337$$

9. Given that $X = 4Y + 5$ and $Y = 4X + 4$ are the lines of regression of X on Y and Y on X respectively. Show that $0 < 4k < 1$. If $k = \frac{1}{16}$ find the means of the two variables and coefficient of correlation between them.

Solution: Given lines are X= 4Y +5 …….(1)

Y = KX+4..... (2)

From (1) & (2), $r\frac{\sigma_x}{\sigma_y} = 4$ and $r\frac{\sigma_y}{\sigma_x} = K$

Multiplying these two equations we get $r^2 = 4K$

Since $0 \leq r^2 \leq 1$, we have $0 \leq 4K \leq \frac{1}{4}$

If $K = \frac{1}{16}$ then we have $X = 4Y + 5$ and

$$Y = X/16 + 4$$

We get $X - 4Y - 5 = 0$

$$\frac{-X}{4} 4Y - 16 = 0$$

Adding we get $3\frac{X}{4} - 21 = 0$

$X = 28$

From(2), we get $Y = \frac{23}{4}$

The regression lines pass through $(\bar{x}, \bar{y})$

We get means $\bar{x} = 28$ and $\bar{y} = \frac{23}{4}$

We have $r^2 = 4k = \frac{4}{16} = \frac{1}{4} \Rightarrow r = \pm \frac{1}{2}$

We consider positive value and take $r = \frac{1}{2}$

10. The difference between the ranks are $0.5, -6, -4.5, -3, -5, -1, 3, 0, 5, 5.5, 0, -0.5$. For refracted ranks x and y. $\frac{\sum m(m^2-1)}{12}$=3.5, $r = 0.44$. Find the number of terms.

Solution: Given difference (d_i) $0.5, -6, -4.5, -3, -5, -1, 3, 0, 5, 5.5, 0, -0.5$

$\sum d_i^2 = 156$

Here $r = 1 - 6\{\frac{\sum d_i^2 + \frac{\sum m(m^2-1)}{12}}{(N^2-N)}\}$

$= \frac{1 - (159.5)6}{(N^2 - N)}$

$= 1 - \frac{957}{N^2-N}$

$\Rightarrow 0.44 = 1 - \frac{957}{N^2-N}$

$\Rightarrow N^2 - N = 1708.92$

$\Rightarrow N = 42$

Multiple Correlation:

In Multiple Correlation, the relationship between three or more variables is studied.
A dependent variable is indicated by X_1 and independent variables by $X_2, X_3, X_4, X_5, \ldots$
The Coefficient of Multiple Correlation id denoted by R and necessary subscripts are added to it.
Suppose there are three variables for X_1, X_2 *and* X_3. Let X_1 be the dependent variable depending on the independent variables X_2 and X_3. Then multiple correlation is defined as follows:

$R_{1.23}$ = Multiple correlation coefficient with X_1 as the dependent variable and X_2, X_3 as independent variables.

$R_{2.13}$ = Multiple correlation coefficient with X_2 as the dependent variable and X_1, X_3 as independent variables.

$R_{3.12}$ = Multiple correlation coefficient with X_3 as the dependent variable and X_1, X_2 as independent variables.

Calculation of Multiple Correlation Coefficient:

The multiple correlation coefficients can be calculated using the following formulae,

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}}$$

$$R_{2.13} = \sqrt{\frac{r_{21}^2 + r_{23}^2 - 2r_{21}r_{23}r_{13}}{1 - r_{13}^2}}$$

$$R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{31}r_{32}r_{12}}{1 - r_{12}^2}}$$

Note:

1. Multiple Correlation Coefficient is non-negative. Its value lies between 0 & 1. It cannot assume a negative value.
2. $R_{1.23} = 0 \Rightarrow r_{12} = 0$ *and* $r_{13} = 0$
3. $R_{1.23} \geq r_{12}, r_{13}, r_{23}$ and $R_{1.23} \geq r_{13}$
4. The position of the subscript to the right of dot does not make a difference i.e. $R_{1.23} = R_{1.32}$ and so on.
5. If $R_{1.23} = 0$ then all the Multiple Correlations involving X_1 are zero.

Problems

1. A single correlation coefficient between yield (x_1) and temperature (x_2) and rainfall (x_3) are given by $r_{12} = 0.6, r_{13} = 0.5, r_{23} = 0.8$. Find the Multiple Correlation Coefficient $R_{1.23}$.

Sol: We know that, $R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - 2r_{12}r_{13}r_{23}}{1 - r_{23}^2}}$

$$R_{1.23} = \sqrt{\frac{(0.6)^2 + (0.5)^2 - 2(0.6)(0.5)(0.8)}{1 - (0.8)^2}}$$

$$= \sqrt{\frac{0.36 + 0.25 - 0.48}{1 - 0.64}} = \sqrt{\frac{0.13}{0.36}} = 0.6$$

2. If $r_{12} = 0.5, r_{23} = 0.45$ *and* $r_{31} = 0.3$ find $R_{3.12}$

Sol: Substituting these values in the formula

$$R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{31}r_{32}r_{12}}{1 - r_{12}^2}}$$

$$R_{3.12} = \sqrt{\frac{(0.3)^2 + (0.45)^2 - 2(0.3)(0.45)(0.5)}{1 - (0.5)^2}}$$

$$R_{3.12} = \sqrt{\frac{0.09 + 0.2025 - 2(0.0675)}{1 - 0.25}} = \sqrt{\frac{0.4275}{0.75}} = \sqrt{0.57} = 0.755$$

3. Given the following data, compute Multiple Coefficient of Correlation of X_3 on X_1 and X_2.

X_1	3	5	6	8	12	14
X_2	16	10	7	4	3	2
X_3	90	72	54	42	30	12

Sol: Here n= 6, $\overline{X_1} = \frac{48}{6} = 8, \overline{X_2} = \frac{42}{6} = 7, \overline{X_3} = \frac{300}{6} = 50$

	$x_1 = X_1 - \overline{X_1}$			$x_2 = X_2 - \overline{X_2}$			$x_3 = X_3 - \overline{X_3}$					
S.No.	X_1	x_1	x_1^2	X_2	x_2	x_2^2	X_3	x_3	x_3^2	x_1x_2	x_2x_3	x_3x_1
1	3	-5	25	16	9	81	90	40	1600	-45	360	-200
2	5	-3	9	10	3	9	72	22	484	-9	66	-66
3	6	-2	4	7	0	0	54	4	16	0	0	-8
4	8	0	0	4	-3	9	42	-8	64	0	24	0
5	12	4	16	3	-4	16	30	-20	400	-16	80	-80
6	14	6	36	2	-5	25	12	-38	1444	-30	190	-228
	48	0	90	42	0	140	300	0	4008	-100	-582	720

$$r_{12} = \frac{\sum x_1x_2}{\sqrt{\sum x_1^2 \sum x_2^2}} = \frac{-100}{\sqrt{90 \times 140}} = -0.89$$

$$r_{13} = \frac{\sum x_1x_3}{\sqrt{\sum x_1^2 \sum x_3^2}} = \frac{-582}{\sqrt{90 \times 4008}} = -0.97$$

$$r_{23} = \frac{\sum x_2x_3}{\sqrt{\sum x_2^2 \sum x_3^2}} = \frac{720}{\sqrt{140 \times 4008}} = 0.96$$

$$R_{3.12} = \sqrt{\frac{r_{31}^2 + r_{32}^2 - 2r_{31}r_{32}r_{12}}{1 - r_{12}^2}} = \sqrt{\frac{(-0.97)^2 + (0.96)^2 - 2(-0.97)(0.96)(-0.89)}{1 - (-0.89)^2}}$$

$R_{3.12} = 0.987$

Multiple Regression Analysis:

In multiple regression analysis, the effect of two or more independent variables on one dependent variable is studied.

Regression Equations:

The procedure for studying multiple regression is similar to the one for simple regression, with the difference that the other variables are added in the regression equation. If there are three variables X_1, X_2 *and* X_3 the multiple regression has the following form:

$$\boldsymbol{X_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3} \quad \text{.....(1)}$$

In the above equation, $a_{1.23}$ is the intercept made by the regression plane. It gives thevalue of the dependent variable when all the independent variables are zero. $b_{12.3}$ indicates the slope of the regression line of X_1 *on* $X_2 when X_3$ is held constant. Similarly, $b_{13.2}$ indicates the slope of the regression line of X_1 *on* X_3 *when* X_2 is held constant.

Normal Equations for Multiple Regression Equations:

(i) The regression plane of X_1 *on* X_2 *and* X_3 is

$$X_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3 \quad \text{..... (1)}$$

In the above equation, the values of $b_{12.3}$ *and* $b_{13.2}$ are determined by solving simultaneously the following three normal equations.

$$\sum X_1 = Na_{1.23} + b_{12.3}\sum X_2 + b_{13.2}\sum X_3$$

$$\sum X_1X_2 = a_{1.23}\sum X_2 + b_{12.3}\sum X_2^2 + b_{13.2}\sum X_2X_3$$

$$\sum X_1X_3 = a_{1.23}\sum X_3 + b_{12.3}\sum X_2X_3 + b_{13.2}\sum X_3^2$$

(ii) The regression plane of X_2 *on* X_1 *and* X_3 is

$$X_2 = a_{2.13} + b_{21.3}X_1 + b_{23.1}X_3 \quad \text{..... (2)}$$

The normal equations for fitting the above equation are:

$$\sum X_2 = Na_{2.13} + b_{21.3}\sum X_1 + b_{23.1}\sum X_3$$

$$\sum X_1X_2 = a_{2.13}\sum X_1 + b_{21.3}\sum X_1^2 + b_{23.1}\sum X_1X_3$$

$$\sum X_2X_3 = a_{2.13}\sum X_3 + b_{21.3}\sum X_1X_3 + b_{23.1}\sum X_3^2$$

(iii) The regression plane of X_3 *on* X_1 *and* X_2 is

$$X_3 = a_{3.12} + b_{31.2}X_1 + b_{32.1}X_2 \quad \text{..... (3)}$$

The normal equations for fitting the above equation are:

$$\sum X_3 = Na_{3.12} + b_{31.2}\sum X_1 + b_{32.1}\sum X_2$$

$$\sum X_1X_3 = a_{3.12}\sum X_1 + b_{31.2}\sum X_1^2 + b_{32.1}\sum X_1X_2$$

$$\sum X_2X_3 = a_{3.12}\sum X_2 + b_{31.2}\sum X_1X_2 + b_{32.1}\sum X_2^2$$

Problems:

Q. Find the multiple linear regression equation of X_1 *on* X_2 *and* X_3 from the data given below:

X_1	2	4	6	8
X_2	3	5	7	9
X_3	4	6	8	10

Sol: The regression plane of X_1 *on* X_2 *and* X_3 is

$X_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3$ ..... (A)

where the values of the three constants are obtained by solving the following three normal equations.

$$\sum X_1 = Na_{1.23} + b_{12.3}\sum X_2 + b_{13.2}\sum X_3$$
$$\sum X_1X_2 = a_{1.23}\sum X_2 + b_{12.3}\sum X_2^2 + b_{13.2}\sum X_2X_3$$
$$\sum X_1X_3 = a_{1.23}\sum X_3 + b_{12.3}\sum X_2X_3 + b_{13.2}\sum X_3^2$$

S.No.	X_1	X_2	X_3	X_1X_2	X_2X_3	X_3X_1	X_1^2	X_2^2	X_3^2
1	2	3	4	6	12	8	4	9	16
2	4	5	6	20	30	24	16	25	36
3	6	7	8	42	56	48	36	49	64
4	8	9	10	72	90	80	64	81	100
$\sum$ =	20	24	28	140	188	160	120	164	216

Substituting the values in the normal equations, we get

$6a_{1.23} + 24b_{12.3} + 28b_{13.2} = 20$ ..... (i)

$24a_{1.23} + 164b_{12.3} + 188b_{13.2} = 140$ ..... (ii)

$28a_{1.23} + 188b_{12.3} + 216b_{13.2} = 160$ ..... (iii)

Multiplying equation (i) by 4 and subtracting it from the equation (ii), we get:

$68b_{12.3} + 76b_{13.2} = 60$ ..... (iv)

Multiplying equation (ii) by 7 and equation (iii) by 6, we get:

$168a_{1.23} + 1148b_{12.3} + 1316b_{13.2} = 980$ ..... (v)

$168a_{1.23} + 1128b_{12.3} + 1296b_{13.2} = 960$ ..... (vi)

Subtracting (vi) from (v), we obtain:

$20b_{12.3} + 20b_{13.2} = 20$ ..... (vii)

Multiplying equation (iv) by 5 and equation (vii) by 7 we get:

$340b_{12.3} + 380b_{13.2} = 300$ ..... (viii)

$340b_{12.3} + 340b_{13.2} = 340$ ..... (ix)

Subtracting (ix) from (viii), we have $40b_{13.2} = -40 \Rightarrow b_{13.2} = -1$ ..... (x)

Substituting the value of $b_{13.2}$ in equation (vii), we have

$20b_{12.3} - 20 = 20 \Rightarrow b_{12.3} = 2$ ..... (xi)

Substituting the values of $b_{12.3}$ and $b_{13.2}$ in equation (i), we get

$6a_{1.23} + 48 - 28 = 20 \Rightarrow a_{1.23} = 0$

Substituting the values of $a_{1.23} = 0$, $b_{12.3} = 2$ and $b_{13.2} = -1$ in equation (A)

The required regression equation of X_1 *on* X_2 *and* X_3 is $X_1 = 0 + 2X_2 - X_3 \Rightarrow X_1 = 2X_2 - X_3$.

TUTORIAL QUESTIONS

1. The heights of mothers & daughters are given in the following table. From the 2 tables of regression estimate the expected average height of daughter when the height of the mother is 64.5 inches.

Ht. of Mother(inches)	62	63	64	64	65	66	68	70
Ht. of the daughter(inches)	64	65	61	69	67	68	71	65

2. The equations of two regression lines are $7x - 16y + 9 = 0$ and $5y - 4x - 3 = 0$.
Find the coefficient of correlation and the means of x& y .

3. The marks obtained by 10 students in mathematics and statistics are given below. Find the coefficient of correlation between the two subjects and the two lines of regression

Marks in mathematics	25	28	30	32	35	36	38	42	45	39
Marks in Statistics	20	26	29	30	25	18	26	35	46	35

4. Fit a straight line $Y=a_0+a_1X$ for the following data and estimate the value of Y when X=25

X	0	5	10	15	20
Y	7	11	16	20	26

5. Find the rank correlation for the following indices of supply and price of an article:

PRICE	80	100	102	91	100	111	109	100	99	104	111	102	98	111
INDEX	124	100	105	112	102	93	99	115	123	104	99	113	121	103

ASSIGNMENT QUESTIONS

1.Fit a curve of the form Y= a +bX by the method of least squares for the following data:

X	1	2	3	4	5
Y	5	2	4.5	8	12.5

\

2.The marks obtained by 10 students in two subjects are given below. Find the correlation coefficient and lines of regression

Subject 1	48	75	30	60	80	53	35	15	40	38
Subject 2	44	85	45	54	91	58	63	35	43	45

3. The following table are the marks obtained by 12 students in economics and statistics:

Economics(X)	78	56	36	66	25	62	75	82
62Statistics(Y)	84	44	51	58	60	58	68	62

Obtain the regression lines.

4.Find the karlpearson's coefficient of correlation for the paired data:

wages	100	101	102	100	99	97	98	96	95	102
Cost of living	98	99	99	95	92	95	94	90	91	97

5. The equations of two regression lines are $7x - 16y + 9 = 0$ and $5y - 4x - 3 = 0$.
Find the coefficient of correlation and the means of x& y .

UNIT –IV

TESTING OF HYPOTHESIS

Introduction: The totality of observations with which we are concerned, whether this number be finite or infinite constitute population. In this chapter we focus on sampling from distributions or populations and such important quantities as the sample mean and sample variance.

Def: Population is defined as the aggregate or totality of statistical data forming a subject of investigation .

EX. The population of the heights of Indian.

The number of observations in the population is defined to be the size of the population. It may be finite or infinite .Size of the population is denoted by N.As the study of entire population may not be possible to carry out and hence a part of the population alone is selected.

Def: A portion of the population which is examined with a view to determining the population characteristics is called a sample . In other words, sample is a subset of population. Size of the sample is denoted by n.

