COURSE MODULE

ON

SIGNALS & SYSTEMS

II Year B. Tech. ECE – I Semester

Prepared by

Mr. KIRAN KUMAR VEMULA

Associate. Professor

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Autonomous Institution-UGC, Govt. of India)

2025-26

Recognized under 2(f) and 12 (B) of UGC ACT 1956
(Affiliated to JNTUH, Hyderabad, Approved by AICTE - Accredited by NBA & NAAC – 'A' Grade - ISO 9001:2015 Certified) Maisammaguda, Dhulapally (Post Via. Kompally), Secunderabad – 500100, Telangana State, India

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

II Year B.Tech. ECE- I Sem **L/T/P/C**

3/1/-/4

(R24A0403) SIGNALS AND SYSTEMS

COURSE OBJECTIVES:

The main objectives of the course are:

1) Knowledge of time-domain representation and analysis concepts of basic elementary signals
2) Knowledge of Fourier Series for Continuous Time Signals
3) Knowledge of frequency-domain representation and analysis concepts F.T., L.T. & Z.T and Concepts of the sampling process.
4) Mathematical and computational skills needed to understand the principal of Linear System and Filter Characteristics of a System.
5) Mathematical and computational skills needed to understand the concepts of auto correlation and cross correlation and power Density Spectrum.

UNIT I:

INTRODUCTION TO SIGNALS: Elementary Signals**-** Continuous Time (CT) signals, Discrete Time (DT) signals, Classification of Signals, Basic Operations on signals.

FOURIER SERIES: Representation of Fourier series, Continuous time periodic signals, Dirichlet's conditions, Trigonometric Fourier Series, Exponential Fourier Series, Properties of Fourier series, Complex Fourier spectrum.

UNIT II:

FOURIER TRANSFORMS: Deriving Fourier transform from Fourier series, Fourier transform of arbitrary signal, Fourier transform of standard signals, Properties of Fourier transforms.

SAMPLING: Sampling theorem **–** Graphical and analytical proof for Band Limited Signals, impulse sampling, Natural and Flat top Sampling, Reconstruction of signal from its samples, effect of under sampling – Aliasing.

UNIT III:

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS: Introduction to Systems, Classification of Systems, Linear Time Invariant (LTI) systems, impulse response, Transfer function of a LTI system. Filter characteristics of linear systems. Distortion less transmission through a system, Signal bandwidth, System bandwidth, Ideal LPF, HPF and BPF characteristics.

UNIT IV:

CONVOLUTION AND CORRELATION OF SIGNALS: Concept of convolution in time domain, Cross correlation and auto correlation of functions, properties of correlation function, Energy density spectrum, Parseval's theorem, Power density spectrum, Relation between convolution and correlation.

UNIT V:

LAPLACE TRANSFORMS: Review of Laplace transforms, Inverse Laplace transform, Concept of region of convergence (ROC) for Laplace transforms, Properties of L.T's relation between L.T's, and F.T. of a signal.

Z–TRANSFORMS: Concept of Z- Transform of a discrete sequence. Distinction between Laplace, Fourier and Z transforms, Region of convergence in Z-Transform, Inverse Z- Transform, and Properties of Z-transforms.

TEXT BOOKS:

1) "Signals & Systems", Special Edition – MRCET, McGraw Hill Publications, 2017
2) Signals, Systems & Communications – B.P. Lathi, BS Publications, 2003.
3) Signals and Systems – A.V. Oppenheim, A.S. Willsky and S.H. Nawab, PHI, 2ndEdn.
4) Signals and Systems – A. Anand Kumar, PHI Publications, 3rd edition.

REFERENCE BOOKS:

1) Signals & Systems – Simon Haykin and Van Veen, Wiley, 2nd Edition.
2) Network Analysis – M.E. Van Valkenburg, PHI Publications, 3rd Edn.,2000.
3) Fundamentals of Signals and Systems Michel J. Robert, MGH International Edition,2008.
4) Signals, Systems and Transforms – C. L. Philips, J. M. Parr and Eve A. Riskin, Pearson education.3rd Edition, 2004.

COURSE OUTCOMES:

After completion of the course, the student will be able to:

1) Understand the basic elementary signals
2) Determine the Fourier Series for Continuous Time Signals
3) Analyze the signals using F.T, L.T & Z.T and study the properties of F.T., L.T. & Z.T.
4) Understand the principal of Linear System and Filter Characteristics of a System.
5) Understand the concepts of auto correlation and cross correlation and power Density Spectrum.

UNIT – I

INTRODUCTION TO SIGNALS

1.1 Introduction to Signals:

In typical applications of science and engineering, we have to process signals, using systems. While the applications can be varied large communication systems to control systems but the basic analysis and design tools are the same. In a signals and systems course, we study these tools: convolution, Fourier analysis, z-transform, and Laplace transform.

The use of these tools in the analysis of linear time-invariant (LTI) systems with deterministic signals. For most practical systems, input and output signals are continuous and these signals can be processed using continuous systems. However, due to advances in digital systems technology and numerical algorithms, it is advantageous to process continuous signals using digital systems by converting the input signal into a digital signal. Therefore, the study of both continuous and digital systems is required. As most practical systems are digital and the concepts are relatively easier to understand, we describe discrete signals and systems, immediately followed by the corresponding description of continuous signals and systems.

Continuous time signal

continuous time views variables as having a particular value for potentially only an infinitesimally short amount of time. Between any two points in time there are an infinite number of other points in time. The variable "time" ranges over the entire real number line, or depending on the context, over some subset of it such as the non-negative reals. Thus time is viewed as a continuous variable.

A continuous signal or a continuous-time signal is a varying quantity (a signal) whose domain, which is often time, is a continuum (e.g., a connected interval of the reals). That is, the function's domain is an uncountable set. The function itself need not be continuous. To contrast, a discrete time signal has a countable domain, like the natural numbers.

A signal of continuous amplitude and time is known as a continuous-time signal or an analog signal. This (a signal) will have some value at every instant of time. The electrical signals derived in proportion with the physical quantities such as temperature, pressure, sound etc. are generally continuous signals. Other examples of continuous signals are sine wave, cosine wave, triangular wave etc.

The signal is defined over a domain, which may or may not be finite, and there is a functional mapping from the domain to the value of the signal. The continuity of the time variable, in connection with the law of density of real numbers, means that the signal value can be found at any arbitrary point in time.

Discrete time signal:

Discrete time views values of variables as occurring at distinct, separate "points in time", or equivalently as being unchanged throughout each non-zero region of time ("time period")—that is, time is viewed as a discrete variable. Thus a non-time variable jumps from one value to another as time moves from one time period to the next. This view of time corresponds to a digital clock that gives a fixed reading of 10:37 for a while, and then jumps to a new fixed reading of 10:38, etc. In this framework, each variable of interest is measured once at each time period. The number of measurements between any two time periods is finite. Measurements are typically made at sequential integer values of the variable "time".

A **discrete signal** or **discrete-time signal** is a time series consisting of a sequence of quantities.

Unlike a continuous-time signal, a discrete-time signal is not a function of a continuous argument; however, it may have been obtained by sampling from a continuous-time signal. When a discrete-time signal is obtained by sampling a sequence at uniformly spaced times, it has an associated sampling rate.

Discrete-time signals may have several origins, but can usually be classified into one of two groups:

- By acquiring values of an analog signal at constant or variable rate. This process is called sampling.[2]
- By observing an inherently discrete-time process, such as the weekly peak value of a particular economic indicator.

1.2 Classification of the Signals:

The Signals can be classified into several categories depending upon the criteria and for its classification. Broadly the signals are classified into the following categories

1. Continuous, Discrete and Digital Signals

1 Periodic and Aperiodic Signals

2 Even and Odd Signals

3 Power and Energy Signals

4 Deterministic and Random signals

Continuous-time and Discrete-time Signals:

Continuous-Time (CT) Signals: They may be de ned as continuous in time and continuous in amplitude as shown in Figure 1.1. Ex: Speech, audio signals etc..
Discrete Time (DT) Signals: Discretized in time and Continuous in amplitude. They may also be defined as sampled version of continuous time signals. Ex: Rail track signals.
Digital Signals: Discretized in time and quantized in amplitude. They may also be defined as quantized version of discrete signals.

Periodic Signals

A CT signal x(t) is said to be periodic if it satisfies the following condition

$$x(t) = x(t + T_0) \qquad (1.1)$$

The smallest positive value of T_0 that satisfies the periodicity condition Eq.(1.1), is referred as the fundamental period of x(t). The reciprocal of fundamental period of a signal is fundamental frequency f_0.
Likewise, a DT signal x[n] is said to be periodic if it satisfies

$$x(n) = x(n + N_0) \qquad (1.2)$$

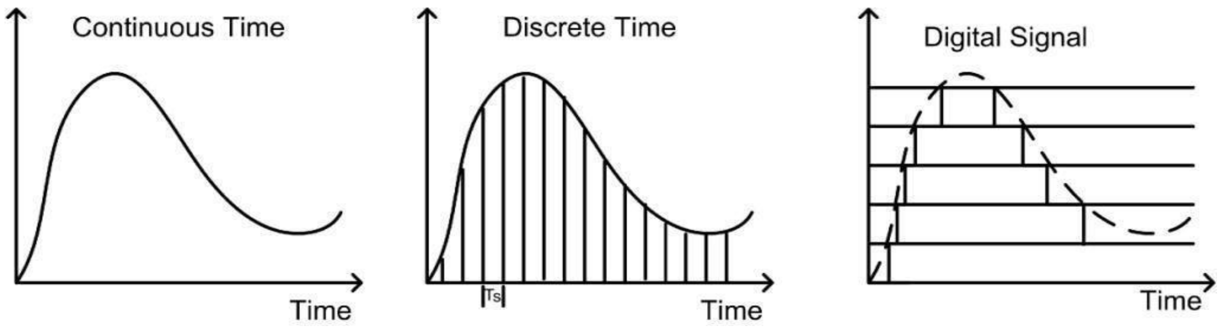


Figure 1.1: Description of Continuous, Discrete and Digital Signals

The smallest positive value of N0 that satis es the periodicity condition Eq.(1.2) is referred to as the fundamental period of x [n].
Note: All periodic signals are ever lasting signals i.e. they start at -1 and end at +1 as shown in below Figure 1.2.

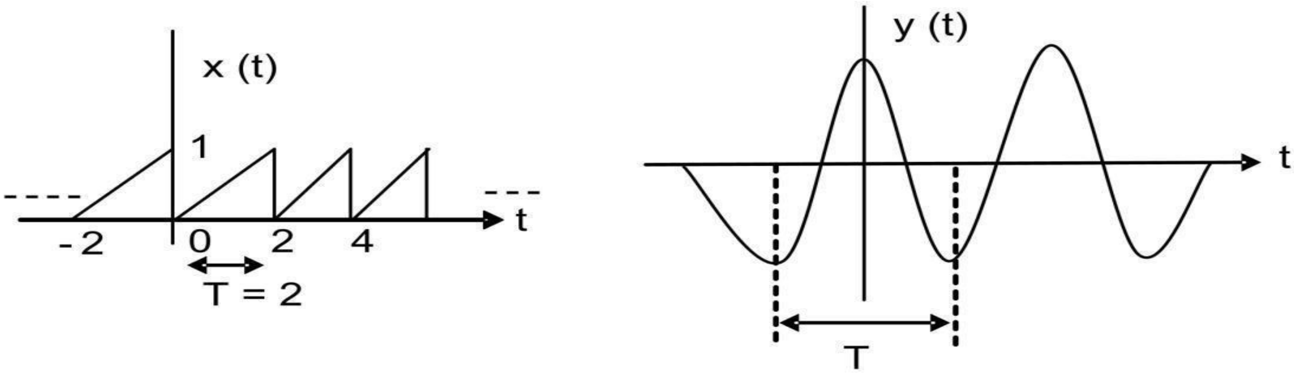
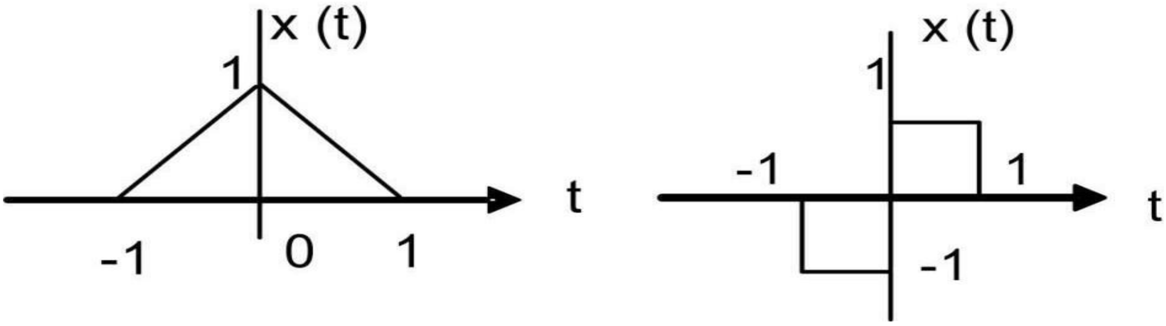


Figure 1.2: A typical periodic signal

Ex.1.1 Consider a periodic signal is a sinusoidal function represented as x (t) = A sin (10t + 20)
The time period of the signal T_0 is 10
Ex.1.2 CT tangent wave: x (t) = tan (10t) is a periodic signal with period T =10
Note: Amplitude and phase difference will not affect the time period.
i.e. 2 sin (3t), 4 sin (3t), 4 sin (3t + 32) will have the same time period

<u>Even and Odd Signal</u>

Any signal can be called even signal if it satisfies x(t) = x(-t) or x(n) = x(-n). Similarly any signal can be called odd signal if it not satisfies x(t) = x(-t) or x(n) = x(-n). Figure 1.2, shows an example of an even and odd signal whereas Figure 1.3 shows neither even nor odd signal.



Any signal X(t) can be expressed in terms of even component $X_e(t)$ and odd component $X_0(t)$.
X(t) =$X_e(t)$+$X_o(t)$, $X_e(t)$+=(X(t) + X(-t)) / 2, $X_o(t)$+=(X(t) - X(-t)) / 2

Energy and Power signals

A signal x(t) (or) x(n) is called an energy signal if total energy has a non - zero finite value i.e. 0 < Ex < 1 and Pavg = 0
A signal is called a power signal if it has non - zero nite power i.e. 0 < Px < 1 and E = 1.
A signal can't be both an energy and power signal simultaneously. The term instantaneous power is reserved for the true rate of change of energy in a system. All periodic signals are power signals and all finite durations signals are energy signals.

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j).$$

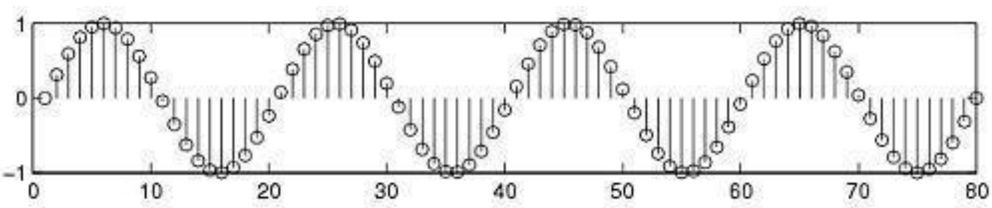
$$P_x = \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2$$

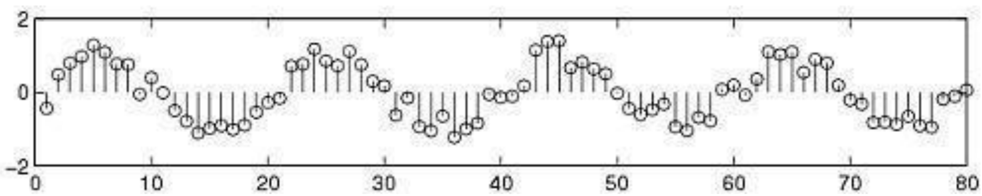
A signal is referred to as a power signal if the power P_x satisfies the condition

$$0 < P_x < \infty$$

Deterministic and Random signal

A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table. Because of this the future values of the signal can be calculated from past values with complete confidence. On the other hand, a random signal has lot of uncertainty about its behaviour. The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals.





1.3 Basic operations on continuous time signal

Basic operations on continuous time signal

In this section we will study some basic operations on continuous time signal:

A) Operation performed on dependent variables:

These operations include sum, product, difference, even, odd, etc.

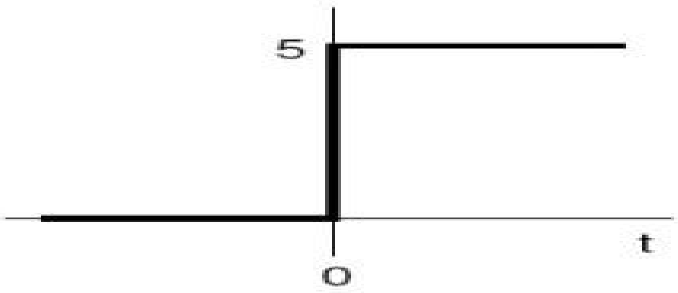
1) Amplitude scaling:

Amplitude scaling means changing an amplitude of given continuous time signal. We will denote continuous time signal by x(t). If it is multiplied by some constant 'B' then resulting signal is,

y(t)= B x(t)

Example: Sketch y(t) = 5u(t)

Solution: we know that u(t) is unit step function. So if we multiply it with 5, its amplitude will become 5 and it shown as follows:



amplitude scaling

2) Sum and difference of two signals:

Consider two signals x1(t) and x2(t). Then addition of these signals is denoted by y(t)=x1(t)+x2(t). similarly subtraction is given by y(t)=x1(t)-x2(t).

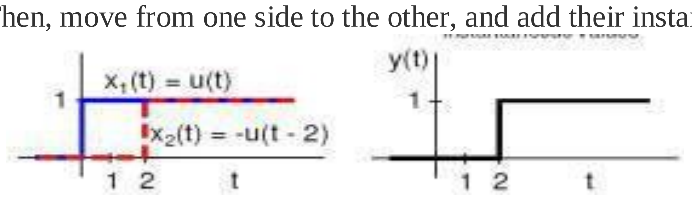
Example: Sketch y(t) = u(t) – u(t – 2)

Solution: First, plot each of the portions of this signal separately

• x1(t) = u(t) …….Simply a step signal

• x2(t) = –u(t-2) ……. Delayed step signal by 2 units and multiplied by -1.

Then, move from one side to the other, and add their instantaneous values:



Sum and difference of two signals

3) Product of two signals:

If x1(t) and x2(t) are two continuous signals then the product of x1(t) and x2(t) is,

Y(t) = x1(t)x2(t).

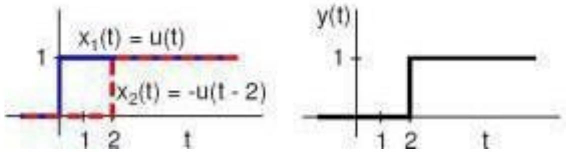
Example: Sketch y(t) = u(t)·u(t – 2)

Solution: First, plot each of the portions of this signal separately

• x1(t) = u(t) _ Simply a step signal

• x2(t) = u(t-2) _ Delayed step signal

Then, move from one side to the other, and multiply instantaneous values:



Multiplication of two signals

4) Even and odd parts:

Even part of signal x(t) is given by,

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

And odd part of x(t) is given by,

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

B) Operations performed on independent variables:

1) Time shifting:

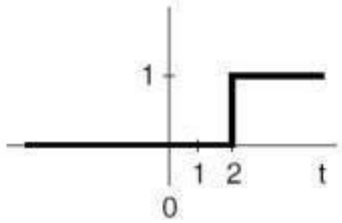
A signal x(t) is said to be 'shifted in time' if we replace t by (t-T). thus x(t-T) represents the time shifted version of x(t) and the amount of time shift is 'T' sec. if T is positive then the shift is to right (delay) and if T is negative then the shift is to the left (advance).

Example: Sketch y(t) = u(t – 2)

Method 1

Let "a" be the argument of "u"

$$y(a)=\begin{cases}1 & a\geq 0\\ 0 & a<0\end{cases}=\begin{cases}1 & t-2\geq 0\\ 0 & t-2<0\end{cases}=\begin{cases}1 & t\geq 2\\ 0 & t<2\end{cases}$$



Method 2 (by inspection)

Simply shift the origin to $t_d = 2$

solution:

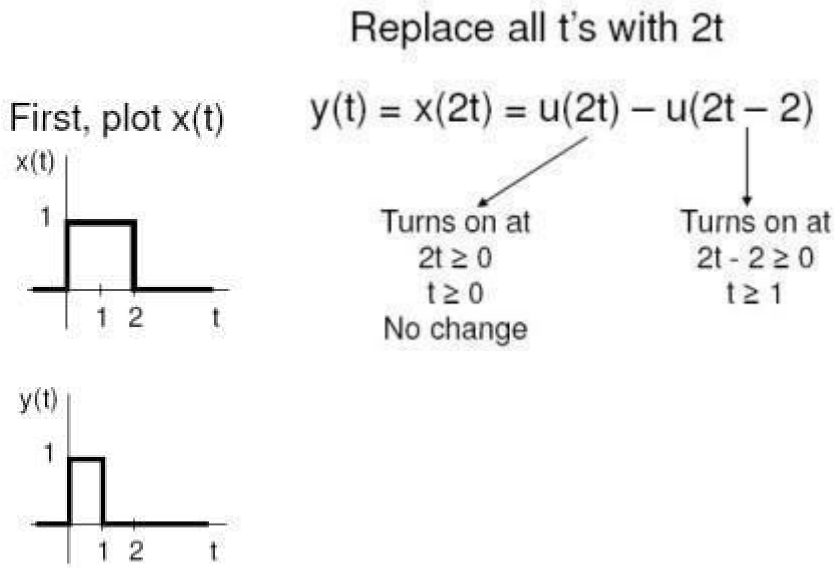
2) Time scaling:

The compression or expression of a signal in time is known as the time scaling. If x(t is the original signal then x(at) represents its time scaled version. Where a is constant.

If a> 1 then x(at) will be a compressed version of x(t) and if a< 1 then it will be a expanded version of x(t).

Example: Let x(t) = u(t) – u(t – 2). Sketch y(t) = x(t/2)

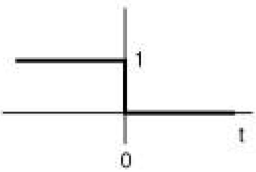
solution:



3) Time reversal (Time inversion):

Flips the signal about the y axis. $y(t) = x(-t)$.

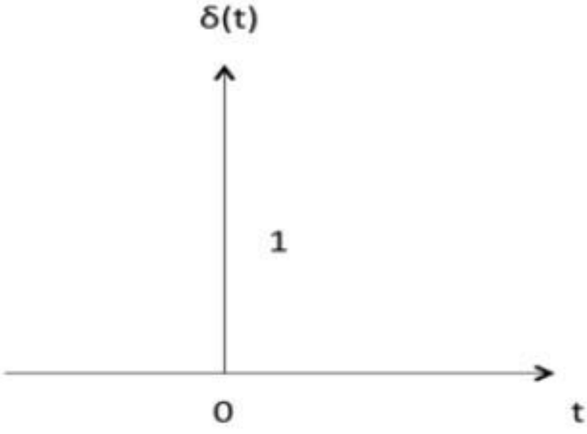
Example: Let $x(t) = u(t)$, and perform time reversal.



1.4. Standard Signals

Unit Impulse Function

Impulse function is denoted by δ(t). and it is defined as δ(t) = $\begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$

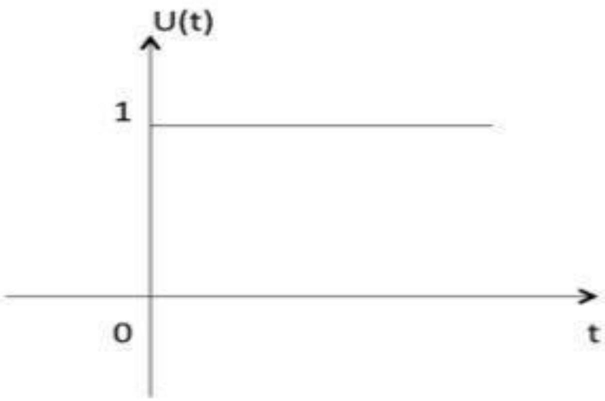


$$\int_{-\infty}^{\infty} \delta(t)dt = u(t)$$

$$\delta(t) = \frac{du(t)}{dt}$$

Unit Step Function

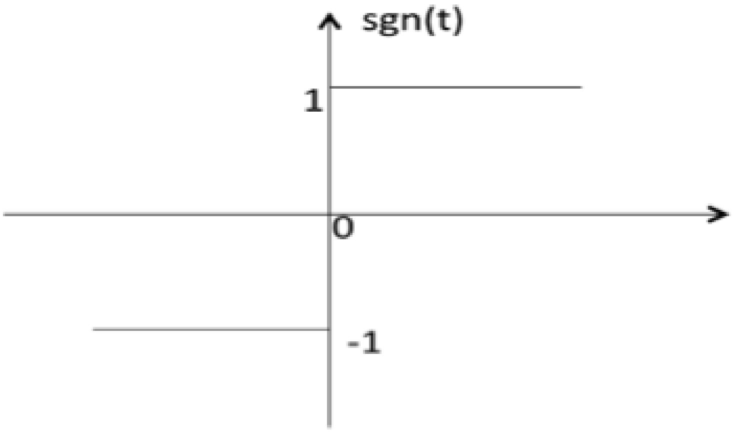
Unit step function is denoted by u(t). It is defined as u(t) = $\begin{cases} 1 & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



- It is used as best test signal.
- Area under unit step function is unity.

Signum Function

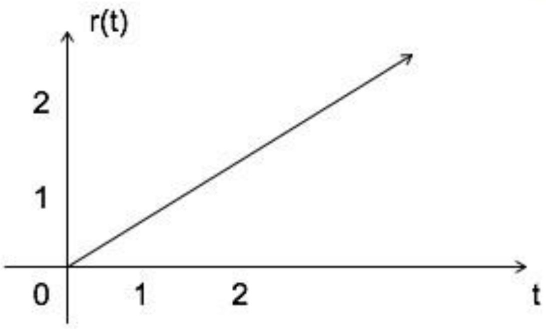
Signum function is denoted as sgn(t). It is defined as sgn(t) = $\begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$



sgn(t) = 2u(t) − 1

Ramp Signal

Ramp signal is denoted by r(t), and it is defined as r(t) = $\begin{cases} t & t \geqslant 0 \\ 0 & t < 0 \end{cases}$



$$\int u(t) = \int 1 = t = r(t)$$

$$u(t) = \frac{dr(t)}{dt}$$

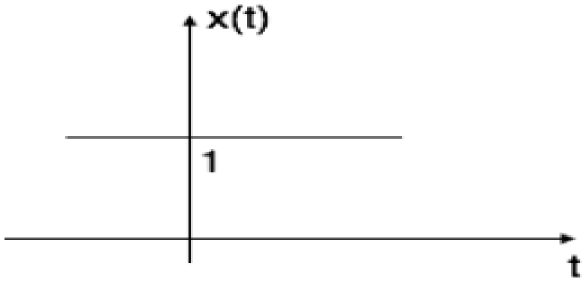
Area under unit ramp is unity.

Exponential Signal

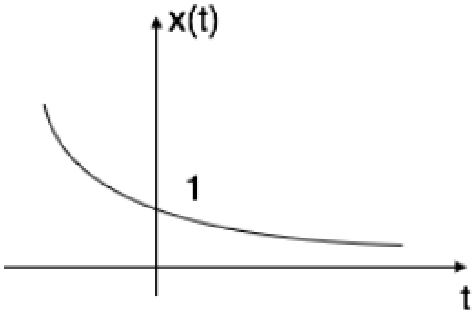
Exponential signal is in the form of $x(t) = e^{\alpha t}$.

The shape of exponential can be defined by α.

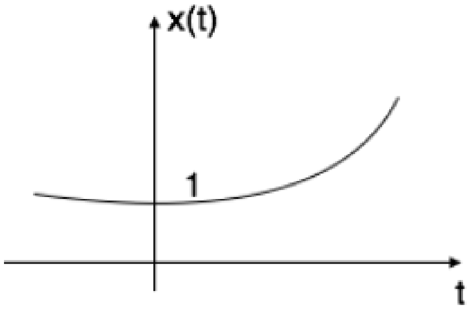
Case i: if $\alpha = 0 \rightarrow x(t) = e^{0} = 1$



Case ii: if $\alpha < 0$ i.e. -ve then $x(t) = e^{-\alpha t}$. The shape is called decaying exponential.



Case iii: if $\alpha > 0$ i.e. +ve then $x(t) = e^{\alpha t}$. The shape is called raising exponential.



1.5. FOURIER SERIES

Fourier series representation of Periodic signals

Introduction:

A signal is said to be a continuous time signal if it is available at all instants of time. A real time naturally available signal is in the form of time domain. However, the analysis of a signal is far more convenient in the frequency domain. These are three important classes of transformation methods available for continuous-time systems. They are:

1. Fourier series 2. Fourier Transform 3. Laplace Transform

Out of these three methods, the Fourier series is applicable only to periodic signals, i.e. signals which repeat periodically over $-\infty < t < \infty$. Not all periodic signals can be represented by Fourier series. Fourier series is to project periodic signals onto a set of basic functions. The basis functions are orthogonal and any periodic signal can be written as a weighted sum of these basis functions.

Representation of Fourier Series

The representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions is called Fourier Series. The Fourier analysis is also sometimes called the harmonic analysis. Fourier series is applicable only for periodic signals. It cannot be applied to non-periodic signals. A periodic signal is one which repeats itself at regular intervals of time, i.e. periodically over $-\infty$ to ∞. Three important classes of Fourier series methods are available.

They are: 1. Trigonometric form 2. Cosine form 3. Exponential form

1.6. Dirichlet's conditions

1) The function x(t) is absolutely integrable over one period, that is

$$\int_0^T |x(t)|\, dt < \infty.$$

2) The function x(t) has only a finite number of maxima and minima.

3) The function x(t) has a finite number of discontinuities.

1.7. Trigonometric Fourier series:

A sinusoidal signal, $x(t)=A \sin \omega_0 t$ is a periodic signal with period $T=2\pi/\omega_0$. Also, the sum of two sinusoids is periodic provided that their frequencies are integral multiples of a fundamental frequency ω_0. We can show that a signal x(t), a sum of sine and cosine functions whose frequencies are integral multiples of ω_0, is a periodic signal.

Let the signal x(t) be

$$x(t)= a_0+a_1\cos\omega_0 t+a_2\cos 2\omega_0 t+\ldots+a_k\cos k\omega_0 t+ b_1\sin\omega_0 t+b_2\sin 2\omega_0 t+\ldots+b_k\sin k\omega_0 t$$

i.e. $$x(t)= a_0+\sum_{n=1}^{k}[a_n \cos \omega_0 nt + b_n \sin\omega_0 nt]$$

where $a_0, a_1, a_2,\ldots\ldots a_k$ and $b_0, b_1, b_2,\ldots\ldots b_k$ are constants, and ω_0 is the fundamental frequency.

For the signal x(t) to be periodic, it must satisfy the condition x(t) = x(t+T) for all t,

i.e. $$x(t+T)= a_0+\sum_{n=1}^{k}[a_n \cos \omega_0 n(t+T) + b_n \sin\omega_0 n(t+T)]$$

$$= a_0+\sum_{n=1}^{k}[a_n \cos \omega_0 n(t+2\pi/\omega_0) + b_n \sin\omega_0 n(t+2\pi/\omega_0)]$$

$$= a_0+\sum_{n=1}^{k}[a_n \cos (\omega_0 nt+2n\pi) + b_n \sin(\omega_0 nt+2n\pi)]$$

$$= a_0+\sum_{n=1}^{k}[a_n \cos \omega_0 nt + b_n \sin\omega_0 nt]$$

$$=x(t)$$

1.8.Exponential Fourier series:

The exponential Fourier series is the most widely used form of Fourier series. In this, the function x(t) is expressed as a weighted sum of the complex exponential functions. The complex exponential form is more general and usually more convenient and more compact. So, it is used almost exclusively, and it finds extensive application in communication theory.

The set of complex exponential functions

$$\{e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2, \ldots\}$$

forms a closed orthogonal set over an interval $(t_0,\ t_0+T)$ where $T= (2\pi/\omega_0)$ for any value of t_0, and therefore it can be used as a Fourier series. Using Euler's identity, we can write

$$A_n \cos(n\omega_0 t + \theta_n) = A_n\left[\frac{e^{j(n\omega_0 t+\theta n)} + e^{-j(n\omega_0 t+\theta n)}}{2}\right]$$

Substituting this in the definition of the cosine Fourier representation, we obtain

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{An}{2}\left[e^{j(n\omega_0 t+\theta n)} + e^{-j(n\omega_0 t+\theta n)}\right]$$

$$= A_0 + \sum_{n=1}^{\infty} \frac{An}{2}\left[e^{jn\omega_0 t}e^{j\theta n} + e^{-jn\omega_0 t}e^{-j\theta n}\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{jn\omega_0 t}e^{j\theta n}\right) + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{-jn\omega_0 t}e^{-j\theta n}\right)$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{j\theta n}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{-j\theta n}\right)e^{-jn\omega_0 t}$$

Letting n = -K in the second summation of the above equation, we have

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}e^{j\theta n}\right)e^{jn\omega_0 t} + \sum_{k=-1}^{-\infty}\left(\frac{A_k}{2}e^{j\theta_k}\right)e^{jk\omega_0 t}$$

on comparing the above two equations for x(t), we get

$$A_n = A_k; \qquad (-\theta_n) = \theta_k \qquad n > 0$$
$$k < 0$$

let us define

$$C_0 = A_0;\ C_n = \frac{An}{2}e^{j\theta n},\ n > 0$$

$\therefore$ $$x(t) = A_0 + \sum_{n=1}^{\infty}\frac{A_n}{2}e^{j\theta n}e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty}\frac{A_n}{2}e^{j\theta_n}e^{jk\omega_0 t}$$

i.e. $$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

This is known as exponential form of Fourier series. The above equation is called the synthesis equation.