The process of selection of a sample is called Sampling. There are different methods of sampling

- Probability Sampling Methods
- Non-Probability Sampling Methods

Probability Sampling Methods:

a) Random Sampling (Probability Sampling):

It is the process of drawing a sample from a population in such a way that each member of the population has an equal chance of being included in the sample.

Ex: A hand of cards from a well shuffled pack of cards is a random sample.

Note : If N is the size of the population and n is the size of the sample, then

- The no. of samples with replacement = N^n
- The no. of samples without replacement = N_{C_n}

b) Stratified Sampling :

In this , the population is first divided into several smaller groups called strata according to some relevant characteristics . From each strata samples are selected at random, all the samples are combined together to form the stratified sampling.

c) Systematic Sampling (Quasi Random Sampling):

In this method , all the units of the population are arranged in some order . If the population size is N, and the sample size is n, then we first define sample interval

denoted by $= \frac{N}{n}$. then from first k items ,one unit is selected at random. Then from first unit every k^{th} unit is serially selected combining all the selected units constitute a systematic sampling.

Non Probability Sampling Methods:

a) **Purposive (Judgment) Sampling :**
In this method, the members constituting the sample are chosen not according to some definite scientific procedure , but according to convenience and personal choice of the individual who selects the sample . It is the choice of the individual items of a sample entirely depends on the individual judgment of the investigator.

b) **Sequential Sampling:**
It consists of a sequence of sample drawn one after another from the population. Depending on the results of previous samples if the result of the first sample is not acceptable then second sample is drawn and the process continues to take proper decision . But if the first sample is acceptable ,then no new sample is drawn.

Classification of Samples:

- **Large Samples :** If the size of the sample n $\geq$ 30 , then it is said to be large sample.
- **Small Samples :** If the size of the sample n $<$ 30 ,then it is said to be small sample or exact sample.

Parameters and Statistics:

Parameter is a statistical measure based on all the units of a population. Statistic is a statistical measure based on only the units selected in a sample.
Note :In this unit , Parameter refers to the population and Statistic refers to sample.

Central Limit Theorem: If $\bar{x}$ be the mean of a random sample of size n drawn from population having mean μ and standard deviation σ , then the sampling distribution of the sample mean $\bar{x}$ is approximately a normal distribution with mean μ and SD = S.E of $\bar{x} = \frac{\sigma}{\sqrt{n}}$ provided the sample size n is large.

Standard Error of a Statistic : The standard error of statistic 't' is the standard deviation of the sampling distribution of the statistic i.e, S.E of sample mean is the standard deviation of the sampling distribution of sample mean.

Formulae for S.E:

- S.E of Sample mean $\bar{x} = \frac{\sigma}{\sqrt{n}}$ i.e, S.E $(\bar{x}) = \frac{\sigma}{\sqrt{n}}$
- S.E of sample proportion $p=\sqrt{\frac{PQ}{n}}$ i.e, S.E (p) = $\sqrt{\frac{PQ}{n}}$ where Q=1-P
- S.E of the difference of two sample means $\bar{x}_1$ and $\bar{x}_2$ i.e, S.E $(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{{\sigma_1}^2}{n_1} + \frac{{\sigma_2}^2}{n_2}}$
- S.E of the difference of two proportions i.e, S.E$(p_1 - p_2) = \sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$

Estimation :

To use the statistic obtained by the samples as an estimate to predict the unknown parameter of the population from which the sample is drawn.

Estimate : An estimate is a statement made to find an unknown population parameter.

Estimator : The procedure or rule to determine an unknown population parameter is called estimator.

Ex. Sample proportion is an estimate of population proportion , because with the help of sample proportion value we can estimate the population proportion value.

Types of Estimation:

- **Point Estimation:** If the estimate of the population parameter is given by a single value , then the estimate is called a point estimation of the parameter.
- **Interval Estimation:** If the estimate of the population parameter is given by two different values between which the parameter may be considered to lie, then the estimate is called an interval estimation of the parameter.

Confidence interval Estimation of parameters:

In an interval estimation of the population parameter θ, if we can find two quantities t_1 and t_2 based on sample observations drawn from the population such that the unknown parameter θ is included in the interval $[t_1,t_2]$ in a specified cases ,then this is called a confidence interval for the parameter θ.

Confidence Limits for Population mean μ

- 95% confidence limits are $\bar{x} \pm 1.96\ (S.E.of\ \bar{x})$
- 99% confidence limits are $\bar{x} \pm 2.58\ (S.E.of\ \bar{x})$
- 99.73% confidence limits are $\bar{x} \pm 3\ (S.E.of\ \bar{x})$
- 90% confidence limits are $\bar{x} \pm 1.645\ (S.E.of\ \bar{x})$

Confidence limits for population proportion P

- 95% confidence limits are $p \pm 1.96$(S.E.of p)
- 99% confidence limits are $p \pm 2.58$(S.E. of p)
- 99.73% confidence limits are $p \pm 3$(S.E.of p)
- 90% confidence limits are $p \pm 1.645$(S.E.of p)

Confidence limits for the difference of two population means μ_1 and μ_2

- 95% confidence limits are $(\bar{x}_1 - \bar{x}_2) \pm 1.96$ (S.E of $(\bar{x}_1 - \bar{x}_2)$)
- 99% confidence limits are $(\bar{x}_1 - \bar{x}_2) \pm 2.58$ (S.E of $(\bar{x}_1 - \bar{x}_2)$)
- 99.73% confidence limits are $(\bar{x}_1 - \bar{x}_2) \pm 3$ (S.E of $(\bar{x}_1 - \bar{x}_2)$)
- 90% confidence limits are $(\bar{x}_1 - \bar{x}_2) \pm 2.58$ (S.E of $(\bar{x}_1 - \bar{x}_2)$)

Confidence limits for the difference of two population proportions

- 95% confidence limits are p_1-$p_2 \pm 1.96$ (S.E. of p_1-p_2)
- 99% confidence limits are p_1-$p_2 \pm 2.58$ (S.E. of p_1-p_2)
- 99.73% confidence limits are p_1-$p_2 \pm 3$ (S.E. of p_1-p_2)
- 90% confidence limits are p_1-$p_2 \pm 1.645$ (S.E. of p_1-p_2)

Determination of proper sample size

Sample size for estimating population mean :

$n = \left(\frac{z_\alpha \sigma}{E}\right)^2$ where z_α – Critical value of z at α Level of significance

σ – Standard deviation of population and

E – Maximum sampling Error = $\bar{x} - \mu$

Sample size for estimating population proportion :

$n = \frac{z_\alpha{}^2 PQ}{E^2}$ where z_α – Critical value of z at α Level of significance

P – Population proportion

Q – 1-P

E – Maximum Sampling error = p-P

Testing of Hypothesis :

It is an assumption or supposition and the decision making procedure about the assumption whether to accept or reject is called hypothesis testing .

Def: Statistical Hypothesis : To arrive at decision about the population on the basis of sample information we make assumptions about the population parameters involved such assumption is called a statistical hypothesis .

Procedure for testing a hypothesis:

Test of Hypothesis involves the following steps:

Step1: Statement of hypothesis :

There are two types of hypothesis :

- **Null hypothesis:** A definite statement about the population parameter. Usually a null hypothesis is written as no difference , denoted by H_0.
  Ex. H_0: $\mu = \mu_0$
- **Alternative hypothesis :** A statement which contradicts the null hypothesis is called alternative hypothesis. Usually an alternative hypothesis is written as some difference , denoted by H_1.
  Setting of alternative hypothesis is very important to decide whether it is two-tailed or one – tailed alternative , which depends upon the question it is dealing.
  Ex.H_1: $\mu \neq \mu_0$ (Two – Tailed test)
  or
  H_1: $\mu > \mu_0$ (Right one tailed test)
  or
  H_1: $\mu < \mu_0$ (Left one tailed test)

Step 2: Specification of level of significance :

The LOS denoted by α is the confidence with which we reject or accept the null hypothesis. It is generally specified before a test procedure ,which can be either 5% (0.05) , 1% or 10% which means that thee are about 5 chances in 100 that we would reject the null hypothesis H_0 and the remaining 95% confident that we would accept the null hypothesis H_0 . Similarly , it is applicable for different level of significance.

Step 3 : Identification of the test Statistic :

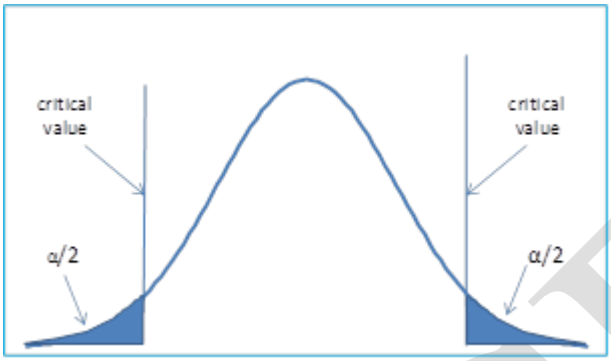
There are several tests of significance like z,t, F etc .Depending upon the nature of the information given in the problem we have to select the right test and construct the test criterion and appropriate probability distribution.

Step 4: Critical Region:

It is the distribution of the statistic .

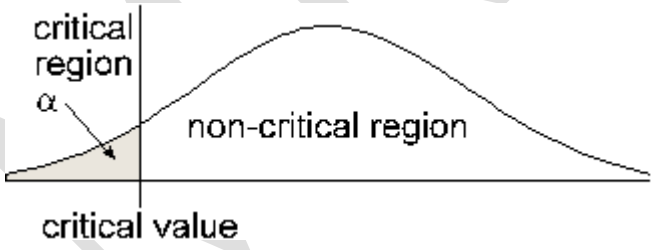
- **Two – Tailed Test : The critical region under the curve is equally distributed on both sides of the mean.**

  If H_1 has $\neq$ sign , the critical region is divided equally on both sides of the distribution.

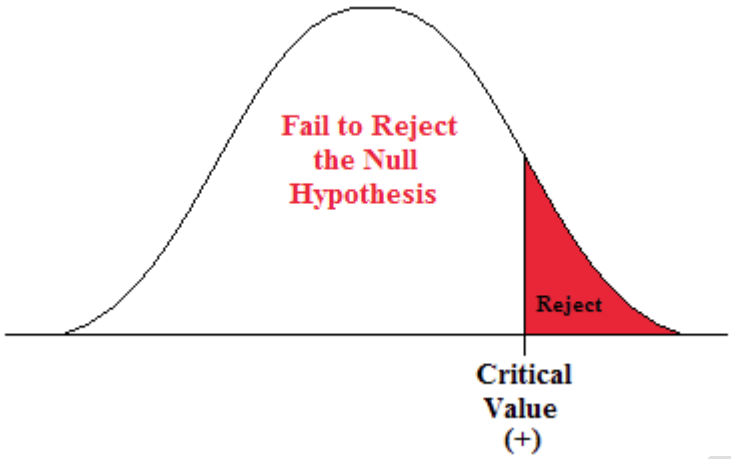


- **One Tailed Test: The critical region under the curve is distributed on one side of the mean.**

Left one tailed test: If H_1 has $<$ sign , the critical region is taken in the left side of the distribution.



Right one tailed test : If H_1 has $>$ sign , the critical region is taken on right side of the distribution.



Step 5 : Making decision:

By comparing the computed value and the critical value decision is taken for accepting or rejecting H_0

If calculated value $\leq$ critical value , we accept H_0, otherwise reject H_0.

Errors of Sampling :

While drawing conclusions for population parameters on the basis of the sample results , we have two types of errors.

- **Type I error :** Reject H_0 when it is true i.e, if the null hypothesis H_0 is true but it is rejected by test procedure .
- **Type II error :** Accept H_0 when it is false i.e, if the null hypothesis $H_0$is false but it is accepted by test procedure.

DECISION TABLE

	H_0 **is accepted**	H_0 **is rejected**
H_0 **is true**	**Correct Decision**	**Type I Error**
H_0 **is false**	**Type II Error**	**Correct Decision**

Problems:

1. **If the population is 3,6,9,15,27**

   a) **List all possible samples of size 3 that can be taken without replacement from finite population**
   b) **Calculate the mean of each of the sampling distribution of means**

c) Find the standard deviation of sampling distribution of means

Sol: Mean of the population , $\mu = \frac{3+6+9+15+27}{5} = \frac{60}{5} = 12$

Standard deviation of the population ,

$$\sigma = \sqrt{\frac{(3-12)^2 + (6-12)^2 + (9-12)^2 + (15-12)^2 + (27-12)^2}{5}}$$

$$= \sqrt{\frac{81+36+9+9+225}{5}} = \sqrt{\frac{360}{5}} = 8.4853$$

a) Sampling without replacement :
The total number of samples without replacement is $N_{C_n} = 5_{C_3} = 10$
The 10 samples are (3,6,9), (3,6,15), (3,9,15), (3,6,27), (3,9,27), (3,15,27), (6,9,15), (6,9,27), (6,15,27), (9,15,27)

b) Mean of the sampling distribution of means is
$\mu_x = \frac{6+8+9+10+12+13+14+15+16+17}{10} = \frac{120}{10} = 12$

c) $\sigma^2 = \frac{(6-12)^2+(8-12)^2+(9-12)^2+(10-12)^2+(12-12)^2+(13-12)^2+(14-12)^2+(15-12)^2+(16-12)^2+(17-12)^2}{10}$

$= 13.3$

$\therefore \sigma_{\bar{x}} = \sqrt{13.3} = 3.651$

2. **A population consist of five numbers 2,3,6,8 and 11. Consider all possible samples of size two which can be drawn with replacement from this population .Find**

a) The mean of the population
b) The standard deviation of the population
c) The mean of the sampling distribution of means and
d) The standard deviation of the sampling distribution of means

Sol: a) Mean of the Population is given by

$$\mu = \frac{2+3+6+8+11}{5} = \frac{30}{5} = 6$$

b) Variance of the population is given by

$$\sigma^2 = \sum \frac{(x_i - \bar{x})^2}{n}$$

$$= \frac{(2-6)^2+(3-6)^2+(6-6)^2+(8-6)^2+(11-6)^2}{5}$$

$$= \frac{16+9+0+4+25}{5} = 10.8 \qquad \therefore \sigma = 3.29$$

c) Sampling with replacement

The total no.of samples with replacement is $N^n = 5^2 = 25$

∴ List of all possible samples with replacement are

$$(2,2),(2,3),(2,6),(2,8),(2,11),(3,2),(3,3)(3,6),(3,8),(3,11)$$
$$\{(6,2),(6,3),(6,6),(6,8),(6,11),(8,2),(8,3),(8,6),(8,8),(8,11)\}$$
$$(11,2),(11,3),(11,6),(11,8),(11,11)$$

Now compute the arithmetic mean for each of these 25 samples which gives rise to the distribution of means of the samples known as sampling distribution of means

The samples means are

2 , 2.5 , 4 , 5 ,6.5
2.5 , 3 ,4.5, ,5.5,7
4,4.5,6,7,8.5
5,5.5,7,8,9.5
{ 6.5,7,8.5,9.5,11 }

And the mean of sampling distribution of means is the mean of these 25 means

$\mu_x = \frac{\text{sum of all above sample means}}{25} = \frac{150}{25} = 6$

d) The variance of the sampling distribution of means is obtained by subtracting the mean 6 from each number in sampling distribution of means and squaring the result ,adding all 25 numbers thus obtained and dividing by 25.

$\sigma^2 = \frac{(2-6)^2+(2.5-6)^2+(4-6)^2+(5-6)^2+\cdots\ldots(11-6)^2}{25} = \frac{135}{25} = 5.4$

$\therefore \sigma = \sqrt{5.4} = 2.32$

3. When a sample is taken from an infinite population , what happens to the standard error of the mean if the sample size is decreased from 800 to 200

Sol: The standard error of mean = $\frac{\sigma}{\sqrt{n}}$

Sample size = n .let n= n_1=800

Then S.E$_1$ = $\frac{\sigma}{\sqrt{800}} = \frac{\sigma}{20\sqrt{2}}$

When $n_1$is reduced to 200

let n= n_2=200

Then S.E$_2$ = $\frac{\sigma}{\sqrt{200}} = \frac{\sigma}{10\sqrt{2}}$

∴ S.E$_2$ = $\frac{\sigma}{10\sqrt{2}} = 2(\frac{\sigma}{20\sqrt{2}}) = 2\ (S.E_1)$

Hence if sample size is reduced from 800to 200, S. E. of mean will be multiplied by 2

4.The variance of a population is 2 . The size of the sample collected from the population is 169. What is the standard error of mean

Sol: n= The size of the sample =169

σ = S.D of population = $\sqrt{\text{Variance}}$ = $\sqrt{2}$

Standard Error of mean = $\frac{\sigma}{\sqrt{n}} = \frac{\sqrt{2}}{\sqrt{169}} = \frac{1.41}{13} = 0.185$

5.The mean height of students in a college is 155cms and standard deviation is 15 . What is the probability that the mean height of 36 students is less than 157 cms.