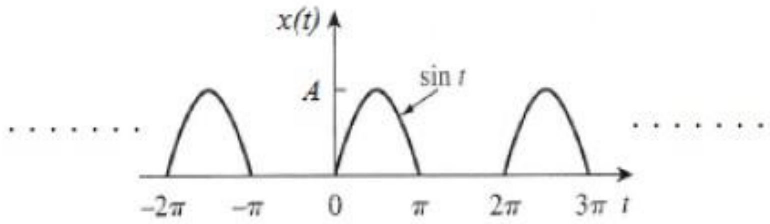
So the exponential series from cosine series is:

$$C_0 = A_0$$
$$C_n = \frac{An}{2}e^{j\theta n}$$

Problems:

Solved Problems:

Problem 1: Find the Fourier series expansion of the half wave rectified sine wave shown in fig below



Solution :

The periodic waveform shown in fig with period 2π is half of a sine wave with period 2π.

$$x(t) = \begin{cases} A \sin\omega t = A \sin\frac{2\pi}{2\pi}t = A\sin t & 0 \le t \le \pi \\ 0 & \pi \le t \le 2\pi \end{cases}$$

Now the fundamental period T=2π

Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Let $t_0 = 0$, $t_0 + T = T = 2\pi$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} A\sin t\,dt = \frac{A}{\pi}$$

$$\therefore a_0 = \frac{A}{\pi}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\omega_0 nt\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi} A\sin t\cos nt\,dt$$

$$= -\frac{A}{2\pi}\left\{\left[\frac{(-1)^{n+1}-1}{1+n}\right] + \left[\frac{(-1)^{n+1}+1}{1-n}\right]\right\}$$

For odd n, $a_n = -\frac{A}{2\pi}\left\{\left[\frac{1-1}{1+n}\right] + \left[\frac{1+1}{1-n}\right]\right\} = 0$

For even n, $a_n = -\frac{A}{2\pi}\left\{\left[\frac{-1-1}{1+n}\right] + \left[\frac{-1-1}{1-n}\right]\right\} = -\frac{2A}{\pi(n^2-1)}$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\omega_0 nt\,dt = b_n = \frac{2}{2\pi}\int_0^{2\pi} x(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} A\sin t\sin nt\,dt$$

$$= \frac{A}{2\pi}\left[\frac{\sin(n-1)t}{n-1} - \frac{\sin(n+1)t}{n+1}\right]_0^{\pi}$$

This is zero for all values of n except for n=1.

For n=1, $b_1 = \frac{A}{2\pi}$

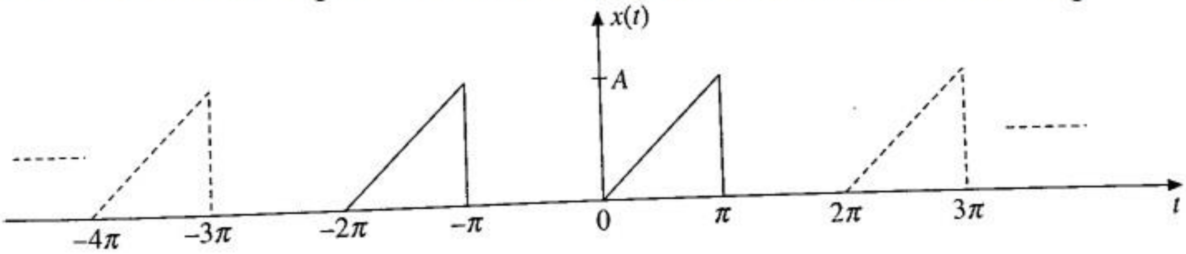
Therefore, the trigonometric Fourier series is:

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n\cos\omega_0 nt + b_n\sin\omega_0 nt]$$

$$= a_0 + b_1\sin t + \sum_{n=1}^{\infty} a_n\cos nt$$

$$= \frac{A}{\pi} + \frac{A}{2\pi}\sin t - \sum_{n=1}^{\infty}\frac{2A}{\pi(n^2-1)}\cos nt$$

Problem 2: Obtain the trigonometric Fourier series for the waveform shown in below fig



Solution:

The waveform shown in fig above is periodic with a period T=2π

Let $t_0 = 0$, $t_0 + T = T = 2\pi$

Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

The waveform is described by

$$x(t) = \begin{cases} \left(\frac{A}{\pi}\right)t & 0 \le t \le \pi \\ 0 & \pi \le t \le 2\pi \end{cases}$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} \frac{A}{\pi}t\,dt = \frac{A}{2\pi^2}\left[\frac{t^2}{2}\right]_0^{\pi} = \frac{A}{4}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos\omega_0 nt\,dt = \frac{2}{2\pi}\int_0^{\pi} \frac{A}{\pi}t\cos nt\;dt = \frac{A}{\pi^2 n^2}(\cos n\pi - \cos 0)$$

$$\therefore \qquad a_n = \begin{cases} -\left(\frac{2A}{\pi^2 n^2}\right) & \text{for odd n} \\ 0 & \text{for even n} \end{cases}$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin\omega_0 nt\,dt = \frac{2}{2\pi}\int_0^{2\pi} x(t)\sin nt\,dt = \frac{1}{2\pi}\int_0^{\pi} \frac{A}{\pi}t\sin nt\,dt = \frac{A}{n\pi}(-1)^{n+1}$$

$$\therefore \qquad b_n = \begin{cases} \left(\frac{A}{\pi n}\right) & \text{for odd n} \\ -\left(\frac{A}{\pi n}\right) & \text{for even n} \end{cases}$$

The trigonometric Fourier series is :

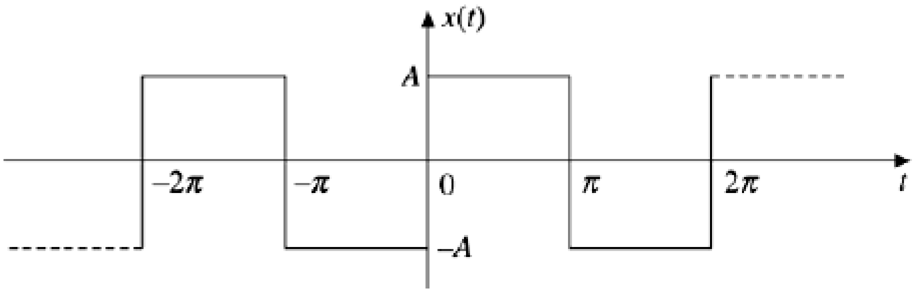
$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\omega_0 nt + b_n \sin\omega_0 nt]$$

$$= \frac{A}{4} - \left(\frac{2A}{\pi^2}\right)\sum_{n=odd}^{\infty}\frac{\cos nt}{n^2} + \frac{A}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nt}{n}$$

$$= \frac{A}{4} - \left(\frac{2A}{\pi^2}\right)\left[\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right]$$

$$+ \frac{A}{\pi}\left[\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \ldots\right]$$

3) Obtain the exponential Fourier Series for the wave form shown in below figure



Solution: The periodic waveform shown in fig with a period T= 2π can be expressed as:

$$x(t) = \begin{cases} A & 0 \le t \le \pi \\ -A & \pi \le t \le 2\pi \end{cases}$$

Let

$$t_0 = 0,\ t_0 + T = 2\pi$$

and Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Exponential Fourier series

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} A\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} -A\,dt = 0$$

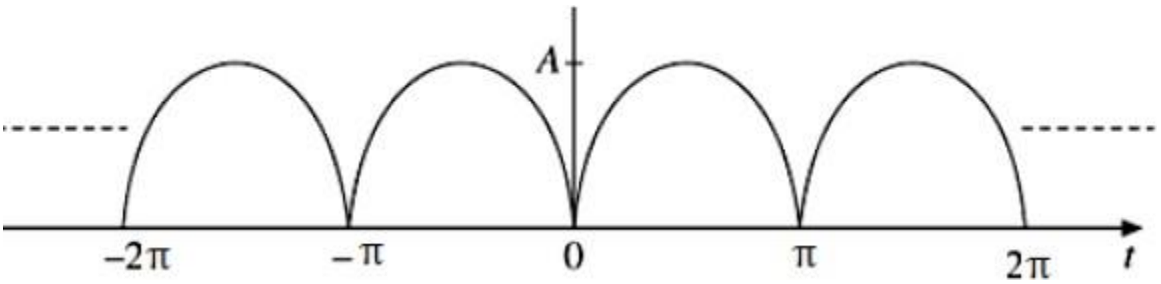
$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} Ae^{-jnt}\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} -Ae^{-jnt}\,dt$$

$$= -\frac{A}{j2n\pi}[(-1)^n - 1] - [1 - (-1)^n] = -j\frac{A}{2n\pi}$$

$$C_n = \begin{cases} \left(-j\frac{2A}{\pi n}\right) & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

4) Find the exponential Fourier series for the full wave rectified sine wave given in below figure.



Solution: The waveform shown in fig can be expressed over one period(0 to π) as:

$$x(t) = A \sin \omega t \text{ where } \omega = \frac{2\pi}{2\pi} = 1$$

because it is part of a sine wave with period = 2π

$$x(t) = A \sin \omega t \quad 0 \le t \le \pi$$

The full wave rectified sine wave is periodic with period T = π

Let

$$t_0 = 0,\ t_0 + T = 0 + \pi = \pi$$

and Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

The exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2nt}$$

---

where $C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t\, e^{-j2nt}\,dt = \frac{A}{\pi}\int_0^{\pi} \sin t\, e^{-j2nt}\,dt$$

$$= \frac{A}{j2\pi}\left[\frac{e^{j(1-2n)t} - e^{0}}{j(1-2n)} - \frac{e^{-j(1-2n)t} - e^{0}}{-j(1-2n)}\right]$$

$$\therefore C_n = \frac{2A}{\pi(1-4n^2)}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t\,dt = \frac{A}{\pi}[-\cos t]_0^{\pi} = \frac{2A}{\pi}$$

The exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{j2nt} = \frac{2A}{\pi} + \frac{2A}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{e^{j2nt}}{1-4n^2}\right)$$

1.9. Properties of continuous- time Fourier series

The Fourier series representation possesses a number of important properties that are useful for various purposes during the transformation of signals from one form to other .

Some of the properties are listed below.

[$x_1(t)$ and $x_2(t)$] are two periodic signals with period T and with Fourier series coefficients C_n and D_n respectively.

1) Linearity property The linearity property states that, if $x_1(t) \overset{Fs}{\leftrightarrow} C_n$ and $x_2(t) \overset{Fs}{\leftrightarrow} D_n$

then $Ax_1(t) + Bx_2(t) \overset{Fs}{\leftrightarrow} AC_n + BD_n$

Proof: From the definition of Fourier series, we have

$$FS[Ax_1(t)+Bx_2(t)] = \frac{1}{T}\int_{t_0}^{t_0+T} [Ax_1(t) + Bx_2(t)]e^{-jn\omega_0 t}\,dt$$

$$= A\left(\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)e^{-jn\omega_0 t}\,dt\right) + B\left(\frac{1}{T}\int_{t_0}^{t_0+T} x_2(t)e^{-jn\omega_0 t}\,dt\right)$$

$$= AC_n + BD_n$$

$$Ax_1(t)+Bx_2(t) \overset{Fs}{\leftrightarrow} AC_n+BD_n$$

2) Time shifting property The time shifting property states that, if $x(t)\ Fs\ Cn$ then

$$x(t-t_0) \overset{Fs}{\leftrightarrow} e^{-jn\omega_0 t_0} C_n$$

Proof: From the definition of Fourier series, we have

$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$

$x(t-t_0)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 (t-t_0)}$

$=\sum_{n=-\infty}^{\infty} [C_n\, e^{-jn\omega_0 t}] e^{jn\omega_0 t}$ $=FS^{-1}[C_n e^{-jn\omega^0 t}]$

3) **Time reversal property** The time reversal property states that, if $x(t) \leftrightarrow C_n$

then $x(-t) \overset{Fs}{\leftrightarrow} C_{-n}$

Proof: From the definition of Fourier series, we have

$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$

$x(-t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 (-t)}$ substituting n = -p in the right hand side, we get

$x(-t)=\sum_{p=\infty}^{-\infty} C_{-p}\, e^{j(-p)\omega_0 (-t)} =\sum_{p=-\infty}^{\infty} C_{-p}\, e^{jp\omega_0 t}$ substituting p = n , we get

$$x(-t)=\sum_{n=-\infty}^{\infty} C_{-n}\, e^{jn\omega_0 t} = FS^{-1}[C_{-n}]$$

$$\boxed{x(-t) \overset{Fs}{\leftrightarrow} C_{-n}}$$

4) Time scaling property The time scaling property states that, if $x(t)\ C_n$

then $x(\alpha t)\ C_n$ with $\omega_0 \rightarrow \alpha\omega_0$

Proof: From the definition of Fourier series, we

$$\boxed{x(t-t_0) \overset{Fs}{\leftrightarrow} e^{-jn\omega_0 t_0} C_n}$$

have

$$x(t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 t}$$

$x(\alpha t)=\sum_{n=-\infty}^{\infty} C_n\, e^{jn\omega_0 \alpha t} =\sum_{n=-\infty}^{\infty} C_n\, e^{jn(\omega_0 \alpha)t} = FS^{-1}[C_n]$ where $\omega 0 \rightarrow \alpha\omega 0$.

$$x(\alpha t) Fs Cn \text{ with fundamental frequency of } \alpha\omega 0$$

5) Time differential property: The time differential property states that, if $x(t)\ Fs \leftrightarrow C_n$

then $\frac{dx(t)}{dt} \overset{Fs}{\leftrightarrow} jn\omega_0 Cn$

Proof: From the definition of Fourier series, we have

x(t)= $\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 t}$ Differentiating both sides with respect to t, we get

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} C_n \frac{d(e^{jn\,\omega_0 t})}{dt} = \sum_{n=-\infty}^{\infty} C_n (\text{j}n\omega_0)\, e^{jn\,\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n (\text{j}n\omega_0)\, e^{jn\,\omega_0 t} = \text{FS}^{-1}[\text{j}n\omega_0 C_n]$$

$$\frac{dx(t)}{dt} \overset{Fs}{\leftrightarrow} \text{j}n\omega_0 Cn$$

6) Time integration property: The time integration property states that, if $x(t) \underset{FS}{\overset{Fs}{\leftrightarrow}} C_n$

then $$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{C_n}{jn\,\omega_0} \quad (\text{if } C_0 = 0)$$

Proof: From the definition of Fourier series, we have

x(t)= $\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 t}$

$$\int_{-\infty}^{t} x(\tau)d\tau = \int_{-\infty}^{t} [\sum_{n=-\infty}^{\infty} C_n\, e^{jn\,\omega_0 \tau}] d\tau$$

Interchanging the order of integration and summation, we get

$$\int_{-\infty}^{t} x(\tau)d\tau = \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{t} e^{jn\,\omega_0 \tau} d\tau$$

$$= \sum_{n=-\infty}^{\infty} C_n \left[\frac{e^{jn\,\omega_0 \tau}}{jn\,\omega_0}\right]_{-\infty}^{t}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{C_n}{jn\,\omega_0}\right) e^{jn\,\omega_0 t} = \text{FS}^{-1}\left[\frac{C_n}{jn\,\omega_0}\right]$$

$$= \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{C_n}{jn\,\omega_0} \quad (\text{if } C_0 = 0)$$

1.10. Complex Fourier Spectrum

The Fourier spectrum of a periodic signal x(t) is a plot of its Fourier coefficients versus frequency ω. It is in two parts: (a) Amplitude spectrum and (b) phase spectrum. The plot of the amplitude of Fourier coefficients verses frequency is known as the amplitude spectra, and the plot of the phase of Fourier coefficients verses frequency is known as phase spectra. The two plots together are known as Fourier frequency spectra of x(t).This type of representation is also called frequency domain representation. The Fourier spectrum exists only at discrete frequencies $n\omega_o$, where n=0,1,2,….. Hence it is known as discrete spectrum or line spectrum. The envelope of the spectrum depends only upon the pulse shape, but not upon the period of repetition.

The below figure (a) represents the spectrum of a trigonometric Fourier series extending from 0 to ∞, producing a one-sided spectrum as no negative frequencies exist here. The figure (b) represents the spectrum of a complex exponential Fourier series extending from - $\infty to \infty$, producing a two-sided spectrum. The amplitude spectrum of the exponential Fourier series is symmetrical Fourier series is symmetrical about the vertical axis. This is true for all periodic functions.

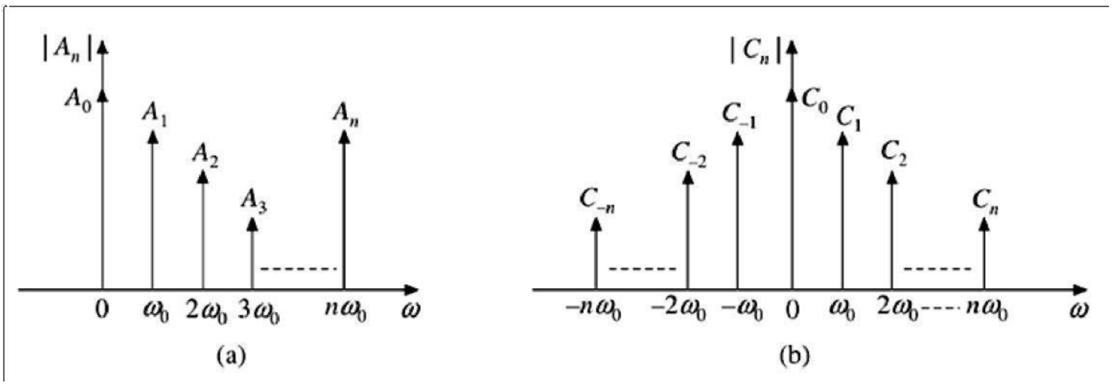


Fig: Complex frequency spectrum for (a) Trigonometric Fourier series and (b) complex exponential Fourier series. If C_n is a general complex number, then

$$C_n = |C_n|\ e^{j\theta n} \ \ \&\ \ C_{-n} = |C_n|\ e^{-} \ \ \&\ \ C_n = |C_{-n}|$$

The magnitude spectrum is symmetrical about the vertical axis passing through the origin, and the phase spectrum is antisymmetrical about the vertical axis passing through the origin. So the magnitude spectrum exhibits even symmetry and phase spectrum exhibits odd symmetry. When x(t) is real , then $C_{-n}=C_n^*$, the complex conjugate of C_n.

UNIT – II
Fourier Transform and Sampling

2.1 Fourier Transform

The Fourier transform is used to analyse aperiodic signals and can be use analyse periodic signals also. So it over comes the limitations of Fourier series. Fourier transform is a transformation technique which transforms signals from the continuous-time domain to the corresponding frequency domain and vice versa, and which applies for both periodic as well as aperiodic signals. Fourier transform can be developed by finding the Fourier series of a periodic function and then tending T to infinity.

The Fourier transform is an extremely useful mathematical tool and is extensively used in the analysis of linear time-invariant systems, cryptography, signal analysis, signal processing, astronomy, etc. Several applications ranging from RADAR to spread spectrum communication employ Fourier transform.

Derivation of the Fourier Transform of a non-periodic signal from the Fourier series of a periodic signal

Let x(t) be a non-periodic function and, $x_T(t)$ be periodic with period T, and let their relation is given by

$$x(t) = \underset{T\to\infty}{Lt}\, x_T(t)$$

The Fourier series of a periodic signal $x_T(t)$ is

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

Where $$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x_T(t)\, e^{-jn\omega_0 t} dt \text{ and } \omega_0 = \frac{2\pi}{T}$$

$$TC_n = \int_{-T/2}^{T/2} x_T(t)\, e^{-jn\omega_0 t} dt$$

Let $n\omega 0 = \omega$ at $T \to \infty$. As $T \to \infty$, we have $\omega_0 = \frac{2\pi}{T} \to 0$ and the discrete Fourier spectrum becomes continuos. Further, the summation becomes integral and $x_T(t) \to x(t)$.

Thus, as $T \to \infty$,

$$TC_n = \underset{T\to\infty}{Lt} \int_{-T/2}^{T/2} x_T(t)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} [\underset{T\to\infty}{Lt}\, x_T(t)]\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t)]\, e^{-j\omega t} dt = X(\omega)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

Hence, $X(\omega)$ is called Fourier transform or the Fourier integral of x(t).

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{T} e^{jn\omega_0 t} \qquad [C_n = \frac{TC_n}{T} = \frac{X(\omega)}{T}]$$

$$= \sum_{n=-\infty}^{\infty} \frac{X(\omega)}{2\pi} e^{jn\omega_0 t}\, \omega_0 \qquad [n\omega_0 = \omega, T\ \frac{2\pi}{\omega_0}]$$

$$x(t) = \underset{T\to\infty}{Lt}\, x_T(t) = Lt\ \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega) e^{jn\omega_0 t}\, \omega_0$$

T→∞

As T →∞, $\omega_0 = \frac{2\pi}{T}$ becomes infinitesimally small and may be represented by dω.

Also the summation becomes integration.

$x(t) = \int_{-\infty}^{\infty} X(\omega)\ e^{j\omega t} d\omega$

Hence, x(t) is called the inverse Fourier transform of X(ω) .

The equations

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t} dt$$

And

$$x(t) = \int_{-\infty}^{\infty} X(\omega)\ e^{j\omega t} d\omega$$

For

X(ω) and x(t) are known as Fourier transform pair and can be denoted as:

X(ω) = F[x(t)]

and $x(t) = F^{-1}[X(\omega)]$

The other notation that can be used to represent the Fourier transform pair is

$x(t) \overset{FT}{\leftrightarrow} X(\omega)$

Magnitude and phase representation of Fourier transform

The magnitude and phase representation of the Fourier transform is the tool used to analyse the transformed signal. In general, X(ω) is a complex valued function of ω. Therefore, X(ω) can be written as: $X(\omega) = X_R(\omega) + jX_I(\omega)$

where $X_R(\omega)$ is real part of X(ω) and $X_I(\omega)$ is the imaginary part of X(ω).

The magnitude of X(ω) is given by $|X_I(\omega)| = \sqrt{X_R(\omega)^2 + X_I(\omega)^2}$

And the pase of X(ω) is given by $\lfloor X(\omega) = \tan^{-1} \frac{X_I(\omega)}{X_R(\omega)}$

The plot of $|X(\omega)|$ versus ω is known as amplitude spectrum, and the plot of $\lfloor X(\omega)$versus ω is known as phase spectrum. The amplitude spectrum and phase spectrum together is called frequency spectrum.

Solved Problems:

Problem 1: Find the Fourier transform of x(t) = f(t-2)+f(t+2)

Solution:

Given x(t) = f(t-2)+f(t+2)

Using linearity property [i.e. $ax_1(t) + bx_2(t) \overset{FT}{\leftrightarrow} aX_1(\omega) + bX(\omega)$], we have

F[x(t)] = F[f(t-2)+f(t+2)]

Using time shifting property [i.e. $x(t\ t_o) \overset{FT}{\leftrightarrow} e^{-j\omega\, t_o} X(\omega)$] , we have

$F[x(t)] = F[f(t)]e^{-j2\omega} + F[f(t)]e^{j2\omega} = e^{-j2\omega}F(\omega) + e^{j2\omega}F(\omega)$

$= F(\omega)[e^{j2\omega} + e^{-j2\omega}]$

$= 2F(\omega)\cos 2\omega$

Problem 2: Find the Fourier transform of the signal $e^{-at}u(t)$

Solution:

Given $x(t) = e^{-at}u(t)$

We know that $F[e^{-at}u(-t)] = \frac{1}{a-j\omega}$

The signal $x(t) = e^{-at}u(t)$ is the time reversal of the signal $e^{-at}u(-t)$. Therefore, using time reversal property [i.e. $x(-t) \overset{FT}{\leftrightarrow} X(-\omega)$], we have

$$F[e^{-at}u(t)] = F[e^{-at}u(-t)]\big|_{\omega=-\omega} = \frac{1}{a-j\omega}\Big|_{\omega=-\omega} = \frac{1}{a+j\omega}$$

$$\therefore \quad e^{-at}u(t) \overset{FT}{\leftrightarrow} \frac{1}{a+j\omega}$$

Problem 3: Find the Fourier transform of the signals $\cos\omega_o t\, u(t)$

Solution:

Given $x(t) = \cos\omega_o t\, u(t)$

i.e. $= \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}u(t)$

$$\therefore X(\omega) = F[\cos\omega_o t\, u(t)] = \int_{-\infty}^{\infty}\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}u(t)\,e^{-j\omega t}dt$$

$$= \frac{1}{2}\left[\int_0^{\infty}e^{-j(\omega-\omega_o)t}dt + \int_0^{\infty}e^{-j(\omega+\omega_o)t}dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-j(\omega-\omega_o)t}}{-j(\omega-\omega_o)} + \frac{e^{-j(\omega+\omega_o)t}}{-j(\omega+\omega_o)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{-e^0}{-j(\omega-\omega_o)} + \frac{-e^0}{-j(\omega+\omega_o)}\right]$$

With impulses of strength π at $\omega = \omega_o$ and $\omega = -\omega_o$

$$\therefore X(\omega) = \frac{1}{2}\left[\frac{1}{j(\omega-\omega_o)} + \frac{1}{j(\omega+\omega_o)} + \pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{1}{2}\left[\frac{j2\omega}{(j\omega)^2+\omega_o^2} + \pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{j\omega}{(j\omega)^2+\omega_o^2} + \frac{1}{2}[\pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)]$$

Problem 4: Find the Fourier transform of the signals $\sin\omega_o t\, u(t)$

Solution:

Given $x(t) = \sin\omega_o t\, u(t)$

i.e. $= \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}u(t)$

$$\therefore X(\omega) = F[\sin\omega_o t\, u(t)] = \int_{-\infty}^{\infty}\frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}u(t)\,e^{-j\omega t}dt$$

$$= \frac{1}{2j}\left[\int_0^{\infty} e^{-j(\omega-\omega_o)t}\, dt - \int_0^{\infty} e^{-j(\omega+\omega_o)t}\, dt\right]$$
$$= \frac{1}{2j}\left[\frac{e^{-j(\omega-\omega_o)t}}{-j(\omega-\omega_o)} - \frac{e^{-j(\omega+\omega_o)t}}{-j(\omega+\omega_o)}\right]_0^{\infty}$$
$$= \frac{1}{2j}\left[\frac{-e^{0}}{-j(\omega-\omega_o)} - \frac{-e^{0}}{-j(\omega+\omega_o)}\right]$$

With impulses of strength π at $\omega = \omega_o$ and $\omega = -\omega_o$

$$\therefore X(\omega) = \frac{1}{2j}\left[\frac{1}{j(\omega-\omega_o)} - \frac{1}{j(\omega+\omega_o)} + \pi\delta(\omega-\omega_o) - \pi\delta(\omega+\omega_o)\right]$$
$$= \frac{1}{2j}\left[\frac{j2\omega_o}{(j\omega)^2+\omega_o^2} + \pi\delta(\omega-\omega_o) - \pi\delta(\omega+\omega_o)\right]$$
$$= \frac{\omega_o}{(j\omega)^2+\omega_o^2} - j\frac{\pi}{2}\left[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)\right]$$

Problem 5: Find the Fourier transform of the signals $e^{-t}\sin 5t\, u(t)$

Solution:

Given $\quad x(t) = e^{-t}\sin 5t\, u(t)$

$$x(t) = e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right) u(t)$$

$$\therefore \quad X(\omega) = F[e^{-t}\sin 5t\, u(t)] = F\left[e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right) u(t)\right]$$
$$= \frac{1}{2j}\int_{-\infty}^{\infty}[e^{-t}(e^{j5t} - e^{-j5t})u(t)]\, e^{-j\omega t}\, dt$$
$$= \frac{1}{2j}\left[\frac{e^{-[1+j(\omega-5)]t}}{-[1+j(\omega-5)]} - \frac{e^{-[1+j(\omega+5)]t}}{-[1+j(\omega+5)]}\right]_0^{\infty}$$
$$= \frac{1}{2j}\left[\frac{1}{[1+j(\omega-5)]} - \frac{1}{[1+j(\omega+5)]}\right]$$
$$= \frac{5}{[1+j(\omega-5)][1+j(\omega+5)]} = \frac{5}{(1+j\omega)^2+25} \text{[neglecting impulses]}$$

Problem 6: Find the Fourier transform of the signals $e^{-2t}\cos 5t\, u(t)$

Solution:

Given $\quad x(t) = e^{-2t}\cos 5t\, u(t)$

$$x(t) = e^{-2t}\left(\frac{e^{j5t} + e^{-j5t}}{2}\right) u(t)$$

$$\therefore \quad X(\omega) = F[e^{-2t}\cos 5t\, u(t)] = F\left[e^{-2t}\left(\frac{e^{j5t} - e^{-j5t}}{2}\right) u(t)\right]$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}[e^{-2t}(e^{j5t} - e^{-j5t})u(t)]\, e^{-j\omega t}\, dt$$
$$= \frac{1}{2}\left[\int_0^{\infty} e^{-[2+j(\omega-5)]t}\, dt + \int_0^{\infty} e^{-[2+j(\omega+5)]t}\, dt\right]$$
$$= \frac{1}{2}\left[\frac{e^{-[2+j(\omega-5)]t}}{-[2+j(\omega-5)]} - \frac{e^{-[2+j(\omega+5)]t}}{-[2+j(\omega+5)]}\right]_0^{\infty}$$
$$= \frac{1}{2}\left[\frac{1}{[1+j(\omega-5)]} - \frac{1}{[1+j(\omega+5)]}\right]$$

$$= \frac{1}{2}\left[\frac{2(2+j\omega)}{(2+j\omega)^2+25}\right] = \left[\frac{2+j\omega}{(2+j\omega)^2+25}\right] \text{[neglecting impulses]}$$

Problem 7: Find the Fourier transform of the signals $e^{-|t|} \sin 5|t|$ for all t

Solution:

Given $\quad x(t) = e^{-|t|} \sin 5|t|$ for all t

i.e. $\quad x(t) = \begin{cases} e^{t} sin5(-t) = -e^{t} sin5t & for\ t < 0 \\ e^{-t} sin5(t) = e^{t} sin5t & for\ t > 0 \end{cases}$

i.e. $\quad x(t) = -e^{t} sin5t\ u(-t) + e^{t} sin5t\ u(t)$

$\therefore \quad X(\omega) = \frac{1}{2j}\int_{-\infty}^{\infty}[-e^{t}(e^{j5t} - e^{-j5t})u(-t) + e^{-t}(e^{j5t} - e^{-j5t})u(t)]\ e^{-j\omega t}\ dt$

$$= -\frac{1}{2j}\left\{\int_{-\infty}^{0}[e^{[1-j(\omega-5)]t} - e^{[1-j(\omega+5)]t}]dt\right\} + \frac{1}{2j}\left\{\int_{0}^{\infty}[e^{-[1+j(\omega-5)]t} - e^{-[1+j(\omega+5)]t}]dt\right\}$$

$$= -\frac{1}{2j}\left\{\int_{0}^{\infty}[e^{-[1-j(\omega-5)]t} - e^{-[1-j(\omega+5)]t}]dt\right\} + \frac{1}{2j}\left\{\int_{0}^{\infty}[e^{-[1+j(\omega-5)]t} - e^{-[1+j(\omega+5)]t}]dt\right\}$$

$$= -\frac{1}{2j}\left[\frac{e^{-[1-j(\omega-5)]t}}{-[1-j(\omega-5)]} - \frac{e^{-[1-j(\omega+5)]t}}{-[1-j(\omega+5)]}\right]_0^{\infty} + \frac{1}{2j}\left[\frac{e^{-[1+j(\omega-5)]t}}{-[1+j(\omega-5)]} - \frac{e^{-[1+j(\omega+5)]t}}{-[1+j(\omega+5)]}\right]_0^{\infty}$$

$$= -\frac{1}{2j}\left[\frac{1}{[1-j(\omega-5)]} - \frac{1}{[1-j(\omega+5)]}\right] + \frac{1}{2j}\left[\frac{1}{[1+j(\omega-5)]} - \frac{1}{[1+j(\omega+5)]}\right]$$

$$= \frac{5}{(1-j\omega)^2+25} + \frac{5}{(1+j\omega)^2+25} \text{[neglecting impulses]}$$

Problem 8: Find the Fourier transform of the signals e^{at} u(-t)

Solution:

Given $\quad x(t) = e^{at}u(-t)$

$\therefore \quad X(\omega) = F[e^{at}u(-t)] = \int_{-\infty}^{\infty} e^{at}u(-t)\ e^{-j\omega t}\ dt$

$$= \int_{-\infty}^{0} e^{(a-j\omega)t}\ dt = \int_{0}^{\infty} e^{-(a-j\omega)t}\ dt = \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty} = \frac{1}{a-j\omega}$$

Problem 9: Find the Fourier transform of the signals te^{at} u(t)

Solution:

Given $\quad x(t) = te^{at}u(t)$

$\therefore \quad X(\omega) = F[te^{at}u(t)] = \int_{-\infty}^{\infty} te^{at}u(t)\ e^{-j\omega t}\ dt$

$$= \int_{-\infty}^{\infty} te^{-(a+j\omega)t}\ dt = \left[t\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} - \int_0^{\infty}\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\ dt = \left[t\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} - \left[\frac{e^{-(a+j\omega)t}}{(a+j\omega)^2}\right]_0^{\infty} = \frac{1}{(a+j\omega)^2}$$

2.2.Fourier Transform of Standard Signals

Introduction:

- The Fourier transform of a finite duration signal can be found using the formula

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$$

  This is called as analysis equation

- The inverse Fourier transform is given by

$$x(t) = \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\, d\omega$$

  This is called as synthesis equation

  Both these equations form the Fourier transform pair.

Existence of Fourier Transform:

The Fourier Transform does not exist for all aperiodic functions. The condition for a function x(t) to have Fourier Transform, called **Dirichlet** conditions are:

1. $x(t)$ is absolutely integrable over the interval -∞ to +∞,that is

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

2. $x(t)$ has a finite number of discontinuities in every finite time interval. Further, each of these discontinuities must be finite.
3. $x(t)$ has a finite number of maxima and minima in every finite time interval.