Sol: μ = Mean of the population

= Mean height of students of a college = 155cms

n = S.D of population = 15cms

$\bar{x}$ = mean of sample = 157 cms

Now z = $\frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{157-155}{\frac{15}{\sqrt{36}}} = \frac{12}{15} = 0.8$

$\therefore$ P ($\bar{x} \leq 157$) = P ($z < 0.8$) = 0.5 + P ($0 \leq z \leq 0.8$)

= 0.5 +0.2881 = 0.7881

Thus the probability that the mean height of 36 students is less than 157 = 0.7881

6.A random sample of size 100 is taken from a population with σ = 5.1 . Given that the sample mean is $\bar{x}$ = 21.6 Construct a 95% confidence limits for the population mean .

Sol: Given $\bar{x}$ = 21.6

$z_{\alpha/2}$ = 1.96, n = 100 , σ = 5.1

$\therefore$ Confidence interval = ($\bar{x} - z_{\alpha/2}.\frac{\sigma}{\sqrt{n}}$, $\bar{x} + z_{\alpha/2}.\frac{\sigma}{\sqrt{n}}$)

$\bar{x} - z_{\alpha/2}.\frac{\sigma}{\sqrt{n}} = 21.6 - \frac{1.96 \times 5.1}{10} = 20.6$

$\bar{x} + z_{\alpha/2}.\frac{\sigma}{\sqrt{n}} = 21.6 + \frac{1.96 \times 5.1}{10} = 22.6$

Hence (20.6,22.6) is the confidence interval for the population mean μ

7.It is desired to estimate the mean time of continuous use until an answering machine will first require service . If it can be assumed that σ = 60 days, how large a sample is

needed so that one will be able to assert with 90% confidence that the sample mean is off by at most 10 days.

Sol: We have maximum error (E) = 10 days , σ = 60 days and $z_{\alpha/2}$ = 1.645

$$\therefore n = \left[\frac{z_{\alpha/2}.\sigma}{E}\right]^2 = \left[\frac{1.645 \times 60}{10}\right]^2 = 97$$

8. A random sample of size 64 is taken from a normal population with $\mu = 51.4$ and σ = 6.8.What is the probability that the mean of the sample will a) exceed 52.9 b) fall between 50.5 and 52.3 c) be less than 50.6

Sol: Given n = the size of the sample = 64

μ = the mean of the population = 51.4

σ = the S.D of the population = 6.8

a) P($\bar{x}$ exceed 52.9) = P ($\bar{x} > 52.9$)

$$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{52.9-51.4}{\frac{6.8}{\sqrt{64}}} = 1.76$$

$\therefore$ P ($\bar{x} > 52.9$) = P($z > 1.76$)

= 0.5 – P(0 < z < 1.76)

= 0.5 – 0.4608 = 0.0392

b) P($\bar{x}$ fall between 50.5 and 52.3)

i.e, P(50.5 < $\bar{x}$ < 52.3) = P ($\bar{x_1} < \bar{x} < \bar{x_2}$)

$$z_1 = \frac{\bar{x_1}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{50.5-51.4}{0.85} = -1.06$$

$$z_2 = \frac{\bar{x_2}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{52.3-51.4}{0.85} = 1.06$$

P(50.5 < $\bar{x}$ < 52.3) = P(-1.06 < z < 1.06)

= P(-1.06 < z < 0) + P(0 < z < 1.06)

= P(0 < z < 1.06) + P(0 < z < 1.06)

= 2(0.3554) = 0.7108

c) P($\bar{x}$ will be less than 50.6) = P($\bar{x} < 50.6$)

$$Z = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{50.6-51.4}{\frac{6.8}{\sqrt{64}}} = -0.94$$

$\therefore$P(z < -0.94) = 0.5 - P(0.94 < z < 0)

= 0.5 - P(0 < z < 0.94) = 0.50-0.3264

= 0.1736

9.The mean of certain normal population is equal to the standard error of the mean of the samples of 64 from that distribution . Find the probability that the mean of the sample size 36 will be negative.

Sol: The Standard error of mean = $\frac{\sigma}{\sqrt{n}}$

Sample size , n =64

Given mean , μ = Standard error of the mean of the samples

$\mu = \frac{\sigma}{\sqrt{64}} = \frac{\sigma}{8}$

We know z = $\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{x} - \frac{\sigma}{8}}{\frac{\sigma}{6}}$

$= \frac{6\bar{x}}{\sigma} - \frac{3}{4}$

If Z < 0.75, $\bar{x}$ is negative

P(z < 0.75) = P($-\infty < z < 0.75$)

$= \int_{-\infty}^{0} \emptyset(z)\, dz + \int_{0}^{0.75} \emptyset(z) dz = 0.50 + 0.2734$

= 0.7734

10.The guaranteed average life of a certain type of electric bulbs is 1500hrs with a S.D of 10 hrs. It is decided to sample the output so as to ensure that 95% of bulbs do not fall short of the guaranteed average by more than 2% . What will be the minimum sample size ?

Sol : Let n be the size of the sample

The guaranteed mean is 1500

We do not want the mean of the sample to be less than 2% of (1500) i.e, 30 hrs

So 1500 – 30 = 1470

$\therefore \bar{x} > 1470$

$\therefore |z| = \left|\frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}\right| = \left|\frac{1470 - 1500}{\frac{120}{\sqrt{n}}}\right| = \frac{\sqrt{n}}{4}$

From the given condition , the area of the probability normal curve to the left of $\frac{\sqrt{n}}{4}$ should be 0.95

$\therefore$ The area between 0 and $\frac{\sqrt{n}}{4}$ is 0.45

We do not want to know about the bulbs which have life above the guranteed life .

$\therefore \frac{\sqrt{n}}{4} = 1.65$ i.e., $\sqrt{n} = 6.6$

$\therefore$ n = 44

11. A normal population has a mean of 0.1 and standard deviation of 2.1 . Find the probability that mean of a sample of size 900 will be negative .

Sol : Given $\mu = 0.1$, $\sigma = 2.1$ and n = 900

The Standard normal variate

$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\bar{x} - \mu}{\frac{2.1}{\sqrt{900}}} = \frac{\bar{x} - 0.1}{0.07}$

$\therefore \bar{x} = 0.1 + 0.007\ z$ where $z \sim N(0, 1)$

$\therefore$ The required probability , that the sample mean is negative is given by

$P(\bar{x} < 0) = P(0.1 + 0.07\ z < 0)$

$= P(0.07\ z < -0.1)$

$= P\left(z < \left(\frac{-0.1}{0.07}\right)\right)$

$= P(z < -1.43)$

$= 0.50 - P(0 < z < 1.43)$

$= 0.50 - 0.4236 = 0.0764$

12. In a study of an automobile insurance a random sample of 80 body repair costs had a mean of Rs 472.36 and the S.D of Rs 62.35. If $\bar{x}$ is used as a point estimator to the true average repair costs , with what confidence we can assert that the maximum error doesn't exceed Rs 10.

Sol : Size of a random sample , n = 80

The mean of random sample , $\bar{x}$ = Rs 472.36

Standard deviation , σ = Rs 62.35

Maximum error of estimate , E_{max} = Rs 10

We have $E_{max} = Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

i.e., $Z_{\alpha/2} = \frac{E_{max} \cdot \sqrt{n}}{\sigma} = \frac{10\sqrt{80}}{62.35} = \frac{89.4427}{62.35} = 1.4345$

$\therefore Z_{\alpha/2} = 1.43$

The area when $z = 1.43$ from tables is 0.4236

$\therefore \frac{\alpha}{2} = 0.4236$ i.e ., $\alpha = 0.8472$

$\therefore$ confidence = $(1 - \alpha)\ 100\% = 84.72\ \%$

Hence we are 84.72% confidence that the maximum error is Rs. 10

13. If we can assert with 95% that the maximum error is 0.05 and P = 0.2 find the size of the sample.

Sol : Given P =0.2 , E = 0.05

We have Q = 0.8 and $Z_{\alpha/2} = 1.96$ (5% LOS)

We know that maximum error , E = $Z_{\alpha/2}\ \sqrt{\frac{PQ}{n}}$

$\Rightarrow 0.05 = 1.96\sqrt{\frac{0.2 \times 0.8}{n}}$

$\Rightarrow$ Sample size , $n = \frac{0.2 \times 0.8 \times (1.96)^2}{(0.05)^2} = 246$

14. The mean and standard deviation of a population are 11,795 and 14,054 respectively . What can one assert with 95 % confidence about the maximum error if $\bar{x}$ = 11,795 and n = 50. And also construct 95% confidence interval for true mean .

Sol: Here mean of population , μ = 11795

S.D of population , σ = 14054

$\bar{x}$ = 11795

$n =$ sample size = 50 , maximum error = $Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$

$Z_{\alpha/2}$ for 95% confidence = 1.96

Max. error , $E = Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} = 1.96 \cdot \frac{14054}{\sqrt{50}} = 3899$

$\therefore$ Confidence interval = $(\bar{x} - Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}} , \bar{x} + Z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}})$

= (11795-3899, 11795+3899)

= (7896, 15694)

15. Find 95% confidence limits for the mean of a normally distributed population from which the following sample was taken 15, 17 , 10 ,18 ,16 ,9, 7, 11, 13 ,14.

Sol: We have $\bar{x} = \frac{15+17+10+18+16+9+7+11+13+14}{10} = 13$

$S^2 = \sum \frac{(x_i - \bar{x})^2}{n-1}$

$= \frac{1}{9}\ [(15-13)^2 + (15-13)^2 + (15-13)^2 + (15-13)^2 + (15-13)^2 + (15-13)^2 + (15-13)^2 + (15-13)^2 + (15-13)^2 + (15-13)^2]$

$= \frac{40}{3}$

Since $Z_{\alpha/2}$ = 1.96 , we have

$Z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 1.96 \cdot \frac{\sqrt{40}}{\sqrt{10}.\sqrt{3}} = 2.26$

$\therefore$ Confidence limits are $\bar{x} \pm Z_{\alpha/2} \cdot \frac{s}{\sqrt{n}} = 13 \pm 2.26 = (10.74 , 15.26)$

16. A random sample of 100 teachers in a large metropolitan area revealed mean weekly salary of Rs. 487 with a standard deviation Rs.48. With what degree of confidence can

we assert that the average weekly of all teachers in the metropolitan area is between 472 to 502 ?

Sol: Given $\mu = 487$, $\sigma = 48$, $n = 100$

$$Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$= \frac{\bar{x} - 487}{\frac{48}{\sqrt{100}}} = \frac{\bar{x} - 487}{4.8}$$

Standard variable corresponding to Rs. 472 is

$$Z_1 = \frac{472 - 487}{4.8} = -3.125$$

Standard vaiable corresponding to Rs. 502

$$Z_2 = \frac{502 - 487}{4.8} = 3.125$$

Let $\bar{x}$ be the mean salary of teacher . Then

$P(472 < \bar{x} < 502) = P(-3.125 < z < 3.125)$

$= 2(0 < z < 3.125)$

$= 2\int_0^{3.125} \emptyset(z)dz$

$= 2(0.4991) = 0.9982$

Thus we can ascertain with 99.82 % confiden

Large Samples: Let a random sample of size n >30 is defined as large sample.

Applications of Large Samples

Test of Significance of a Single Mean

Let a random sample of size n, $\bar{x}$ be the mean of the sample and μ be the population mean.

1. **Null hypothesis**: H_0: There is no significant difference in the given population mean value say 'μ_0'.

   i.e $H_0: \mu = \mu_0$

2. **Alternative hypothesis**: H_1 :There is some significant difference in the given population mean value.

i.e

a) $H_1 : \mu \neq \mu_0$ (Two –tailed)

b) $H_1 : \mu > \mu_0$ (Right one tailed)

c) $H_1 : \mu > \mu_0$ (Left one tailed)

3. **Level of significance**: Set the LOS α
4. **Test Statistic**: $z_{cal} = \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}$ (OR) $z_{cal} = \frac{\bar{x}-\mu_0}{s/\sqrt{n}}$
5. **Decision /conclusion** : If z_{cal}value $<$ $z_{\propto}$value , accept H_0 otherwise reject H_0

CRITICAL VALUES OF Z

LOS ∝	1%	5%	10%
$\mu \neq \mu_0$	/Z/>2.58	/Z/>1.96	/Z/>1.645
$\mu > \mu_0$	Z>2.33	z>1.645	Z>1.28
$\mu < \mu_0$	Z<-2.33	Z<-1.645	Z<-1.28

NOTE: Confidence limits for the mean of the population corresponding to the given sample.

$\mu = \bar{X} \pm Z_{\propto/2}$ (S.E of $\bar{X}$) i.e,

$\mu = \bar{X} \pm Z_{\propto/2} (\frac{\sigma}{\sqrt{n}})$ (or) $\mu = \bar{X} \pm Z_{\propto/2} (\frac{\varepsilon}{\sqrt{n}})$

2. Test of Significance for Difference of Means of two Large Samples

Let $\bar{x}_1 \& \bar{x}_2$ be the means of the samples of two ramdom sizes n_1 & n_2 drawn from two populations having means $\mu_1 \& \mu_2$ and SD's $\sigma_1 \& \sigma_2$

i) **Null hyopothesis**: $H_0: \mu_1 = \mu_2$

ii) **Alternative hypothesis** :. a) $H_1 : \mu_1 \neq \mu_2$(Two Tailed)

b) H_1: $\mu_1 < \mu_2$ (Left one tailed)

c) H_1: $\mu_1 > \mu_2$ (Right one tailed)

iii) **Level of Significance**: Set the LOS α

iv) **Test Statistic** : $Z_{cal} = \frac{(\bar{x}_1-\bar{x}_2)-\delta}{SE\ of(\bar{x}_1-\bar{x}_2)} = -\frac{(\bar{x}_1-\bar{x}_2)-\delta}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$

Where $\delta = \mu_1 - \mu_2$(where given constant)

Other wise $\delta = \mu_1 - \mu_2$=0

$Z_{cal} = \frac{\bar{x}_1-\bar{x}_2-\delta}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$ if $\sigma_1^2 = \sigma_2^2 = \sigma^2$ then $Z_{cal} = \frac{\bar{X}_1-\bar{X}_2}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$

Critical value of Z from normal table at the LOS α

v) **Decision:** If $|Z_{cal}| < Z_{tab}$, accept H_0 otherwise reject H_0

CRITICAL VALUES OF Z

LOS ∝	1%	5%	10%
μ≠ μ0	/Z/>2.58	/Z/>1.96	/Z/>1.645
μ> μ0	Z>2.33	z>1.645	Z>1.28
μ< μ0	Z<-2.33	Z<-1.645	Z<-1.28

NOTE: Confidence limits for difference of means

$$\mu_1 - \mu_2 = (\bar{X}_1 - \bar{X}_2) \pm z_{\propto/2}[S.E\,of\,(\bar{X}_1 - \bar{X}_2)]$$

$$= (\bar{X}_1 - \bar{X}_2) \pm z_{\propto/2}\left[\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_1}}\right]$$

3. Test of Significance for Single Proportions

Suppose a random sample of size n has a sample proportion p of members possessing a certain attribute (proportion of successes). To test the hypothesis that the proportion P in the population has a specified value P_0.

i) **Null hyopothesis** : $H_0: P = P_0$

ii) **Alternative hypothesis** : a) H_1 : P≠ P_0(Two Tailed test)

b) H_1 : $P < P_0$ (Left one- tailed)

c) H_1 : P $> P_0$ (Right one tailed)

iii) **Test statistic** $Z_{cal} = \frac{p-P}{\sqrt{\frac{PQ}{n}}}$ when P is the Population proportion $Q = 1 - P$

iv) At specified LOS ∝, critical value of Z

v) **Decision**: If $|z_{cal}| < Z_{tab}$, accept H_0 otherwise reject H_0

CRITICAL VALUES OF Z

LOS ∝	1%	5%	10%
μ≠ μ0	/Z/>2.58	/Z/>1.96	/Z/>1.645
μ> μ0	Z>2.33	z>1.645	Z>1.28
μ< μ0	Z<-2.33	Z<-1.645	Z<-1.28

NOTE : Confidence limits for population proportion

$$P = P \pm Z_{\frac{\alpha}{2}}(\text{S E of P})$$

$$= P \pm Z_{\frac{\alpha}{2}} \left(\sqrt{\frac{pq}{n}}\right)$$

4. Test for Equality of Two Proportions (Populations)

Let p_1 and p_2 be the sample proportions in two large random samples of sizes n_1 & n_2 drawn from two populations having proportions P_1 & P_2

i) **Null hyopothesis** : $H_0: P_1 = P_2$

ii) **Alternative hypothesis** : a) $H_1 : P_1 \neq P_2$(Two Tailed)

b) $H_1 : P_1 < P_2$ (Left one tailed)

c) $H_1 : P_1 > P_2$ (Right one tailed)

iii) **Test statistic** $Z_{cal} = \frac{(P_1 - P_2) - (P_1 - P_2)}{\sqrt{\frac{P_1 Q_1}{n_1} + \frac{P_1 Q_1}{n_2}}}$ if $(P_1 - P_2)$ is given.