1. **Impulse Function $\delta(t)$**

Given $x(t) = \delta(t)$,

$$\delta(t) = \begin{cases} 1 \text{ for } t = 0 \\ 0 \text{ for } t \neq 0 \end{cases}$$

Then

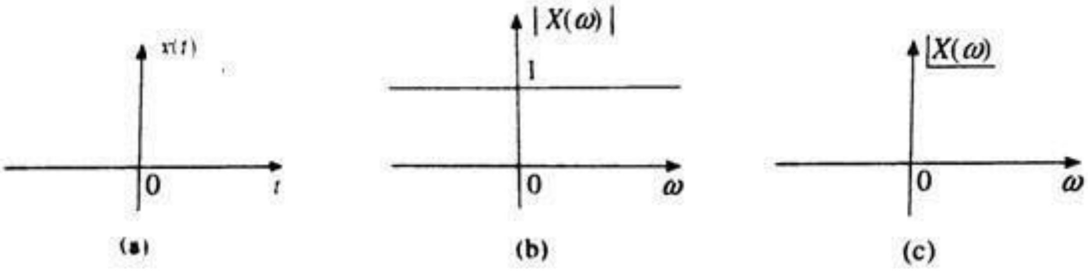
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\, dt \quad = \quad e^{-j\omega t}|_{t=0} \quad = \quad 1$$

$$\therefore \qquad F[\delta(t)] = 1 \qquad \text{or } \delta(t) \overset{FT}{\leftrightarrow} 1$$

Hence , the Fourier Transform of a unit impulse function is unity.

$$|X(\omega)| = 1 \quad \text{for all} \quad w$$
$$\lfloor X(w) = 0 \quad \text{for all} \quad w$$

The impulse function with its magnitude and phase spectra are shown in below figure:



Similarly,

$$F[\delta(t-t_o)] = \int_{-\infty}^{\infty} \delta(t-t_o)e^{-j\omega t}dt = e^{-j\omega t_0} \text{ i.e. } \delta(t-t_o) \overset{FT}{\leftrightarrow} e^{-j\omega t_o}$$

2. Single Sided Real exponential function $e^{-at}u(t)$

Given $x(t) = e^{-at}u(t)$, $\quad u(t) = \begin{cases} 1 & \text{for} t \geq 0 \\ 0 & \text{for} t < 0 \end{cases}$

Then

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \left[\frac{e^{(-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{e^{-\infty}-e^0}{-(a+j\omega)}$$

$$= \frac{0-1}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

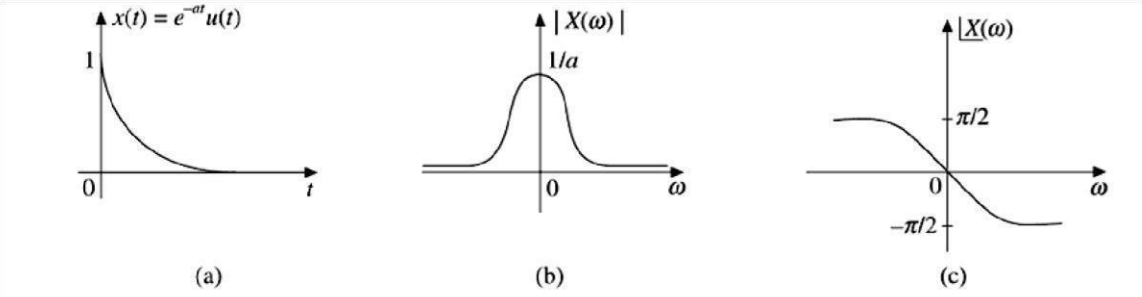
$\therefore F[e^{-at}u(t)] = \frac{1}{a+j\omega}$ or $e^{-at}u(t) \overset{FT}{\leftrightarrow} \frac{1}{a+j\omega}$

Now, $X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{(a+j\omega)(a-j\omega)}$

$$= \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2} = \frac{1}{\sqrt{a^2+\omega^2}} \lfloor -\tan^{-1}\frac{\omega}{a}$$

$$\therefore |X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \lfloor X(\omega) = -\tan^{-1}\frac{\omega}{a} \text{ for all} \omega$$

Figure shows the single-sided exponential function with its magnitude and phase spectra.



3. Double sided real exponential function $e^{-a\,t}$

Given $x(t) = e^{-a|t|}$

$$\therefore x(t) = e^{-a|t|} = \begin{cases} e^{-a(-t)=}e^{at} \text{for} t \le 0 \\ e^{-at} = e^{-at} \text{for} t \ge 0 \end{cases}$$

$$= e^{-a(-t)}u(-t) + e^{-at}u(t)$$

$$= e^{at}u(-t) + e^{-at}u(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \int_0^{\infty} e^{-(a-j\omega)t}dt + \int_0^{\infty} e^{-(a+j\omega)t}dt = \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

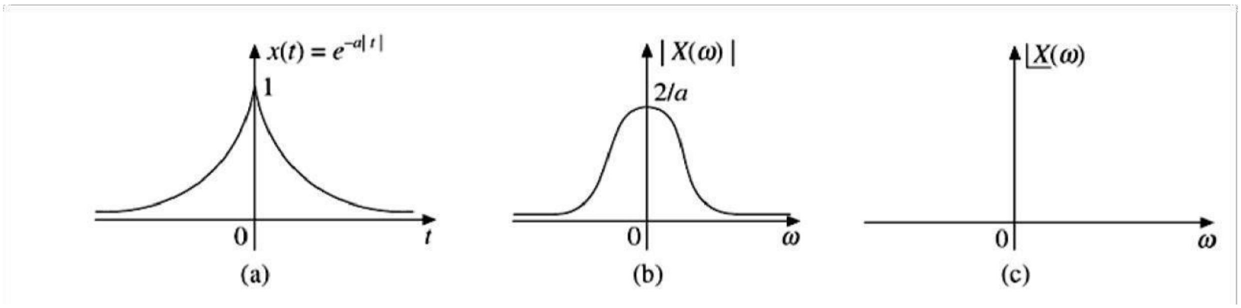
$$= \frac{e^{-\infty} - e^{-0}}{-(a-j\omega)} + \frac{e^{-\infty} - e^{-0}}{-(a+j\omega)} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$\therefore F\left(e^{-a[t]}\right) = \frac{2a}{a^2 + \omega^2} \text{or} e^{-a[t]} \overset{FT}{\leftrightarrow} \frac{2a}{a^2 + \omega^2}$$

$$\therefore [X(\omega)] = \frac{2a}{a^2 + \omega^2} \text{for all} \omega$$

And $\lfloor X(\omega) = 0 \quad \text{for all} \omega$

A Two sided exponential function and its amplitude and phase spectra are shown in figures below:

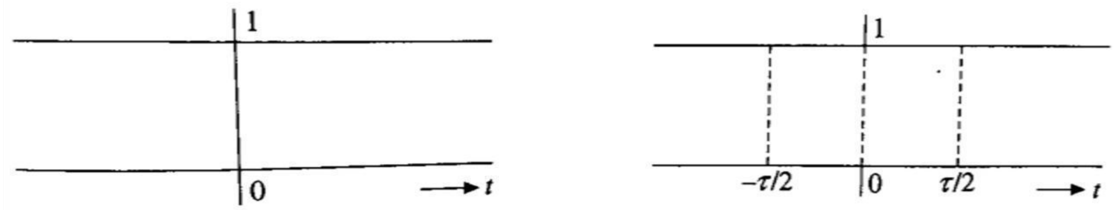


4. **Constant Amplitude (1)**

Let $\quad x(t) = 1 \qquad -\infty \le t \le \infty$

The waveform of a constant function is shown in below figure .Let us consider a small section of constant function, say, of duration τ.If we extend the small duration to infinity, we will get back the original function.Therefore

$$x(t) = \underset{t \to \infty}{Lt} \left[\text{rect}\left(\frac{t}{\tau}\right)\right]$$



(a) (b)

Where $\quad \text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for} \frac{-\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & \text{elesewhere} \end{cases}$

By definition, the Fourier transform of x(t) is:

$$X(\omega) = F[x(t)] = F\left[\underset{t\to\infty}{\text{Lt}}\ \text{rect}\left(\frac{t}{\tau}\right)\right] = \underset{t\to\infty}{\text{Lt}}\ F\left[\text{rect}\left(\frac{t}{\tau}\right)\right]$$

$$= \underset{t\to\infty}{\text{Lt}} \int_{-\tau/2}^{\tau/2} (1)\, e^{-j\omega t}\, dt = \underset{t\to\infty}{\text{Lt}} \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2}$$

$$= \underset{t\to\infty}{\text{Lt}} \left[\frac{e^{-j\omega(\tau/2)} - e^{j\omega(\tau/2)}}{-j\omega}\right] = \underset{t\to\infty}{\text{Lt}} \left\{\frac{2\sin[\omega\left(\frac{\tau}{2}\right)]}{\omega}\right\} = \underset{t\to\infty}{\text{Lt}} \left\{\tau \frac{\sin[\omega\left(\frac{\tau}{2}\right)]}{\omega\left(\frac{\tau}{2}\right)}\right\}$$

$$= \underset{t\to\infty}{\text{Lt}}\ \tau\, \text{sa}\left(\frac{\omega\tau}{2}\right) = 2\pi\left[\underset{t\to\infty}{\text{Lt}} \frac{\tau/2}{\pi} \text{sa}\left(\frac{\omega\tau}{2}\right)\right]$$

Using the sampling property of the delta function $\left\{\text{i. e.} \left[\underset{t\to\infty}{\text{Lt}} \frac{\tau/2}{\pi} \text{sa}\left(\frac{\omega\tau}{2}\right)\right] = \delta(\omega)\right\}$, we get

$$X(\omega) = F\left[\underset{t\to\infty}{\text{Lt}}\ \text{rect}\left(\frac{t}{\tau}\right)\right] = 2\pi\delta(\omega)$$

5. **Signum function (sgn(t))**

The signum function is denoted by sgn(t) and is defined by

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

This function is not absolutely integrable. So we cannot directly find its Fourier transform. Therefore, let us consider the function $e^{-a|t|}$ sgn(t) and substitute the limit $a \to 0$ to obtain the above sgn(t)

Given $x(t) = \text{sgn}(t) = \underset{a\to 0}{\text{Lt}}\ e^{-a|t|}\, \text{sgn}(t) = \underset{a\to 0}{\text{Lt}} [\, e^{-at}u(t) - e^{-at}u(-t)$

$\therefore X(\omega) = F[\text{sgn}(t)] = \int_{-\infty}^{\infty} \underset{a\to 0}{\text{Lt}} [\, e^{-at}u(t) - e^{-at}u(-t)e^{-j\omega t}dt$

$$= \underset{a\to 0}{\text{Lt}} \left[\int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}u(t)dt - \int_{-\infty}^{\infty} e^{at}e^{-j\omega t}u(-t)dt\,\right]$$

$$= \underset{a\to 0}{\text{Lt}} \left[\int_{0}^{\infty} e^{-(a+j\omega)t}dt - \int_{-\infty}^{0} e^{(a-j\omega)t}dt\,\right] = \underset{a\to 0}{\text{Lt}} \left[\int_{0}^{\infty} e^{-(a+j\omega)t}dt - \int_{0}^{\infty} e^{-(a-j\omega)t}dt\,\right]$$

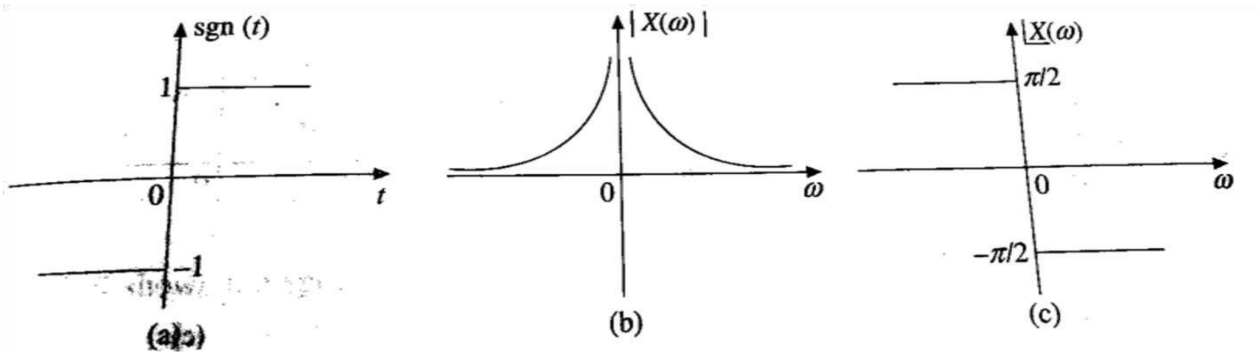
$$= \underset{a\to 0}{\text{Lt}} \left\{\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} - \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty}\right\} = \underset{a\to 0}{\text{Lt}} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \frac{1}{j\omega} - \frac{1}{-j\omega} = \frac{2}{j\omega}$$

$$F[\text{sgn}(t)] = \frac{2}{j\omega}$$

$$\text{sgn}(t) \overset{FT}{\leftrightarrow} \frac{2}{j\omega}$$

∴ $|X(\omega)| = \frac{2}{\omega}$ and $\angle X(\omega) = \frac{\pi}{2} for\ \omega < 0\ and - \frac{\pi}{2} for\ \omega > 0$

Figure below shows the signum function and its magnitude and phase spectra



6. **Unit step function u(t)**

The unit step function is defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

since the unit step function is not absolutely integrable, we cannot directly find its Fourier transform. So express the unit step function in terms of signum function as: $u(t) = \frac{1}{2} + \frac{1}{2}\, sgn(t)$ $\quad x(t) = u(t) = \frac{1}{2}[1 + sgn(t)]$

$$X(\omega) = F[u(t)] = F\left\{\frac{1}{2}[1 + sgn(t)]\right\}$$

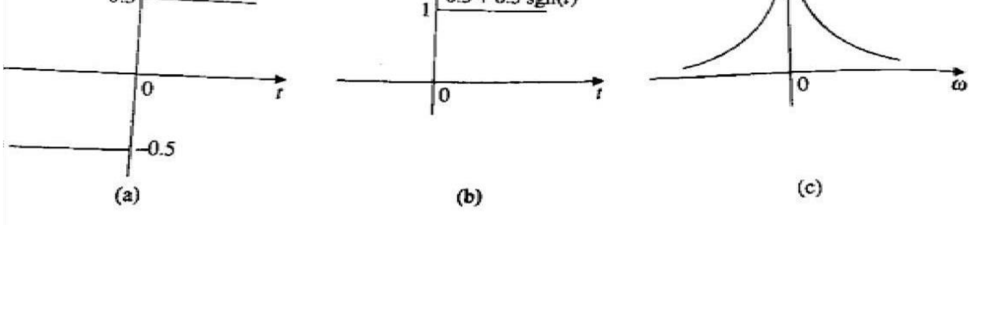
$$= \frac{1}{2}\{F[1] + F[sgn(t)]\}$$

We know that $F[1] = 2\pi\delta(\omega)$ and $F[sgn(t)] = \frac{2}{j\omega}$

$$F[u(t)] = \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right] = \pi\delta(\omega) + \frac{1}{j\omega}$$

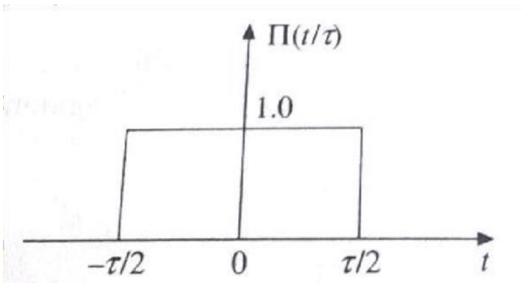
$$u(t) \overset{FT}{\leftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega}$$

∴ $|X(\omega)| = \infty$ at $\omega = 0$ and is equal to 0 at $\omega = -\infty$ and $\omega = \infty$



7. Rectangular pulse (Gate pulse) $\prod\left(\frac{t}{\tau}\right)$ or rect$\left(\frac{t}{\tau}\right)$

Consider a rectangular pulse as shown in below figure. This is called a unit gate function and is defined as



$$x(t) = \text{rect}\left(\frac{t}{\tau}\right) = \prod\left(\frac{t}{\tau}\right) = \begin{cases} 1 & for\ |t| < \frac{\tau}{2} \\ 0 & otherwise \end{cases}$$

Then $X(\omega) = F[\,x(t)] = F\left[\prod\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} \prod\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$

$$= \int_{-\tau/2}^{\tau/2} (1) e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{\tau/2}^{\tau/2} = \frac{e^{-j\omega(\tau/2)} - e^{j\omega(\tau/2)}}{-j\omega}$$

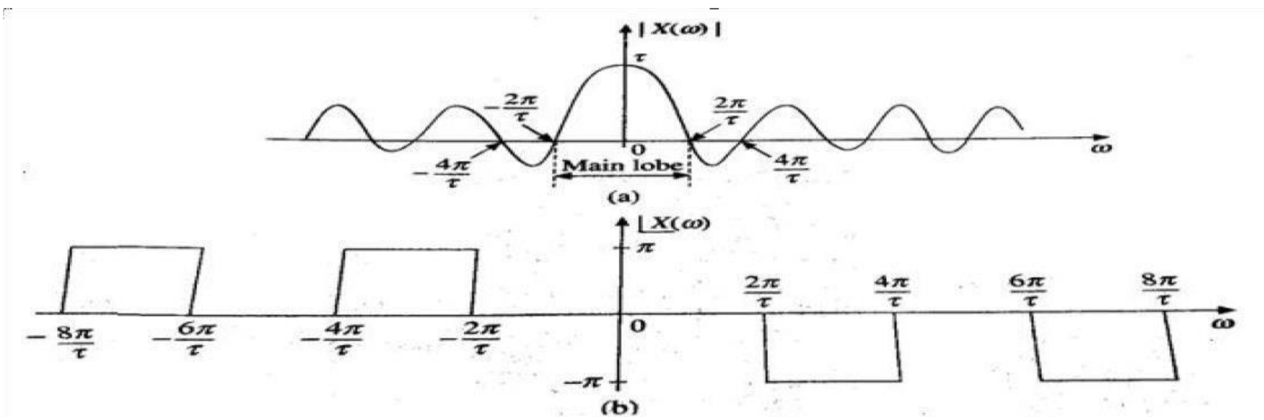
$$= \frac{\tau}{\omega(\tau/2)}\left[\frac{e^{j\omega(\tau/2)} - e^{-j\omega(\tau/2)}}{2j}\right] = \tau\left[\frac{\sin\,\omega(\tau/2)}{\omega(\tau/2)}\right]$$

$$= \tau\, sinc\, \omega(\tau/2)$$

∴ $F\left[\prod\left(\frac{t}{\tau}\right)\right] = \tau\, sinc\, \omega(\tau/2)$, that is

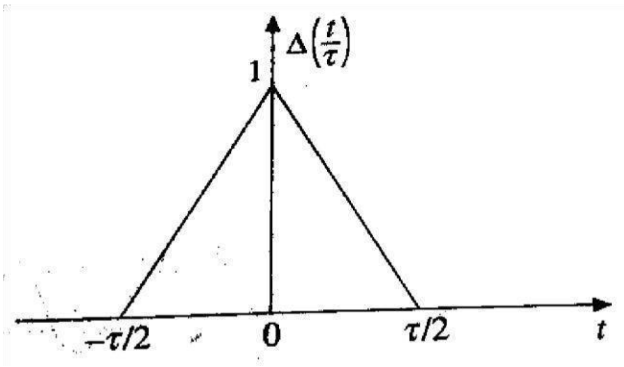
$$\text{rect}\left(\frac{t}{\tau}\right) = \prod\left(\frac{t}{\tau}\right) \overset{FT}{\leftrightarrow} \tau\, sinc\, \omega(\tau/2)$$

Figure shows the spectra of the gate function



8. Triangular Pulse $\Delta\left(\frac{t}{\tau}\right)$

Consider the triangular pulse as shown in below figure. It is defined as:



$$x(t) = \Delta\left(\frac{t}{\tau}\right) = \begin{cases} \frac{1}{\tau/2}\left(t+\frac{\tau}{2}\right) = \left(1+2\frac{t}{\tau}\right) & \text{for } -\frac{\tau}{2} < t < 0 \\ \frac{1}{\tau/2}\left(t-\frac{\tau}{2}\right) = \left(1-2\frac{t}{\tau}\right) & \text{for } 0 < t < \frac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases}$$

i.e. as $x(t) = \Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1-\frac{2|t|}{\tau} & \text{for } |t| < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$

Then $X(\omega) = F[x(t)] = F\left[\Delta\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} \Delta\left(\frac{t}{\tau}\right) e^{-j\omega t}\,dt$

$$= \int_{-\tau/2}^{0}\left(1+\frac{2t}{\tau}\right)e^{-j\omega t}\,dt + \int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{j\omega t}\,dt + \int_{0}^{\tau/2}\left(1-\frac{2t}{\tau}\right)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\tau/2} e^{j\omega t}\,dt - \int_{0}^{\tau/2}\left(\frac{2t}{\tau}\right)e^{j\omega t}\,dt + \int_{0}^{\tau/2} e^{-j\omega t}\,dt - \int_{0}^{\tau/2}\left(\frac{2t}{\tau}\right)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\tau/2}[e^{j\omega t} + e^{-j\omega t}]dt - \frac{2}{\tau}\int_{0}^{\tau/2} t[e^{j\omega t} + e^{-j\omega t}]dt$$

$$= \int_{0}^{\tau/2} 2\cos\omega t\,dt - \frac{2}{\tau}\int_{0}^{\tau/2} 2t\cos\omega t\,dt$$

$$= 2\left[\frac{\sin\omega t}{\omega}\right]_0^{\tau/2} - \frac{4}{\tau}\left[\left[t\frac{\sin\omega t}{\omega}\right]_0^{\tau/2} + \left[\frac{\cos\omega t}{\omega^2}\right]_0^{\tau/2}\right]$$

$$= \frac{2}{\omega}\left[\sin\omega\frac{\tau}{2}\right] - \frac{4}{\omega\tau}\left[\frac{\tau}{2}\sin\frac{\omega\tau}{2}\right] - \frac{4}{\omega^2\tau}\left[\cos\frac{\omega\tau}{2} - 1\right]$$

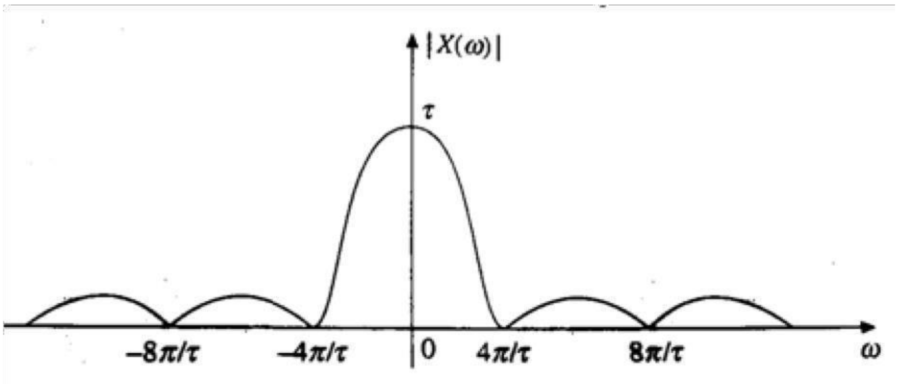
$$= \frac{4}{\omega^2\tau}\left[1-\cos\frac{\omega\tau}{2}\right] = \frac{4}{\omega^2\tau}\left[2\sin^2\frac{\omega\tau}{4}\right]$$

$$= \frac{8}{\omega^2\tau}\left(\frac{\omega\tau}{4}\right)^2 \frac{\sin^2\left(\frac{\omega\tau}{4}\right)}{\left(\frac{\omega\tau}{4}\right)} = \frac{\tau}{2}\,sinc^2\left(\frac{\omega\tau}{4}\right)$$

$$F\left[\Delta\left(\frac{t}{\tau}\right)\right] = \frac{\tau}{2}\,sinc^2\left(\frac{\omega\tau}{4}\right)$$

Or $\quad \Delta\left(\frac{t}{\tau}\right) \overset{FT}{\leftrightarrow} \frac{\tau}{2}\,sinc^2\left(\frac{\omega\tau}{4}\right)$

Figure shows the amplitude spectrum of a triangular pulse.



2.3. Fourier Transform of Periodic Signal

The periodic functions can be analysed using Fourier series and that non-periodic function can be analysed using Fourier transform. But we can find the Fourier transform of a periodic function also. This means that the Fourier transform can be used as a universal mathematical tool in the analysis of both non-periodic and periodic waveforms over the entire interval. Fourier transform of periodic functions may be found using the concept of impulse function.

We know that using Fourier series , any periodic signal can be represented as a sum of complex exponentials. Therefore, we can represent a periodic signal using the Fourier integral. Let us consider a periodic signal x(t) with period T. Then, we can express x(t) in terms of exponential Fourier series as:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

The Fourier transform of x(t) is:

$$X(\omega) = F[x(t)] = F\left[\sum_{n=\infty}^{\infty} C_n \, e^{jn\omega_0 t}\right]$$

$$= \sum_{n=\infty}^{\infty} C_n \, F\left[e^{jn\omega_0 t}\right]$$

Using the frequency shifting theorem, we have

$$F\left[1e^{jn\omega_0 t}\right] = F\,[1]\,\big|_{\omega=\omega-n\omega_0} = s\pi\delta(\omega - n\omega_0)$$

$$X(\omega) = 2\pi \sum_{n=\infty}^{\infty} C_n \, \delta(\omega - n\omega_0)$$

Where $C_n s$ are the Fourier coefficients associated with x(t) and are given by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t} dt$$

Thus, the Fourier transform of a periodic function consists of a train of equally spaced impulses. These impulses are located at the harmonic frequencies of the signal and the strength of each impulse is given as 2 .

Solved Problems:

Problem 1: Find the Fourier transform of the signals $e^{3t}u(t)$

Solution:

Given $x(t) = e^{3t}u(t)$

The given signal is not absolutely integrable.

That is $\int_{-\infty}^{\infty} e^{3t}u(t) = \infty$.

Therefore, Fourier transform of $x(t) = e^{3t}u(t)$ does not exist.

Problem 2: Find the Fourier transform of the signals $\cos\omega_o t u(t)$

Solution:

Given $x(t) = \cos\omega_o t\, u(t)$

i.e. $= \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} u(t)$

$$\therefore \quad X(\omega) = F[\cos\omega_o t\, u(t)] = \int_{-\infty}^{\infty} \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2} u(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-j(\omega-\omega_o)t}\, dt + \int_0^{\infty} e^{-j(\omega+\omega_o)t}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-j(\omega-\omega_o)t}}{-j(\omega-\omega_o)} + \frac{e^{-j(\omega+\omega_o)t}}{-j(\omega+\omega_o)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{-e^0}{-j(\omega-\omega_o)} + \frac{-e^0}{-j(\omega+\omega_o)}\right]$$

With impulses of strength π at $\omega=\omega_o$ and $\omega=-\omega_o$

$$\therefore X(\omega) = \frac{1}{2}\left[\frac{1}{j(\omega-\omega_o)} + \frac{1}{j(\omega+\omega_o)} + \pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{1}{2}\left[\frac{j2\omega}{(j\omega)^2 + \omega_o^2} + \pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{j\omega}{(j\omega)^2 + \omega_o^2} + \frac{1}{2}[\pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)]$$

Problem 3: Find the Fourier transform of the signals $\sin\omega_o t\, u(t)$

Solution:

Given $x(t) = \sin\omega_o t\, u(t)$

i.e. $= \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j} u(t)$

$\therefore \quad X(\omega) = F[\sin\omega_o t\, u(t)] = \int_{-\infty}^{\infty} \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}\, u(t)\, e^{-j\omega t}\, dt$

$$= \frac{1}{2j}\left[\int_0^{\infty} e^{-j(\omega-\omega_o)t}\, dt - \int_0^{\infty} e^{-j(\omega+\omega_o)t}\, dt\right]$$

$$= \frac{1}{2j}\left[\frac{e^{-j(\omega-\omega_o)t}}{-j(\omega-\omega_o)} - \frac{e^{-j(\omega+\omega_o)t}}{-j(\omega+\omega_o)}\right]_0^{\infty}$$

$$= \frac{1}{2j}\left[\frac{-e^{0}}{-j(\omega-\omega_o)} - \frac{-e^{0}}{-j(\omega+\omega_o)}\right]$$

With impulses of strength π at $\omega=\omega_o$ and $\omega=-\omega_o$

$$\therefore X(\omega) = \frac{1}{2j}\left[\frac{1}{j(\omega-\omega_o)} - \frac{1}{j(\omega+\omega_o)} + \pi\delta(\omega-\omega_o) - \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{1}{2j}\left[\frac{j2\omega_o}{(j\omega)^2+\omega_o^2} + \pi\delta(\omega-\omega_o) - \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{\omega_o}{(j\omega)^2+\omega_o^2} - j\frac{\pi}{2}[\,\delta(\omega-\omega_o) + \delta(\omega+\omega_o)]$$

Problem 4: Find the Fourier transform of the signals $e^{-t}\sin 5t\, u(t)$

Solution:

Given $\quad x(t) = e^{-t}\sin 5t\, u(t)$

$$x(t) = e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right) u(t)$$

$\therefore \quad X(\omega) = F[e^{-t}\sin 5t\, u(t)] = F\left[e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right) u(t)\right]$

$$= \frac{1}{2j}\int_{-\infty}^{\infty}[\,e^{-t}(e^{j5t} - e^{-j5t})u(t)]\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\left[\frac{e^{-[1+j(\omega-5)]t}}{-[1+j(\omega-5)]} - \frac{e^{-[1+j(\omega+5)]t}}{-[1+j(\omega+5)]}\right]_0^{\infty}$$

$$= \frac{1}{2j}\left[\frac{1}{[1+j(\omega-5)]} - \frac{1}{[1+j(\omega+5)]}\right]$$

$= \frac{5}{[1+j(\omega-5)][1+j(\omega+5)]} = \frac{5}{(1+j\omega)^2+25}$ [neglecting impulses]

Problem 5: Find the Fourier transform of the signals $e^{-2t}\cos 5t\, u(t)$

Solution:

Given $\quad x(t) = e^{-2t}\cos 5t\, u(t)$

$$x(t) = e^{-2t}\left(\frac{e^{j5t} + e^{-j5t}}{2}\right) u(t)$$

$\therefore \quad X(\omega) = F[e^{-2t}\cos 5t\, u(t)] = F\left[e^{-2t}\left(\frac{e^{j5t} - e^{-j5t}}{2}\right) u(t)\right]$

$$= \frac{1}{2}\int_{-\infty}^{\infty} [\, e^{-2t}(e^{j5t} - e^{-j5t})u(t)]\; e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{-[2+j(\omega-5)]t}\, dt + \int_{0}^{\infty} e^{-[2+j(\omega+5)]t}\, dt\, \right]$$

$$= \frac{1}{2}\left[\frac{e^{-[2+j(\omega-5)]t}}{-[2+j(\omega-5)]} - \frac{e^{-[2+j(\omega+5)]t}}{-[2+j(\omega+5)]}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{1}{[1+j(\omega-5)]} - \frac{1}{[1+j(\omega+5)]}\right]$$

$$= \frac{1}{2}\left[\frac{2(2+j\omega)}{(2+j\omega)^2+25}\right] = \left[\frac{2+j\omega}{(2+j\omega)^2+25}\right] \text{[neglecting impulses]}$$

2.4.Properties of Fourier Transform

These properties provides significant amount of insight into the transform and into the relationship between the time-domain and frequency domain descriptions of a signal. Many of these properties are useful in reducing the complexity Fourier transforms or inverse transforms.

Linearity

If x (t) ⟷ f X (jw)

y (t) ⟷ f Y (jw)

Then

a x(t) + b y(t) ⟷ f (aX (jw) + Y b (jw))

Time Shifting

If x (t) ⟷ X (jw)

Then

x (t - t_0) ⟷ fX (jw)e^{-jwt0}

To establish this property, consider x(t)=1/2π∫-∞ X(jw)e^{-jwtO} dw

Replacing t by t-to in this equation, we obtain

x(t-to)=1/2π∫-∞ X(jw)e-jw(t-to)dw

x(t)=1/2π∫-∞e-jwt_o X(jw)ejwt dw

Recognizing this as the synthesis equation for x(t-to) ,we conclude that

F{x(t-to)}= e^{-jwto} X(jw)

Conjugation and Conjugate symmetry

The conjugation property states that if

x (t) $\longleftrightarrow$ fX(j*w*) Then

x* (t) $\longleftrightarrow$ f X*(-j*w*)	….. (i)

This property follows from the evaluation of the complex conjugate

$$X^*(-jw) = [\ x(t) \int_{-\infty}^{\infty} e^{-jwt}\, dt\ ]^*$$

$$= \int_{-\infty}^{\infty} x^*(t)e^{jwt}\, dt.$$

Replacing *w* by –*w*, we see that

$$X^*(-jw) \quad = \int_{-\infty}^{\infty} x^*(t)e^{jwt}\, dt. \qquad \text{(ii)}$$

The conjugate property allows us to show that if x(t) is real ,then X(jw) has *conjugate symmetry:* that is

X(-j*w*)=X*(j*w*)	x(t) real] …... (iii)

If x(t) is real so that x*(t) = x(t), we have ,from eq.(ii)

$X^*(-jw) = \int_{-\infty} x^*(t)e^{jwt}\, dt = X(jw)$. Follows by replacing *w* by –*w*

<u>Differentiation and Integration</u>

If x (*t*) $\longleftrightarrow$ f X(j*w*)

then differentiating both sides of the Fourier transform synthesis equation we have

$$dx(t)/dt = 1/2\pi \int_{-\infty}^{\infty} jwX(jw)e^{-jwto}\, dw$$ Therefore,

dx(t)/dt $\longleftrightarrow$ f *jwX* (jw)

This important property replaces the operation of the differentiation in time domain with that of multiplication by *jw* in the frequency domain similarly integration should involve division by *jw*in frequency domain.

$$\int_{-\infty}^{t} x(t)dt \longleftrightarrow f\ (\ 1/jw(X(jw)) + \pi X(0)\ \delta(w)\ )$$

The impulse term on the right-hand side above equation reflects the dc or average value that can result from integration.

Time and Frequency Scaling

If x (t) $\longleftrightarrow$ fX(jw)

Then x(at) $\longleftrightarrow$ f1/|a|X(jw/a).. (v)

Where a is real constant. This property follows directly from the definition of the Fourier transform If a = -1 we have,

x (t) $\longleftrightarrow$ f X (-jw)

PART-2 SAMPLING THEOREM

2.5.Statement of the sampling theorem

- A band limited signal of finite energy , which has no frequency components higher than W hertz , is completely described by specifying the values of the signal at instants of time separated by 1/2W seconds and
- A band limited signal of finite energy, which has no frequency components higher than W hertz, may be completely recovered from the knowledge of its samples taken at the rate of 2W samples per second.

The first part of above statement tells about sampling of the signal and second part tells about reconstruction of the signal. Above statement can be combined and stated alternately as follows:

A continuous time signal can be completely represented into samples and recovered back if the sampling frequency is twice of the highest frequency content of the signal i.e., $f_s \geq 2W$ Here f_s is the sampling frequency and W is the higher frequency content.

2.6.Proof of sampling theorem

There are two parts:

I) Representation of x(t) in terms of its samples

II) Reconstruction of x(t) from its samples

PART I: Representation of x(t) in its samples x(nTs)

Step 1: Define $x_\delta(t)$

Step 2 : Fourier transform of $x_\delta(t)$ i.e. $X_\delta(f)$

Step 3: Relation between X(f) and $X_\delta(f)$

Step 4 : Relation between x(t) and x(nTs)

Step 1: Define $x_\delta(t)$

The sampled signal $x_\delta(t)$ is given as ,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)$$

--------- (1)

Here, observe that $x_\delta(t)$ is the product of x(t) and impulse train δ(t) as shown in figure.
In the above equation $\delta(t-nT_s)$ indicates the samples placed at $\pm T_s, \pm 2T_s, \pm 3T_s \ldots$ and so on

Step 2: Fourier transform of $x_\delta(t)$i.e. $X_\delta(f)$
Taking FT of equation (1)

$$X_\delta(f) = \text{FT}\left\{\sum_{n=-\infty}^{\infty} x(t)\,\delta(t-nT_s)\right\}$$

$$= \text{FT}\ \{\text{Product of } x(t) \text{ and impulse train}\}$$

-------- (2)

We know that FT of product in time domain becomes convolution in frequency domain i.e.,

$$X_\delta(f) = \text{FT}\ \{x(t)\} * FT\ \{\delta(t-nT_s)\}$$

By definitions, $x(t) \xleftrightarrow{FT} X(f)$ and

$$\delta(t-nT_s) \xleftrightarrow{FT} f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Hence equation becomes,

$$X_\delta(f) = X(f) * f_s \sum_{n=-\infty}^{\infty} \delta(f-nf_s)$$

Since convolution is linear,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f-nf_s)$$

$$= f_s \sum_{n=-\infty}^{\infty} X(f-nf_s) \qquad \textit{By shifting property of impulse function}$$

$$= \cdots f_s\, X(f-2f_s) + f_s\, X(f-f_s) + f_s\, X(f) + f_s\, X(f-f_s) + f_s\, X(f-2f_s) + \cdots$$

--------- (3)

Conclusions:

(i) The RHS of above equation shows that X(f) is placed at $\pm f_s, \pm 2f_s, \pm 3f_s, \cdots$

(ii) This means X(f) is periodic in f_s.

(iii) If sampling frequency is $f_s = 2W$, then the spectrums $X(f)$ just touch each other.

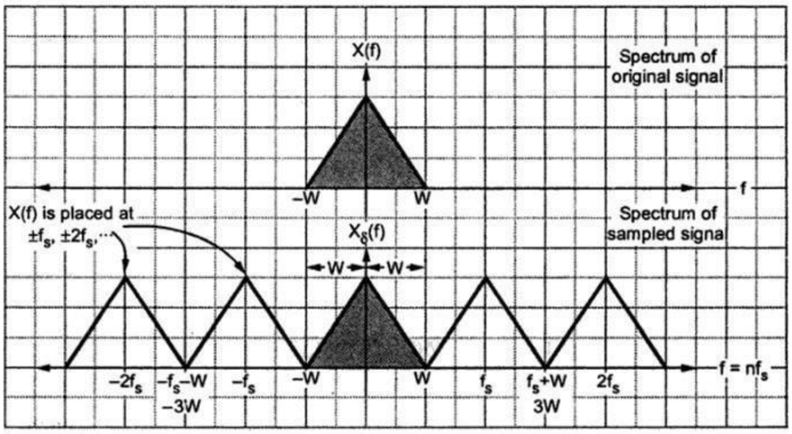


Figure 1.10 Spectrum of Original Signal & Sampled Signal

Step 3: Relation between X(f) and X_δ(f)

Important assumption

Let us assume that f_s=2W , then as per above diagram

$$X_\delta(f) = f_s X(f)$$

$$X(f) = \frac{1}{f_s} X_\delta(f)$$

-------- (4)

Step 3: Relation between x(t) and x(nT_s): From DTFT,

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n}$$

--------- (5)

In above equation 'f' is the frequency of DT signal. If we replace $X(f)$ by $X_\delta(f)$, then 'f' becomes frequency of CT signal. i.e.,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n}$$

In above equation 'f' is frequency of CT signal. And $\frac{f}{f_s}$ = Frequency of DT signal in equation Since $x(n) = x(nT_s)$, i.e. samples of $x(t)$, then we have,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \quad \text{since } \frac{1}{f_s} = T_s$$

Putting above expression in equation (4)

$$X(f) = \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi fnT_s}$$

Inverse Fourier Transform (IFT) of above equation gives $x(t)$ i.e.,

$$x(t) = IFT\left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi fnT_s}\right\}$$

--------- (6)

Conclusions:

- Here x(t) is represented completely in terms of $x(nT_s)$
- Above equation holds for $f_s=2W$.This means if the samples are taken at the rate of 2W or higher, x(t) is completely represented by its samples.
- First part of the sampling theorem is proved by above two conclusions.

II) Reconstruction of x(t)from its samples

Step 1 : Take inverse Fourier transform of X(f) which is in terms of $X_\delta(f)$

Step 2 : Show that x(t) is obtained back with the help of interpolation function.

Step 1 : Take inverse Fourier transform of equation (6) becomes ,

$$x(t) = \int_{-\infty}^{\infty} \left\{\frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi fnT_s}\right\} e^{j2\pi ft}\, df$$

Here the integration can be taken from $-W \leq f \leq W$. Since $X(f) = \frac{1}{f_s} X_\delta(f)$ for $-W \leq f \leq W$.

$$\therefore \quad x(t) = \int_{-W}^{W} \frac{1}{f_s} \sum_{n=-\infty}^{\infty} x(nT_s) e^{-j2\pi f n T_s} \cdot e^{j2\pi f t}\, df$$

Interchanging the order of summation and integration,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{1}{f_s} \int_{-W}^{W} e^{j2\pi f(t-nT_s)}\, df$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \left[\frac{e^{j2\pi f(t-nT_s)}}{j2\pi(t-nT_s)} \right]_{-W}^{W}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \left\{ \frac{e^{j2\pi W(t-nT_s)} - e^{-j2\pi W(t-nT_s)}}{j2\pi(t-nT_s)} \right\}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \frac{1}{f_s} \cdot \frac{\sin 2\pi W(t-nT_s)}{\pi(t-nT_s)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi(2Wt - 2W nT_s)}{\pi(f_s t - f_s nT_s)}$$

Here $f_s = 2W$, hence $T_s = \frac{1}{f_s} = \frac{1}{2W}$. Simplifying above equation,

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)}$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s) \operatorname{sinc}(2Wt-n) \qquad \text{since } \frac{\sin \pi\theta}{\pi\theta} = \operatorname{sinc} \theta$$

Step 2: Let us interpret the above equation and exapanding we get,

$$x(t) = \cdots + x(-2T_s)\operatorname{sinc}(2Wt+2) + x(-T_s)\operatorname{sinc}(2Wt+1) + x(0)\operatorname{sinc}(2Wt) + x(T_s)\operatorname{sinc}(2Wt-1) + \cdots$$

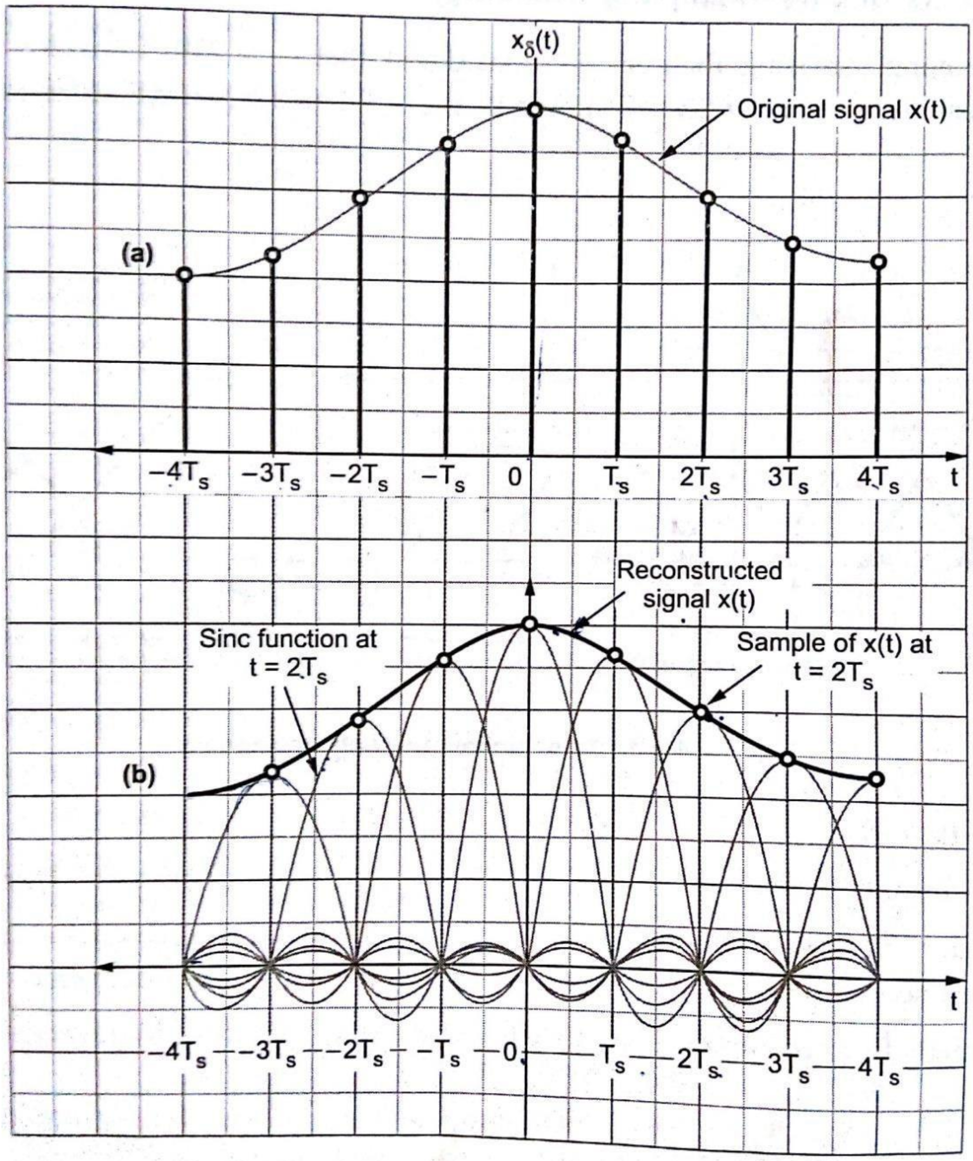


Figure 1.11 Sampled Version of Signal & Reconstruction of x(t) from its samples

Conclusions:

The samples x(nTs(nTs)are weighted by sinc functions.
The sinc function is the interpolating function above figure shows, how x(t) is interpolated.

Step 3: Reconstruction of x(t) by low pass filter

When the interpolated signal of equation (6) is passed through the low pass filter of bandwidth $-W \leq f \leq W$, then the reconstructed waveform shown in figure is obtained. The individual sinc functions are interpolated to get smooth x(t).

2.7 Aliasing

When high frequency interferes with low frequency and appears as low frequency, then the phenomenon is called aliasing.

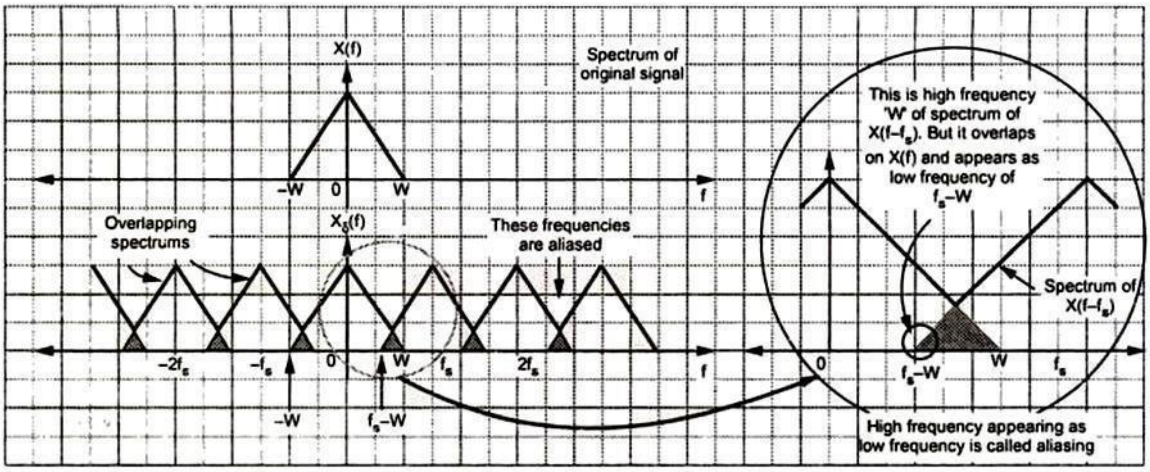


Figure 1.12 Effects of under sampling or aliasing

Effects of aliasing:

- Since high and low frequencies interfere with each other, distortion is generated.
- The data is lost and it cannot be recovered.

Different ways to avoid aliasing:

Aliasing can be avoided by two methods

- Sampling rate fs≥2W
- Strictly band limit the signal to 'W'

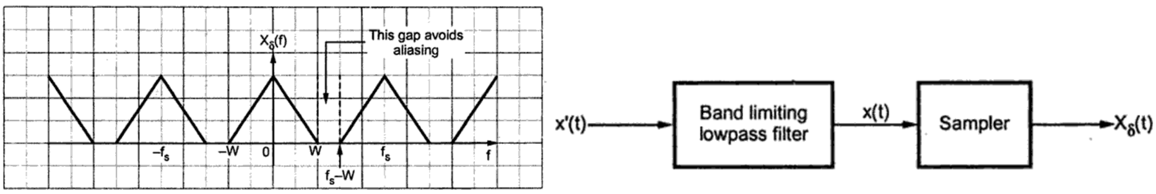


Figure 1.13 Methods to avoid aliasing

2.8.NYQUIST RATE & NYQUIST INTERVAL

Nyquist rate : When the sampling rate becomes exactly equal to '2W' samples/sec, for a given bandwidth of W Hertz, then it is called Nyquist rate.

Nyquist interval : It is the time interval between any two adjacent samples when sampling rate is Nyquist rate.

$$\text{Nyquist rate} = 2W \text{ Hz}$$

$$\text{Nyquist interval} = \frac{1}{2W} \text{ seconds}$$

UNIT-III

SIGNAL TRANSMISSION THROUGH LINEAR SYSTEMS

3.1 Introduction to System:

System is a device or combination of devices, which can operate on signals and produces corresponding response. Input to a system is called as excitation and output from it is called as response.

For one or more inputs, the system can have one or more outputs.

Example: Communication System

3.2 Classification of Systems: Systems are classified into the following categories:

- Linear and Non-linear Systems
- Time Variant and Time Invariant Systems
- Liner Time variant and Liner Time invariant systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Stable and Unstable System

Linear and Non-linear Systems

A system is said to be linear when it satisfies superposition and homogeneity principles. Consider two systems with inputs as $x_1(t)$, $x_2(t)$, and outputs as $y_1(t)$, $y_2(t)$ respectively. Then, according to the superposition and homogeneity principles,

$$T[a_1 x_1(t) + a_2 x_2(t)] = a_1 T[x_1(t)] + a_2 T[x_2(t)]$$

$$\therefore T[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

From the above expression, is clear that response of overall system is equal to response of individual system.

Example: $x(t) = x^2(t)$

Solution: $y_1(t) = T[x_1(t)] = x_1^2(t)$

$y_2(t) = T[x_2(t)] = x_2^2(t)$

$T[a_1 x_1(t) + a_2 x_2(t)] = [a_1 x_1(t) + a_2 x_2(t)]^2$

Which is not equal to $a_1 y_1(t) + a_2 y_2(t)$. Hence the system is said to be non linear.

Time Variant and Time Invariant Systems

A system is said to be time variant if its input and output characteristics vary with time. Otherwise, the system is considered as time invariant.

The condition for time invariant system is: $y(n, t) = y(n-t)$

The condition for time variant system is: $y(n, t) \neq y(n-t)$

Where $y(n, t) = T[x(n-t)]$ = input change y (n-t) = output change

Example: y(n) = x(-n)

y(n, t) = T[x(n-t)] = x(-n-t)

y(n-t) = x(-(n-t)) = x(-n + t)

∴ y(n, t) ≠ y(n-t). Hence, the system is time variant.

Liner Time variant (LTV) and Liner Time Invariant (LTI) Systems

If a system is both liner and time variant, then it is called liner time variant (LTV) system.

If a system is both liner and time Invariant then it is called liner time invariant (LTI) system.

Static and Dynamic Systems

Static system is memory-less whereas dynamic system is a memory system.

Example 1: y(t) = 2 x(t)

For present value t=0, the system output is y(0) = 2x(0).

Here, the output is only dependent upon present input. Hence it is memory less or static.

Example 2: y(t) = 2 x(t) + 3 x(t-3)

For present value t=0, the system output is y(0) = 2x(0) + 3x(-3).

Here x(-3) is past value for the present input for which the system requires memory to get this output. Hence, the system is a dynamic system.

Causal and Non-Causal Systems

A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input.

For non-causal system, the output depends upon future inputs also.

Example 1: y(n) = 2 x(t) + 3 x(t-3)

For present value t=1, the system output is y(1) = 2x(1) + 3x(-2).

Here, the system output only depends upon present and past inputs. Hence, the system is causal.

Example 2: y(n) = 2 x(t) + 3 x(t-3) + 6x(t + 3)

For present value t=1, the system output is y(1) = 2x(1) + 3x(-2) + 6x(4) Here, the system output depends upon future input. Hence the system is non-causal system.

Stable and Unstable Systems

The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system then it is said to be unstable.

Note: For a bounded signal, amplitude is finite.

Example 1: y (t) = $x^2(t)$

Let the input is u(t) (unit step bounded input) then the output y(t) = u2(t) = u(t) = bounded output. Hence, the system is stable.

Example 2: y (t) = $\int x(t)dt$

Let the input is u (t) (unit step bounded input) then the output y(t) = $\int u(t)dt$

= ramp signal (unbounded because amplitude of ramp is not finite, it goes to infinite when t ⟶ infinite). Hence, the system is unstable.

3.3. Transfer Function of an LTI System:

The transfer function of a continuous-time LTI system may be defined using Fourier transform or Laplace transform. The transfer function is defined only under zero initial conditions.

A continuous time system is shown in fig:

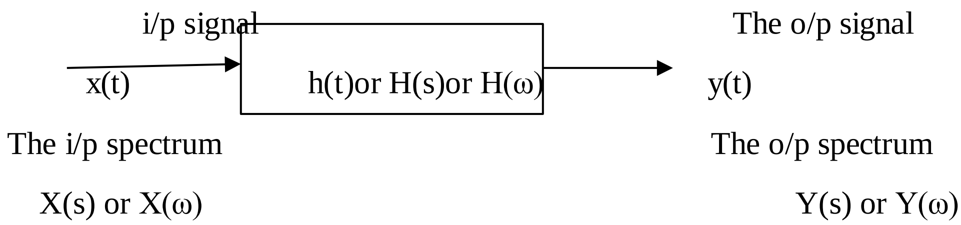


Fig: A system

The transfer function of a LTI system H(ω) is defined as the ratio of the Fourier transform of the output signal to the Fourier Transform of the input signal when the initial conditions are zero.

$$H(\omega) = Y(\omega)/X(\omega)$$

H(ω) is a complex quantity having magnitude and phase.

$$H(\omega) = |H(\omega)|e^{j\theta(\omega)}$$

The transfer function in frequency domain H(ω) is also called frequency response of the system. The frequency response is amplitude response plus phase response.

$|H(\omega)|$= Amplitude response of the system.

$\theta(\omega)=\angle H(\omega)$= Phase response of the system.

We can say that H(ω) is a frequency domain representation of a system.

Since $$Y(\omega)=H(\omega)X(\omega)$$

$$|Y(\omega)| = |H(\omega)||X(\omega)|$$

$\angle Y(\omega) = \angle H(\omega) + \angle X(\omega)$

H(ω) has conjugate symmetry property.

$$H(-\omega)=H^*(\omega)$$

i.e. $H(-\omega)| = |H(\omega)|$ and

$(\omega)=-\angle H(\omega)$

The impulse response h(t) of a system is the inverse Fourier transform of its transfer function H(ω). H(ω)=F[h(t)] & h(t)=F^{-1}[H(ω)]

3.4. Filter characteristics of Linear Systems:

A filter is a frequency selective network. It allows transmission of signals of certain frequencies with no attenuation or with very little attenuation, and it rejects or heavily attenuates signals of all other frequencies. Filters are usually classified according to their frequency response characteristics as lowpass filter (LPF), high- pass filter (HPF), band-pass filter (BPF) and band-elimination or band stop or band reject filter (BEF, BSF, BRF).

The system modifies the spectral density function of the input. The system acts as a kind of filter for various frequency components. Some frequency components are boosted in strength, i.e. they are amplified. Some frequency components are weakened in strength, i.e. they are attenuated and some may remain unaffected. Similarly, each frequency component suffers a different amount of phase shift in the process of transmission. The system, therefore, modifies the spectral density function of the input according to its filter characteristics. The modification is carried out according to the transfer function H(s) or H(ω), which represents the response of the system to various frequency components. H(ω) acts as a weighting function or spectral shaping function to the different frequency components in the input signal. An LTI system, acts as a filter. A filter is a basically a frequency selective network.

(i) Some LTI systems allow the transmission of only low frequency components and stop all high frequency components. They are called low – pass filters (LPFs).

(ii) Some LTI systems allow the transmission of only high frequency components and stop all low frequency components. They are called high – pass filters (HPFs).

(iii) Some LTI systems allow the transmission of only a particular band of frequencies and stop all other frequency components. They are called band pass filters (BPFs).

(iv) Some LTI systems reject the transmission of only a particular band of frequencies and allow all other frequency components. They are called band-rejection filters (BRFs).

The band of frequency that is allowed by the filter is called pass-band. The band of frequency that is severely attenuated and not allowed to pass through the filter is called stop-band or rejection-band. An LTI system may be characterized by its pass-band, stopband and half power band width.

3.5. Distortion less transmission through a system

The change of shape of the signal when it is transmitted through a system is called distortion. Transmission of a signal through a system is said to be distortion less if the output is an exact replica of the input signal. This replica may have different magnitude and also it may have different time delay. Mathematically, We can say that a signal x(t) is transmitted without distortion if the output

$$y(t) = kx(t-t_d)$$

Where k is a constant representing the change in amplitude and t_d is delay time.

Taking Fourier transform on both sides of the equation for y(t) and using the shifting property, we have

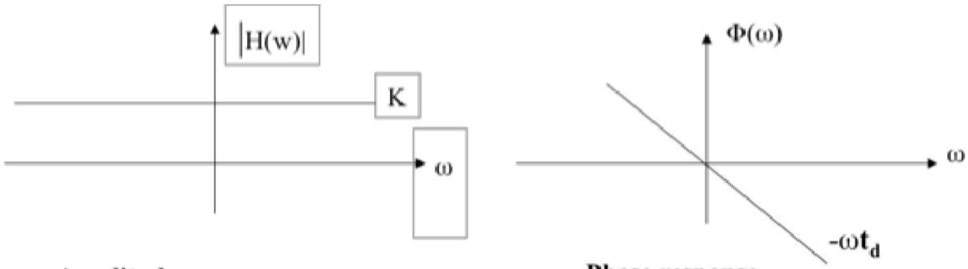
$$Y(\omega)= k\, e^{-j\omega td}$$

Therefore inverse Fourier transform, the corresponding impulse response must be
$h(t)=k\delta(t-t_d)$

The magnitude of the transfer function is given by $|H(\omega)|=k$ for all w

The phase shift is given by $\theta(\omega) = L(\omega) = -\omega t_d$

and it varies linearly with frequency given by $\theta(\omega) = n\pi - \omega t_d$ (n integral)



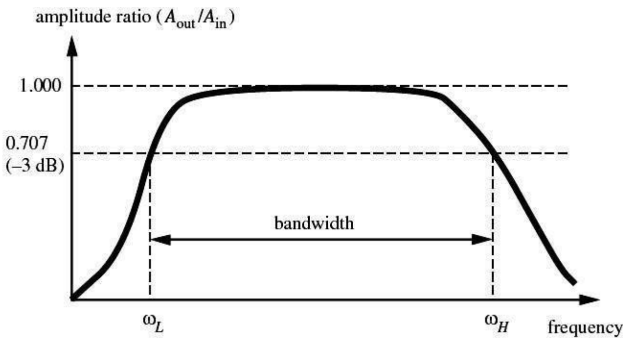
Amplitude response **Phase response**

So for distortion less transmission of a signal through a system, the magnitude $|H(\omega)|$should be a constant, i.e. all the frequency components of the input signal must undergo the same amount of amplification or attenuation, i.e. the system bandwidth is infinite and the phase spectrum should be proportional to frequency as shown in above figure. But, in practice, no system can have infinite bandwidth and hence distortion less conditions are never met exactly.

3.6. Signal bandwidth & System bandwidth:

Signal *Bandwidth:*

The spectral components of a signal extend from $-\infty$ *to*∞. Any practical signal has finite energy. As a result, the spectral components approach zero as ω tends to ∞. Therefore, we neglect the spectral components which have negligible energy and select only a band of frequency components which have most of the signal energy is known as the bandwidth of the signal. Normally, the band is selected such that it contains 95% of total energy depending on the precision.

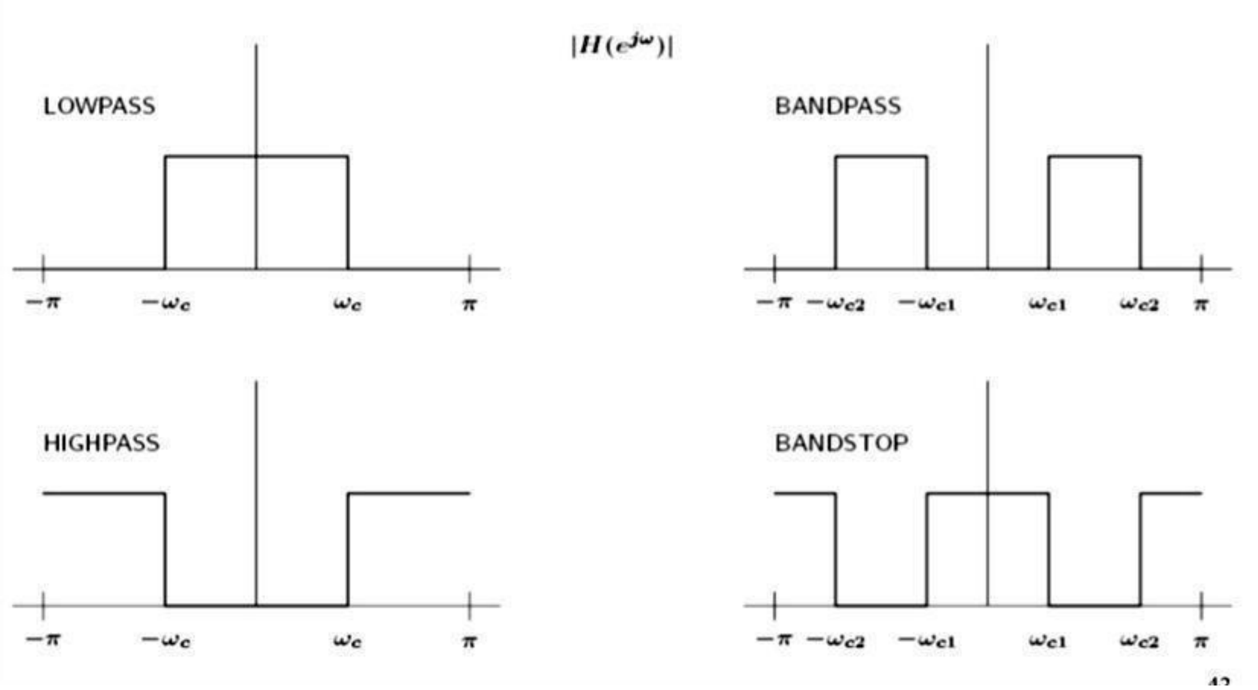


System Bandwidth*:*

For distortion less transmission, we need a system with infinite bandwidth. Due to physical limitations, it is impossible to construct a system with infinite bandwidth. Actually a satisfactory distortion less transmission can be achieved by a system with finite, but fairly large band widths, if the magnitude $|H(\omega)|$ is constant over this band. The bandwidth of a system is defined as the range of frequencies over which the magnitude $|H(\omega)|$ remain within $1/\sqrt{2}$ times (within 3 dB) of its value at mid band . The bandwidth of a system $H(\omega)|$ plot is shown in above figure is($\omega 2$-$\omega 1$) where $\omega 2$ is called the upper cut off frequency or upper 3 dB frequency or upper half power frequency and $\omega 1$ is called lower cut off frequency or lower 3dB frequency or lower half frequency. The band limited signals can be transmitted without distortion, if the system bandwidth is atleast equal to the signal bandwidth.

3.7. Ideal LPF, HPF and BPF characteristics

An ideal filter has very sharp cutoff characteristics, and it passes signals of certain specified band of frequencies exactly and totally rejects signals of frequencies outside this band.



Its phase spectrum is linear.

Ideal LPF

An ideal low-pass filter transmits, without any distortion, all of the signals of frequencies
below a certain frequency ω_c radians per second. The signals of frequencies above ω_c radians/second are completely attenuated. ω_c is called the cutoff frequency. The corresponding phase function for distortion less transmission is $-\omega t_d$. the transfer function of an ideal LPF is given by

$$|H(\omega)| = 1, \quad |\omega| < \omega_c$$
$$= 0, \quad |\omega| > \omega_c$$

The frequency response characteristics of an ideal LPF are shown in figure (a).
It is a gate function.

Ideal HPF

An ideal high-pass filter transmits, without any distortion, all of the signals of frequencies above a certain frequency ω_c radians per second and attenuates completely the signals of frequencies below ω_c radians per second, where ω_c is called the cutoff frequency. The corresponding phase function for distortion less transmission is $-\omega t_d$. the transfer function of an ideal LPF is given by

$$|H(\omega)| = 0, \quad |\omega| < \omega_c$$
$$= 1, \quad |\omega| > \omega_c$$

The frequency response characteristics of an ideal HPF are shown in figure (b).

Ideal BPF

An ideal band-pass filter transmits, without any distortion, all of the signals of frequencies within a certain frequency band ω_2-ω_1 radians per second and attenuates completely the signals of frequencies outside this band. (ω_2-ω_1) is the bandwidth of the band-pass filter. The corresponding phase function for distortion less transmission is $-\omega t_d$.

An ideal BPF is given by $H(\omega)| = 1, \quad |\omega_1| < \omega < |\omega_2|$
$$= 0, \ \omega < |\omega_1| \text{ and } \omega > |\omega_2|$$

The frequency response characteristics of an ideal BPF are shown in figure (c).

Ideal BRF

An ideal band-rejection filter rejects totally all of the signals of frequencies within a certain frequency band ω_2-ω_1 radians per second and transmits without any distortion all signals of frequencies outside this band. (ω_2-ω_1) is the rejection band. The corresponding phase function for distortion less transmission is $-\omega t_d$.

An ideal BRF is given by $|H(\omega)| = 0, \quad |\omega_1| < \omega < |\omega_2|$
$$= 1, \ \omega < |\omega_1| \text{ and } \omega > |\omega_2|$$

The frequency response characteristics of an ideal BRF are shown in figure (d).
All ideal filters are non-causal systems. Hence none of them is physically realizable.

Unit-IV
CONVOLUTION AND CORRELATION OF SIGNALS

Convolution in time and frequency domain

Convolution of signals may be done either in time domain or frequency domain. So there are following two theorems of convolution associated with Fourier transforms:

1.Time convolution theorem

2.Frequency convolution theorem

Time convolution theorem

The time convolution theorem states that convolution in time domain is equivalent to multiplication of their spectra in frequency domain.

Mathematically, if $x_1(t)\leftrightarrow X_1(\omega)$, $x_2(t)\leftrightarrow X_2(\omega)$ then, $x_1(t) * x_2(t)\leftrightarrow X_1(\omega)X_2(\omega)$.

Proof: $F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}[x_1(t) * x_2(t)]\, e^{-jwt}\, dt$

we have $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\boldsymbol{\tau}$

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty}\{\int_{-\infty}^{\infty}[x_1(\tau)\, x_2(t-\tau)d\tau]\, e^{-jwt}\, dt\}$$

Interchanging the order of integration, we have

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty}[\, x_2(t-\tau)\, e^{-jwt}\, dt]d\tau$$

Letting t-τ = p, in the second integration, we have

$$t=p+\tau \text{ and } dt = dp$$

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty}[\, x_2(p)\, e^{-jw\,(p+\tau)} dp]d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \int_{-\infty}^{\infty}[\, x_2(p)\, e^{-jwp}\, dp]e^{-jw\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) X_2(\omega) e^{-jw\tau}\, d\tau = \int_{-\infty}^{\infty} x_1(\tau) e^{-jw\tau}\, d\tau X_2(\omega)$$

$$= X_1(\omega)X_2(\omega)$$

$x_1(t) * x_2(t)\leftrightarrow X_1(\omega)X_2(\omega)$ This is time convolution theorem.

Frequency convolution theorem

The frequency convolution theorem states that the multiplication of two functions in time domain is equivalent to convolution of their spectra in frequency domain.