If given only sample proportions then

$$Z_{cal} = \frac{p_1 - p_2}{\sqrt{\frac{P_1 q_1}{n_1} + \frac{P_2 q_1}{n_2}}} \quad \text{where } p_1 = \frac{x_1}{n_1} \text{ \& } p_1 = \frac{x_2}{n_2}$$

OR

$$Z_{cal} = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \quad \text{Where } p = \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{x1 + x2}{n_1 + n_2} \text{ and } q = 1 - p$$

iv) At specified LOS $\propto$ critical value of 'Z'

v) **Decision**: If $|Z_{cal}| < Z_{Tab}$, accept H_0 otherwise reject H_0

CRITICAL VALUES OF Z

LOS $\propto$	1%	5%	10%
$\mu \neq \mu_0$	/Z/>2.58	/Z/>1.96	/Z/>1.645
$\mu > \mu_0$	Z>2.33	z>1.645	Z>1.28
$\mu < \mu_0$	Z<-2.33	Z<-1.645	Z<-1.28

NOTE: Confidence limits for difference of population proportions

$$P_1 - P_2 = (p_1 - p_2) \pm Z_{\frac{\alpha}{2}} (S.E\ of\ P_1 - P_2)$$

Problems:

1. **A sample of 64 students have a mean weight of 70 kgs . Can this be regarded as asample mean from a population with mean weight 56 kgs and standard deviation 25 kgs.**
   Sol : Given $\bar{x}$= mean of he sample = 70 kgs
   μ = Mean of the population = 56 kgs
   σ = S.D of population = 25 kgs

   and n = Sample size = 64

   i) **Sol**: Null Hypothesis H_0 : A Sample of 64 students with mean weight 70 kgs be regarded as a sample from a population with mean weight 56 kgs and standard deviation 25 kgs. i.e., $H_0 : \mu =$ 70 kgs

   ii) Alternative Hypothesis H_1 : Sample cannot be regarded as one coming from the population . i.e., $H_1 : \mu \neq$ 70 kgs (Two –tailed test)

   iii) Level of significance : $\alpha = 0.05$ (Z_α = 1.96)

   iv) Test Statistic : $Z_{cal} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{70-56}{\frac{25}{\sqrt{64}}} = 4.48$

   v) Conclusion: Since $|Z_{cal}|\ value > Z_\alpha$ value , we reject H_0
   ∴ Sample cannot be regarded as one coming from the population

2. **In a random sample of 60 workers , the average time taken by them to get to work is 33.8 minutes with a standard deviation of 6.1 minutes . Can we reject the null hypothesis μ = 32.6 in favor of alternative null hypothesis μ > 32.6 at α = 0.05 LOS**
   Sol : Given n = 60 , $\bar{x}$ = 33.8 , μ = 32.6 and σ = 6.1

   i) Null Hypothesis $H_0 : \mu$ = 32.6

   ii) Alternative Hypothesis $H_1 : \mu$ > 32.6 (Right one tailed test)

   iii) Level of significance : $\alpha = 0.01$ (Z_α = 2.33)

   iv) Test Statistic : $Z_{cal} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{33.8-32.6}{\frac{6.1}{\sqrt{60}}} = \frac{1.2}{0.7875} = 1.5238$

   v) Conclusion: Since $Z_{cal}\ value < Z_\alpha$ value , we accept H_0

3. **A sample of 400 items is taken from a population whose standard deviation is 10 . The mean of the sample is 40 . Test whether the sample has come from a population with mean 38 . Also calculate 95% confidence limits for the population .**

Sol : Given n = 400 , $\bar{x}$ = 40 , μ = 38 and σ = 10

i) Null Hypothesis $H_0 : \mu = 38$

ii) Alternative Hypothesis $H_1 : \mu \neq 38$ (Two –tailed test)

iii) Level of significance : $\alpha = 0.05$ (Z_α = 1.96)

iv) Test Statistic : $Z_{cal} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{38-40}{\frac{10}{\sqrt{400}}} = \frac{-2}{0.5} = -4$

v) Conclusion: Since $|Z_{cal}|\ v\_lue > Z_\alpha$ value , we reject H_0

i.e., the sample is not from the population whose is 38.

$\therefore$ 95% confidence interval is $(\bar{x} - 1.96.\frac{\sigma}{\sqrt{n}}, \bar{x} + 1.96.\frac{\sigma}{\sqrt{n}})$

i.e., $(40 - \frac{1.96(10)}{\sqrt{400}}, 40 + \frac{1.96(10)}{\sqrt{400}})$

$= (40 - \frac{1.96(10)}{20}, 40 + \frac{1.96(10)}{20})$

= (40 – 0.98 , 40 + 0.98)

= (39.02 , 40.98)

4. An insurance agent has claimed that the average age of policy holders who issue through him is less than the average for all agents which is 30.5. A random sample of 100 policy holders who had issued through him gave the following age distribution .

Age	**16-20**	**21-25**	**26-30**	**31-35**	**36-40**
No# of persons	**12**	**22**	**20**	**30**	**16**

Calculate the arithmetic mean and standard deviation of this distribution and use these values to test his claim at 5% los.

Sol : Take A = 28 where A – Assumed mean

$d_i = x_i$ – A

$\bar{x} = A + \frac{h \sum f_i d_i}{N}$

$= 28 + \frac{5 \times 16}{100} = 28.8$

S.D : $S = h\sqrt{\frac{\sum fd^2}{N} - (\frac{\sum fd}{N})^2} = 5.\sqrt{\frac{164}{100} - (\frac{16}{100})^2} = 6.35$

i) Null Hypothesis H_0 : The sample is drawn from population with mean μ

ii) i.e., $H_0 : \mu$ = 30.5 years

iii) Alternative Hypothesis $H_1 : \mu$ < 30.5 (Left one –tailed test)

iv) Level of significance : $\alpha = 0.05$ (Z_α = 1.645)

v) Test Statistic : $Z_{cal} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{28.8-30.5}{\frac{6.35}{\sqrt{100}}} = -2.677$

vi) Conclusion: Since $|Z_{cal}|\ value > Z_\alpha$ value , we reject H_0
i.e., the sample is not drawn from the population with μ = 30.5 years .

5. An ambulance service claims that it takes on the average less than 10 minutes to reach its destination in emergency calls . A sample of 36 calls has a mean of 11 minutes and the variance of 16 minutes .Test the claim at 0.05 los?

Sol : Given n = 36 , $\bar{x}$ =11 , μ = 10 and $\sigma = \sqrt{16} = 4$

i) Null Hypothesis $H_0 : \mu = 10$

ii) Alternative Hypothesis $H_1 : \mu < 10$ (Left one –tailed test)

iii) Level of significance : $\alpha = 0.05$ (Z_α = 1.645)

iv) Test Statistic : $Z_{cal} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{11-10}{\frac{4}{\sqrt{36}}} = \frac{6}{4} = 1.5$

v) Conclusion: Since $|Z_{cal}|\ value < Z_\alpha$ value , we accept H_0

6. The means of two large samples of sizes 1000 and 2000 members are 67.5 inches and 68 inches respectively . Can the samples be regarded as drawn from the same population of S.D 2.5 inches.

Sol: Let μ_1 and μ_2 be the means of the two populations
Given n_1 = 1000 , n_2 = 2000 and $\bar{x}_1$ = 67.5 inches , $\bar{x}_2$ = 68 inches
Population S.D, σ = 2.5 inches

i) Null Hypothesis H_0 :The samples have been drawn from the same population of S.D 2.5 inches
i.e., $H_0 : \mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 \neq \mu_2$ (Two – Tailed test)

iii) Level of significance : $\alpha = 0.05$ (Z_α = 1.96)

iv) Test Statistic : $Z_{cal} = \frac{\bar{X}_1 - \bar{X}_2}{\sigma\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}} = \frac{67.5-68}{\sqrt{(2.5)^2\left(\frac{1}{1000}+\frac{1}{2000}\right)}} = \frac{-0.5}{0.0968} = -5.16$

v) Conclusion: Since $|Z_{cal}|\ value > Z_\alpha$ value , we reject H_0
Hence , we conclude that the samples are not drawn from the same population of S.D 2.5 inches.

7. Samples of students were drawn from two universities and from their weights in kilograms , mean and standard deviations are calculated and shown below. Make a large sample test to test the significance of the difference between the means.

	Mean	**S .D**	**Size of the sample**
University A	**55**	**10**	**400**
University B	**57**	**15**	**100**

Sol: Let μ_1 and μ_2 be the means of the two populations

Given $n_1 = 400$, $n_2 = 100$ and $\bar{x}_1 = 55$ kgs , $\bar{x}_2 = 57$ kgs

$\sigma_1 = 10$ and $\sigma_2 = 15$

i) Null Hypothesis $H_0 : \mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 \neq \mu_2$ (Two – Tailed test)

iii) Level of significance : $\alpha = 0.05$ $(Z_\alpha = 1.96)$

iv) Test Statistic : $Z_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{55 - 57}{\sqrt{\frac{10^2}{400} + \frac{15^2}{100}}} = \frac{-2}{\sqrt{\frac{1}{4} + \frac{9}{4}}} = -1.26$

v) Conclusion: Since $|Z_{cal}|\ value < Z_\alpha$ value , we accept H_0

Hence , we conclude that there is no significant difference between the means

8. The average marks scored by 32 boys is 72 with a S.D of 8 . While that for 36 girls is 70 with a S.D of 6. Does this data indicate that the boys perform better than girls at 5% los ?

Sol: Let μ_1 and μ_2 be the means of the two populations

Given $n_1 = 32$, $n_2 = 36$ and $\bar{x}_1 = 72$, $\bar{x}_2 = 70$

$\sigma_1 = 8$ and $\sigma_2 = 6$

i) Null Hypothesis $H_0 : \mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 > \mu_2$ (Right One Tailed test)

iii) Level of significance : $\alpha = 0.05$ $(Z_\alpha = 1.645)$

iv) Test Statistic : $Z_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{72 - 70}{\sqrt{\frac{8^2}{32} + \frac{6^2}{36}}} = \frac{2}{\sqrt{2+1}} = 1.1547$

v) Conclusion: Since $|Z_{cal}|\ value < Z_\alpha$ value , we accept H_0

Hence , we conclude that the performance of boys and girls is the same

9. A sample of the height of 6400 Englishmen has a mean of 67.85 inches and a S.D of 2.56 inches while another sample of heights of 1600 Austrians has a mean of 68.55 inches and S.D of 2.52 inches. Do the data indicate that Austrians are on the average taller than the Englishmen ? (Use $\alpha\ as\ 0.01$)

Sol : Let μ_1 and μ_2 be the means of the two populations

Given $n_1 = 6400$, $n_2 = 1600$ and $\bar{x}_1 = 67.85$, $\bar{x}_2 = 68.55$

$\sigma_1 = 2.56$ and $\sigma_2 = 2.52$

i) Null Hypothesis $H_0 : \mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 < \mu_2$ (Left One Tailed test)

iii) Level of significance : $\alpha = 0.01$ $(Z_\alpha = -2.33)$

iv) Test Statistic : $Z_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{67.85 - 68.55}{\sqrt{\frac{2.56^2}{6400} + \frac{2.52^2}{1600}}}$

$$= \frac{67.85 - 68.55}{\sqrt{\frac{6.5536}{6400} + \frac{6.35}{1600}}}$$

$= \frac{-0.7}{\sqrt{0.001+0.004}} = \frac{-0.7}{0.0707}$ - 9.9

v) Conclusion: Since $|Z_{cal}|$ value $> Z_\alpha$ value , we reject H_0

Hence , we conclude that Australians are taller than Englishmen.

10. At a certain large university a sociologist speculates that male students spend considerably more money on junk food than female students. To test her hypothesis the sociologist randomly selects from records the names of 200 students . Of thee , 125 are men and 75 are women . The mean of the average amount spent on junk food per week by the men is Rs. 400 and S.D is 100. For the women the sample mean is Rs. 450 and S.D is 150. Test the hypothesis at 5 % los ?

Sol: Let μ_1 and μ_2 be the means of the two populations

Given $n_1 = 125$, $n_2 = 75$ and $\bar{x}_1$ = Mean of men = 400 , $\bar{x}_2$ = Mean of women = 450

$\sigma_1 = 100$ and $\sigma_2 = 150$

i) Null Hypothesis $H_0 : \mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 > \mu_2$ (Right One Tailed test)

iii) Level of significance : $\alpha = 0.05$ $(Z_\alpha = 1.645)$

iv) Test Statistic : $Z_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{400 - 450}{\sqrt{\frac{100^2}{125} + \frac{150^2}{75}}}$

$$= \frac{-50}{\sqrt{80 + 300}}$$

$= \frac{-50}{\sqrt{380}} = \frac{-50}{19.49}$ = - 2.5654

v) Conclusion: Since Z_{cal}value $< Z_\alpha$ value , we accept H_0

Hence , we conclude that difference between the means are equal

11. The research investigator is interested in studying whether there is a significant difference in the salaries of MBA grads in two cities. A random sample of size 100 from city A yields an average income of Rs. 20,150 . Another random sample of size 60 from city B yields an average income of Rs. 20,250. If the variance are given as σ_1^2 = 40,000 and

σ_2^2 = 32,400 respectively . Test the equality of means and also construct 95% confidence limits.

Sol: Let μ_1 and μ_2 be the means of the two populations

Given $n_1 = 100$, $n_2 = 60$ and $\bar{x}_1$ = Mean of city A = 20,150 , $\bar{x}_2$ = Mean of city B = 20,250

$\sigma_1^2 = 40{,}000$ and $\sigma_2^2 = 32{,}400$

i) Null Hypothesis $H_0 : \mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 \neq \mu_2$ (Two -Tailed test)

iii) Level of significance : $\alpha = 0.05$ $(Z_\alpha = 1.96)$

iv) Test Statistic : $Z_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = \frac{20,150 - 20,250}{\sqrt{\frac{40000}{100} + \frac{32400}{60}}}$

$$= \frac{100}{\sqrt{400 + 540}}$$

$= \frac{100}{30.66} = 3.26$

v) Conclusion: Since Z_{cal}value $> Z_\alpha$ value , we reject H_0

Hence , we conclude that there is a significant difference in the salaries of MBA grades two cities.

$\therefore$ 95% confidence interval is $\mu_1 - \mu_2 = (\bar{x}_1 - \bar{x}_2) \pm 1.96 \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

$= (20,150 - 20,250)) \pm 1.96 \sqrt{\frac{40000}{100} + \frac{32400}{60}} = (39.90, 160.09)$

12. A die was thrown 9000 times and of these 3220 yielded a 3 or 4. Is this consistent with the hypothesis that the die was unbiased?