Mathematically, if $x_1(t)\leftrightarrow X_1(\omega)$, $x_2(t)\leftrightarrow X_2(\omega)$, then, $x_1(t)\, x_2(t) \leftrightarrow 1/2\pi\, [\, X_1(\omega) * X_2(\omega)]$

Proof: $F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} [x_1(t)\, x_2(t)]\, e^{-j\omega t} dt$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} x_1(\lambda) e^{j\lambda t} d\lambda\right] x_2(t)\, e^{-j\omega t} dt$$

Interchanging the order of integration, we get

$$F[x_1(t)\ x_2(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} x_1(\lambda) \left[\int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} e^{j\lambda t} dt\right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x_1(\lambda) \left[\int_{-\infty}^{\infty} x_2(t) e^{-j(\omega-\lambda)t} dt\right] d\lambda$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x_1(\lambda)\, x_2(\omega - \lambda)\, d\lambda$$

$$= \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)]$$

$$x_1(t)\ x_2(t) \leftrightarrow \frac{1}{2\pi} [X_1(\omega) * X_2(\omega)]$$

$$2\pi\, x_1(t)\ x_2(t) \leftrightarrow X_1(\omega) * X_2(\omega)$$

This is frequency convolution theorem in radian frequency in terms of frequency, we get

$$F[x_1(t)\ x_2(t)] = X_1(f) * X_2(f)$$

3.10. Graphical representation of convolution:

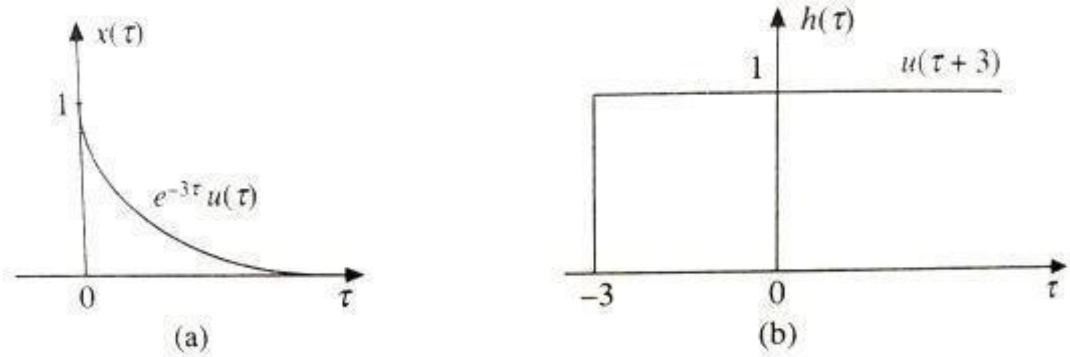
The convolution of two signals can be performed using graphical method. The procedure is:

1. For the given signals x(t) and h(t), replace the independent variable t by a dummy variable τ and plot the graph for x(τ) and h(τ).
2. Keep the function x(τ) fixed. visualize the function h(τ) as a rigid wire frame and rotate (or invert) this frame about the vertical axis (τ = 0) to obtain h(-τ).
3. Shift the frame along the τ-axis by t sec. the shifted frame now represents h(t-τ).
4. Plot the graph for x(τ) and h(t-τ) on the same axis beginning with very large negative time shift t.
5. For a particular value of t=a , integration of the product x(τ)h(t-τ) represents the area under the product curve (common area). this common area represents the convolution of x(t) and h(t) for a shift of t=a.

$$\text{i.e.,} \quad \int_{-\infty}^{\infty} x(\tau) h(a - \tau) d\tau = [x(t) * h(t)]$$

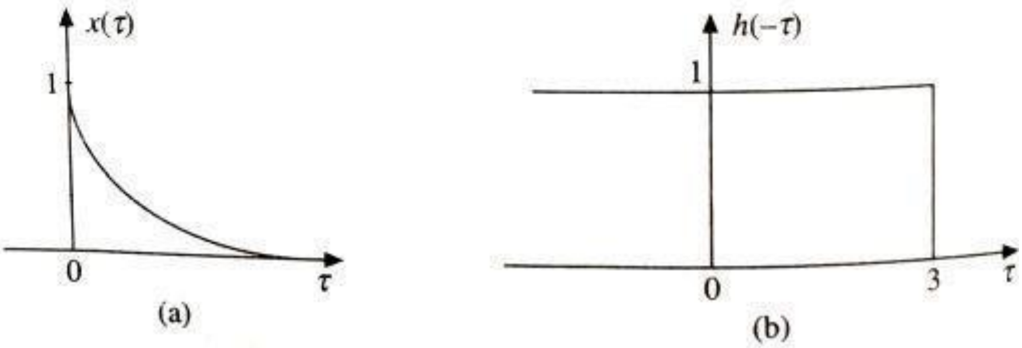
6. Increase the time shift t and take the new interval whenever the function either x(τ) or h(t-τ) changes. the value of t at which the change occurs defines the end of the current interval and the beginning of a new interval. calculate y(t) using step5.
7. The value of convolution obtained at different values of t (both positive and negative values) may be plotted on a graph to get the combined convolution.

Example : Find the convolution of the signals by graphical method
x(t) = $e^{-3t}u(t)$; h(t)=$u(t+3)$
Solution: Given x(t) = $e^{-3t}u(t)$; h(t)=$u(t+3)$
The output y(t)= x(t) * h(t) = $\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$
The two functions x(τ) and h(τ) will be
x(τ) = $e^{-3\tau}u(\tau) = e^{-3\tau}$ for $\tau \geq 0$
h(τ)=$u(\tau+3)$=1 for $\tau \geq$ -3
h(-τ)=$u(-\tau+3)$
h(-τ) can be obtained by folding h(τ) about τ=0. Figure shows the plots of x(τ) and h(τ).

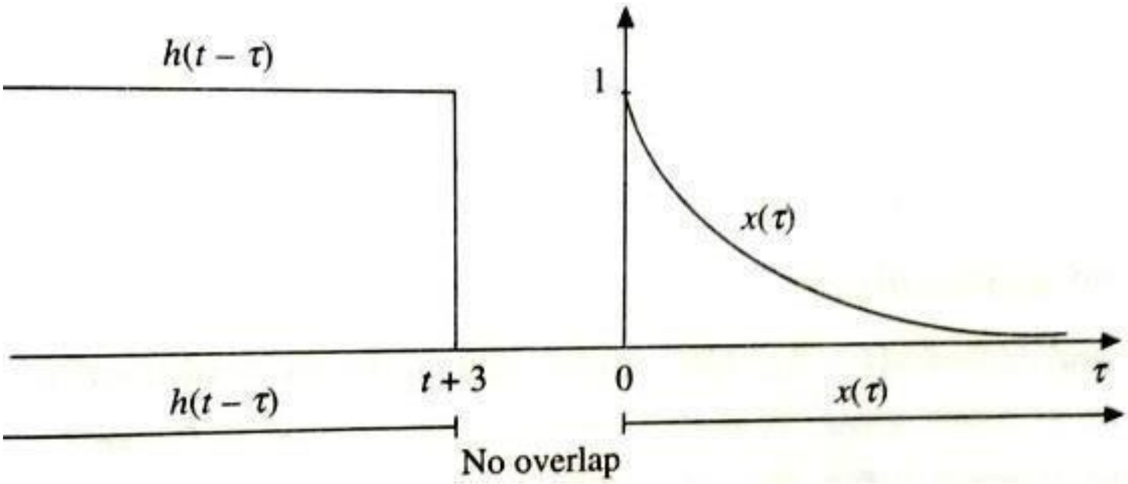


Plots of (a) $x(\tau)$, and (b) $h(\tau)$.

shows the plots of $x(\tau)$ and $h(-\tau)$.

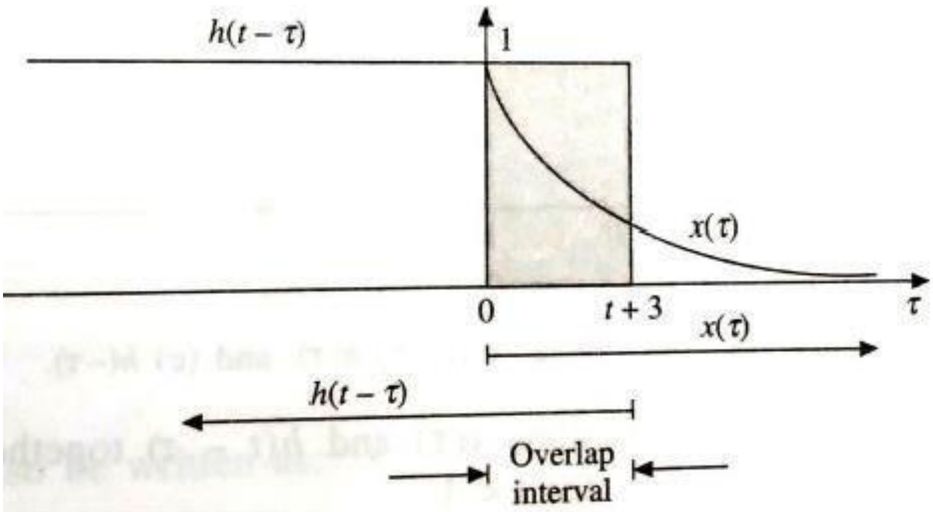


The above figure shows the plots of x(τ) and h(t-τ) together on the same time axis. Here the signal h(t-τ) is sketched for t<-3. x(τ) and h(t-τ) do not overlap. Therefore, the product x(τ) h(t-τ) is equal to zero. y(t)=0 (for t<-3)



Plots of (a) x(τ), and (b)h(t-τ)when there is no overlap

Now, increase the time shift t until the signal h(t-τ) intersects x(τ). Figure below shows the situation for t>-3. Here x(τ) and h(t-τ) overlapped.[This overlapping continuous for all values for t>-3 up to t=∞ because x(τ) exists for all values of τ>0]. But x(τ)=0 to τ=t+3.



Plot of x(τ), and h(t-τ) with overlap

$$y(t)= \int_0^{t+3} x(\tau)h(t-\tau)d\tau$$

$$= \int_0^{t+3} e^{-3\tau} d\tau = [\, e^{-3\tau}/\text{-}3]_0^{t+3} = [e^{-3(t+3)} - 1]/-3 \;\; = \left[1 - e^{-3(t+3)}\right]/3$$

$y(t) = 0$ for $t<-3$

$= \left[1 - e^{-3(t+3)}\right]/3$ for $t>-3$

3.11 Convolution properties of Fourier transform:

With two functions h(t) and g(t), and their corresponding Fourier transforms H(f) and G(f), we can form two special combinations – The convolution, denoted f = g * h, defined by

$$f(t) = g * h \equiv \int_{-\infty}^{\infty} g(\tau)h(t-\tau)\, d\tau$$

Convolution: g*h is a function of time, and g*h = h*g. The convolution is one member of a transform pair g*h ↔ G(f) H(f)

The Fourier transform of the convolution is the product of the two Fourier transforms. This is the Convolution Theorem.

Problems:

Find the convolution of the signals using Fourier transform.

1 $x_1(t) = e^{-at}u(t)$; $x_2(t)= e^{-bt}u(t)$

2 $x_1(t) = 2e^{-2t}u(t)$; $x_2(t)= u(t)$

3 $x_1(t) = e^{-t}u(t)$; $x_2(t)= e^{-t}\ u(t)$

4 $x_1(t) = 2e^{-2t}u(t)$; $x_2(t)= u(t)$

1)

Solution: Given $x_1(t) = e^{-at}u(t)$

$$X_1(\omega)=\frac{1}{a+j\omega}$$

$$x_2(t) = e^{-bt}u(t)$$

$$X2(\omega)=\frac{1}{b+j\omega}$$

$$F[x_1(t) * x_2(t)] = X_1(\omega)X_2(\omega)$$

$$x_1(t) * x_2(t) = F^{-1}[X_1(\omega)X_2(\omega)]$$

$$x_1(t) * x_2(t) = F^{-1}[\frac{1}{(a+j\omega)(b+j\omega)}] = F^{-1}\left[\frac{1}{(b-a)}\left(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right)\right]$$

$$=\frac{1}{(b-a)}\left[F^{-1}\left(\frac{1}{a+j\omega}\right) - F^{-1}\left(\frac{1}{b+j\omega}\right)\right]$$

$$=\frac{1}{(b-a)}[e^{-at}u(t) - e^{-bt}u(t)\ ]$$

2)

Solution: Given $x_1(t) = 2e^{-2t}u(t)$

$$X_1(\omega) = \frac{2}{2+j\omega}$$

$$x_2(t) = u(t)$$

$$X2(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}$$

$$X_1(\omega)X_2(\omega) = \frac{2}{2+j\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \frac{2}{j\omega(2+j\omega)} + \frac{2\pi\delta(\omega)}{2+j\omega}$$

Since $x_1(t) * x_2(t) = F^{-1}[X_1(\omega)X_2(\omega)]$, we have

$$x_1(t) * x_2(t) = F^{-1}\left[\frac{2}{j\omega(2+j\omega)} + \frac{2\pi\delta(\omega)}{2+j\omega}\right] = F^{-1}\left[\frac{1}{j\omega} - \frac{1}{(2+j\omega)} + \frac{2\pi\delta(\omega)}{2+j\omega}\right]$$

Since $\delta(\omega) = 1$ for $\omega=0$ and $\delta(\omega) = 0$ for $\omega \neq 0$, we have $\frac{2\pi\delta(\omega)}{2+j\omega} = \pi\delta(\omega)$.

$$x_1(t) * x_2(t) = F^{-1}\left[\frac{1}{j\omega} + \pi\delta(\omega) - \frac{1}{(2+j\omega)}\right] = F^{-1}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - F^{-1}\left(\frac{1}{(2+j\omega)}\right)$$

$$= u(t) - e^{-2t}u(t) = (1 - e^{-2t})u(t)$$

3)

Solution : Given $x_1(t) = e^{-t}u(t)$

$$X_1(\omega) = \frac{1}{1+j\omega}$$

$$x_1(t) = e^{-t}u(t)$$

$$X_1(\omega) = \frac{1}{1+j\omega}$$

$$X_1(\omega)X_2(\omega) = \frac{1}{1+j\omega}\frac{1}{1+j\omega} = \frac{1}{(1+j\omega)^2}$$

Since $x_1(t) * x_2(t) = F^{-1}[X_1(\omega)X_2(\omega)]$, we have

$$x_1(t) * x_2(t) = F^{-1}\left[\frac{1}{(1+j\omega)^2}\right] = te^{-t}u(t)$$

4)

2)**Solution:** Given $x_1(t) = 2e^{-2t}u(t)$

$$X_1(\omega)= \frac{2}{2+j\omega}$$

$$x_2(t) = u(t)$$

$$X2(\omega)= \pi\delta(\omega) + \frac{1}{j\omega}$$

$$X_1(\omega)X_2(\omega) = \frac{2}{2+j\omega}\left(\pi\delta(\omega) + \frac{1}{j\omega}\right) = \frac{2}{j\omega(2+j\omega)} + \frac{2\pi\delta(\omega)}{2+j\omega}$$

Since $x_1(t) * x_2(t) = F^{-1}[X_1(\omega)X_2(\omega)]$, we have

$$x_1(t) * x_2(t) = F^{-1}\left[\frac{2}{j\omega(2+j\omega)} + \frac{2\pi\delta(\omega)}{2+j\omega}\right] = F^{-1}\left[\frac{1}{j\omega} - \frac{1}{(2+j\omega)} + \frac{2\pi\delta(\omega)}{2+j\omega}\right]$$

Since $\delta(\omega) = 1$ for $\omega=0$ and $\delta(\omega) = 0$ for $\omega \neq 0$, we have $\frac{2\pi\delta(\omega)}{2+j\omega} = \pi\delta(\omega)$.

$$x_1(t) * x_2(t) = F^{-1}\left[\frac{1}{j\omega} + \pi\delta(\omega) - \frac{1}{(2+j\omega)}\right] = F^{-1}\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - F^{-1}\left(\frac{1}{(2+j\omega)}\right)$$

$$= u(t) - e^{-2t}u(t) = (1- e^{-2t})u(t)$$

Correlation

Correlation is a measure of similarity between two signals. The general formula for correlation is

$$\int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)dt$$

There are two types of correlation: •

Auto correlation

- Cros correlation

Auto Correlation Function

It is defined as correlation of a signal with itself. Auto correlation function is a measure of similarity between a signal & its time delayed version. It is represented with $R(\tau)$.

Consider a signals $x(t)$. The auto correlation function of $x(t)$ with its time delayed version is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \qquad \text{[-ve shift]}$$

Where τ = searching or scanning or delay parameter.

If the signal is complex then auto correlation function is given by

$$R_{11}(\tau) = R(\tau) = \int_{-\infty}^{\infty} x(t)x^*(t-\tau)dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x(t+\tau)x^*(t)dt \qquad \text{[-ve shift]}$$

Properties of Auto-correlation Function of Energy Signal

- Auto correlation exhibits conjugate symmetry i.e. $R(\tau) = R^*(-\tau)$
- Auto correlation function of energy signal at origin i.e. at τ=0 is equal to total energy of that signal, which is given as:

  $R(0) = E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$
- Auto correlation function $\infty \frac{1}{\tau}$,
- Auto correlation function is maximum at τ=0 i.e $|R(\tau)| \leq R(0)\ \forall\ \tau$
- Auto correlation function and energy spectral densities are Fourier transform pairs. i.e.

  $F.T\,[R(\tau)] = \Psi(\omega)$

  $\Psi(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau}\, d\tau$
- $R(\tau) = x(\tau) * x(-\tau)$

Auto Correlation Function of Power Signals

The auto correlation function of periodic power signal with period T is given by

$$R(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)x^*(t-\tau)dt$$

Properties

- 2) Auto correlation of power signal exhibits conjugate symmetry i.e. $R(\tau) = R*(-\tau)$
- Auto correlation function of power signal at $\tau = 0$ $at origin$ is equal to total power of that signal. i.e.

  $R(0) = \rho$
- Auto correlation function of power signal $\infty \frac{1}{\tau}$,
- Auto correlation function of power signal is maximum at $\tau = 0$ i.e.,

  $|R(\tau)| \leq R(0) \, \forall \, \tau$
- Auto correlation function and power spectral densities are Fourier transform pairs. i.e.,

  $F.T[R(\tau)] = s(\omega)$

  $s(\omega) = \int_{-\infty}^{\infty} R(\tau)e^{-j\omega\tau} d\tau$
- $R(\tau) = x(\tau) * x(-\tau)$

Density Spectrum

Let us see density spectrums:

Energy Density Spectrum

Energy density spectrum can be calculated using the formula:

$$E = \int_{-\infty}^{\infty} | x(f) |^2 df$$

Power Density Spectrum

Power density spectrum can be calculated by using the formula:

$$P = \Sigma_{n=-\infty}^{\infty} | C_n|^2$$

Cross Correlation Function

Cross correlation is the measure of similarity between two different signals.

Consider two signals x_1t and x_2t. The cross correlation of these two signals $R_{12}(\tau)$ is given by

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)x_2(t-\tau)\, dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2(t)\, dt \qquad \text{[-ve shift]}$$

If signals are complex then

$$\boldsymbol{R_{12}(\tau)} = \int_{-\infty}^{\infty} x_1(t)x_2^*(t-\tau)\, dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_1(t+\tau)x_2^*(t)\, dt \qquad \text{[-ve shift]}$$

2) $$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)x_1^*(t-\tau)\,dt \qquad \text{[+ve shift]}$$

$$= \int_{-\infty}^{\infty} x_2(t+\tau)x_1^*(t)\,dt \qquad \text{[-ve shift]}$$

Properties of Cross Correlation Function of Energy and Power Signals

- Auto correlation exhibits conjugate symmetry i.e. $R_{12}(\tau) = R^*_{21}(-\tau)$.
- Cross correlation is not commutative like convolution i.e.

$$R_{12}(\tau) \neq R_{21}(-\tau)$$

- If $R_{12}(0) = 0$ means, if $\int_{-\infty}^{\infty} x_1(t)x_2^*(t)\,dt = 0$, then the two signals are said to be orthogonal.

  For power signal if $\lim_{T\to\infty} \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} x(t)x^*(t)\,dt$ then two signals are said to be orthogonal.

- Cross correlation function corresponds to the multiplication of spectrums of one signal to the complex conjugate of spectrum of another signal. i.e.

$$R_{12}(\tau) \rightarrow\rightarrow X_1(\omega)X_2^*(\omega)$$

  This also called as correlation theorem.

Parsvel's Theorem

Parsvel's theorem for energy signals states that the total energy in a signal can be obtained by the spectrum of the signal as

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Note: If a signal has energy E then time scaled version of that signal $x(at)$ has energy E/a.

UNIT – V
Laplace Transforms

5.1 Introduction:

Laplace Transform is a powerful tool for analysis and design of Continuous Time signals and systems. The Laplace Transform differs from Fourier Transform because it covers a broader class of CT signals and systems which may or may not be stable. Most of the LTI Systems act in time domain but they are more clearly described in the frequency domain instead. It is important to understand Fourier analysis in solving many problems involving signals and LTI systems. Now, we shall deal with signals and systems which do not have a Fourier transform.

We found that continuous-time Fourier transform is a tool to represent signals as linear combinations of complex exponentials. The exponentials are of the form e^{st} with $s = j\omega$and $e^{j\omega}$ is an eigen function of the LTI system. Also, we note that the Fourier Transform only exists for signals which can absolutely integrated and have a finite energy. This observation leads to generalization of continuous-time Fourier transform by considering a broader class of signals using the powerful tool of "Laplace transform".

Bilateral Laplace Transform

The Laplace transform of a general signal x(t) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

It is a function of complex variable,„s" and is written as $s = \sigma + j\omega$, with σ and ω, the real and imaginary parts, respectively.

The transform relationship between *x(t)* and *X(s)* is indicated as

$$x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$$

Existence of Laplace Transform

In general,

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

The ROC consists of those values of „s" (i.e., those points in the s-plane) for which X(s) converges i.e., value of s for which

$$\int_{-\infty}^{\infty} |x(t)e^{-st}|dt < \infty$$

Since $s = \sigma + j\omega$ the condition for existence is

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < \infty$$

Thus, ROC of the Laplace transform of an x(t) consists of all values of s for which $x(t)\, e^{-\sigma t}$ is absolutely integrable.

5.2 Relation between Laplace and Fourier transform

When $s = j\omega,\ X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ corresponds to the Fourier transform of x(t), i.e., $X(s)|s = j\omega = \mathcal{F}\{x(t)\}$. The Laplace transform also bears a straight forward relationship to the Fourier transform when the complex variable „s" is not purely imaginary. To see this relationship, consider X(s) with $s = j\omega$.

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t}\,dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

or

The real exponential $e^{-\sigma t}$may be decaying or growing in time, depending on σ being positive or negative.

Unilateral Laplace Transform This Transform have considerable value in analyzing causal systems and particularly, systems specified by linear constant coefficient differential equations with nonzero initial conditions(i.e., systems that are not initially at rest)

The Unilateral Laplace transform of a continuous time signal x(t) is defined as

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}\,dt$$

Problem 1: Find the Laplace transform of (t)=$e^{-at}\,u(t)$

Solution:

The Fourier transform *X(jω)* converges for a>0 and is given by

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}\,u(t)e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-at}\,e^{-j\omega t}\,dt = \frac{1}{a+j\omega},\ a > 0$$

Now, the Laplace transform is

$$X(s) = \int_{-\infty}^{\infty} e^{-at}\,u(t)e^{-st}\,dt$$

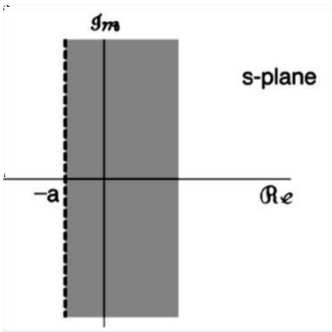
$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} e^{-at}\,u(t)e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-(\sigma+j\omega)t}\,dt = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}\,dt = \frac{1}{(\sigma + a) + j\omega}$$

with $(\sigma + a) > 0$ or equivalently, since $s = \sigma + j\omega$ and $\sigma = Re\{s\}$,

$$X(s) = \frac{1}{s+a},\ \text{Re}\{s\} > -a$$

That is $e^{-at}\ u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}$,Re{s} > -*a*.



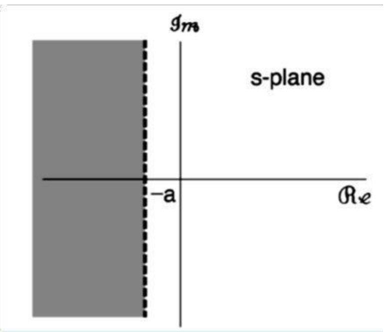
For example *a*=*0*, *x(t)* is the unit step with Laplace transform $X(s) = \frac{1}{s}$, Re{s} > 0

Problem 2: Find the Laplace transform of $x(t) = -e^{-at}\ u(-t)$
Solution: The Laplace transform is

$$X(s) = \int_{-\infty}^{\infty} [-e^{-at}\ u(-t)]e^{-st} dt = -\int_{-\infty}^{0} e^{-(s+a)t} dt = \frac{1}{s+a}$$

This result converges only when *Re{s+a} <0, or Re{s}<-a*

That is $-e^{-at}\ u(-t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}$,Re{s} < -*a*



Comparing the two results in the above two problems, we see that Laplace transform is identical for both the signals. But the range of values of „s" for which the transform converges is different i.e., *Region of Convergence* (ROC) is different for the above said signals. ***Note:*** same X(s) may correspond to different x(t) depending on ROC

Problem 3: Find the Laplace transform of $(t)=\delta(t)$. What is the region of convergence?

Solution: The Laplace transform of a general signal x(t) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt$$

$$\boldsymbol{\mathcal{L}}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-st} dt$$

From the property of impulse function

$$\delta(t).x(t) = \delta(t)x(0)$$

$$\boldsymbol{\mathcal{L}}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-st}|t = 0dt = \int_{-\infty}^{\infty} \delta(t)e^{0}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1$$

As we know that area under impulse function is unity.

Since $\boldsymbol{\mathcal{L}}\{\delta(t)\} = 1$, which is a constant, Laplace transform converges for all values of s i.e., ROC is entire s-plane

Problem 4: What is the Laplace transform of $cos(\omega_o t)u(t)$? What is its Region of Convergence?
Solution: The Laplace transform of a general signal x(t) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

From Euler"s relation $cos(\omega_o t) = \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}$

$$\boldsymbol{\mathcal{L}}\{cos(\omega_o t)u(t)\} = \int_{-\infty}^{\infty} cos(\omega_o t)u(t)e^{-st}dt = \int_{-\infty}^{\infty} \left(\frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}\right)e^{-st}dt$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{\infty} e^{j\omega_o t}e^{-st}dt + \int_{-\infty}^{\infty} e^{-j\omega_o t}e^{-st}dt\right\}$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{\infty} e^{-(s-j\omega_o)t}dt + \int_{-\infty}^{\infty} e^{-(s+j\omega_o)t}dt\right\} = \frac{1}{2}\left\{\frac{1}{(s-j\omega_o)} + \frac{1}{(s+j\omega_o)}\right\}$$

$$= \frac{s}{s^2 + \omega_o{}^2}$$

Problem 5: Find the unilateral Laplace transform of $(t)=e^{-a(t+1)}u(t+1)$

Solution: The Unilateral Laplace transform is

$$X(s) = \int_0^{\infty} x(t)\,e^{-st}dt = \int_0^{\infty} e^{-a(t+1)}u(t+1)\,e^{-st}dt = \int_0^{\infty} e^{-a}\,e^{-(s+a)t}dt = \frac{e^{-a}}{s+a}$$

ROC: Re{s}>-a

Problem 6: Determine the Laplace transform of the ramp function.

Solution: The unit ramp function is given as

$$r(t) = \begin{cases} t\,; & t \geq 0 \\ 0 & otherwise \end{cases} \qquad \text{or } r(t)=t\,u(t)$$

The Laplace transform of a general signal x(t) is defined as

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$\boldsymbol{\mathcal{L}}\{tu(t)\} = \int_0^{\infty} te^{-st}dt = \left[\frac{te^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s}dt = \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = \frac{1}{s^2}$$

With ROC: Re{s}>0

Problem 7: A damped sine wave is given by $(t)=e^{-at}sin(\omega t)$. Find the LT of this signal
Solution: With the help of Euler‟s Identity,

$$x(t) = e^{-at}\left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right] = \frac{1}{2j}\left[e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right]$$

Applying the Laplace transform

$$X(s) = \frac{1}{2j}\mathcal{L}\left[e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right] = \frac{1}{2j}\left[\frac{1}{s+(a-j\omega)} - \frac{1}{s+(a+j\omega)}\right] = \frac{\omega}{(s+a)^2+\omega^2}$$

ROC : Re{s} > -a

Problem 8: Find the Transfer function of a system with impulse response

$$h(t) = 3.e^{-2t}u(t) - e^{-3t}u(t)$$

Solution: Transfer function is obtained by applying Laplace Transform to the impulse response h(t)

$$H(s) = \frac{3}{s+2} - \frac{1}{s+3} = \frac{2s+7}{s^2+5s+6}$$

Problem 9: Find the Laplace transform of $x(t) = \cosh(\omega t)u(t)$

Solution: With the help of Euler‟s Identity,

$$x(t) = \left[\frac{e^{\omega t} + e^{-\omega t}}{2}\right]u(t) = \frac{1}{2}\left[e^{\omega t}u(t) + e^{-\omega t}u(t)\right]$$

Therefore,

$$X(s) = \frac{1}{2}\mathcal{L}\left[e^{\omega t}u(t) + e^{-\omega t}u(t)\right] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s+\omega}\right] = \frac{s}{s^2-\omega^2}$$

Problem 10: Find the Unilateral Laplace transform of $x(t) = \delta(t+1)$

Solution:

The Unilateral Laplace transform of a general signal x(t) is defined as

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

$$\mathcal{L}\{\delta(t+1)\} = \int_0^\infty \delta(t+1)e^{-st}dt = 0$$

5.3 Region of Convergence (ROC) of LT

The Laplace transform of a continuous signal *x(t)* is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

The Laplace transform has two parts which are, the expression and Region of Convergence respectively.

Whether the Laplace transform X(s) of a signal x(t) exists or not depends on the complex variable „s" as well as the signal itself. All complex values of „s" for which the integral in the definition converges form a *region of convergence (ROC)* in the s-plane

The concept of ROC can be understood easily by finding Laplace transform of two functions given below:

a) $x(t) = e^{-at}u(t)$

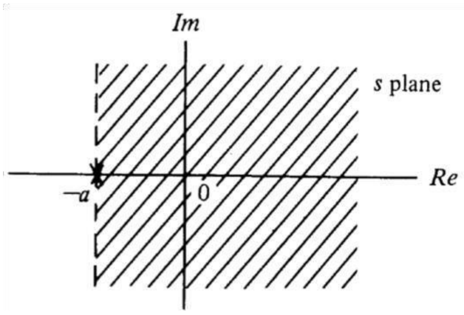
$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt$$

$$= \int_0^{\infty} e^{-at}e^{-st}dt$$

$$= \int_0^{\infty} e^{-(s+a)t}\, dt$$

with $s = \sigma + j\omega$ the integral converges only when $Re(s+a) > 0$, i.e., $\sigma = Re(s) > -a$

therefore $X(s) = \frac{1}{s+a}$. The ROC is shown in figure below.



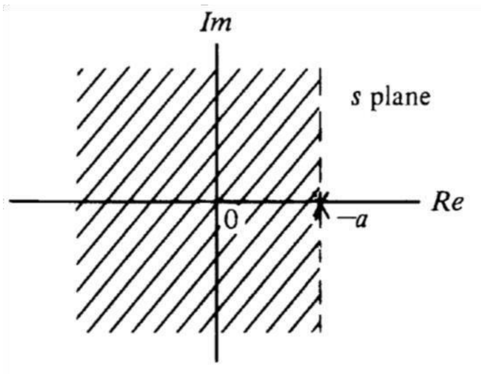
b) $x(t) = -e^{-at}u(-t)$

$$X(s) = \int_{-\infty}^{\infty} \{-e^{-at}u(-t)\}\, e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-at}e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

with $s = \sigma + j\omega$ the integral converges only when $Re(s+a) < 0$, i.e., $\sigma = Re(s) < -a$ therefore $X(s) = \frac{1}{s+a}$. The ROC is shown in figure below.



The plane in which ROC is shown is known as s-plane.As $s = \sigma + j\omega$, s-plane consists of real and imaginary axes. The region towards left side of the imaginary axis is called Left Half Plane and towards right is called Right Half Plane.

Zeros and Poles of the Laplace Transform

Laplace transforms in the above examples are rational, i.e., they can be written as a ratio of polynomials of variable „s"in the general form

$$X(s) = \frac{N(s)}{D(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{\prod_{k=1}^{M} s - s_{zk}}{\prod_{k=1}^{N} s - s_{pk}}.$$

- *N(s)* is the numerator polynomial of order M with s_{zk},(k=1,2,…,M) roots
- *D(s)* is the denominator polynomial of order N with s_{pk}(k=1,2,…,N) roots

Roots of numerator polynomial are called zeros and the roots of denominator polynomial are called poles. Poles in s-plane are indicated with „x" and zeros with"o". The representation of X(s) through its poles and zeros in the s-plane is referred to as the *pole-zero plot* of X(s).

In general, we assume the order of the numerator polynomial is always lower than that of the denominator polynomial, i.e., M<N. If this is not the case, we can always expand X(s) into multiple terms so that M<N is true for each of terms.

5.4 Properties of ROC:

Property 1: The ROC of X(s) consists of strips parallel to the jω-axis in the s-plane.

$$\int_{-\infty}^{\infty} |x(t)| e^{-\sigma t} dt < \infty$$

Property 2: For rational Laplace transforms, the ROC does not contain any poles but it is bounded by poles or extends to infinity.

Property 3: If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

Property 4: If x(t) is right sided, and if the line $Re\{s\}=\sigma_o$ is in the ROC, then all values of s for which $Re\{s\}>\sigma_o$ will also be in the ROC.

Property 5: If x(t) is left sided, and if the line Re{s}=σ_o is in the ROC, then all values of s for which Re{s}<σ_o will also be in the ROC.