Sol : Given n = 9000

P = Population of proportion of successes

= P(getting a 3 or 4) = $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$ 0.3333

Q = 1- P = 0.6667

P = Proportion of successes of getting 3 or 4 in 9000 times = $\frac{3220}{9000} = 0.3578$

i) Null Hypothesis H_0 : The die is unbiased

i.e., H_0 : P = 0.33

ii) Alternative Hypothesis H_1 : The die is biased

i.e., H_1 : P $\neq$ 0.33 (Two –Tailed test)

iii) Level of Significance : $\alpha = 0.05$ (Z_α = 1.96)

iv) Test Statistic : $Z_{cal} = \frac{p - P}{\sqrt{\frac{PQ}{n}}} = \frac{0.3578 - 0.3333}{\sqrt{\frac{(0.3333)(0.6667)}{9000}}} = 4.94$

v)Conclusion: Since Z_{cal}value $> Z_\alpha$ value , we reject H_0

Hence , we conclude that the die is biased.

13. In a random sample of 125 cool drinkers , 68 said they prefer thumsup to Pepsi . Test the null hypothesis P = 0.5 against the alternative hypothesis hypothesis P > 0.5?

Sol : Given n = 125 , x = 68 and p = $\frac{x}{n} = \frac{68}{125} = 0.544$

i) Null Hypothesis H_0 : P = 0.5

ii) Alternative Hypothesis H_1 : P > 0.5(Right One Tailed test)

iii) Level of Significance : $\alpha = 0.05$ (Z_α = 1.645)

iv) Test Statistic : $Z_{cal} = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.544-0.5}{\sqrt{\frac{(0.5)(0.5)}{125}}} = 0.9839$

v) Conclusion: Since Z_{cal}value $<$ Z_α value , we accept H_0

14. A manufacturer claimed that at least 95% of the equipment which he supplied to a factory conformed to specifications . An experiment of a sample of 200 piece of equipment revealed that 18 were faulty .Test the claim at 5% los ?

Sol : Given n = 200

Number of pieces confirming to specifications = 200-18 = 182

$\therefore$ p = Proportion of pieces confirming to specification = $\frac{182}{200}$ = 0.91

P = Population proportion = $\frac{95}{100}$ = 0.95

i) Null Hypothesis H_0 : P = 0.95

ii) Alternative Hypothesis H_1 : P < 0.95(Left One Tailed test)

iii) Level of Significance : $\alpha = 0.05$ (Z_α = -1.645)

iv) Test Statistic : $Z_{cal} = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.91-0.95}{\sqrt{\frac{0.95 \times 0.05}{200}}} = -2.59$

v) Conclusion: We reject H_0

Hence , we conclude that the manufacturer's claim is rejected.

15. Among 900 people in a state 90 are found to be chapatti eaters . Construct 99% confidence interval for the true proportion and also test the hypothesis for single proportion ?

Sol: Given x = 90 , n = 900

$\therefore$ p = $\frac{x}{n} = \frac{90}{100} = \frac{1}{10} = 0.1$

And q = 1- p= 0.9

Now $\sqrt{\frac{pq}{n}} = \sqrt{\frac{(0.1)(0.9)}{900}} = 0.01$

Confidence interval is $P = p \pm Z_{\frac{\alpha}{2}} (\sqrt{\frac{pq}{n}})$

i.e., (0.1- 0.03 , 0.1 + 0.03)

= (0.07 , 0.13)

i) Null Hypothesis H_0 : P = 0.5

ii) Alternative Hypothesis H_1 : P $\neq$0.5(Two Tailed test)

iii) Level of Significance : $\alpha = 0.01$ (Z_α = 2.58)

iv) Test Statistic : $Z_{cal} = \frac{p-P}{\sqrt{\frac{PQ}{n}}} = \frac{0.1-0.5}{\sqrt{\frac{0.5 \times 0.5}{900}}} = -24.39$

v) Conclusion: Since $|Z_{cal}|value > Z_\alpha$ value , we reject H_0

16. Random samples of 400 men and 200 women in a locality were asked whether they would like to have a bus stop a bus stop near their residence . 200 men and

40 women in favor of the proposal . Test the significance between the difference of two proportions at 5% los ?

Sol: Let P_1 and P_2 be the population proportions in a locality who favor the bus stop

Given n_1 = Number of men = 400

n_2 = number of women = 200

x_1 = Number of men in favor of the bus stop = 200

x_2 = Number of women in favor of the bus stop 40

$\therefore p_1 = \frac{x_1}{n_1} = \frac{200}{400} = \frac{1}{2}$ and $p_2 = \frac{x_2}{n_2} = \frac{40}{200} = \frac{1}{5}$

i) Null Hypothesis $H_0 : P_1 = P_2$

ii) Alternative Hypothesis $H_1 : P_1 \neq P_2$ (Two Tailed test)

iii) Level of Significance : $\alpha = 0.05$ (Z_α = 1.96)

iv) Test Statistic : $Z_{cal} = \frac{p_1 - p_2}{\sqrt{pq(\frac{1}{n_1} + \frac{1}{n_2})}}$

We have p = $\frac{n_1p_1+n_2p_2}{n_1+n_2} = \frac{x_1+x_2}{n_1+n_2} = \frac{200+40}{400+200} = \frac{240}{600} = \frac{2}{5}$

q = 1- p = $\frac{3}{5}$

$= \frac{0.5-0.2}{\sqrt{(0.4)(0.6)(\frac{1}{400} + \frac{1}{200})}} = 7.07$

v) Conclusion: Since $|Z_{cal}|$value > Z_α value , we reject H_0

Hence we conclude that there is difference between the men and women in their attitude towards the bus stop near their residence.

17. A machine puts out 16 imperfect articles in a sample of 500 articles . After the machine is overhauled it puts out 3 imperfect articles in a sample of 100 articles . Has the machine is improved ?

Sol : Let P_1 and P_2 be the proportions of imperfect articles in the proportion of articles manufactured by the machine before and after overhauling , respectively.

Given n_1 = Sample size before the machine overhauling = 500

n_2 = Sample size after the machine overhauling = 100

x_1 = Number of imperfect articles before overhauling = 16

x_2 = Number of imperfect articles after overhauling = 3

$\therefore p_1 = \frac{x_1}{n_1} = \frac{16}{500} = 0.032$ and $p_2 = \frac{x_2}{n_2} = \frac{3}{100} = 0.03$

i) Null Hypothesis $H_0 : P_1 = P_2$

ii) Alternative Hypothesis $H_1 : P_1 > P_2$ (Left one Tailed test)

iii) Level of Significance : $\alpha = 0.05$ (Z_α = 1.645)

iv) Test Statistic : $Z_{cal} = \frac{p_1 - p_2}{\sqrt{pq(\frac{1}{n_1} + \frac{1}{n_2})}}$

We have p = $\frac{n_1p_1+n_2p_2}{n_1+n_2} = \frac{x_1+x_2}{n_1+n_2} = \frac{16+3}{500+100} = \frac{19}{600} = 0.032$

q = 1- p = 0.968

$$= \frac{0.032-0.03}{\sqrt{(0.032)(0.968)\left(\frac{1}{500}+\frac{1}{100}\right)}}$$

$$\frac{0.002}{0.019} = 0.104$$

v) Conclusion: Since $|Z_{cal}|$value $< Z_\alpha$ value , we accept H_0
Hence we conclude that the machine has improved.

18. In an investigation on the machine performance the following results are obtained .

	No# of units inspected	**No# of defectives**
Machine 1	**375**	**17**
Machine 2	**450**	**22**

Test whether there is any significant performance of two machines at α = 0.05

Sol: Let P_1 and P_2 be the proportions of defective units in the population of units inspected in machine 1 and Machine 2 respectively.

Given n_1 = Sample size of the Machine 1 = 375
n_2 = Sample size of the Machine 2 = 450
x_1 = Number of defectives of the Machine 1 = 17
x_2 = Number of defectives of the Machine 2 = 22
$\therefore p_1 = \frac{x_1}{n_1} = \frac{17}{375} = 0.045$ and $p_2 = \frac{x_2}{n_2} = \frac{22}{450} = 0.049$

i) Null Hypothesis $H_0 : P_1 = P_2$
ii) Alternative Hypothesis $H_1 : P_1 \neq P_2$ (Two Tailed test)
iii) Level of Significance : $\alpha = 0.05$ (Z_α = 1.96)
iv) Test Statistic : $Z_{cal} = \frac{p_1 - p_2}{\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$

We have p = $\frac{n_1p_1+n_2p_2}{n_1+n_2} = \frac{x_1+x_2}{n_1+n_2} = \frac{17+22}{375+450} = \frac{39}{825} = 0.047$

q = 1- p = 1- 0.047 = 0.953

$$= \frac{0.045-0.049}{\sqrt{(0.047)(0.953)\left(\frac{1}{375}+\frac{1}{450}\right)}}$$

= - 0.267

v) Conclusion: Since $|Z_{cal}|$value $< Z_\alpha$ value , we accept H_0
Hence we conclude that there is no significant difference in performance of machines.

19. A cigarette manufacturing firm claims that its brand A line of cigarettes outsells its

brand B by 8% . If it is found that 42 out of 200 smokers prefer brand A and 18 out of another sample of 100 smokers prefer brand B . Test whether 8% difference is a valid claim?

Sol: Given n_1 = 200

n_2 = 100

x_1 = Number of smokers preferring brand A= 42

x_2 = Number of smokers preferring brand B = 18

$\therefore p_1 = \frac{x_1}{n_1} = \frac{42}{200} = 0.21$ and $p_2 = \frac{x_2}{n_2} = \frac{18}{100} = 0.18$

and $P_1 - P_2 = 8\% = 0.08$

i) Null Hypothesis $H_0 : P_1 - P_2 = 0.08$

ii) Alternative Hypothesis $H_1 : P_1 - P_2 \neq 0.08$ (Two Tailed test)

iii) Level of Significance : $\alpha = 0.05$ $(Z_\alpha = 1.96)$

iv) Test Statistic : $Z_{cal} = \frac{(p_1 - p_2) - (P_1 - P_2)}{\sqrt{pq(\frac{1}{n_1} + \frac{1}{n_2})}}$

We have p = $\frac{n_1 p_1 + n_2 p_2}{n_1 + n_2} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{42+18}{200+100} = \frac{60}{300} = 0.2$

q = 1- p = 1- 0.2 = 0.8

$Z_{cal} = \frac{(0.21-0.18)-0.08}{\sqrt{(0.2)(0.8)(\frac{1}{200} + \frac{1}{100})}}$

$= \frac{-0.05}{0.0489} = -1.02$

v) Conclusion: Since $|Z_{cal}| value < Z_\alpha$ value , we accept H_0

Hence we conclude that 8% difference in the sale of two brands of cigarettes is a valid claim.

20. In a city A , 20% of a random sample of 900 schoolboys has a certain slight physical defect . In another city B ,18.5% of a random sample of 1600 school boys has the same defect . Is the difference between the proportions significant at 5% los?

Sol: Given n_1 = 900

n_2 = 1600

x_1 = 20% of 900 = 180

x_2 = 18.5% of 1600 = 296

$\therefore p_1 = \frac{x_1}{n_1} = \frac{180}{900} = 0.2$ and $p_2 = \frac{x_2}{n_2} = \frac{296}{1600} = 0.185$

i) Null Hypothesis $H_0 : P_1 = P_2$

ii) Alternative Hypothesis $H_1 : P_1 \neq P_2$ (Two Tailed test)

iii) Level of Significance : $\alpha = 0.05$ $(Z_\alpha = 1.96)$

iv) Test Statistic : $Z_{cal} = \frac{(p_1 - p_2)}{\sqrt{pq(\frac{1}{n_1} + \frac{1}{n_2})}}$

We have p = $\frac{n_1p_1+n_2p_2}{n_1+n_2} = \frac{x_1+x_2}{n_1+n_2} = \frac{180+296}{900+1600} = \frac{476}{2500} = 0.19$

q = 1- p = 1- 0.19 = 0.81

$Z_{cal} = \frac{0.2-0.185}{\sqrt{(0.19)(0.81)\left(\frac{1}{900}+\frac{1}{1600}\right)}}$

$= \frac{-0.015}{0.01634} = -0.918$

v) Conclusion: Since $|Z_{cal}|value < Z_\alpha$ value , we accept H_0

Hence we conclude that there is no significant difference between the proportions.

SMALL SAMPLES

Introduction When the sample size n < 30, then if is referred to as small samples. In this sampling distribution in many cases may not be normal ie., we will not be justified in estimating the population parameters as equal to the corresponding sample values.

Degree Of Freedom The number of independent variates which make up the statistic is known as the degrees of freedom (d.f) and it is denoted by ϑ.

For Example**:** If $x_1 + x_2 + x_3 = 50$ and we assign any values to two os the variables (say x_1, x_2), then the values of x_3 will be known. Thus, the two variables are free and independent choices for finding the third.

In general, the number of degrees of freedom is equal to the total number of observations less the number of independent constraints imposed on the observations.

For example: in a set of data of n observations, if K is the number of independent constraints then $\vartheta = n - k$

Student's t-Distribution Or t-Distribution

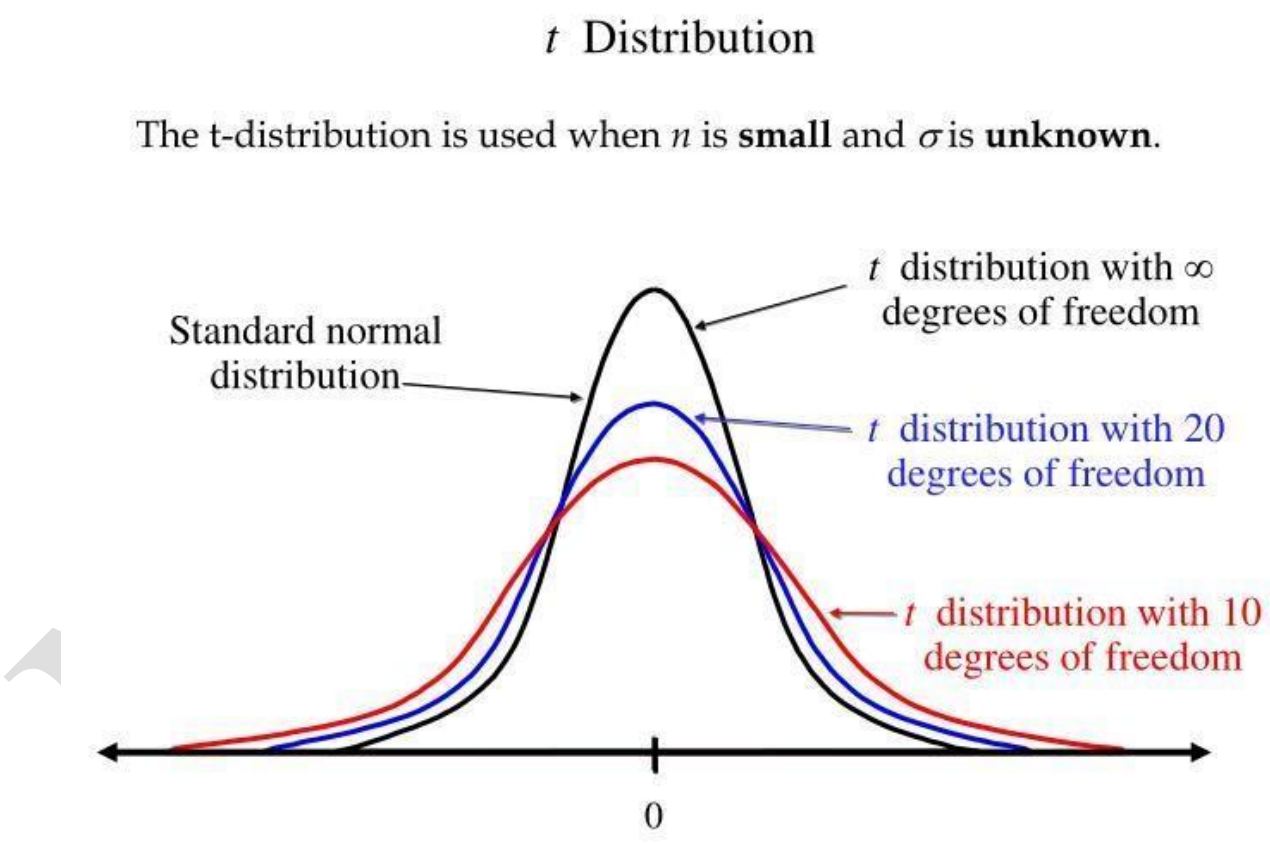
Let $\bar{X}$ be the mean of a random sample of size n, taken from a normal population having the mean μ and the variance σ^2, and sample variance $S^2 = \sum \frac{(X_i-\bar{X})^2}{n-1}$, then

$t = \frac{\bar{x}-\mu}{S/\sqrt{n}}$ is a random variable having the $t-$ distribution with $\vartheta = n - 1$ degrees of freedom.