Property 6: If x(t) is two sided, and if the line Re{s}=σ_o is in the ROC, then the ROC consist of a strip in the s-plane that includes the line Re{s}=σ_o.

Property 7: Ifx(t) is right sided and its Laplace transform X(s) is rational, then the ROC in s-plane is right of the rightmost pole.

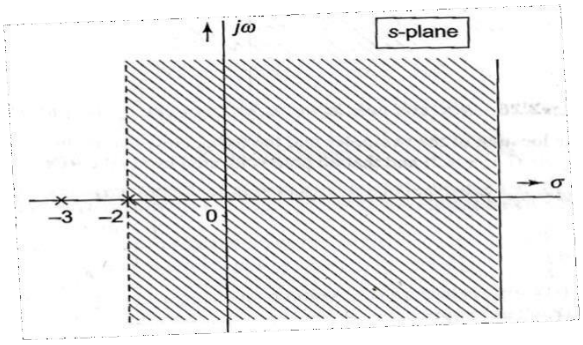
Property 8: If x(t) is left sided and its Laplace transform X(s) is rational, then the ROC in s-plane is left of the leftmost pole.

Solved Problems:

Problem 1: Find the Laplace transform of x(t) = $[2e^{-2t}+3e^{-3t}]u(t)$.Also indicate locations of poles and zeros and Plot Region of Convergence.

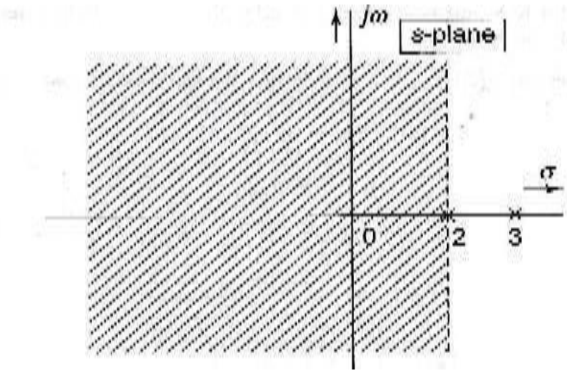
Solution: The signal given x(t) = $[2e^{-2t}+3e^{-3t}]u(t)$ is a right-sided signal and its Laplace transform is $X(s) = \frac{2}{s+2} + \frac{3}{s+3}$ with ROC: Re{s}>-2 ∩ Re{s}>-3 = Re{s}>-2

ROC is right of the right most pole (Property 7) and the plot is shown below



Problem 2: Find the Laplace transform of x(t) = $[2e^{2t}+3e^{3t}]u(-t)$.Also indicate locations of poles and zeros and Plot Region of Convergence.

Solution: The signal given x(t) = $[2e^{2t}+3e^{3t}]u(-t)$ is a left-sided signal and its Laplace transform is $X(s) = \frac{-2}{s-2} + \frac{-3}{s-3}$ with ROC: Re{s}<2 ∩ Re{s}<3 = Re{s}<2

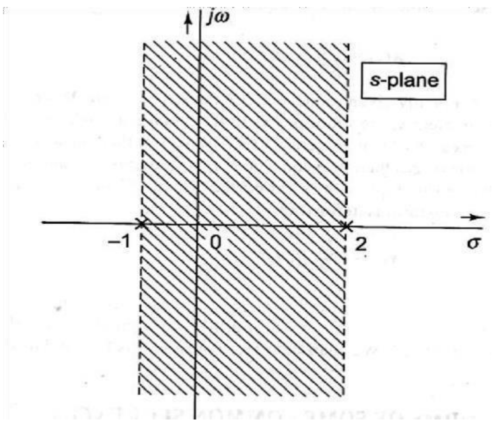


ROC is left of the left most pole (Property 8) and the plot is shown below

Problem 3: Find the Laplace transform of $x(t) = e^{-t}u(t)+e^{2t}u(-t)$.Also indicate locations of poles and zeros and Plot Region of Convergence.

Solution: The signal given $x(t) = e^{-t}u(t)+e^{2t}u(-t)$ is a two-sided signal and its Laplace transform is $X(s) = \frac{1}{s+1} - \frac{1}{s-2}$ with ROC: Re{s}>-1 ∩ Re{s}<2 = -1<Re{s}<2

ROC is strip in the s-plane and lies to the right of the pole at s=-1 and to the left of the pole at s=2. Theplot is shown below



5.5Properties of Laplace Transform

1. Linearity of the Laplace Transform ***Statement:***

If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$ with a region of convergence denoted as R_1 and

$x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$ with a region of convergence denoted as R_2

then $ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\leftrightarrow} aX_1(s) + bX_2(s)$, with ROC containing $R_1 \cap R_2$

Proof:

Consider the linear combination of two signals $x_1(t)$ and $x_2(t)$ as $z(t)=ax_1(t)+bx_2(t)$. Now, take the Laplace transform of $z(t)$ as

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{ax_1(t) + bx_2(t)\} = \int_{-\infty}^{\infty} \{ax_1(t) + bx_2(t)\}e^{-st}dt$$

$$= a\int_{-\infty}^{\infty} x_1(t)e^{-st}dt + b\int_{-\infty}^{\infty} x_2(t)e^{-st}dt$$

$$= aX_1(s) + bX_2(s)$$

2. Time Shifting: If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$**with ROC= R**

then $x(t-\tau) \overset{\mathcal{L}}{\leftrightarrow} e^{-s\tau}X(s)$ with ROC= R

Proof:

$$\mathcal{L}\{x(t-\tau)\} = \int_{-\infty}^{\infty} x(t-\tau)e^{-st}dt$$

Let t-τ=p

$$= \int_{-\infty}^{\infty} x(p)e^{-s(p+\tau)}dt$$

$$= e^{-s\tau} \int_{-\infty}^{\infty} x(p)e^{-sp}dt$$

$$= e^{-s\tau}X(s)$$

3. Shifting in s-Domain*:* If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R then $e^{s_o t}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s-s_o)$ with ROC= R+Re{s_o}

Proof:

$$\mathcal{L}\{e^{s_o t}x(t)\} = \int_{-\infty}^{\infty} e^{s_o t}x(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-(s-s_o)t}dt$$

$$= X(s-s_o)$$

4. Time Scaling: If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ **with ROC= R then** $x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|}X\left(\frac{s}{a}\right)$ **with ROC=** $\mathbf{R_1}$**=*a*R**

Proof:

To prove this we have to consider two cases: *a (real)* is positive and *a* is negative. Case 1: For *a*>0:

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt$$

Using the substitution of *λ=at; dt=adλ*

$$= \frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-\left(\frac{s}{a}\right)\lambda}d\lambda$$

$$= \frac{1}{a}X\left(\frac{s}{a}\right)$$

Case 2: For *a*< 0 :

$$\mathcal{L}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-st}dt$$

Using the substitution of *λ=at; dt=adλ*

$$= -\frac{1}{a}\int_{-\infty}^{\infty} x(\lambda)e^{-\left(\frac{s}{a}\right)\lambda} d\lambda$$

$$= -\frac{1}{a} X\left(\frac{s}{a}\right)$$

Combining the two cases, we get $x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{s}{a}\right)$ with ROC= R_1=aR

5. Convolution Property: If $x_1(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)$ with ROC = R_1 and $x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_2(s)$ with ROC = R_2 then $x_1(t) * x_2(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s).X_2(s)$, with ROC containing $R_1 \cap R_2$

Proof:

$$\mathcal{L}\{z(t)\} = \mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} \{x_1(t) * x_2(t)\} e^{-st} dt$$

$$= \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)d\tau\right\} e^{-st} dt$$

Interchanging the order of integrations

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} x_1(\tau)\left\{\int_{-\infty}^{\infty} x_2(t-\tau)e^{-st} dt\right\} d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau)\{e^{-s\tau} X_2(s)\}\, d\tau$$ (Since from Time shifting property)

$$= X_2(s) \int_{-\infty}^{\infty} x_1(\tau)e^{-s\tau}\, d\tau$$

$$= X_1(s).X_2(s)$$

6. Differentiation in the Time Domain: If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= then $\frac{dx(t)}{dt} \overset{\mathcal{L}}{\leftrightarrow} s\, X(s)$ with ROC containing R

Proof: Inverse Laplace transform is given by

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$$

Differentiating above on both sides with respect to „t"

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \{sX(s)\}e^{st} ds$$

Comparing both equations $s\, X(s)$ is the Laplace transform of $\frac{dx(t)}{dt}$.

7. Integration in the Time Domain: If $x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s)$ with ROC= R

then $\int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s} X(s)$ with ROC containing $R \cap \{Re\{s\} > 0\}$

Proof: This can be derived using convolution property as

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$\mathcal{L}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \mathcal{L}\{x(t) * u(t)\} = X(s).\mathcal{L}\{u(t)\} = X(s)\frac{1}{s}$$

8. The Initial and Final Value Theorem:

If x(t) and $\frac{dx(t)}{dt}$ are Laplace transformable, and under the specific constraints that x(t)=0 for t<0 containing no impulses at the origin,one can directly calculate, from the Laplace transform, the initial value *x(0⁺)*, i.e., x(t) as t approaches zero from positive values of t. Specifically the ***initial - value theorem*** states that

$$x(0\ \ ) = \lim_{s\to\infty} sX(s)^{+}$$

Also, if x(t)=0 for t<0 and, in addition, x(t) has a finite limit as t→∞, then the ***final value theorem*** says that

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Proof*:*

To prove these theorems, we need to consider the Unilateral Laplace transform of $\frac{dx(t)}{dt}$

Unilateral Laplace transform of $\{x(t)\} = \int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st}dt$

$$= [x(t)e^{-st}]_{0^+}^{\infty} + s\int_{0^+}^{\infty} x(t)e^{-st}dt$$

$$= x(\infty)e^{-\infty} - x(0^+) + sX(s)$$

$$= sX(s) - x(0^+)$$

Initial value theorem

From the above discussion, we know that

$$\int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st}dt = sX(s) - x(0^+)$$

Applying the $\lim_{s\to\infty}$ *on both sides*

$$0 = \lim_{s\to\infty} sX(s) - x(0^+)$$

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Final value theorem we know that

$$\int_{0+}^{\infty} \frac{dx(t)}{dt} e^{-st}dt = sX(s) - x(0^+)$$

Applying the $\lim_{s\to 0}$ *on both sides*

$$[x(t)]_{0^+}^{\infty} = \lim_{s\to 0} sX(s) - x(0^+)$$

$$\lim_{t\to\infty} x(t) - x(0^+) = \lim_{s\to\infty} sX(s)$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) - x(0^+)$$

Solved Problems:

Problem 1: Find the Laplace transform and ROC of $x(t) = e^{-4t}u(t-2)$

Solution:

We know that $e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \ ROC: Re\{s\} > -a$

$$e^{-4t}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+4}, \ ROC: Re\{s\} > -4$$

Signal $e^{-4t}u(t)$is now delayed by 2 units to get $e^{-4(t-2)}u(t-2)$

Therefore, applying time shifting property

$$e^{-4(t-2)}u(t-2) = e^{8}e^{-4t}u(t-2) \overset{\mathcal{L}}{\leftrightarrow} \frac{e^{-2s}}{s+4}, \ ROC: Re\{s\} > -4$$

$$e^{-4t}u(t-2) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{e^8}\frac{e^{-2s}}{s+4}, \ ROC: Re\{s\} > -4$$

Problem 2: Given $F(s) = \frac{s+8}{s^2+6s+13}$, find *f(0)*

Solution:

Consider $sF(s) = \frac{s(s+8)}{s^2+6s+13} = 1 + \frac{2s-13}{s^2+6s+13}$

From the initial value theorem, we know that

$$initial\ value\ of\ f(t) = f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left(1 + \frac{2s-13}{s^2+6s+13}\right) = 1$$

5.6 Inverse Laplace Transform

Inverse Laplace transform maps a function in s-domain back to the time domain. One application is to convert a system response to an input signal from s-domain back to the time domain. The Laplace transform converts the differential equations that describe system behaviour to a polynomial. Also the convolution operation which describes the system action on the input signals is converted to a multiplication operation. These two properties make it much easier to do systems analysis in the s-domain. Inverse Laplace transform is performed using Partial Fraction Expansion that split up a complicated fraction into forms that are in the Laplace Transform table. The Laplace transform of a continuous signal *x(t)* is given by

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

Since $s=\sigma+j\omega$

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{\infty} \{x(t)e^{-\sigma t}\}e^{-j\omega t}dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$$

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

We can recover x(t) from its Laplace transform evaluated for a set of values of $s=\sigma+j\omega$ in the ROC, with σ fixed and ω varying from -∞ to +∞. Recovering s(t) from X(s) is done by changing the variable of integration in the above equation from ω to s and using the fact that σ is constant, so that $ds=jd\omega$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{\sigma t}e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}d\omega = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds$$

The contour of integration in above equation is a straight line in the s-plane corresponding to all points s satisfying Re{s}=σ. This line is parallel to the $j\omega$-axis. Therefore, we can choose any value of σ such that $X(\sigma + j\omega)$ converges.

5.7 Partial Fraction Expansion

As we know that the rational form of X(s) can be expanded into partial fractions, Inverse Laplace transform can be taken according to location of poles and ROC of X(s). The roots of denominator polynomial, i.e., poles can be simple and real, complex or multiple.

We know that X(s) is expanded in partial fractions as

$$X(s) = \frac{c_0}{s - s_0} + \frac{c_1}{s - s_1} + \frac{c_2}{s - s_2} + \cdots + \frac{c_n}{s - s_n}$$

Here the roots $s_0,s_1,s_2,...s_n$ can be real, complex or multiple. Then the values of $k_0,k_1,k_2,...k_n$ constants are calculated accordingly.

In order to find the appropriate time domain function, ROC should be indicated for the s-domain function. Otherwise we may have multiple time-domain functions based on different possible ROCs.

Example for Real roots:

Problem 1:Find out the partial fraction expansion and hence Inverse Laplace transform of the function $X(s) = \frac{s^2+2s-2}{s(s+2)(s-3)}$, ROC: Re{s} > 3

Solution:

The function $X(s) = \frac{s^2+2s-2}{s(s+2)(s-3)}$ can be written as,

$$X(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-3}$$

The constants calculated are A=1/3, B=-1/5, C=13/15

$$X(s) = \frac{1/3}{s} + \frac{1/5}{s+2} + \frac{13/15}{s-3}$$

$$x(t) = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{13}{15}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

From the given ROC: Re{s}>3, the resultant signal x(t) should be right sided.

Therefore, $x(t) = \frac{1}{3}u(t) + \frac{1}{5}e^{-2t}u(t) + \frac{13}{15}e^{3t}u(t)$

Example for Complex roots:

Problem 2: Obtain right sided time domain signal for the function $X(s) = \frac{s^2+2s+1}{(s+2)(s^2+4)}$ **Solution:**

We can write the given function as $X(s) = \frac{s^2+2s+1}{(s+2)(s^2+4)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+2^2}$

The constants can be calculated as A=1/8, B=0.874, C=0.5

Therefore, $X(s) = \frac{1}{8}\frac{1}{s+2} + \frac{0.874s+0.5}{s^2+2^2} = \frac{1}{8}\left\{\frac{1}{s+2}\right\} + 0.874\left\{\frac{s}{s^2+2^2}\right\} + 0.25\left\{\frac{2}{s^2+2^2}\right\}$

Finally $x(t) = \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 0.874\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + 0.25\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$

i.e., $x(t) = \frac{1}{8}e^{-2t}u(t) + 0.874\cos(2t)\,u(t) + 0.25sin(2t)u(t)$

Example for Multiple roots:

Problem 3: Find out the inverse Laplace transform of $X(s) = \frac{s-2}{s(s+1)^3}$, ROC:Re{s}<-1

Solution:

We can write the given function as $X(s) = \frac{s-2}{s(s+1)^3} = \frac{A}{(s+1)^3} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)} + \frac{D}{s}$

The constants can be calculated as A=3, B=2, C=2, D=-2

Therefore, $X(s) = 3\left\{\frac{1}{(s+1)^3}\right\} + 2\left\{\frac{1}{(s+1)^2}\right\} + 2\left\{\frac{1}{(s+1)}\right\} - 2\left\{\frac{1}{s}\right\}$

From the given ROC: Re{s}<-1, the resultant signal x(t) should be left sided.

Finally, $x(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$

From the result of Laplace transform

$$\frac{-t^n}{n!}e^{-at}u(-t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{(s+a)^{n+1}}, \;\; ROC: Re\{s\} < -a$$

i.e. ,$x(t) = 3\frac{-t^2}{2}e^{-t}u(-t) - 2te^{-t}u(-t) - 2e^{-t}u(-t) + 2u(-t)$

Problem 4: Find the Inverse Laplace transform of $X(s) = \frac{3s+7}{s^2-2s-3}$ ROC: Re{s}>3

Solution:

We know that $e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \ ROC: Re\{s\} > -a$

Writing X(s) in the form of partial fraction expansion

$$X(s) = \frac{A}{s-3} + \frac{B}{s+1}$$

The constants can be calculated as A=4, B=-1

Therefore, $X(s) = \frac{A}{s-3} + \frac{B}{s+1} = 4\left\{\frac{1}{s-3}\right\} - \left\{\frac{1}{s+1}\right\}$

From the given ROC: Re{s}>3, the resultant signal x(t) should be right sided.

i.e., $x(t) = 4\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = 4e^{3t}u(t) - e^{-t}u(t)$

Problem 5: Find the Inverse Laplace transform of $X(s) = \frac{-3}{(s+2)(s-1)}$ ROC: Re{s}<-2

Solution:

We know that $-e^{-at}u(-t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \ ROC: Re\{s\} < -a$ Writing X(s) in the form of partial fraction expansion

$$X(s) = \frac{A}{s+2} + \frac{B}{s-1}$$

The constants can be calculated as A=1, B=-1

Therefore, $X(s) = \frac{A}{s+2} + \frac{B}{s-1} = \left\{\frac{1}{s+2}\right\} - \left\{\frac{1}{s-1}\right\}$

From the given ROC: Re{s}<-2, the resultant signal x(t) should be left sided.

i.e., $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} = -e^{-2t}u(-t) + e^{t}u(-t)$

Problem 6: Find the Inverse Laplace transform of $X(s) = \frac{1}{s^2+3s+2}$ ROC: -2<Re{s}<-1

Solution:

We know that

$$e^{-at}u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \ ROC: Re\{s\} > -a$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s+a}, \ ROC: Re\{s\} < -a$$

Writing X(s) in the form of partial fraction expansion

$$X(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

The constants can be calculated as A=1, B=-1

Therefore, $X(s) = \frac{A}{s+1} + \frac{B}{s+2} = \left\{\frac{1}{s+1}\right\} - \left\{\frac{1}{s+2}\right\}$

From the given ROC: -2<Re{s}<-1, the two derived conditions are Re{s}<-1 which suits for $\left\{\frac{1}{s+1}\right\}$ and Re{s}>-2 which suits for $\left\{\frac{1}{s+2}\right\}$. Therefore, the resultant signal x(t) should be two-sided.

i.e., $x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = -e^{-t}u(-t) - e^{-2t}u(t)$

$$\mathcal{L}\{f(t)\} = A\frac{1}{s^2} - 2A\frac{e^{-s}}{s^2} + A\frac{e^{-2s}}{s^2} = \frac{A}{s^2}(1 - 2e^{-s} + e^{-2s})$$

Z–TRANSFORMS

5.6 Introduction: Digital signals are discrete in both time (the independent variable) and amplitude (the dependent variable). Signals that are discrete in time but continuous in amplitude are referred to as discrete-time signals.

Z Transform is a powerful tool for analysis and design of Discrete Time signals and systems. The Z Transform differs from Fourier Transform because it covers a broader class of DT signals and systems which may or may not be stable. Fourier Transform only exists for signals which can absolutely integrated and have a finite energy. Z-transforms is a generalization of Discrete-time Fourier transform by considering a broader class of signals.

Existence of z Transform

In general,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The ROC consists of those values of „z" (i.e., those points in the z-plane) for which X(z) converges i.e., value of z for which

$$\sum_{n=-\infty}^{\infty} |x(n) z^{-n}| < \infty$$

Since $z = re^{j\omega}$ the condition for existence is

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n} e^{-j\omega n}| < \infty$$

Since $e^{-j\omega n} = 1$

Therefore, the condition for which z-transform exists and converges is $\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$

Thus, ROC of the z transform of an x(n) consists of all values of z for which $x(n)r^{-n}$ is absolutely summable.

5.7.Relation between Z and Discrete Time Fourier transform

When $z = e^{j\omega}$, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$ corresponds to the Discrete Time Fourier transform (DTFT) of x(n), i.e., $X(z)|z = e^{j\omega} = \mathcal{F}\{x(n)\}$. Thez transform also bears a straight forward relationship to the DTFT when the complex variable $z =$. To see this relationship, consider X(z) with $z = re^{j\omega}$.

$$X(z = re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

$$X(z = re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n} = \mathcal{F}\{x(n) r^{-n}\}$$

or

Unilateral Z Transform have considerable value in analyzing causal systems and particularly, systems specified by linear constant coefficient difference equations with non-zero initial conditions(i.e., systems that are not initially at rest).
The Unilateral z- transform of a discrete time signal x(n) is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

5.8.Relation between Laplace, Fourier and z- transforms

Let x(t) be a continuous signal sampled with a sampling time of T units. Call this sampled signal as $x_s(t)$. We represent this sampled signal by

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(n - kT)$$

Applying the Laplace transform to $x_s(t)$ results

$$\mathcal{L}\big(x_s(t)\big) = \int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x(kT)\delta(n - kT) \right\} e^{-st} dt$$

Interchanging the order of integration and summation

$$\mathcal{L}\big(x_s(t)\big) = \sum_{k=-\infty}^{\infty} x(kT) \left\{ \int_{-\infty}^{\infty} \delta(n - kT) e^{-st} dt \right\} = \sum_{k=-\infty}^{\infty} x(kT) \left\{ \int_{-\infty}^{\infty} \delta(n - kT) e^{-skT} dt \right\}$$

$$= \sum_{k=-\infty}^{\infty} x(kT) e^{-skT} \left\{ \int_{-\infty}^{\infty} \delta(n - kT) dt \right\} = \sum_{k=-\infty}^{\infty} x(kT) e^{-skT}$$

For uniform sampling x(kT)≡x(k)

Then $\mathcal{L}\big(x_s(t)\big) = \sum_{k=-\infty}^{\infty} x(k) e^{-skT} = \sum_{k=-\infty}^{\infty} x(k) (e^{sT})^{-k}$

Comparing this with the z-transform formula

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

We get a relation that **$z = e^{sT}$**

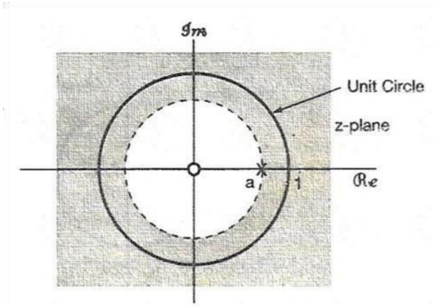
Problems

Problem 1. Finding the z-transform of

a) $\boldsymbol{x(n) = a^n u(n)}$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of X(z), we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$. Consequently, the region of convergence is that range of values of z for which $az^{-1} < 1$, or equivalently, $z > a$ and is shown in figure below



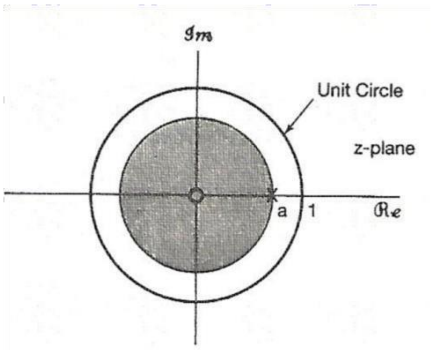
Then $X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

b) $\boldsymbol{x(n) = -a^n u(-n-1)}$

$$X(z) = \sum_{n=-\infty}^{\infty} \{-a^n u(-n-1)\} z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

This result converges only when $|a^{-1}z| < 1$,or equivalently, $|z| < |a|$. The ROC is shown below



If we consider the signals $a^n u(n)$ and $-a^n u(-n-1)$, we note that although the signals are differing, their z Transforms are identical which is $\frac{z}{z-a}$.Thus, we conclude that to distinguish z-Transforms uniquely their ROC's must be specified.

Problem 2: Find the z-transform of (n)=$\delta(n)$. What is the region of convergence?

Solution:

The z transform of a general signal x(n) is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\boldsymbol{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

Since $\delta(n) = \begin{cases} 1, & n = 0 \\ 0 & n \neq 0 \end{cases}$

$$\boldsymbol{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} = (1) z^{0} = 1$$

Since $\boldsymbol{Z}\{\delta(n)\} = 1$, which is a constant, the result of z transform converges for all values of „z", i.e., ROC is entire z-plane

Problem 3: Determine the z-transform of $x(n) = (0.2)^n\{u(n) - u(n-4)\}$

Solution:

$$u(n) - u(n-4) = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & n \geq 4\,, n \leq -1 \end{cases}$$

Therefore,

$$X(z) = \sum_{n=0}^{3} (0.2)^n\, z^{-n} = \sum_{n=0}^{3} (0.2 z^{-1})^n = 1 - \frac{(0.2z^{-1})^4}{1 - 0.2z^{-1}}$$

Problem 4: Find the z-transform of $x(n) = 2^n u(n) + 3^n u(n-1)$

Solution:

We know that $a^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1-az^{-1}}; ROC\colon |z| > |a|$

AlsoWe know that $-a^n u(-n-1) \overset{Z}{\leftrightarrow} \frac{1}{1-az^{-1}}; ROC\colon |z| < |a|$

Therefore, $Z\{x(n)\} = Z\{2^n u(n)\} - Z\{-3^n u(n-1)\} = \frac{1}{1-2z^{-1}} - \frac{1}{1-3z^{-1}} = -\frac{z^{-1}}{1-5z^{-1}+6z^{-2}}$

ROC: 2<|z|<3

Problem 5: Find the two sided z-transform of the signal

$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^n & n \geq 0 \\ (-2)^n & n \leq -1 \end{cases}$$

Solution:

Here x(n) can be written as

$$x(n) = \left(\frac{1}{3}\right)^n u(n) + (-2)^n u(-n-1)$$

Therefore, the z-transform of x(n) is

$$X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{1}{1 + 2z^{-1}}$$

ROC: 1/3 < |z| < 2

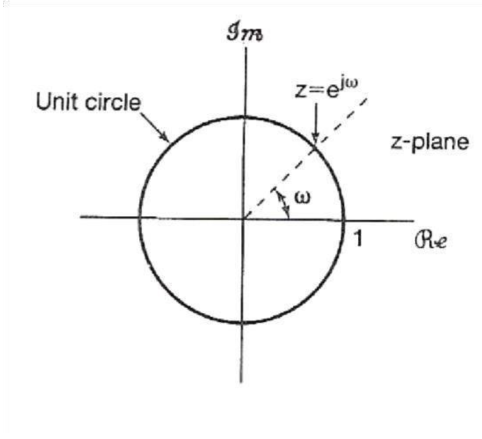
5.9. Region of Convergence (ROC) of Z-Transforms

The Z- transform of a discrete signal *x(n)* is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

The Z-transform has two parts which are, the expression and Region of Convergence respectively.

Whether the Z-transform X(z) of a signal x(n) exists or not depends on the complex variable „z" as well as the signal itself. All complex values of „$z=re^{j\omega}$" for which the summation in the definition converges form a *region of convergence (ROC)* in the z-plane. A circle with r=1 is called unit circle and the complex variable in z-plane is represented as shown below.



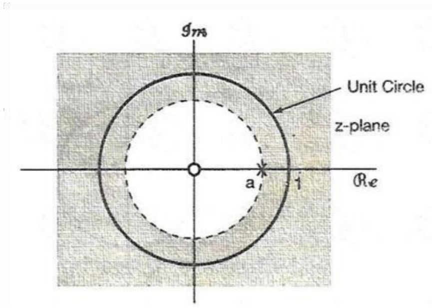
Description :

The concept of ROC can be understood easily by finding z transform of two functions given below:

a) $x(n) = a^n u(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of X(z), we require that $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$. Consequently, the region of convergence is that range of values of z for which az^{-1} <1, or equivalently, $z > a$ and is shown in figure below



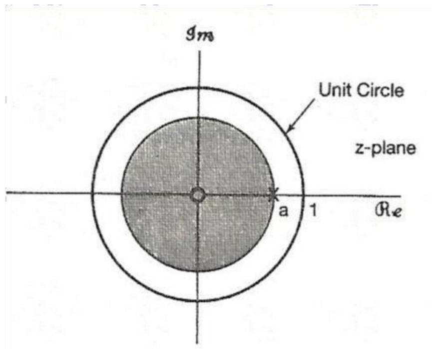
Then $X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

b) $x(n) = -a^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty}\{-a^n u(-n-1)\}z^{-n} = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty}(a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty}(a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

This result converges only when $|a^{-1}z| < 1$,or equivalently, $|z| < |a|$. The ROC is shown below



5.10. Properties of ROC:

Property 1: *The ROC of X(z) consists of a ring in the z-plane centered about the origin.*

$$\sum_{n=-\infty}^{\infty}|x(n)|r^{-n} < \infty$$

Property 2: If the z-transform X(z) of x(n) is rational, then the ROC does not contain any poles but is bounded by poles or extend to infinity.

Property 3: If x(n) is of finite duration, then the ROC is the entire z-plane, except possibly z=0 and / or z=∞

$$X(z) = \sum_{n=N}^{M} x(n)z^{-n}$$

Property 4: If x(nt) is a right sided sequence, and if the circle $|z|=r_o$ is in the ROC, then all finite values of z for which $|z|>r_o$ will also be in the ROC.

Property 5: If x(n) is a left sided sequence, and if the circle$|z|=r_o$ is in the ROC, then all values of z for which $0<|z|<r_o$ will also be in the ROC.

Property 6: If x(n) is two sided, and if the circle $|z|=r_o$ is in the ROC, then the ROC will consist of a ring in the z-plane that includes the circle $|z|=r_o$.

Property 7: Ifthe z-transform X(z) of x(n) is rational, and if x(n) is right sided, then the ROC is the region in the z-plane outside the outermost pole i.e., outside the circle of radius equal to the largest magnitude of the poles of X(z).

Property 8: If the z-transform X(z) of x(n) is rational, and if x(n) is left sided, then the ROC is the region in the z-plane inside the innermost pole i.e., inside the circle of radius equal to the smallest magnitude of the poles of X(z) other than any at z=0 and extending inward to and possibly including z=0.

Solved Problems:

Problem 1: Find the Z-transform and plot the ROC of $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$

Solution:

Given signal $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$ is right sided

We know that $b^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1-bz^{-1}}; |z| > b$

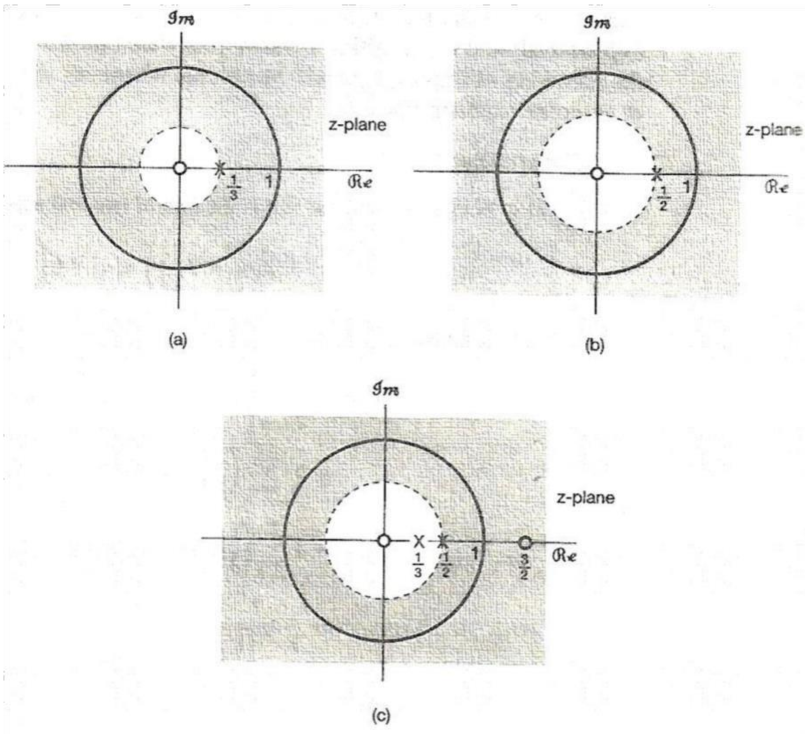
Therefore, $\left(\frac{1}{3}\right)^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1-\left(\frac{1}{3}\right)z^{-1}}; ROC: |z| > \left(\frac{1}{3}\right)$ {shown in figure a} and

$$\left(\frac{1}{2}\right)^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1-\left(\frac{1}{2}\right)z^{-1}}; ROC: |z| > \left(\frac{1}{2}\right) \text{\{Shown in figure b\}}$$

$$X(z) = 7\frac{1}{1-\left(\frac{1}{3}\right)z^{-1}} - 6\frac{1}{1-\left(\frac{1}{2}\right)z^{-1}} = \frac{1-\frac{3}{2}z^{-1}}{\left(1-\left(\frac{1}{3}\right)z^{-1}\right)\left(1-\left(\frac{1}{2}\right)z^{-1}\right)} = \frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}$$

For convergence of X(z), both sums must converge, which requires that the ROC should be an intersection of $|z| > \left(\frac{1}{3}\right)$ and $|z| > \left(\frac{1}{2}\right)$. i.e., $|z| > \left(\frac{1}{2}\right)$ {shown in figure c}

The pole zero plot and ROC are shown in the figure below

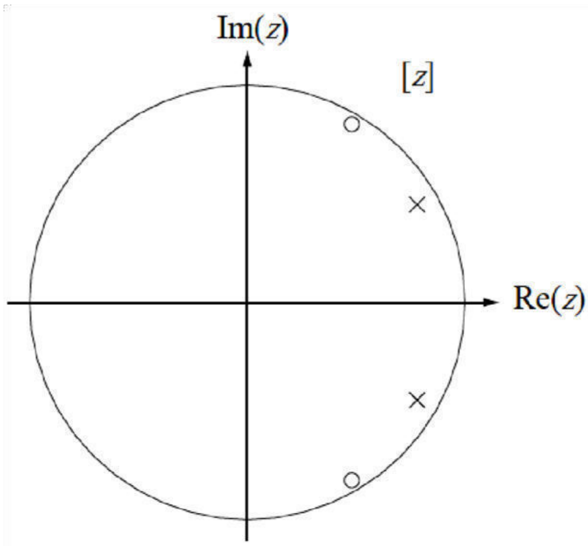


Problem 2: Draw the pole-zero plot and graph the frequency response for the system whose transfer function is $H(z) = \frac{z^2-0.96z+0.9028}{z^2-1.56z+0.8109}$. Is the system both causal and stable?