Properties of $t-$ Distribution

1. The shape of $t-$distribution is bell shaped, which is similar to that of normal distribution and is symmetrical about the mean.

2. The mean of the standard normal distribution as well as t −distribution is zero, but the variance of t −distrubution depends upon the parometer ϑ which is called the degrees of freedom.
3. The variance of t −distribution exceeds 1, but approaches 1 as $n \to \infty$.



Applications Of t – Distributions

1. To test the significance of the sample mean, When population variance is not given:

Let $\bar{x}$ be the mean of the sample and n be the size of the sample 'σ' be the standard deviation of the population and μ be the mean of the population.

Then the student t − distribution is defined by the statistic

$t = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n-1}}}$ if s is given directly

If $'\sigma'$ is unknown, then $t = \frac{\bar{X}-\mu}{S/\sqrt{n}}$ where

$S^2 = \sum \frac{(X_i - \bar{X})^2}{n-1}$

Note : Confidence limits for mean $\mu = \bar{x} \pm t_\alpha(S/\sqrt{n})$ or $\mu = \bar{x} \pm t_\alpha(S/\sqrt{n-1})$

2. To test the significance of the difference between means of the two independent samples :

To test the significant difference between the sample means $\bar{x}_1$ and $\bar{x}_2$ of two independent samples of sizes n_1 and n_2, with the same variance .

We use statistic

$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{S^2(\frac{1}{n_1}+\frac{1}{n_2})}}$ ---------- (1) where

$\bar{x}_1 = \frac{\sum x_1}{n_1}$, $\bar{x}_2 = \frac{\sum x_2}{n_2}$ and

$S^2 = \frac{1}{n_1+n_2-2}\left[\sum(x_1 - \bar{x}_1)^2 + \sum(x_2 - \bar{x}_2)^2\right]$

OR $S^2 = \frac{1}{n_1+n_2-2}[(n_1 s_1^2) + (n_2 s_2^2)]$

Where s_1 and s_2 are sample standard deviations.

Note: Confidence limits for difference of means : $\mu_1 - \mu_2 = (\bar{x}_1 - \bar{x}_2) \pm t_\alpha\left(\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}\right)$

Paired t- test (Test the significance of the difference between means of two dependent samples) :

Paired observations arise in many practical situations where each homogenous experimental unit receives both population condition.

For Example: To test the effectiveness of 'drug' some // person's blood pressure is measured before and after the intake of certain drug. Here the individual person is the experimental unit and the two populations are blood pressure "before" and "after" the drug is given

Paired t-test is applied for n paired observations by taking the differences d_1, d_2 ---------- d_n of the paired data. To test whether the differences d_i from a random sample of a population with mean μ.

$t = \frac{\bar{d}}{s/\sqrt{n}}$ where $\bar{d} = \frac{1}{n}\epsilon\, d_i$ and $s^2 = \frac{1}{n-1}\sum(d - \bar{d})^2$

Problems:

1. **A sample of 26 bulbs gives a mean life of 990 hours with a S.D of 20 hours. The manufacturer claims that the mean life of bulbs is 1000 hours . Is the sample not upto the standard?**

   Sol: Given n = 26

   $\bar{x}$ = 990

   μ = 1000 and S.D i.e., s = 20

   i) Null Hypothesis : $H_0 : \mu = 1000$

   ii) Alternative Hypothesis**:** $H_1 : \mu < 1000$(Left one tailed test)
   (Since it is given below standard)

   iii) Level of significance : $\alpha = 0.05$
   t tabulated value with 25 degrees of freedom for left tailed test is 1.708

   iv) Test Statistic : $t_{cal} = \frac{\bar{x}-\mu}{\frac{s}{\sqrt{n-1}}} = \frac{990-1000}{\frac{20}{\sqrt{25}}} = -2.5$

   v) Conclusion**:** Since $|t_{cal}|$ value $> t_\alpha$ value , we reject H_0
   Hence we conclude that the sample is not upto the standard.

2. **A random sample of size 16 values from a normal population showed a mean of 53 and sum of squares of deviations from the mean equals to 150 . Can this sample be regarded as taken from the population having 56 as mean ? Obtain 95% confidence limits of the mean of the population.?**

   Sol: a) Given n = 16

   $\bar{x}$ = 53

   μ = 56 and $\sum(x_i - \bar{x})^2 = 150$

   $\therefore S^2 = \frac{\sum(x_i-\bar{x})^2}{n-1} = \frac{150}{15} = 10 \Rightarrow S = \sqrt{10}$

   Degrees of freedom ϑ = n-1 = 16-1 =15

   i) Null Hypothesis $H_0 : \mu = 56$

   ii) Alternative Hypothesis $H_1 : \mu \neq 56$ (Two tailed test)

   iii) Level of significance : $\alpha = 0.05$

t tabulated value with 15 degrees of freedom for two tailed test is 2.13

iv) Test Statistic : $t_{cal} = \frac{\bar{x}-\mu}{\frac{S}{\sqrt{n}}} = \frac{53-56}{\frac{\sqrt{10}}{\sqrt{15}}} = -3.79$

v) Conclusion: Since $|t_{cal}|\ value > t_\alpha$ value , we reject H_0

Hence we conclude that the sample cannot be regarded as taken from population.

b) The 95% confidence limits of the mean of the population are given by

$\bar{x} \pm t_{0.05} \frac{S}{\sqrt{n}} = 53 \pm 2.13 \times 0.79$

$= 53 \pm 1.6827$

$= 54.68$ and 51.31

∴ 95% confidence limits are(51.31, 54.68)

3. A random sample of 10 boys had the following I.Q's : 70, 120 ,110, 101,88, 83,95,98,107 and 100.

a) Do these data support the assumption of a population mean I.Q of 100?

b) Find a reasonable range in which most of the mean I.Q values of samples of 10 boys lie

Sol: Since mean and s.d are not given

We have to determine these

x	$x - \bar{x}$	$(x - \bar{x})^2$
70	-27.2	739.84
120	22.8	519.84
110	12.8	163.84
101	3.8	14.44
88	-9.2	84.64
83	-14.2	201.64
95	-2.2	4.84
98	0.8	0.64
107	9.8	96.04
100	2.8	7.84

$\sum x = 972$		$\sum(x - \bar{x})^2 = 1833.60$

Mean , $\bar{x} = \frac{\sum x}{n} = \frac{972}{10} = 97.2$ and

$S^2 = \frac{1}{n-1}\sum(x - \bar{x})^2 = \frac{1833.6}{9}$

$\therefore S = \sqrt{203.73} = 14.27$

i) Null Hypothesis $H_0 : \mu = 100$

ii) Alternative Hypothesis $H_1 : \mu \neq 100$ (Two tailed test)

iii) Level of significance : $\alpha = 0.05$

t tabulated value with 9 degrees of freedom for two tailed test is 2.26

iv) Test Statistic : $t_{cal} = \frac{\bar{x} - \mu}{\frac{S}{\sqrt{n}}} = \frac{97.2 - 100}{\frac{14.27}{\sqrt{10}}} = -0.62$

v) Conclusion: Since $|t_{cal}|\ value < t_\alpha$ value , we accept H_0

Hence we conclude that the data support the assumption of mean I.Q of 100 in the population.

b) The 95% confidence limits of the mean of the population are given by

$\bar{x} \pm t_{0.05}\frac{S}{\sqrt{n}} = 97.2 \pm 2.26 \times 4.512$

$= 97.2 \pm 10.198$

$= 107.4$ and 87

$\therefore$ 95% confidence limits are(87, 107.4)

4. Samples of two types of electric bulbs were tested for length of life and following data were obtained

Type 1	**Type 2**
Sample number , $n_1 = 8$	**$n_2 = 7$**
Sample mean , $\bar{x}_1 = 1234$	**$\bar{x}_2 = 1036$**
Sample S.D , $s_1 = 36$	**$s_2 = 40$**

Is the difference in the mean sufficient to warrant that type 1 is superior to type 2 regarding length of life .

Sol: i) Null Hypothesis H_0 : The two types of electric bulbs are identical

i.e., H_0: $\mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 \neq \mu_2$

iii) Test Statistic : $t_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{s\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$

Where $S^2 = \frac{n1s1^2+n_1s_1^2}{n_1+n_2}$

$= \frac{1}{8+7-2}(8(36)^2 + 7(40)^2) = 1659.08$

$\therefore t = \frac{1234-1036}{\sqrt{1659.08\,(\frac{1}{8}+\frac{1}{7})}} = 9.39$

iv) Degrees of freedom = 8+7-2 =13 ,tabulated value of t for 13 d.f at 5% los is 2.16

v)Conclusion: Since $|t_{cal}|\ value > t_\alpha$ value , we reject H_0

Hence we conclude that the two types 1 and 2 of electric bulbs are not identical .

5. Two horses A and B were tested according to the time to run a particular track with the following results .

Horse A	**28**	**30**	**32**	**33**	**33**	**29**	**34**
Horse B	**29**	**30**	**30**	**24**	**27**	**29**	

Test whether the two horses have the same running capacity

Sol: Given $n_1 = 7$, $n_2 = 6$

We first compute the sample means and standard deviations

$\bar{x}$ = Mean of the first sample = $\frac{1}{7}$ (28 + 30 + 32 + 33 + 33 + 29 + 34)

$= \frac{1}{7}(219) = 31.286$

$\bar{y}$= Mean of the second sample = $\frac{1}{6}$ (29 + 30 + 30 + 24 + 27 + 29)

$= \frac{1}{6}(169) = 28.16$

x	$x - \bar{x}$	$(x-\bar{x})^2$	y	$y-\bar{y}$	$(y-\bar{y})^2$
28	-3.286	10.8	29	0.84	0.7056
30	-1.286	1.6538	30	1.84	3.3856
32	0.714	0.51	30	1.84	3.3856
33	1.714	2.94	24	-416	17.3056
33	1.714	2.94	27	-1.16	1.3456
29	-2.286	5.226	29	0.84	0.7056
34	2.714	7.366			

$\sum x$ $= 219$		$\sum(x - \bar{x})^2$ $= 31.4358$	$\sum y$ $= 169$		$\sum(y - \bar{y})^2$ $= 26.8336$

Now $S^2 = \frac{1}{n_1+n_2-2}[(\sum(x-\bar{x})^2 + \sum(y-y)^2]$

$= \frac{1}{11}[31.4358 + 26.8336]$

$= \frac{1}{11}(58.2694)$

$= 5.23$

$\therefore$ S = $\sqrt{5.23}$ = 2.3

i) Null Hypothesis H_0: $\mu_1 = \mu_2$

ii) Alternative Hypothesis H_1 : $\mu_1 \neq \mu_2$

iii) Test Statistic : $t_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})}}$

$$= \frac{31.286 - 28.16}{2.3\,(\sqrt{\frac{1}{7} + \frac{1}{6}})} = 2.443$$

$\therefore$ t_{cal} = 2.443

iv) Degrees of freedom = 7+6-2 =11

Tabulated value of t for 11 d.f at 5% los is 2.2

Conclusion: Since $|t_{cal}|$ value > t_α value , we reject H_0

Hence we conclude that both horses do not have the same running capacity.

6. Ten soldiers participated in a shooting competition in the first week. After intensive training they participated in the competition in the second week . Their scores before and after training are given below :

Scores before	**67**	**24**	**57**	**55**	**63**	**54**	**56**	**68**	**33**	**43**
Scores after	**70**	**38**	**58**	**58**	**56**	**67**	**68**	**75**	**42**	**38**

Do the data indicate that the soldiers have been benefited by the training.

Sol: Given n_1 = 10 , n_2 = 10

We first compute the sample means and standard deviations

$\bar{x}$ = Mean of the first sample = $\frac{1}{10}$ (67 + 24 + 57 +55+63+54+56+68+33+43)

$= \frac{1}{10}(520) = 52$

$\bar{y}$= Mean of the second sample = $\frac{1}{10}$ (70+38+58+58+56+67+68+75+42+38)

$= \frac{1}{10}(570) = 57$

x	$x - \bar{x}$	$(x - \bar{x})^2$	y	$y - \bar{y}$	$(y - \bar{y})^2$
67	15	225	70	13	169
24	-28	784	38	-19	361
57	5	25	58	1	1
55	3	9	58	1	1
63	11	121	56	-1	1
54	2	4	67	10	100
56	4	16	68	11	121
68	16	256	75	18	324
33	-19	361	42	-15	225
43	-9	81	38	-19	361
$\sum x = 520$		$\sum(x - \bar{x})^2 = 1882$	$\sum y = 570$		$\sum(y - \bar{y})^2 = 1664$

Now $S^2 = \frac{1}{n_1+n_2-2}[(\sum(x - \bar{x})^2 + \sum(y - y)^2]$

$= \frac{1}{18}[1882 + 1664]$

$= \frac{1}{18}(3546)$

$= 197$

$\therefore$ S = $\sqrt{197}$ = 14.0357

i) Null Hypothesis H_0: $\mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 < \mu_2$ (Left one tailed test)

iii) Test Statistic : $t_{cal} = \frac{\bar{x}_1 - \bar{x}_2}{S\sqrt{(\frac{1}{n_1} + \frac{1}{n_2})}}$

$$= \frac{52-57}{14.0357\left(\sqrt{\frac{1}{10}+\frac{1}{10}}\right)}$$

$= \frac{3546}{18} = -0.796$

$\therefore\ t_{cal}$ = -0.796

iv) Degrees of freedom = 10+10-2 =18

Tabulated value of t for 18 d.f at 5% los is -1.734

Conclusion: Since $|t_{cal}|$ value $<$ $|t_{\alpha}|$ value , we accept H_0

Hence we conclude that the soldiers are not benefited by the training.

7. The blood pressure of 5 women before and after intake of a certain drug are given below:

Before	**110**	**120**	**125**	**132**	**125**
After	**120**	**118**	**125**	**136**	**121**

Test whether there is significant change in blood pressure at 1% los?

Sol: Given n = 5

i) Null Hypothesis H_0: $\mu_1 = \mu_2$

ii) Alternative Hypothesis H_1 : $\mu_1 < \mu_2$ (Left one tailed test)

iii) Test Statistic $t_{cal} = \frac{\bar{d}}{s/\sqrt{n}}$

where $\bar{d} = \frac{\sum d}{n}$ and $S^2 = \frac{1}{n-1}\sum(d-\bar{d})^2$

B.P before training	B.P after training	$d = y - x$	$d - \bar{d}$	$(d-\bar{d})^2$
110	120	10	8	64
120	118	-2	-4	16
123	125	2	0	0
132	136	4	2	4
125	121	-4	-6	36
		$\sum d = 10$		$\sum(d-\bar{d})^2 = 120$

$\therefore \bar{d} = \frac{10}{5} = 2$ and $S^2 = \frac{120}{4} = 30$

$\therefore$ S = 5.477

$t_{cal} = \frac{\bar{d}}{s/\sqrt{n}} = \frac{2}{5.477/\sqrt{5}} = 0.862$

iv) Degrees of freedom = 5-1= 4

Tabulated value of t for 4 d.f at 1% los is 4.6

Conclusion: Since $|t_{cal}|$ value $<$ $|t_\alpha|$ value , we accept H_0

Hence we conclude that there is no significant difference in Blood pressure after intake of a certain drug.

8. Memory capacity of 10 students were tested before and after training . State whether the training was effective or not from the following scores.