Solution:
The transfer function can be factored into

$$H(z) = \frac{z^2 - 0.96z + 0.9028}{z^2 - 1.56z + 0.8109} = \frac{(z - 0.48 + j0.82)(z - 0.48 - j0.82)}{(z - 0.78 + j0.45)(z - 0.78 - j0.45)}$$

The Pole-zero diagram is shown below



The magnitude and phase frequency responses of the system are obtained by substituting $z=e^{j\omega}$and varying the frequency variable ω over a range of 2π. The frequency response plots are illustrated in Figure below

The system is both causal and stable because all the poles are inside the unit circle

Problem 3: For the following algebraic expression for the z-transform of a signal, determine the number of zeros in the finite z-plane and the number of zeros at infinity

$$\frac{z^{-1}(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{4}z^{-1})}$$

Solution:

The given z-transform may be written as

$$X(z) = \frac{z-\frac{1}{2}}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}$$

Clearly, X(z) has a zero at z=1/2. Since the order of the denominator polynomial exceeds the order of the numerator polynomial by 1, X(z) has a zero at infinity. Therefore, X(z) has one zero in the finite z-plane and one zero at infinity.

5.11. Properties of Z-Transform

The z transform of a discrete signal *x(n)* is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

And inverse z transform is given by

$$x(n) = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

1. **Linearity:**

   If $x_1(n) \overset{Z}{\leftrightarrow} X_1(z)$ with ROC = R_1 and $x_2(n) \overset{Z}{\leftrightarrow} X_2(z)$ with ROC = R_2 then

   $ax_1(n) + bx_2(n) \overset{Z}{\leftrightarrow} aX_1(z) + bX_2(z)$, with ROC containing $R_1 \cap R_2$

   Proof:
   Taking the z-transform

$$Z\{ax_1(n) + bx_2(n)\} = \sum_{n=-\infty}^{\infty} \{ax_1(n) + bx_2(n)\}z^{-n}$$

$$= a\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

2. **Time Shifting:**

   If $x(n) \overset{Z}{\leftrightarrow} X(z)$with ROC= R

   then $x(n-m) \overset{Z}{\leftrightarrow} z^{-m}X(z)$with ROC= R, except for the possible addition or deletion of the origin or infinity ***Proof:***

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)z^{-n}$$

Let n-m=p

$$= \sum_{p=-\infty}^{\infty} x(p)z^{-(p+m)}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p)z^{-p}$$

$$= z^{-m}X(z)$$

3. **Scaling in the z-Domain:**

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $z_o{}^n x(n) \overset{Z}{\leftrightarrow} X\left(\frac{z}{z_o}\right)$ with ROC= $|z_o|$R where, $|z_o|$R is the scaled version of R.

Proof:

$$Z\{z_o{}^n x(n)\} = \sum_{n=-\infty}^{\infty} z_o{}^n x(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{z_o}\right)^{-n} = X\left(\frac{z}{z_o}\right)$$

4. **Time Reversal:**

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R then

$x(-n) \overset{Z}{\leftrightarrow} X\left(\frac{1}{z}\right)$ with ROC= $\frac{1}{R}$

Proof:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)z^{-n}$$

Let -n=p

$$= \sum_{p=-\infty}^{\infty} x(p)(z)^{p} = \sum_{p=-\infty}^{\infty} x(p)(z^{-1})^{-p} = X\left(\frac{1}{z}\right)$$

5. **The Initial Value Theorem:**

If x(n)=0, for n < 0 then initial value of x(n) i.e., $x(0) = \lim_{z\to\infty} X(z)$ ***Proof:***

We know that Z{x(n)}= $X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$ as x(n) is causal.

Expanding the summation

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Applying the $\lim_{z\to\infty}$ *on bot□ sides*

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\{x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots\}$$

i.e.,

$$\lim_{z\to\infty} X(z) = x(0)$$

6. The Final Value Theorem:

If x(n) is causal and X(z) is the z-transform of x(n) and if all the poles of X(z) lie strictly inside the unit circle except possibly for a first order pole at z=1 then

$$\lim_{N\to\infty} x(n) = \lim_{z\to 1}(1-z^{-1})X(z)$$

Proof:

Consider the z-transform of x(n)-x(n-1)

$$\mathrm{x(n)} - \mathrm{x(n-1)} \overset{Z}{\leftrightarrow} (1-z^{-1})X(z)$$

$$\boldsymbol{Z}\{\, x(n) - x(n-1)\} = \sum_{n=0}^{\infty} \{x(n) - x(n-1)\} z^{-n} = (1-z^{-1})X(z)$$

Also, the above can be written as

$$\lim_{N\to\infty} \sum_{n=0}^{N} \{x(n) - x(n-1)\} z^{-n} = (1-z^{-1})X(z)$$

Applying the limit z→1 on both sides

$$\lim_{z\to 1} \left\{ \lim_{N\to\infty} \sum_{n=0}^{N} \{x(n) - x(n-1)\} z^{-n} \right\} = \lim_{z\to 1}(1-z^{-1})X(z)$$

LHS after applying the limit z→1 becomes

$$\left\{ \lim_{N\to\infty} \sum_{n=0}^{N} \{x(n) - x(n-1)\} \right\} = \lim_{N\to\infty} \begin{Bmatrix} x(0) - x(-1) + x(1) - x(0) + x(2) - x(1) + \cdots \\ +x(N-1) - x(N-2) + x(N) - x(N-1) \end{Bmatrix}$$

All terms cancel except x(N). Therefore,

$$\lim_{N\to\infty} x(n) = \lim_{z\to 1}(1-z^{-1})X(z)$$

7. Differentiation in the z-Domain:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R then

$nx(n) \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}$ with ROC = R

Proof: z transform is given by

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Differentiating above on both sides with respect to „z"

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left\{\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right\} = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}\{z^{-n}\} = \sum_{n=-\infty}^{\infty} -nx(n)z^{-n-1}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx(n)z^{-n}$$

Comparing both equations $-z\frac{dX(z)}{dz}$ is the z transform of (n).

5.12.Inverse Z-Transform

Inverse z-transform maps a function in z-domain back to the time domain. One application is to convert a discrete system response to an input sequence from z-domain back to the time domain. The z-transform converts the Linear Constant Coefficient Difference Equations(LCCDE) that describe system behaviour to a polynomial. Also the convolution operation which describes the system action on the input signals is converted to a multiplication operation. These two properties make it much easier to do systems analysis in the z-domain.

Inverse z-transform is performed using Long Division Method(Power Series Expansion method), Partial Fraction Expansion and Residue method (Contour Integral Method).

The z- transform of a discrete signal *x(n)* is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Since $z=re^{j\omega}$ $X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\omega n} = \mathcal{F}\{x(n)r^{-n}\}$

$$x(n)r^{-n} = \mathcal{F}^{-1}\{X(re^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

$$\Rightarrow x(n) = r^n\frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^n d\omega$$

Long Division Method (Power Series Expansion)

Z-transform of the sequence x(n) is given as,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

From above expansion of z-transform, the sequence x(n) can be obtained as,

$$x(n) = \{\dots, x(-2), x(-1), x(0), x(1), x(2), \dots\}$$

The Power series expansion can be obtained directly or by long division method.

Example: Determine inverse z-transform of the following:

i) $X(z) = \frac{1}{1-az^{-1}}, ROC : |z| > |a|$ ii)

$X(z) = \frac{1}{1-az^{-1}}, ROC : |z| < |a|$

Solution:

i) $X(z) = \frac{1}{1-az^{-1}}$, ROC : $|z| > |a|$

$$1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} \leftarrow \text{Negative power of 'z'}$$

$$1 - az^{-1} \overline{)\, 1 \,}$$

$$\begin{array}{l} 1 - az^{-1} \\ - \quad + \\ \hline az^{-1} \\ az^{-1} - a^2 z^{-2} \\ - \quad\quad + \\ \hline a^2 z^{-2} \\ a^2 z^{-2} - a^3 z^{-3} \\ \quad\quad + \\ \hline a^3 z^{-3} \\ a^3 z^{-3} - a^4 z^{-4} \\ - \quad\quad + \\ \hline a^4 z^{-4} \ \ldots \end{array}$$

Thus we have, $X(z) = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$

Taking inverse z-transform, $x(n) = \{1, a, a^2, a^3, \cdots\} = a^n u(n)$

ii) $$X(z) = \frac{1}{1 - a z^{-1}}, \text{ ROC} : |z| < |a|$$

$$= \frac{1}{-a z^{-1} + 1} \quad \text{Equation rearranged to get positive powers of 'z'.}$$

$$\begin{array}{r|l}
 & -a^{-1} z - a^{-2} z^2 - a^{-3} z^3 - a^{-4} z^4 \leftarrow \text{Positive powers of 'z'} \\
-a z^{-1} + 1 & 1 \\
 & 1 - a^{-1} z \\
 & - \quad + \\
\hline
 & a^{-1} z \\
 & a^{-1} z - a^{-2} z^2 \\
 & - \quad + \\
\hline
 & a^{-2} z^2 \\
 & a^{-2} z^2 - a^{-3} z^3 \\
 & - \quad + \\
\hline
 & a^{-3} z^3 \\
 & a^{-3} z^3 - a^{-4} z^4 \\
 & - \quad + \\
\hline
 & a^{-4} z^4 \ldots
\end{array}$$

Thus we have, $$X(z) = \frac{1}{1 - a z^{-1}} = -a^{-1} z - a^{-2} z^2 - a^{-3} z^3 - a^{-4} z^4 \ldots$$

Rearranging above equation, $$= \ldots\ldots - a^{-4} z^4 - a^{-3} z^3 - a^{-2} z^2 - a^{-1} z$$

Taking inverse z-transform, $$x(n) = \{\cdots - a^{-4}, -a^{-3}, -a^{-2}, -a^{-1}\}$$
$$\uparrow$$
$$= -a^n u(-n-1)$$

Partial Fraction Expansion

As we know that the rational form of X(z) can be expanded into partial fractions, Inverse z-transform can be taken according to location of poles and ROC of X(z).

Following steps are to be performed for partial fraction expansions:

Step 1: Arrange the given X(z) as,

$$\frac{X(z)}{z} = \frac{numerator\ polynomial}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

Step 2:

$$\frac{X(z)}{z} = \frac{A_1}{(z - p_1)} + \frac{A_2}{(z - p_2)} + \frac{A_3}{(z - p_3)} + \cdots + \frac{A_N}{(z - p_N)}$$

Where A_k for k=1, 2,…N are the constants to be found in partial fractions. Poles may be of multiple order. The coefficients will be calculated accordingly.

Step 3: Above equation can be written as

$$X(z) = \frac{A_1 z}{(z - p_1)} + \frac{A_2 z}{(z - p_2)} + \frac{A_3 z}{(z - p_3)} + \cdots + \frac{A_N z}{(z - p_N)}$$

$$= \frac{A_1}{1 - p_1 z^{-1}} + \frac{A_2}{1 - p_2 z^{-1}} + \frac{A_3}{1 - p_3 z^{-1}} + \cdots + \frac{A_N}{1 - p_N z^{-1}}$$

Step 4: All the terms in above step are of the form $\frac{A_k}{1-p_k z^{-1}}$. Depending upon ROC, following standard pairs must be used.

$$p_k{}^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1-p_k z^{-1}} \text{ with ROC: } |z| > |p_k|\text{, i.e., causal response}$$

$$-p_k{}^n u(-n-1) \overset{Z}{\leftrightarrow} \frac{1}{1-p_k z^{-1}} \text{ with ROC: } |z| < |p_k|\text{, i.e., non-causal response}$$

Example: Determine the inverse z-transform of $X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$

For ROC i) |z| > 1 ii)|z| < 0.5 iii)0.5 < |z| < 1

Solution:

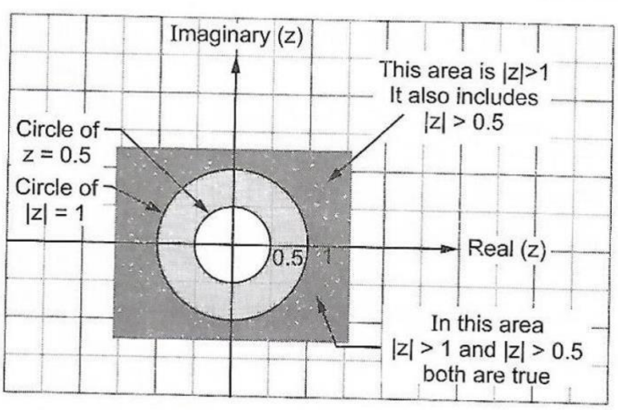
Given $X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$

Which can be written as $X(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-0.5z^{-1}}$

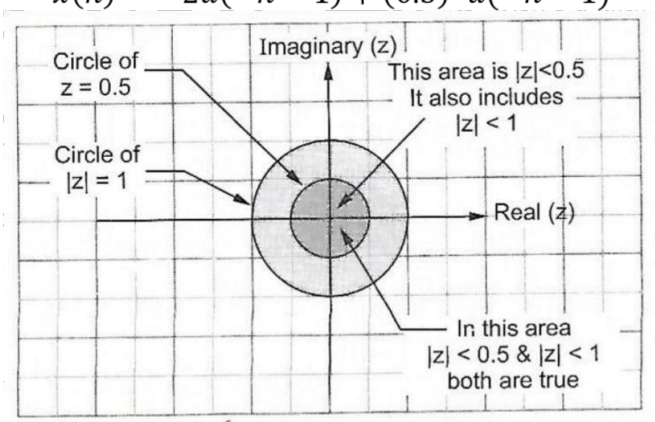
After finding the constants as A=2 and B=-1

$$X(z) = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

i) For ROC |z| > 1 i.e., causal or right sided
$x(n) = 2u(n) - (0.5)^n u(n)$



ii) For ROC |z| < 0.5 i.e., non-causal or left sided
$x(n) = -2u(-n-1) + (0.5)^n u(-n-1)$



iii) For ROC 0.5 < |z| < 1 i.e., two sided

$$x(n) = -2u(-n-1) - (0.5)^n u(n)$$

The ROC is a circular strip between z>0.5 and z<1

Residue Method (Contour Integral Method)

Cauchy integral theorem is used to calculate inverse z-transform. Following steps are to be followed:

Step 1: Define the function $(z)=X(z)z^{n-1}$, which is rational and its denominator is expanded into product of poles.

$$X_o(z) = X(z)z^{n-1} = \frac{N(z)}{\prod_{i=1}^{m}(z-p_i)^m}$$

Here „m" is order of the pole

Step 2: For poles of order „m", the residue of $X_o(z)$ can be calculated as,

$$\underset{z=p_i}{Res} X_o(z) = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}(z-p_i)^m X_o(z)\right\} z = p_i$$

Step 3:

i) Using residue theorem, calculate x(n) for poles inside the unit circle, i.e.,

$$x(n) = \sum_{i=1}^{N} \underset{z=p_i}{Res} X_o(z)$$

ii) For poles outside the contour of integration,

$$x(n) = -\sum_{i=1}^{N} \underset{z=p_i}{Res} X_o(z) \text{ with } n < 0$$

Example: Determine the inverse z-transform of $X(z) = \frac{z^2}{(z-a)^2}$,

ROC : $|z| > |a|$ using contour integration

Solution:

Step 1: $X_o(z) = X(z)z^{n-1} = \frac{z^2}{(z-a)^2} z^{n-1} = \frac{z^{n+1}}{(z-a)^2}$

Step 2: Here the pole is at $z=a$ and it has order m=2. Hence finding the residues at $z=a$

$$\underset{z=p_i}{Res} X_o(z) = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{dz^{m-1}}(z-p_i)^m X_o(z)\right\} z = p_i$$

$$\underset{z=a}{Res} X_o(z) = \frac{1}{(2-1)!}\left\{\frac{d^{2-1}}{dz^{2-1}}(z-a)^2\frac{z^{n+1}}{(z-a)^2}\right\} at\ z = a$$

$$= \frac{d}{dz} z^{n+1}\ at\ z = a$$

$$= (n+1)z^n\ at\ z = a$$

$$= (n+1)a^n$$

Step 3: The sequence x(n) is given as

$$x(n) = \sum_{i=1} \underset{z=p_i}{Res} X_o(z)$$

$x(n) = (n+1)a^n u(n)$ Since ROC: $z > |a|$

Examples:

Solved Problems:

Problem 1: Find the inverse z-transform of $X(z) = \log\left(\frac{1}{1-az^{-1}}\right)$, $|z|>|a|$

Solution:

$$X(z) = \log\left(\frac{1}{1-az^{-1}}\right) = -\log(1-az^{-1})$$

log(1-p) is expanded with the help of power series. It is given as

$$\log(1-p) = -\sum_{n=1}^{\infty}\frac{p^n}{n}, |p| < 1$$

Therefore,

$X(z) = -\log(1-az^{-1}) = -\left[-\sum_{n=1}^{\infty}\frac{(az^{-1})^n}{n}\right] = \sum_{n=1}^{\infty}\frac{a^n}{n}z^{-n}$ for $|z| > |a|$

Hence $x(n) = \begin{cases} \frac{a^n}{n}, & n \geq 1 \\ 0, & otherwise \end{cases}$

i.e.,

$$x(n) = \frac{a^n}{n}u(n-1)$$

Problem 2: Find the inverse z-transform of $X(z) = \frac{z}{(z+2)(z-3)}$ using partial fraction method when ROC is

(i) |z| <- 2
(ii) |z| > 3 (iii) -2 < |z| < 3

Solution: Given $X(z) = \frac{z}{(z+2)(z-3)}$

Applying partial fractions $\frac{X(z)}{z} = \frac{1}{5}\left(\frac{1}{z-3}\right) - \frac{1}{5}\left(\frac{1}{z+2}\right)$

$$\Rightarrow X(z) = \frac{1}{5}\left(\frac{z}{z-3}\right) - \frac{1}{5}\left(\frac{z}{z+2}\right)$$

i) For ROC |z| < -2

$$x(n) = \frac{1}{5}[-3^n + (-2)^n]u(-n-1)$$

ii) For ROC |z| > 3

$$x(n) = \frac{1}{5}[3^n - (-2)^n]u(n)$$

iii) For ROC -2 < |z| <3

$$x(n) = \frac{1}{5}[-3^n u(-n-1) - (-2)^n u(n)]$$

Problem 3: Determine the sequence whose z-transform is,

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} \qquad \text{ROC : |z|> 1}$$

Solution: To arrange X(z) in proper form suitable for partial fraction expansion
The highest power of denominator polynomial should be atleast one less than that of numerator polynomial

Let us arrange X(z) as follows:

$$X(z) = \frac{z^{-2} + 2z^{-1} + 1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

Now perform one step division so that order of numerator polynomial is reduced by one unit.

$$\begin{array}{r|l} & \quad 2 \\ \frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & z^{-2} + 2z^{-1} + 1 \\ & z^{-2} - 3z^{-1} + 2 \\ & \;\; - \qquad + \\ \hline & 5z^{-1} - 1 \end{array}$$

$$X(z) = 2 + \frac{5z^{-1} - 1}{\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1} = 2 + \frac{-1 + 5z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

$$= 2 + X_1(z), \qquad \text{where } X_1(z) = \frac{-1 + 5z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

Now

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$$

Writing in partial fractions

$$X(z) = 2 + \frac{A}{1 - z^{-1}} + \frac{B}{1 - \frac{1}{2}z^{-1}}$$

Finding the constants A =8 and B=-9

$$X(z) = 2 + \frac{8}{1 - z^{-1}} - \frac{9}{1 - \frac{1}{2}z^{-1}}$$

As ROC is |z| > 1, representing the signal as causal signal which is right sided

$$x(n) = 2\delta(n) + 8u(n) - 9\left(\frac{1}{2}\right)^n u(n)$$

Problem 4: An LTI system is characterized by the system function

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}}$$

Specify the ROC of H(z) and determine h(n) for the following conditions

a) The system is causal and unstable
b) The system is non-causal and stable
c) The system is non-causal and unstable

Solution: Given that

$$H(z) = \frac{3 - 4z^{-1}}{1 - 3.5z^{-1} + 1.5z^{-2}} = \frac{3z - 4}{(z - 0.5)(z - 3)}$$

Using partial fraction expansion, we obtain

$$\frac{H(z)}{z} = \frac{3z - 4}{(z - 0.5)(z - 3)} = \frac{A}{z - 0.5} + \frac{B}{z - 3}$$

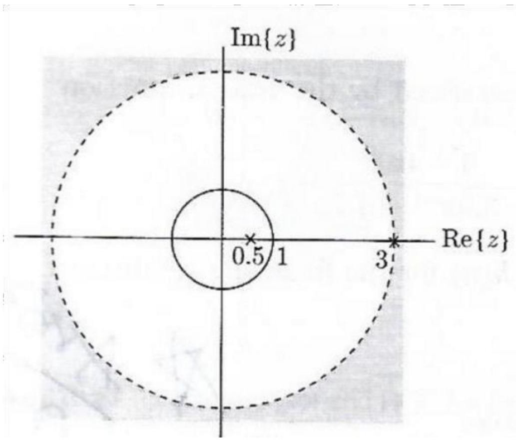
Finding the constants A =1 and B=2

$$\frac{H(z)}{z} = \frac{1}{z - 0.5} + \frac{2}{z - 3}$$

$$H(z) = \frac{z}{z - 0.5} + 2\frac{z}{z - 3}$$

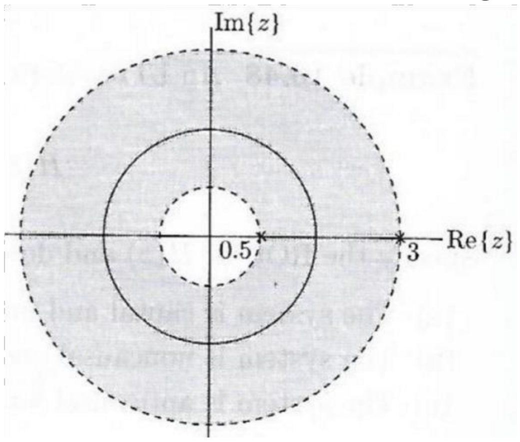
The system has poles at z=0.5 and z=3

a) For the system to be causal and unstable, the ROC of H(z) is the region in the z-plane outside the outermost pole and it must not include the unit circle. Therefore, the ROC is the region |z| > 3



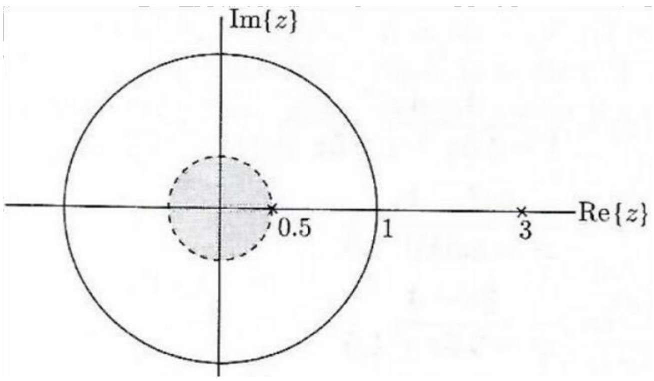
Hence $h(n) = 0.5^n u(n) + 2(3)^n u(n)$

b) For the system to be non-causal and stable, the ROC of H(z) is the ring in the z-plane and it must include the unit circle. Therefore, the ROC is the region, 0.5 < |z| < 3



Hence $h(n) = 0.5^n u(n) - 2(3)^n u(-n-1)$

c) For the system to be non-causal and unstable, the ROC of H(z) is the ring in the zplane inside the inner most pole and it must not include the unit circle. Therefore, the ROC is the region, |z| < 0.5



Hence $h(n) = -0.5^n u(-n-1) - 2(3)^n u(-n-1)$

$$Y(z) = -\frac{1}{1+z^{-1}} + \frac{16}{1-4z^{-1}}$$

Therefore, for n≥0 $y(n) = [-(-1)^n + 16(4)^n]u(n)$

TUTORIAL QUESTIONS

1. a) Write short notes on "Orthogonal Vector Space".
   b) A rectangular function f(t) is defined by

$$f(t) = \begin{cases} 1 & (0 < t < \Pi) \\ -1 & (0 < t < 2\Pi) \end{cases}$$

   Approximate the above function by a finite series of Sinusoidal functions.

2. a) Define and sketch the following elementary signals
   i. Unit impulse signal
   ii. Unit step signal
   iii. Signum function
   b) Explain the Analogy of vectors and signals in terms of orthogonality and evaluation of constant.

3. a) Sketch the single sided and double sided spectra of the following signal
   $x(t) = 2 \sin(10\Pi t - \Pi/6)$.
   b) Show that the functions Sin $n\omega_0 t$ and Sin $m\omega_0 t$ are orthogonal over any interval $(t_0, t_0 + 2\Pi / \omega_0)$ for integer values of n and m.

4. a) Write short notes on "Orthogonal functions".
   b) Define the following Elementary signals.
   i. Real Exponential Signal
   ii. Continuous time version of sinusoidal signal and Bring out the relation between Sinusoidal and complex exponential signals.

5. a) Define Orthogonal signal space and bring out clearly its application in representing a signal.
   b) Obtain the condition under which two signals f1 (t) & f2 (t) are said to be orthogonal to each other. Hence, prove that sin $n\omega_0 t$ and cos $m\omega_0 t$ are orthogonal to each other for all integer values of m, n.

6. a) A rectangular function defined as

$$f(t) = \begin{cases} A & 0 < t < (\Pi/2) \\ -A & (\Pi/2) < t < (3\Pi/2) \\ A & (3\Pi/2) < t < 2\Pi \end{cases}$$

   Approximate the above function by A cos (t) between the intervals (0, 2Π) such that the mean square error is minimum.
   b) Prove the following.

   i. $\delta(n) = u(n) - u(n-1)$ ii. $u(n) = \sum_{k=\infty}^{n} \delta(K)$

7. a) Sketch the single sided and double sided spectra of the following signal

   $x(t) = 2 \text{ Sin } (10\Pi t - \Pi/6)$
   b) Derive polar Fourier series from the exponential Fourier series representation and hence prove that $D_n = 2 |C_n|$.

8. a) Distinguish between Orthogonal vectors and Orthogonal functions.

b) Consider the complex valued exponential signal

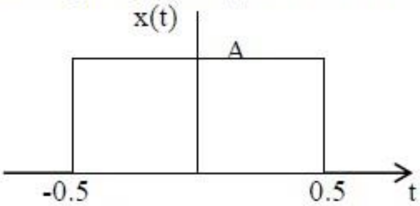
$x(t) = A\, e^{a+J\omega t}$, $a > 0$. Evaluate the real and imaginary components of x(t) for the following cases.

i. a real, $a = a_1$.

ii. a imaginary, $a = j\omega_1$

iii. a complex, $a = a_1 + j\omega$

c) Consider the rectangular pulse x(t) as shown in the below figure.



Repeat the above rectangular pulse in terms of weighted sum of two step functions.

9. a) Sketch the following signals.

i. $\Pi\,[(t-1)/2] + \Pi(t-1)$

ii. $f(t) = 3u(t) + tu(t) - (t-1)\,u\,(t-1) - 5u(t-2)$

b) Evaluate the following integrals.

i. $\int_{0}^{5} \delta(t) \sin 2\Pi t\, dt$

ii. $\int_{-\infty}^{\infty} e^{-at^2}\, \delta(t-10)\, dt$

10. a) Define and sketch the following signals.

i. Real exponential signals for $C \neq 0$, $a > 0$

ii. Even continuous time signal

iii. Unit doublet

iv. Real part of damped complex exponential for $a = 0$

b) Evaluate the following integrals.

i. $\int_{\infty}^{\infty} \delta(t+3)^{-t}\, edt$

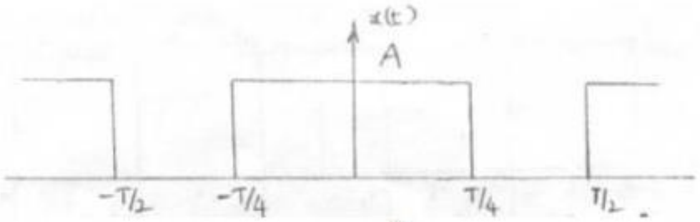
ii. $\int_{\infty}^{\infty} [\delta(t)\cos t + \delta\,(t-1)\sin t]dt$

c) Discuss the signal with a neat sketch.

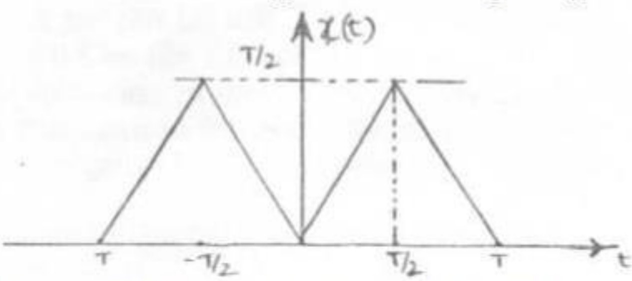
1. a) Prove that Sine(0) = 1 and plot Sine function.
   b) Determine the Fourier series representation of that Signal x(t) = 3 Cos (Πt/2 + Π/4) using the method of inspection.
2. Prove the following properties.
   a) The FS symmetry properties for
      i. Real valued time signals
      ii. Real and even time signals
   b) Obtain the Fourier series representation of an impulse train given by

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0)$$

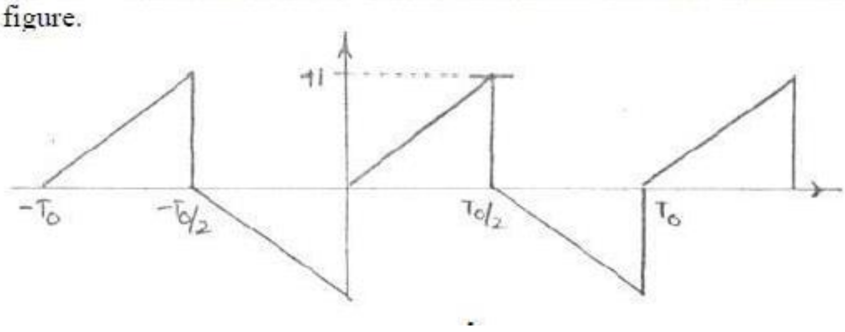
3. a) Explain about even and odd functions.
   b) Obtain the trigonometric Fourier series for the periodic waveform as shown in the below figure.



4. a) Prove that the normalized is given by p = $\sum_{n=-\infty}^{\infty} |C_n|^2$, where $|C_n|$ are complex Fourier coefficients for the periodic wave form.
   b) Determine the Fourier series expansion for the signal x(t) shown in the below figure.



5. a) State the three important spectral properties of periodic power signals.
   b) Assuming $T_0 = 2$, determine the Fourier series expansion of the signal shown in the below figure.



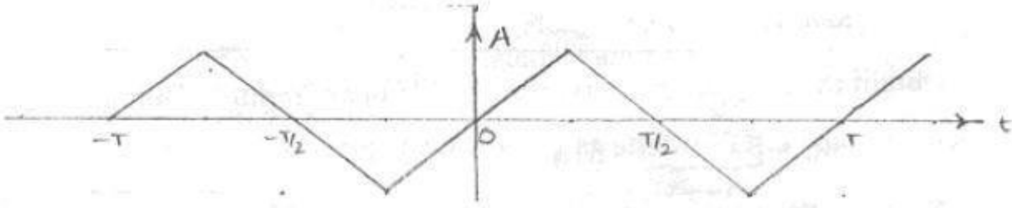
6. a) Show that the magnitude spectrum of every periodic function is Symmetrical about the vertical axis passing through the origin.
   b) With regard to Fourier series representation, justify the following statements.
      i. Odd functions have only Sine term.
      ii. Even functions have no sine terms.
      iii. Functions with half wave Symmetry have only odd harmonics.

7. Show that the Fourier series for a real valued signal can be written as

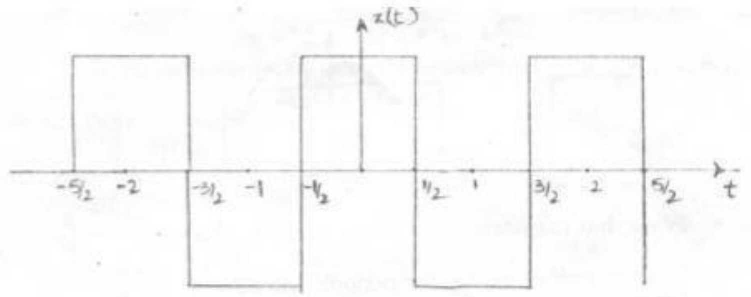
$$x(t) = B(o) + \sum_{n=1}^{\infty} B(n)\cos(n\omega_o t) + A(n)\sin(n\omega_o t)$$

Where B(n) and A(n) are real valued coefficients and express c_n in terms of b(n) and A(n).

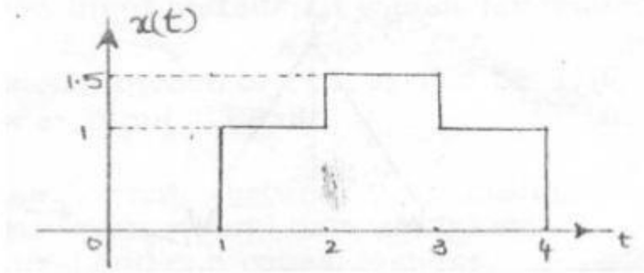
8. a) Write short notes on "Exponential Fourier Spectrum".
b) Find the Fourier series expansion of the periodic triangular wave shown in the below figure.



9. a) State the properties of Complex Fourier series.
b) Determine the Fourier series of the function shown in the below figure.