Sol : i) Null Hypothesis H_0: $\mu_1 = \mu_2$

ii) Alternative Hypothesis $H_1 : \mu_1 < \mu_2$ (Left one tailed test)

iii) Test Statistic $t_{cal} = \frac{\bar{d}}{s/\sqrt{n}}$

where $\bar{d} = \frac{\Sigma d}{n}$ and $S^2 = \frac{1}{n-1}\sum(d - \bar{d})^2$

Before(x)	After(y)	$d = y - x$	d^2
12	15	-3	9
14	16	-2	4
11	10	1	1
8	7	1	1
7	5	2	4
10	12	-2	4
3	10	-7	49
0	2	-2	4
5	3	2	4
6	8	-2	4
		$\sum d = -12$	$\sum d^2 = 84$

$\bar{d} = \frac{-12}{10}$ = -1.2

$S^2 = \frac{84 - (-1.2)^2 \text{x } 10}{9}$ = 7.73

$\therefore$ S = 2.78

$t_{cal} = \frac{\bar{d}}{s/\sqrt{n}} = \frac{-1.2}{2.78/\sqrt{10}}$ = -1.365 and d.f = n-1 = 9

Tabulated value of t for 9 d.f at 5% los is 1.833

Conclusion: Since $|t_{cal}|$ value $<$ $|t_\alpha|$ value , we accept H_0

Hence we conclude that there is no significant difference in memory capacity after the training program.

Chi-Square (χ^2) Distribution

Chi square distribution is a type of cumulative probability distribution . probability distributions provide the probability of every possible value that may occur . Distributions that are cumulative give the probability of a random variable being less than or equal to a particular value. Since the sum of the probabilities of every possible value must equal one , the total area under the curve is equal to one . Chi square distributions vary depending on the degrees of freedom. The degrees of freedom is found by subtracting one from the number of categories in the data .

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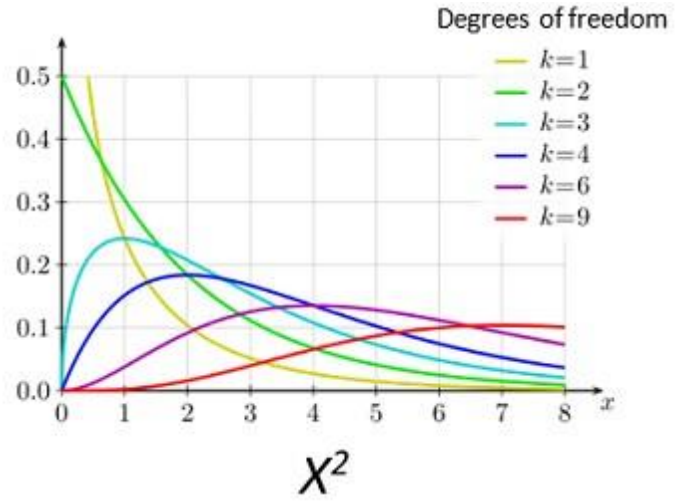


X^2

Applications of Chi – Square Distribution:

Chi – Square test as a test of goodness of fit :

χ^2– test enables us to ascertain how well the theoretical distributions such as binomial, Poisson, normal etc, fit the distributions obtained from sample data. If the calculated value of $\chi^2$is less than the table value at a specified level of generally 5% significance, the fit is considered to be good.

If the calculated value of χ^2 is greater than the table value, the fit is considered to be poor.

i) Null hypothesis: H_0 : There is no difference in given values and calculated values

ii) Altenative hypothesis: H_1 : There is some difference in given values and calculated values

iii) Test Statistic $\chi^2_{cal} = \sum \frac{(O-E)^2}{E}$

iv) At specified level of significance for n-1 d.f if the given problem is binomial distribution

At specified level of significance for n-2 d.f if the given problem is Poisson distribution

v)Conclusion :If χ^2_{cal} value < χ^2_{tab} value , then we accept H_0 , Otherwise reject H_0.

2. Chi – Square test for independence of attributes :

Definition : An attribute means a quality or characteristic
Eg: Drinking, Smoking, blindness, Honesty, beauty etc.,

An attribute may be marked by its presence or absence in a number of a given population.

Let us consider two attributes A and B.

A is divided into two classes and B is divided into two classes. The various cell frequencies can be expressed in the following table known as 2x2 contingency table.

a	b	a + b
c	d	c+ d
a + c	b + d	N =a + b + c + d

The expected frequencies are given by

$$E(a) = \frac{(a+c)(a+b)}{N}$$

$$E(b) = \frac{(b+c)(a+b)}{N}$$

$$E(c) = \frac{(a+c)(c+d)}{N}$$

$$E(d) = \frac{(b+d)(c+d)}{N}$$

$$\chi^2_{cal} = \sum \frac{(O-E)^2}{E}$$

χ^2_{cal} value to be compared with χ^2_{tab} value at 1% (5.1 or10%) level of significance for

(r-1) (c-1) d.f where r- number of rows

c-number of columns.

Note: In χ^2 distribution for independence of attributes, we test if two attributes A and B are independent or not.

i) Null Hypothesis: H_0 : The two attributes are independent

ii) Alternative hypothesis: H_1 : The two attributes are not independent

iii) Test Statistic $\chi^2_{cal} = \sum \frac{(O-E)^2}{E}$

where E = $\frac{\textbf{Row total x Column total}}{\textbf{Grand total}}$

iv) At specified level of significance for (m-1) (n-1) d.f where m- no. of rows and n- no. of columns

v) Conclusion : If $\boldsymbol{\chi^2}_{cal}$ value < $\boldsymbol{\chi^2}_{tab}$ value , then we accept H_0 , Otherwise reject H_0.

Problems :

1. Fit a Poisson distribution to the following data and test for its goodness of fit at 5% los

x	**0**	**1**	**2**	**3**	**4**
f	**419**	**352**	**154**	**56**	**19**

Sol:

X	f	fx
0	419	0
1	352	352
2	154	308
3	56	168
4	19	76
	N=1000	$\sum fx = 904$

Mean $\lambda = \frac{\sum fx}{N} = \frac{904}{1000} = 0.904$

Theoretical distribution is given by

= N x p(x) = 1000 x $\frac{e^{-\lambda}\lambda^x}{x!}$

Hence the theoretical frequencies are given by

x	0	1	2	3	4	Total
f = 1000 x $\frac{e^{-\lambda}\lambda^x}{x!}$	406.2	366	165.4	49.8	12.6	1000

Since Given frequencies total is equal to Calculated frequencies total.

To test for goodness of fit:

i) H_0 : There is no difference in given values and calculated values

ii) H_1 : There is some difference in given values and calculated values

iii) $\chi^2_{cal} = \sum \frac{(O-E)^2}{E}$

O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
419	406.2	$(419-406.2)^2$	$\frac{(419-406.2)^2}{406.2}$
352	366	$(352-366)^2$	$\frac{(352-366)^2}{366}$
154	165.4	$(154-165.4)^2$	$\frac{(154-165.4)^2}{165.4}$
56	49.8	$(56-49.8)^2$	$\frac{(56-49.8)^2}{49.8}$
19	12.6	$(19-12.6)^2$	$\frac{(19-12.6)^2}{12.6}$

$\sum \frac{(O-E)^2}{E} = 5.748$

Degrees of freedom = 5-2 = 3

χ^2_{tab} at 5% LOS = 7.82

Since χ^2_{cal} value < χ^2_{tab} ,we accept H_0 .

3. A die is thrown 264 times with following results. Show that the die is biased [Given $\chi^2_{0.05}$ = 11.07 for 5 d.f]

No. appeared on the die	**1**	**2**	**3**	**4**	**5**	**6**
Frequency	**40**	**32**	**28**	**58**	**54**	**52**

Sol: i) H_0 : The die is unbiased

ii) H_1 : The die is not unbiased

iii)$\chi^2_{\text{cal}} = \sum \frac{(O-E)^2}{E}$

The expected frequency of each of the number 1,2,3,4,5,6 is $\frac{264}{6} = 44$

Calculation of χ^2**:**

O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
40	44	16	0.3636
32	44	144	3.2727
28	44	256	5.8181
58	44	196	4.4545
54	44	100	2.2727
52	44	64	1.4545

$\sum \frac{(O-E)^2}{E} = 17.6362$

$\chi^2_{cal} = 17.6362$

The number of degrees of freedom = n-1 = 5

$\chi^2_{0.05}$ = 11.07 for 5 d.f

Since χ^2_{cal} value > χ^2_{tab} value , we reject H_0

Hence the die is biased

4. On the basis of information given below about the treatment of 200 patients suffering from disease , state whether the new treatment is comparatively Superior to the conventional treatment.

Treatment	**Favorable**	**Not Favorable**	**Total**

New	**60**	**30**	**90**
Conventional	**40**	**70**	**110**

Sol: i) H_0 : The two attributes are independent

ii) H_1 : The two attributes are not independent

iii) $\chi^2_{cal} = \sum \frac{(O-E)^2}{E}$

where E = $\frac{\textbf{Row total x Column total}}{\textbf{Grand total}}$

$\frac{90 \times 100}{200} = 45$	$\frac{90 \times 100}{200} = 45$	90
$\frac{100 \times 110}{200} = 55$	$\frac{100 \times 110}{200} = 55$	11
100	100	200

Calculation of χ^2:

O	E	$(O-E)^2$	$\frac{(O-E)^2}{E}$
60	45	225	5
30	45	225	5
40	55	225	4.09
70	55	225	4.09

$\sum \frac{(O-E)^2}{E} = 18.18$

$\chi^2_{cal} = 18.18$

χ^2_{tab} for 1 d.f . at 5% los is 3.841

since χ^2_{cal} value > χ^2_{tab} value , we reject H_0

Hence we conclude that new and conventional treatment are not independent.

Snedecor's F- Test of Significance

The F-Distribution is also called as Variance Ratio Distribution as it usually defines the ratio of the variances of the two normally distributed populations. The F-distribution got its name after the name of R.A. Fisher, who studied this test for the first time in 1924.

Symbolically, the quantity is distributed as F-distribution with and degrees of freedom $\vartheta_1 = n_1 - 1$and $\vartheta_2 = n_2 - 1$ is represented as:

$$F_{cal} = \frac{\text{Greater Variance}}{\text{Smaller Varinace}}$$

$$F_{cal} = \frac{{S_1}^2}{{S_2}^2} \text{ Or } \frac{{S_2}^2}{{S_1}^2}$$

Where,

${S_1}^2$ is the unbiased estimator of σ_1^2 and is calculated as: ${S_1}^2 = \frac{n_1 {s_1}^2}{n_1 - 1} = \frac{1}{n_1 - 1} \sum (x_1 - \bar{x}_1)^2$

${S_2}^2$ is the unbiased estimator of σ_2^2 and is calculated as: ${S_2}^2 = \frac{n_2 {s_2}^2}{n_2 - 1} = \frac{1}{n_2 - 1} \sum (x_2 - \bar{x}_2)^2$

To test the hypothesis that the two population variances $\boldsymbol{\sigma_1^2}$ and $\boldsymbol{\sigma_2^2}$ are equal

i) $H_0 : \sigma_1^2 = \sigma_2^2$

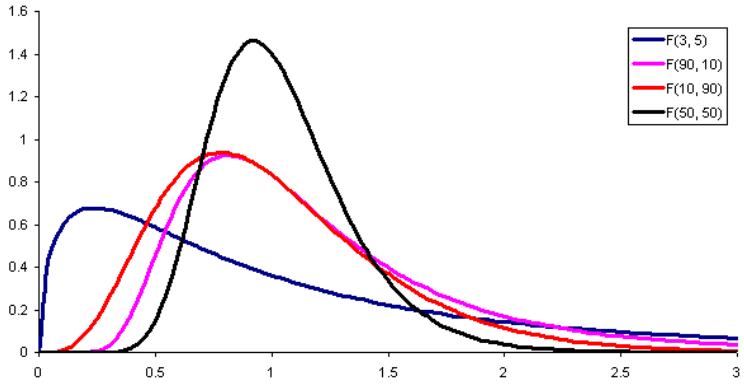
ii) $H_1 : \sigma_1^2 \neq \sigma_2^2$

iii) $F_{cal} = \frac{\textbf{Greater Variance}}{\textbf{Smaller Varinace}}$

iv) At specified level of significance (1% or 5 %) for $(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2)$ d.f

v) If $\mathbf{F}_{cal}$ value < $\mathbf{F}_{tab}$ value , then we accept H_0 , Otherwise reject H_0.

$F_{cal}(\vartheta_1, \vartheta_2)$ is the value of F with ϑ_1 and ϑ_2 degrees of freedom such that the area under the F – distribution to the right of F_α is α.



Problems:

1. **In one sample of 8 observations from a normal population, the sum of the squares of deviations of the sample values from the sample mean is 84.4 and in another sample of 10 observations it was 102.6. Test at 5% level whether the populations have the same varience**.

   Sol: Let σ_1^2 and σ_2^2 be the variances of the two normal populations from which the samples are drawn.

   Let the Null Hypothesis be H_0: $\sigma_1^2 = \sigma_2^2$

   Then the Alternative Hypothesis is H_1: $\sigma_1^2 \neq \sigma_2^2$

   Here $n_1 = 8, n_2 = 10$

   Also $\sum(x_i - \bar{x})^2 = 84.4, \sum(y_i - \bar{y})^2 = 102.6$

   If $S_1^2 and S_2^2$ be the estimates of σ_1^2 and σ_2^2 then

$$S_1^2 = \frac{1}{n_1 - 1}\sum(x_i - \bar{x})^2 = \frac{84.4}{7} = 12.057$$

   and

$$S_2^2 = \frac{1}{n_2 - 1}\sum(y_i - \bar{y})^2 = \frac{102.6}{9} = 11.4$$

   Let H_0 be true. Since $S_1^2 > S_2^2$, the test statistic is

$$F = \frac{S_1^2}{S_2^2} = \frac{12.057}{11.4} = 1.057$$

i.e., calculated F = 1.057.

Degrees of freedom are given by $v_1 = n_1 - 1 = 8 - 1 = 7$

and $v_2 = n_2 - 1 = 10 - 1 = 9$

Tabulated value of F at 5% level for $(7,9)$ degrees of freedom is 3.29

i.e.,$F_{0.05}(7,9) = 3.29$

Since calculated $F <$ tabulated F, we accept the Null Hypothesis H_0 and conclude that the populations have the same variance.

2.**The time taken by workers in performing a job by method I and method II is given below**

Method I	20	16	26	27	23	22	-
Method II	27	33	42	35	32	34	38

Do the data show that the variances of time distribution from population from which these samples are drawn do not differ significantly?

Sol: Let the Null Hypothesis be H_0: $\sigma_1^2 = \sigma_2^2$ where σ_1^2 and σ_2^2 are the variances of the two populations from with the samples are drawn.

The Alternative Hypothesis is H_1: $\sigma_1^2 \neq \sigma_2^2$.

Calculation of sample variances.

x	$x - \bar{x}$	$(x - \bar{x})^2$	y	$y - \bar{y}$	$(y - \bar{y})^2$
20	-2.3	5.29	27	-7.4	54.76
16	-6.3	39.69	33	-1.4	1.96
26	3.7	13.69	42	7.6	57.76
27	4.7	22.09	35	0.6	0.36
23	0.7	0.49	32	-2.4	5.76
22	-0.3	0.09	34	-0.4	0.16
			38	3.6	12.96

134		81.34	241		133.72

Given $n_1 = 6, n_2 = 7$

$$\therefore \bar{x} = \frac{\sum x}{n_1} = \frac{134}{6} = 22.3, \bar{y} = \frac{\sum y}{n_2} = \frac{241}{67} = 34.4$$

And

$$\sum(x_i - \bar{x})^2 = 81.34, \sum(y_i - \bar{y})^2 = 133.72$$

If $S_1^2 and S_2^2$ be the estimates of σ_1^2 and σ_2^2, then

$$S_1^2 = \frac{1}{n_1 - 1}\sum(x_i - \bar{x})^2 = \frac{81.34}{5} = 16.26$$

and

$$S_2^2 = \frac{1}{n_2 - 1}\sum(y_i - \bar{y})^2 = \frac{133.72}{6} = 22.29$$

Let H_0 be true

Since $S_2^2 > S_1^2$, the statistic is

$$F = \frac{S_2^2}{S_1^2} = \frac{22.29}{16.268} = 1.3699 = 1.37$$

$F_{0.05}(5,6)\ d.f = 4.39$

Since calculated F < tabulated F , we accept the null hypothesis H_0 at 5% los i.e., there is no significant difference between the variances of the distribution by the workers.

UNIT-V

QUEUING THEORY

Introduction:

Queues are essential in every day real life circumstances. Queues need not necessarily be only for human centered systems. These are applicable for other than human centered systems such as jobs in a computer system managed by its job manager or operating system.

Definition: If the jobs (people) arrive frequently they should wait to get their turn than the only way is to form a queue called the waiting line.

Definition: The jobs (people) arriving are called customers and the person (job manager) attending the job is called a server.

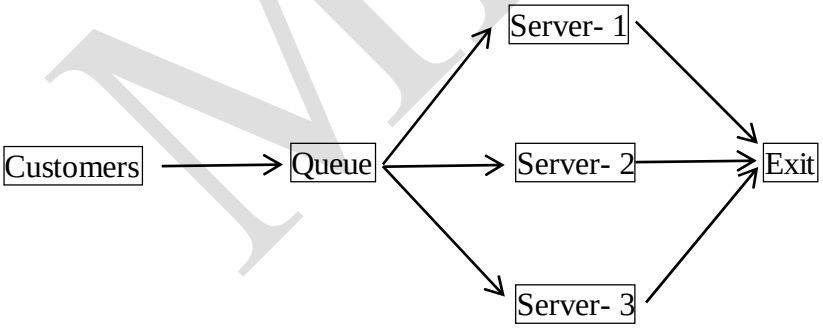
Note: The arriving units may form one line and be observed through only one server may form only one and be served through several servers, may form several lines and served through as many servers.

Servers may be in parallel or in servers. When in parallel the arriving units may form a single queue or individual queues in front of each server.

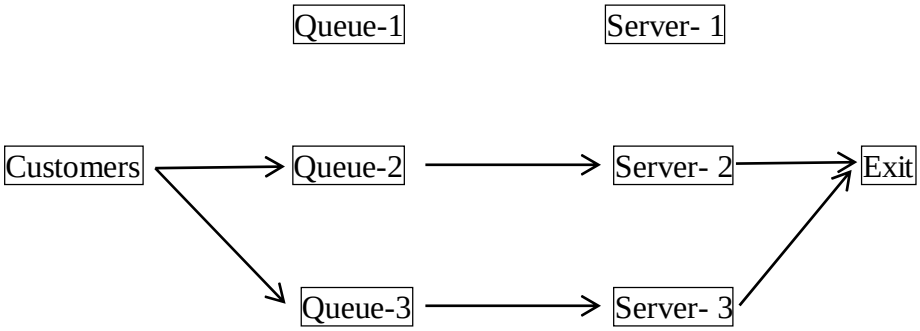
Example:

Queuing system with single queue and single server

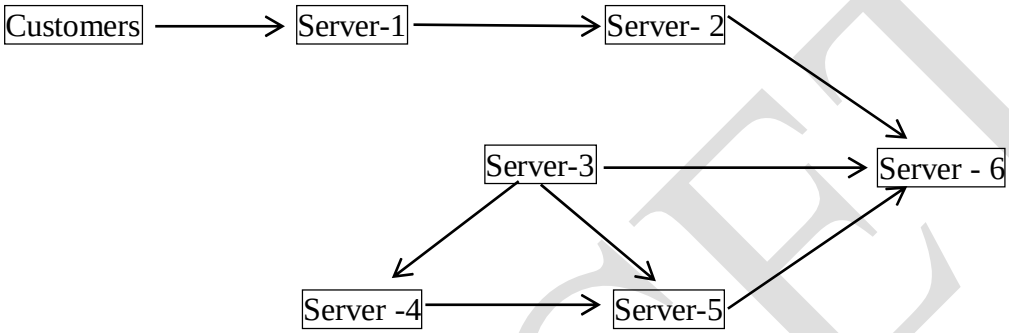
Example:



Queuing system with single queue and multiple servers.



Queuing system with multiple queues and multiple servers.



A complex Queuing system.

In general, a queuing system consists of one or more queues and one or more servers and operating procedures.

In any queuing system, the aim is to design the system in such a way, that the mean waiting time of the customers is minimized and with utilization of the server is managed above a desired level.

To understand and model queuing systems, the behaviour of the queuing system in the sense of waiting time for a customer, servicing time and the server's busy period should be studied.

Queuing theory is about the statistical description of the behaviour of queues. i.e., the probability distribution of the umber in the queue from which the mean and variance of queue length and the probability distribution of waiting time for a customer or the distribution of a server's busy time.

Queuing System:

A queuing system consists of

(i) The input (arrival pattern)

(ii) The service procedure (service pattern)

(iii) The queue discipline

(iv) Customer's behaviour.

The arrival pattern:

Definition: the arrival pattern tells the way in which the customers arrive and join the system. In general, the customers arrive randomly not suitable for prediction. Thus, the arrival pattern can be described in terms of probabilities for inter arrivals times (i.e., the time between two successive arrivals) or the distribution of number of customers arriving in unit time.

The service pattern:

Definition: The service pattern tells how many customers can be served at a time, and related statistical distribution of service time. It is a fact in most scenarios that service time is a random variable with the same distribution for all arrivals apart from the cases of different classes of customers.

The queue discipline:

Definition: It tells about the information of the queue, waiting and service. The basic one is first come, first served other are last come, first served and "service in random orders". Attributes and properties of a queuing system which are concerned with waiting times, in general, depend on queue discipline.

Notations:

FIFO → First in First out

LIFO → Last in, First out

SIRO → Service in Random order

We will discuss about FIFO only.

Customer's behaviour:

Generally, the behaviour of customer is of four ways:

(i) **Balking:** A customer may leave the queue because the queue is too long and he has no time to wait or there is not enough waiting space.

(ii) **Reneging:** this happens when a waiting customer leaves the queue due to impatience

(iii) Priorities: In some applications, some customers are served before others regardless of their other of arrival. These customers have priority over others.

Queuing problem:

The general problem of queuing is to determine the following:

(i) **Probability distribution of queue length:** if the nature of probability distribution of the arrival and service pattern is given, the probability distribution of queue length can be obtained.

(ii) Probability distribution of waiting time of customers: waiting is the time spent by a customer in the queue before the commencement of his service. The total time spent by him in the system is the waiting time plus service time.

(iii) The busy period distribution: suppose that the service is free initially and customer arrives, he will be served immediately. During his service time, some more customers may arrive and will be served. This will until no customer is left un-served and the server becomes free again. Then we say that busy period has just over.

Definition: A system is in "**transient state**", when its operating characteristics are dependent on time.

Definition: a system "**steady state**" when its operating characteristics are independent of time. We consider only steady state analysis.

Notation to be followed:

N=number of units (services or customers) in the system

$P_n(t)$ =transient state probability that exactly n calling units are in the queue are in the queue at time t.

E_n=the state in which n calling units in the system

P_n=steady state probability of having n units in the state

λ_n=mean arrival rate of customers

μ_n=mean service rate

λ=mean arrival rate (when λ_n constant for all n)

M=mean service rate (when μ_n is constant for all n)

s=number of parallel service stations

$\rho = \frac{\lambda}{\mu_s}$ traffic intensity for services facility

Queue size = number of customers in the queue

Queue length = queue size – number of units being served

$\psi(\omega)$ = p.d.f of waiting time in the system

L_s = expected line length(expected no. Of customers in the system)

L_q= expected queue length (expected no. Of customers in the queue)

W_s=expected waiting time per customer in the system

W_q= expected waiting time per customer in the queue

$(\omega/\omega>0)$ =expected waiting time of a customer

$(L/L>0)$ = expected no. Of customers in the queue when there is queue

$P(\omega>0)$ =probability of a customer to wait for service.

Pure Birth and Death process:

Definition: the term "birth" refers to the arrival of a new calling unit in the system .

Definition: the term "death" refers to the departure of a served unit.

Definition: the model in which only arrivals are counted and no departures take place is called "**pure birth model**".

Definition: the model with talks about departures is called "**pure death model**".

Theorem: Arrival probability distribution: when the arrivals are completely random, then the probability distribution of number of arrivals in a fixed time interval follows a Poisson distribution.

Theorem: If n is the number of arrivals in time, which follows Poisson distribution $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$ then T (the inter arrival time) follows the negative exponential law.

$A(T) = \lambda e^{-\lambda T}$ and vice versa. Which is the exponential law of probability for T with mean $\frac{1}{\lambda}$ and variance $\frac{1}{\lambda^2}$ i.e. $E(T) = \frac{1}{\lambda}$ and $V(T) = \frac{1}{\lambda^2}$.

Probability distribution of departures [Pure death process]

Assume that there are N customers in the system at time t =0.

Also, consider that no arrivals (births) can occur in the system. Departments occur at a rate μ per unit time. We would like to determine the distribution of departures from the system on the basis of the following three assumptions.

(i) Probability (one departure during δt) = $\mu\delta t + O(\delta t)^2 = \mu\delta$ [$O(\delta t)^2$ is negligible]

(ii) Probability (more than one departure during δt) = $O(\delta t)^2 = 0$

(iii) The number of departures in non-overlapping intervals is statistically independent and identically distributed random variable.i.e. The process N(t) has independent increments.

Considering different scenarios when n = N, 0 < n < N, n = 0.

We can have the following equations

$P'_N = -\mu P_N(t)$; n =N

$P'_n(t) = -\mu P_N(t) + \mu P_{n+1}(t)$; 0 < n < N

$P'_0(t) = \mu P_1(t)$; n = 0

Solving the above differential equations we have

$$P_n(t) = \begin{cases} \dfrac{(\mu t)^{N-n}}{(N-n)!} \text{ for n =1,2, ...,N} \\ 1 - \sum \dfrac{(\mu t)^{N-n}}{(N-n)!} \text{ for n = 0} \end{cases}$$

Hence the number of departures in time t follows the "truncated Poisson distribution".

(M/M/1): (∞/FIFO) OR (∞FCFS QUEUING SYSTEM (INFINITE QUEUE MODEL)

Arrivals are assumed to be births and departures are assumed to be deaths. Let λ and μ respectively denote the number of arrivals per unit time and number of departures per unit time. If *n* units are there in the system, we state that the system is in the state E_n. If now an

arrival occurs, the state of the system is E_{n+1}. If the system is in state E_n and a departure occurs, the state of the system will be E_{n-1}.

Let $P_n(t)$ be the probability that there are *n* customers in the system at time *t*.

If *n* customers are there in the system at time *t*, it means that then it is in state E_n. If *n* customers are there in the system at time $t + \delta t$, it means that then it is in state E_n.

Characteristics of (M/M/1):($\infty/FIFS$)

1. Probability that there are n customers in the system = $\rho^n(1-\rho)$
2. Probability that there are n or more customers in the system = ρ^n
3. Average number of customers in the system $E(n) = L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$
4. Average queue length = $L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$
5. Average length of non-empty queue $E(m/m > 0) = \frac{\mu}{\mu-\lambda}$
6. Variance of n where n is the number of customers in the system $V(n) = \frac{2\rho^2}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$
7. Average waiting time in the queue = $w_q = \frac{\rho}{\mu-\lambda} = \frac{\lambda}{\mu(\mu-\lambda)}$
8. Average waiting time in the system = $w_s = \frac{1}{\mu-\lambda}$

PROBLEMS

1. A TV/P.C repair man finds that the time spent on his jobs has an exponential distribution with mean 30 minutes. He repairs sets in the order in which they arrive. The arrival of the sets is approximately Poisson with an average of 10 per an eight hour day. Find the repairman's idle time each day .How many jobs are head of the average set just brought in?

   Solution: The repair man spends 30 minutes per job on an average

   Hence the service rate $\mu = 2\ sets\ /hour$

   The sets arrive at the average rate of 10 per 8 hours

   Hence the arrival rate, $\lambda = \frac{10}{8} = \frac{5}{4}\ sets/hour$

   Hence traffic intensity, $\rho = \frac{\lambda}{\mu} = \frac{5}{4} \cdot \frac{1}{2} = \frac{5}{8}$

   The repairman will be idle if there is no set in the system

$\therefore$ Probability that there is no set in the system = $P_0 = 1 - \rho = 1 - \frac{5}{8} = \frac{3}{8}$

Hence the expected idle time for the repairman in an 8 hour day $= \frac{3}{8}.8 = 3\ hours$

We have to find the number of jobs ahead of the average set just brought in. This is same as the average number of sets in the system i.e, $E(n) = \frac{\rho}{1-\rho} = \frac{5/8}{3/8} = \frac{5}{3} =$ one and $\frac{2}{3}$ jobs .Hence $1\frac{2}{3}$ jobs are ahead of the average set just brought in.

2. Patients arrive at a clinic in a Poisson manner at an average rate of 6 per hour. The doctor on average can attend to 8 patients per hour. Assuming that the service time distribution is exponential , find
   a) Average number of patients waiting in the queue
   b) Average time spent by a patient in the clinic

Sol. Given λ = Average arrival rate = 6 patients/hour

μ = Average service rate = 8 patients/hour

$\therefore$ Traffic intensity , $\rho = \frac{\lambda}{\mu} = \frac{6}{8} = \frac{3}{4}$

a) Average number of patients waiting in the queue = $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{36}{8(8-6)}$

$$= \frac{36}{8 \times 2} = \frac{36}{16} = \frac{9}{4}$$

b)Average time spent by a patient in the clinic = $W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6}$

$= \frac{1}{2}$ hour = 30minutes

(M/M/1): (N/FIFO) MODEL (FINITE MODEL)

Let assume that there are λ arrivals per unit time following a Poisson process and let there be μ services per unit time following a Poisson process. Let the maximum capacity in the system

be N. In this case, we have to formulate the difference equations concerning the probabilities P_n carefully.

Characteristics of the model

1. Average number of customers in the system = $\frac{N}{2}$(if $\rho = 1$)
   $= \frac{\rho[1-(N+1)\rho^N+N.\rho^{N+1}}{(1-\rho)(1-\rho^{N+1})}$(if $\rho \neq 1$)
2. Average queue length is given by-
   $= \frac{\rho^2[1-N.\rho^{N-1}+(N-1).\rho^N}{(1-\rho)(1-\rho^{N+1})}$(if $\rho \neq 1$)
   $= \frac{N(N-1)}{2(N+1)}$ (if$\rho = 1$)
3. Average waiting time in the system
   $= W_s = \frac{E(n)}{\lambda'}$where $\lambda' = \lambda(1 - P_N)$
   Here $\lambda' = \lambda(1 - P_N)$ is the mean rate of customers entering into the system.
   Average waiting time in the queue $= \frac{E(m)}{\lambda'}$

Problems

1. Consider a single server queuing system with Poisson input and exponential service time. Suppose the mean arrival rate is 3 calling units per hour with the expected service time as 0.25 hours and the maximum permissible number of calling units in the system is two. Obtain the steady state probability distribution of the number of calling units in the system and then calculate the expected number in the system.

   Solution: $\lambda =$ arrival rate $= 3$ units/hour
   Since expected service time is 0.25 hours, the service rate $\mu = 4$ units/hour.
   Traffic intensity $\rho = \frac{\lambda}{\mu} = \frac{3}{4}$
   Number of units allowed in the system $= N = 2$
   Let P_n be the probability for *n* units to be in the system.

$$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{1-3/4}{1-(3/4)^3} = \frac{16}{37}$$

$$P_1 = \rho.P_0 = \frac{12}{37}$$

$$P_2 = \rho^2.P_0 = \frac{9}{37}$$

The expected number in the system $= E(n) = \sum_{n=0}^{2} n.P_n = 0.P_0 + 1.P_1 + 2.P_2$

$$= \frac{30}{37}$$

There will be $\frac{30}{37}$units on average in the system.

2. A car park contains 5 cars. The arrival of cars is Poisson with a mean rate of 10 per hour. The length of time each car spends in the car park has negative exponential distribution with mean 2 hours. How many cars are in the car park on average and what is the probability of a newly arriving customer finding the car park full and having to park his car elsewhere?

Solution: The capacity of the car park $N = 5$

Cars arrive at the rate of 10 per hour

Hence, $\lambda = \frac{10}{6} = \frac{1}{6}$ per minute

A car stays in the system on average for 2hours

(i.e.) hence$\mu = \frac{1}{2}$ car/hour (i.e.) $\mu = \frac{1}{2\times60}$per minute

Hence, $\rho = \frac{10/60}{1/(2\times60)} = 20$

In this case, $P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{20-1}{20^6-1} = \frac{19}{(20^6-1)}$

Number of cars in the system on average $= \sum_{n=1}^{5} nP_n$

$$= 1.P_1 + 2.P_2 + 3.P_3 + 4.P_4 + 5.P_5$$

$$= (\rho + 2\rho^2 + 3\rho^3 + 4\rho^4 + 5\rho^5)P_0$$

And this can be simplified. This is left to the students as an exercise.