1. a) Find the Fourier Transform of the signal shown in the below figure.



b) Find the Fourier Transform of the signal given below.

$$y(t) = \begin{cases} \cos 10t & -2 \le t \le 2 \\ 0 & otherwise \end{cases}$$

2. a) Obtain the Fourier transform of the following functions :
   i. Impulse function $\delta(t)$
   ii. DC Signal
   iii. Unit step function

   b) State and prove time differentiation property of Fourier Transform.

3. a) State and prove properties of correlation function.

   b) If V(f) = AT sin 2Π fT / 2Π fT find the energy contained in V(t).

   c) Obtain the Fourier Transform of the following :
   i. $x(t) = A \sin(2\Pi f_c t)\, u(t)$
   ii. $x(t) = f(t) \operatorname{Cos}(2\Pi f_c t) + \varphi)$

   d) State and prove the following properties of Fourier Transform.
   i. Multiplication in time domain
   ii. Convolution in time domain

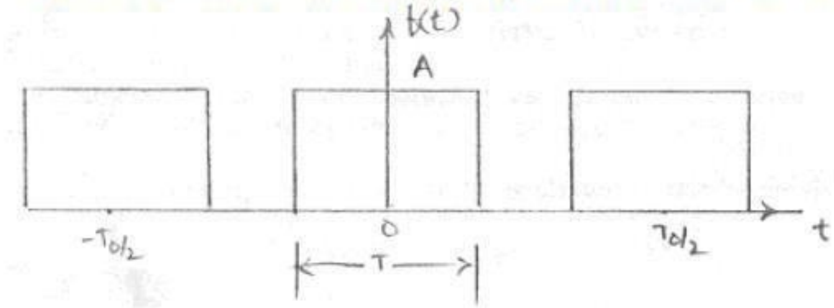
   a) Find Fourier Transform of the following time function.

   $x(t) = e^{-3t}\,[u(t+2) - u(t-3)]$

   b) State and prove frequency and time shifting properties of Fourier Transform.

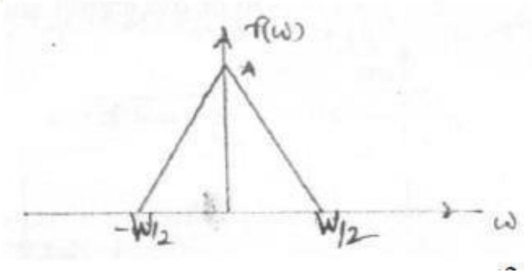
4. a) Explain the concept of Fourier transform for periodic signals.

   b) Find out the Fourier Transform of the periodic pulse train shown in the below figure.



5. a) State and prove time convolution and time differentiation properties of Fourier transform.

   b) Find and sketch the Inverse Fourier transform of the Waveform shown in the below figure.



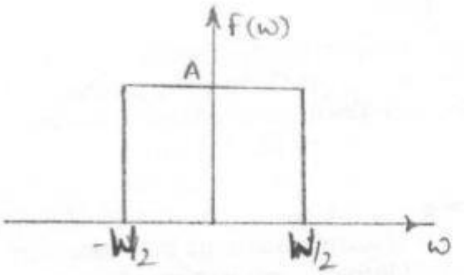
6. a) Find the Fourier transform of the signal $x(t) = \frac{2}{1+t^2}$.

   b) Explain how Fourier transform can be derived from Fourier series.

7. Find the Fourier Transform of the following functions.
   a) A single symmetrical Triangle Pulse.
   b) A single symmetrical Gate Pulse.
   c) A single cosine wave at t = 0.

8. a) Distinguish between Fourier series and Fourier transform.
   b) State the conditions for the existence of Fourier transform of signal.
   c) Find the Fourier transform of the signum function and plot it's amplitude and phase spectra.

9. a) Find and sketch the Inverse Fourier transform of the waveform shown in the below figure.



1. a) Explain how input and output signals are related to impulse response of a LTI system.

   b) Let the system function of a LTI system be $\frac{1}{j\omega + 2}$. What is the output of the system for an input $(0.8)^t\, u(t)$?

2. a) Explain the difference between the following systems.
   i. Time invariant and time variant systems.
   ii. Causal and non-causal systems.

   b) Consider a stable LTI system characterized by the differential equation :

   $$\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{d\,x(t)}{dt}+2x(t)$$

   Find its impulse response and transfer function.

3. a) Explain the difference between the following systems.
   i. Linear and Non-linear systems.
   ii. Causal and Non-causal systems.

   b) Consider a stable LTI system characterized by the differential equation :

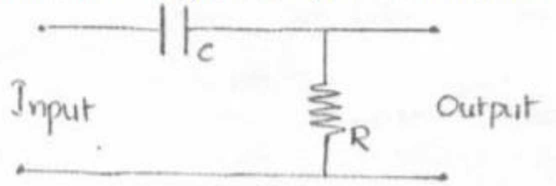
   $$\frac{dy(t)}{dt}+2y(t)=x(t)$$

   Find its impulse response.

4. a) There are several possible ways of estimating an essential bandwidth of non-band limited signal. For a low pass signal, for example, the essential bandwidth may be chosen as a frequency where the amplitude spectrum of the signal decays to k percent of its value. The choice of k depends on the nature of application. Choosing k = 5 determine the essential bandwidth of g(t) = exp (-at) u(t).

   b) Differentiate between signal bandwidth and system bandwidth.

5. a) Find the impulse response of the system shown in the below figure. Find the transfer function. What would be its frequency response ? Sketch the response.

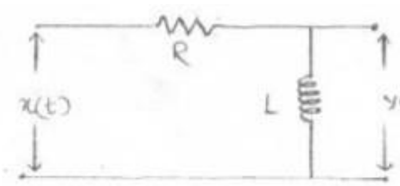


   b) Explain the characteristics of an ideal LPF, Explain why it can't be realized.

6. a) What is a LTI system ? Explain its properties. Derive an expression for the transfer function of a LTI system.

   b) Obtain the conditions for the distortion less transmission through a system. What do you understand by the term signal bandwidth ?

7. a) Find the impulse response to the RL filter shown in the below figure.



1. a) State and prove properties of auto correlation function ?

b) A filter has an impulse response h(t) as shown in figure 5b. The input to the network is a pulse of unit amplitude extending from t = 0 to t = 2. By graphical means determine the output of the filter.

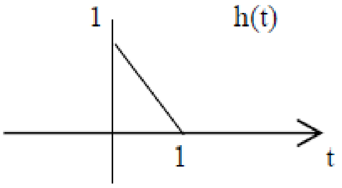


Figure 5b

2. a) Find the energies of the signals shown in figures 1, 2.

b) Determine the power of the following signals.

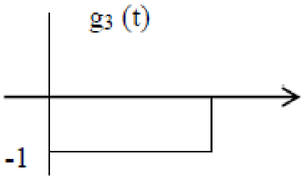


Figure 1

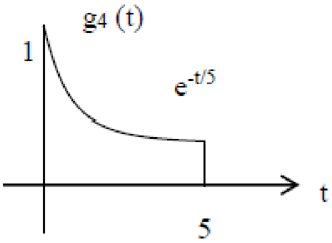


Figure 2

i. (10 + 2 sin 3t) Cos 10 t.

ii. 10 Cos 5t cos 10t.

3. a) A signal y(t) given by y(t) = $C_0 + \sum_{n=1} C_n \cos(n\omega_o t + \theta_n)$. Find the auto correlation and PSD of y(t).

b) Find the mean square value (or power) of the output voltage y(t) of the system shown in figure 2. If the input voltage PSD. $S_2(\omega) = \text{rect}(\omega/2)$. Calculate the power (mean square value) of input signal x(t).

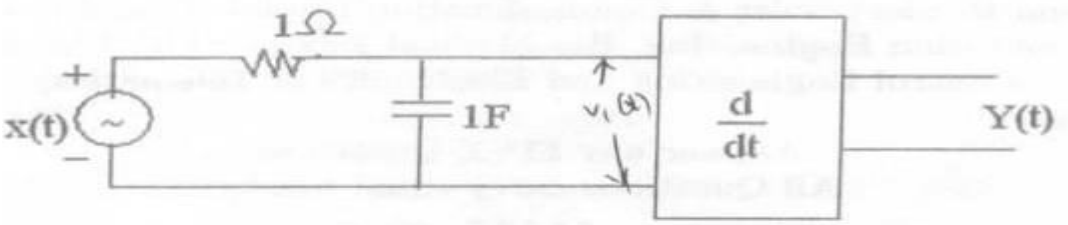


Figure 2

4. a) A waveform m(t) has a Fourier transform M(f) whose magnitude is as shown in figure 2. Find the normalized energy content of the waveform.

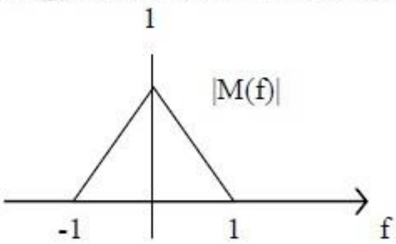


Figure 2

b) The signal $V(t) = \cos \omega_0 t + 2\sin 3\omega_0 t + 0.5 \sin 4\omega_0 t$ is filtered by an RC low pass filter with a 3 dB frequency.
$f_c = 2f_0$. Find the output power S_o.

c) State parseval's theorem for energy X power signals.

5. a) The signal $v(t) = \cos \omega_0 t + 2 \sin 3\omega_0 t + 0.5 \sin 4\omega_0 t$ is filtered by an RC low pass filter with 3 dB frequency $f_C = 2f_0$.
   i. Find $G_i(f)$
   ii. Find $G_0(f)$

   b) Let G(f) denote the Fourier transform of real valued energy signal g(t), and $R_g(\tau)$ its autocorrelation function, show that

$$\int_{-\infty}^{\infty} \left[\frac{dR_g(\tau)}{d\tau}\right]^2 d\tau = 4\pi^2 \int_{-\infty}^{\infty} f^2 |G(f)|^4 df$$

6. a) State and prove properties of cross correlation function.
   b) If $V(f) = AT \sin 2\pi fT / 2\pi fT$ find the energy contained in V(t).

7. a) Explain briefly detection of periodic signals in the presence of noise by correlation.
   b) Explain briefly extraction of a signal from noise by filtering.

8. a) For the signal $g(t) = 2a / (t^2+a^2)$. Determine the essential band width B Hz of g(t) such that the energy contained in the spectral components of g(t) of frequencies below B Hz is 99% of signal energy E_g.

   b) Show that the auto correlation function of $g(t) = C \cos(\omega_0 t + \theta_0)$ is given by
   $R_g(\tau) = (c^2/2) \cos \omega_0 \tau$, and the corresponding PSD is
   $S_g(\omega) = (c^2\pi/2) [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$.

9. a) Using frequency domain convolution, find X(f) for $x(t) = A \text{ Sinc}^2 2wt$.
   b) Show that correlation can be written in terms of convolution as

$$R(\tau) = \underset{T \to \infty}{Lt\ 1/T} [x(t) * x(-t)]$$

   c) The input to an RC Low pass filter is $x(t) = \text{sinc}^2 wt$. Find output energy Ey.

10. Find the power of periodic signal g(t) shown in figure 5c. Find also the powers of
    (a) $-g(t)$ b) $2g(t)$ c) $g(t)$

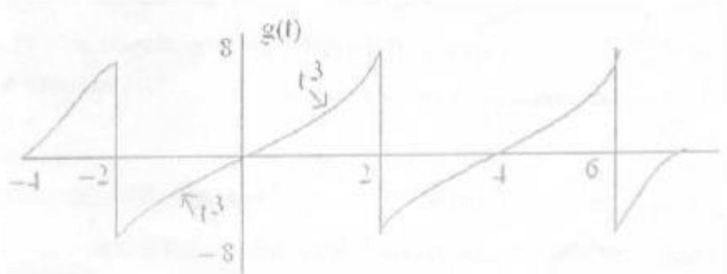


Figure 5c

11. a) Determine an expression for the correlation function of a square wave having the values 1 or 0 and a period T.

b) The energy of a non periodic wave form v(t) is E = $\int_{-\infty}^{\infty} v^2(t)\,dt$.

i. Show that this can be written as E = $\int_{-\infty}^{\infty} dt\, v(t) \int_{-\infty}^{\infty} v(f)\, e^{j2\pi ft} df$.

ii. Show that by interchanging the order by integration we have E = $\int_{-\infty}^{\infty} v(f)\, v^*$ (f) df = $\int_{-\infty}^{\infty} |v(f)|^2\, df$.

12. a) Derive Parsavel's theorem from the frequency convolution property.

b) Find the cross correlation between [u(t) + u(t - τ)] and e^{-t} u(t).

13. a) Find the Z-transform X(n).

i. $x[n] = (1/2)^n u[n] + (1/3)^n u[n]$

ii. $x[n] = (1/3)^n u[n] + (1/2)^n u[-n-1]$

b) Find inverse z transform of x(z) using long division method

$x(z) = (2 + 3z)^{-1} / (1 + z^{-1})\ (1 + 0.25\, z^{-1} - (z-2) / 8)$

14. The complex exponential representation of a signal f(t) over the interval (0, T)

f(t) = $\sum_{n=-\alpha}^{\alpha} \left(\frac{3}{4} + (n\pi)^2 \right) e^{j\pi nt}$

(a) What is the numerical value of T

(b) One of the components of f(t) is A Cos 3πt. Determine the value of A.

(c) Determine the minimum no. of terms which must be maintained in representation of f(t) in order to include 99.9% of the energy in the interval (0, T).

1. a) Consider the signal $x(t) = \left(\frac{\sin 50\Pi t}{\Pi t}\right)^2$ which to be sampled with a sampling frequency of $\omega_s = 150\Pi$ to obtain a signal g(t) with Fourier transform $G(j\omega)$. Determine the maximum value of ω_0 for which it is guaranteed that $G(j\omega) = 75 \times (j\omega)$ for $|\omega| \le \omega_0$ where $X(j\omega)$ is the Fourier transform of x(t).

b) The signal $x(t) = u(t + T_0) - u(t - T_0)$ can undergo impulse train sampling without aliasing, provided that the sampling period $T < 2T_0$. Justify.

c) The signal x(t) with Fourier transform $X(j\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$ can undergo impulse train sampling without aliasing, provided that the sampling period $T < \pi/\omega_0$. Justify.

2. a) With the help of graphical example explain sampling theorem for Band limited signals.
b) Explain briefly band pass sampling.

3. a) Explain briefly impulse sampling.
b) Define sampling theorem for time limited signal and find the nyquist rate for the following signals.
   i. rect300t
   ii. $-10 \sin 40\pi t \cos 300\pi t$

4. a) Determine the Nyquist rate corresponding to each of the following signals.
   i. $x(t) = 1 + \cos 200\ pt + \sin 4000\ \pi t$
   ii. $x(t) = \frac{\sin 4000\pi t}{\pi t}$

b) The signal Y(t) is generated by convolving a band limited signal $x_1(t)$ with another band limited signal $x_2(t)$ that is

$y(t) = x_1(t) * x_2(t)$
where
$x_1(j\omega) = 0$ for $|\omega| > 1000\Pi$
$x_2(j\omega) = 0$ for $|\omega| > 2000\Pi$
Impulse train sampling is performed on y(t) to obtain

$$y_p(t) = \sum_{n=-\infty}^{\infty} y(nT)\,\delta(t - nT)$$

Specify the range of values for sampling period T which ensures that y(t) is recoverable from $y_p(t)$.

5. Determine the Nyquist sampling rate and Nyquist sampling interval for the signals.
(a) sin c(100Πt)   (b) sin τ (100Πt)   (c) sin c (100Πt) + sin c(50Πt)
(d) sin c(100Πt) + 3 sin c^2 (60Πt)

6. a) Explain Flat top sampling.

b) A Band pass signal with a spectrum shown in figure 6b is ideally sampled. Sketch the spectrum of the sampled signal when f_s = 20, 30 and 40 Hz. Indicate if and how the signal can be recovered.

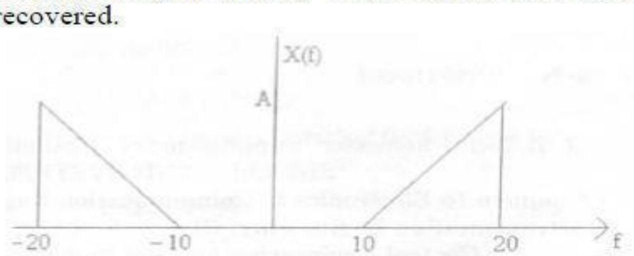


Figure 6b

7. a) A signal $x(t) = 2 \cos 400 \pi t + 6 \cos 640 \pi t$ is ideally sampled at $f_s = 500$Hz. If the sampled signal is passed through an ideal low pass filter with a cut off frequency of 400 Hz, what frequency components will appear in the output.

b) A rectangular pulse waveform shown in figure 6b is sampled once every T_S seconds and reconstructed using an ideal LPF with a cutoff frequency of $f_s/2$. Sketch the reconstructed waveform for $T_s = \frac{1}{6}$ sec and $T_s = \frac{1}{12}$ sec.

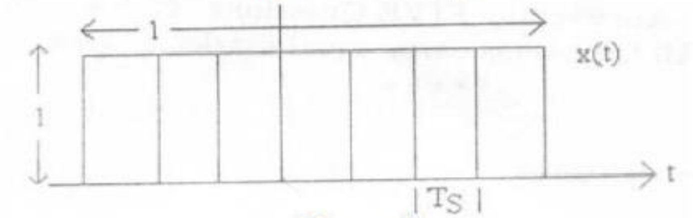


Figure 6b

8. a) A periodic waveform is formed by eliminating the alternate cycle of a Sinusoidal waveform as shown in figure 6a.

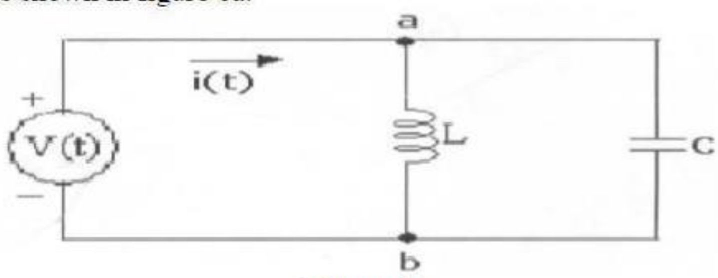


Figure 6a

i. Find the Fourier series (exponentially) by direct evaluation of the coefficients.

ii. If the waveform is shifted to the left by π seconds, the new waveform $f(t + \pi)$ is odd function of the time whose Fourier series contains only sine terms. Find the Fourier series of $f(t + \pi)$. From this series, write down the Fourier series for f(t).

9. a) Let x(t) be a signal with Nyquist rate ω_0. Also let $y(t) = x(t)\, p(t-1)$, where $p(t) = \sum_{n=-\infty}^{\infty} \delta_1(t - nT)$, and $T < \frac{2\pi}{\omega_0}$

Specify the constraints on the magnitude and phase of the frequency response of a filter that gives x(t) as its output when y(t) is the input.

b) Explain the Sampling theorem for Band Limited Signals with analytical proof.

10. a) Find the output voltage v(t) of the network shown in figure 6a when the voltage applied to the terminals a b is given by $f(t) = e^{-t/4} u(t) + e^{-t/2} u(-t)$

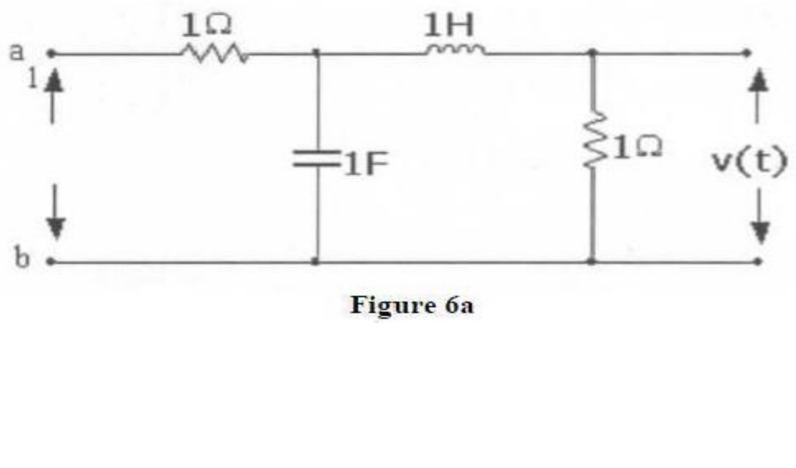


Figure 6a

1. a) Obtain the inverse laplace transform of F(s) = $\frac{1}{s^2(s+2)}$ by convolution integral.

b) Using convolution theorem find inverse laplace transform of $\frac{s}{(s^2+a^2)^2}$.

c) Define laplace transform of signal f(t) and its region of convergence.

2. a) When a function f(t) is said to be laplace transformable.

b) What do you mean by region of convergence.

c) List the advantages of Laplace transform.

d) If δ(t) is a unit impulse function find the laplace transform of $\frac{d^2}{dt^2}[\delta(t)]$.

3. a) State and prove the properties of Laplace transforms.

b) Derive the relation between Laplace transform and Fourier transform of signal.

4. a) State the properties of the ROC of L.T.

b) Determine the function of time x(t) for each of the following laplace transforms and their associated regions of convergence.

i) $\frac{(s+1)^2}{s^2-s+1}$ Re {S} > ½

ii) $\frac{s^2-s+1}{(s+1)^2}$ Re {S} > -1

5. a) Determine the Laplace transform and the associate region convergence for each of the following functions of time.

i) x(t) = 1 $0 \le t \le 1$

ii) x(t) = $\begin{matrix} t & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \end{matrix}$

b) State and prove initial value theorem of L.T.

6. a) The system function of a causal LTI system is H(s) = $\frac{s+1}{s^2+2s+2}$. Determine the response y(t) when the input is x(t) = $e^{-|t|}$.

b) State and prove initial value and final value theorems.

7. Consider the following signals, find laplace transform and region of convergence for each signal.

(a) e^{-2t} u(t) + e^{-3t} u(t)

(b) e^{-4t} u(t) + e^{-5t} sin 5t u(t)

(c) State properties of laplace transform.

8. Determine the function of time x(t) for each of the following Laplace transforms and their associated regions of convergence.

(a) $\frac{1}{s^2+9}$ Re {S} > 0

(b) $\frac{S}{S^2+9}$ Re {S} < 0

(c) $\frac{s+1}{(s+1)^2+9}$ Re{S} < -1

9. a) The two periodic functions $f_1(t)$ and $f_2(t)$ with zero dc components have arbitrary waveforms with periods T and $\sqrt{2T}$ respectively. Show that the component in $f_1(t)$ of waveform $f_2(t)$ is zero in the interval $(-\alpha < t < a)$.

b) Complex Sinusoidal signal x(t) has the following components.
$\text{Re}\{x(t)\} = x_R(t) = A\text{Cos}(\omega t + \theta)$
$\text{I}_m\{x(t)\} = x_I(t) = A\text{Sin}(\omega t + \theta)$

The amplitude of x(t) is given by the square root of $x_R^2(t) + x_I^2(t)$. Show that this amplitude equals A and is therefore independent of the phase angle θ.

10. a) Find the initial values and final values of the function $F(s) = \frac{17s^3 + 7s^2 + s + 6}{s^5 + 3s^4 + 5s^3 + 4s^2 + 2s}$.

b) Explain the Step and Impulse responses of Series R-C circuit using Laplace transforms.

11. a) Consider the signal $x(t) = (\sin 50\pi t / \pi t)^2$ which to be sampled with a sampling frequency of $\omega_s = 150\pi$ to obtain a signal g(t) with Fourier transform $G(j\omega)$. Determine the maximum value of ω_0 for which it is guaranteed that $G(j\omega) = 75 \times j(\omega)$ for $|\omega| - \omega_0$. where $X(j\omega)$ is the Fourier transform of x(t).

b) The signal $x(t) = u(t + T_0) - u(t - T_0)$ can undergo impulse train sampling without aliasing, provided that the sampling period $T < 2T_0$. Justify.

1. a) A finite sequence x[n] is defined as x[n] = {5, 3, -2, 0, 4, -3} Find X[Z] and its ROC.

b) Consider the sequence $x[n] = \begin{cases} a^n & 0 \le n \le N-1,\ a > 0 \\ 0 & otherwise \end{cases}$ Find X[Z].

c) Find the Z-transform of $x(n) = \cos(n\omega)\,u(n)$.

2. a) Find the inverse Z-transform of

$$X(Z) = \frac{2Z^3 - 5Z^2 + Z + 3}{(Z-1)(Z-2)} \qquad |Z| < 1$$

b) Find the inverse Z-transform of $X(Z) = \frac{3}{Z-2}$ $|Z| > 2$

c) Find the Z-transform of $a^n \sin(n\omega)\,u(n)$.

3. a) State and Prove the properties of the z-transform.

b) Find the Z-transform of the following Sequence.
$x[n] = a^n u[n]$

4. a) Find the Z-transform and ROC of the signal $x[n] = [4.(5^n) - 3(4^n)]\,u(n)$

b) Find the Z-transform as well as ROC for the following sequences.

i) $\left(\frac{1}{3}\right)^n u(-n)$ and ii) $\left(\frac{1}{3}\right)^n [u(-n) - u(n-8)]$

5. a) Using the Power Series expansion technique, find the inverse Z-transform of the following X(Z) :

i. $X(Z) = \dfrac{Z}{2Z^2 - 3Z + 1} \quad |Z| < \dfrac{1}{2}$

ii. $X(Z) = \dfrac{Z}{2Z^2 - 3Z + 1} \quad |Z| > 1$

b) Find the inverse Z-transform of

$$X(Z) = \frac{Z}{Z(Z-1)(Z-2)^2} \qquad |Z| > 2$$

6. a) Find the Z-transform of the following Sequences.

i. $x[n] = a^{-n}\,u[-n-1]$

ii. $x[n] = u[-n]$

iii. $x[n] = -\,a^n\,u[-n-1]$

b) Derive relationship between z and Laplace Transform.

7. Using the method indicated, determine the sequence that goes with each of the following Z transforms :

a) Partial fractions :

$X(z) = \dfrac{1-2z^{-1}}{\left(1+\frac{5}{2}z^{-1} - z^{-2}\right)}$, and x[n] is absolutely summable.

b) Long division :

$X(z) = \dfrac{1-\frac{1}{2}z^{-1}}{1+\frac{1}{2}z^{-1}}$, and x[n] is right sided.

c) Partial fractions :

$X(z) = \dfrac{3}{z-\frac{1}{4}-\frac{1}{8}z^{-1}}$, and x[n] is absolutely summable.

8. a) Determine and sketch the auto correlation function of the following exponential pulses

i. $f(t) = e^{-at} u(t)$

ii. $f(t) = e^{-a|t|}$

b) Determine the cross correlation function $R_{12}(\tau)$ of the two signals $g_1(t)$ and $g_2(t)$ denoted by

$$g_1(t) = A \cos(2\pi f_1 t + \theta_1), \quad 0 \le t \le T$$
$$= 0 \quad , \text{ elsewhere}$$
$$g_2(t) = A \cos(2\pi f_2 t + \theta_2), \quad 0 \le t \le T$$
$$= 0 \quad , \text{ elsewhere}$$

9. a) Which of the following signals or functions are periodic and if what is its fundamental period.

i. $g(t) = e^{-j60\pi t}$

ii. $g(t) = 10 \sin(12\pi t) + 4 \cos(18t)$

b) Let two functions be defined by :

$$x_1(t) = 1, \ \sin(20\pi t) \ge 0$$
$$-1, \ \sin(20\pi t) < 0$$
$$x_2(t) = t, \ \sin(2\pi t) \ge 0$$
$$-t \ \sin(2\pi t) < 0$$

Graph the product of these two functions vs time over the time interval $-2 < t < 2$.

MULTIPLE CHOICE QUESTIONS

1) Which mathematical notation specifies the condition of periodicity for a continuous time signal?
a. $x(t) = x(t + T_0)$
b. $x(n) = x(n+ N)$
c. $x(t) = e^{-\alpha t}$
d. None of the above
ANSWER: (a) $x(t) = x(t + T_0)$

2) Which property of delta function indicates the equality between the area under the product of function with shifted impulse and the value of function located at unit impulse instant?
a. Replication
b. Sampling
c. Scaling
d. Product
ANSWER: (b) Sampling

3) Which among the below specified conditions/cases of discrete time in terms of real constant 'a', represents the double-sided decaying exponential signal?
a. $a > 1$
b. $0 < a < 1$
c. $a < -1$
d. $-1 < a < 0$
ANSWER: (d) $-1 < a < 0$

4) Damped sinusoids are _____
a. sinusoid signals multiplied by growing exponentials
b. sinusoid signals divided by growing exponentials
c. sinusoid signals multiplied by decaying exponentials
d. sinusoid signals divided by decaying exponentials
ANSWER: (c) sinusoid signals multiplied by decaying exponentials

5) An amplitude of sinc function that passes through zero at multiple values of an independent variable 'x' _______
a. Decreases with an increase in the magnitude of an independent variable (x)
b. Increases with an increase in the magnitude of an independent variable (x)
c. Always remains constant irrespective of variation in magnitude of 'x'
d. Cannot be defined
ANSWER: (a) Decreases with an increase in the magnitude of an independent variable (x)

6) A system is said to be shift invariant only if_______
a. a shift in the input signal also results in the corresponding shift in the output
b. a shift in the input signal does not exhibit the corresponding shift in the output
c. a shifting level does not vary in an input as well as output
d. a shifting at input does not affect the output
ANSWER: (a) a shift in the input signal also results in the corresponding shift in the output

7) Which condition determines the causality of the LTI system in terms of its impulse response?
a. Only if the value of an impulse response is zero for all negative values of time
b. Only if the value of an impulse response is unity for all negative values of time
c. Only if the value of an impulse response is infinity for all negative values of time
d. Only if the value of an impulse response is negative for all negative values of time
ANSWER: (a) Only if the value of an impulse response is zero for all negative values of time

8) Under which conditions does an initially relaxed system become unstable?
a. only if bounded input generates unbounded output
b. only if bounded input generates bounded output
c. only if unbounded input generates unbounded output
d. only if unbounded input generates bounded output
ANSWER: (a) only if bounded input generates unbounded output

9) Which among the following are the stable discrete time systems?
1. y(n) = x(4n)
2. y(n) = x(-n)
3. y(n) = ax(n) + 8
4. y(n) = cos x(n)

a. 1 & 3
b. 2 & 4
c. 1, 3 & 4
d. 1, 2, 3 & 4
ANSWER: (d) 1, 2, 3 & 4

10) An equalizer used to compensate the distortion in the communication system by faithful recovery of an original signal is nothing but an illustration of __________
a. Static system
b. Dynamic system
c. Invertible system
d. None of the above
ANSWER: (c) Invertible system

11) Which block of the discrete time systems requires memory in order to store the previous input?
a. Adder
b. Signal Multiplier
c. Unit Delay
d. Unit Advance
ANSWER: (c)Unit Delay

12) Which type/s of discrete-time system do/does not exhibit the necessity of any feedback?
a. Recursive Systems
b. Non-recursive Systems
c. Both a & b
d. None of the above

ANSWER: (b) Non-recursive Systems

13) Which among the following belongs to the category of non-recursive systems?
a. Causal FIR Systems
b. Non-causal FIR Systems
c. Causal IIR Systems
d. Non-causal IIR Systems
ANSWER: (a) Causal FIR Systems

14) Recursive Systems are basically characterized by the dependency of its output on ________
a. Present input
b. Past input
c. Previous outputs
d. All of the above
ANSWER: (d) All of the above

15) What does the term y(-1) indicate especially in an equation that represents the behaviour of the causal system?
a. initial condition of the system
b. negative initial condition of the system
c. negative feedback condition of the system
d. response of the system to its initial input
ANSWER: (a) initial condition of the system

16) Which type of system response to its input represents the zero value of its initial condition?
a. Zero state response
b. Zero input response
c. Total response
d. Natural response
ANSWER: (a) Zero state response

17) Which is/are the essential condition/s to get satisfied for a recursive system to be linear?
a. Zero state response should be linear
b. Principle of Superposition should be applicable to zero input response
c. Total Response of the system should be addition of zero state & zero input responses
d. All of the above
ANSWER: (d) All of the above

18) Which among the following operations is/are not involved /associated with the computation process of linear convolution?
a. Folding Operation
b. Shifting Operation
c. Multiplication Operation
d. Integration Operation
ANSWER: (d) Integration Operation

19) A LTI system is said to be initially relaxed system only if _____
a. Zero input produces zero output
b. Zero input produces non-zero output
c. Zero input produces an output equal to unity
d. None of the above
ANSWER:(a) Zero input produces zero output

20) What are the number of samples present in an impulse response called as?
a. string
b. array
c. length
d. element
ANSWER: (c) length

21) Which are the only waves that correspond/ support the measurement of phase angle in the line spectra?
a. Sine waves
b. Cosine waves
c. Triangular waves
d. Square waves
ANSWER: (b) Cosine waves

22) Double-sided phase & amplitude spectra ______
a. Possess an odd & even symmetry respectively
b. Possess an even & odd symmetry respectively
c. Both possess an odd symmetry
d. Both possess an even symmetry
ANSWER: (a) Possess an odd & even symmetry respectively

23) What does the first term 'a_0' in the below stated expression of a line spectrum indicate?
$x(t) = a_0 + a_1 \cos w_0 t + a_2 \cos 2 w_0 t + \ldots\ldots + b_1 \sin w_0 t + b_2 \sin w_0 t + \ldots..$
a. DC component
b. Fundamental component
c. Second harmonic component
d. All of the above
ANSWER: (a) DC component

24) Which kind of frequency spectrum/spectra is/are obtained from the line spectrum of a continuous signal on the basis of Polar Fourier Series Method?
a. Continuous in nature
b. Discrete in nature
c. Sampled in nature
d. All of the above
ANSWER: (b) Discrete in nature

25) Which type/s of Fourier Series allow/s to represent the negative frequencies by plotting the double-sided spectrum for the analysis of periodic signals?
a. Trigonometric Fourier Series
b. Polar Fourier Series

c. Exponential Fourier Series
d. All of the above
ANSWER: (c) Exponential Fourier Series

26) What does the signalling rate in the digital communication system imply?
a. Number of digital pulses transmitted per second
b. Number of digital pulses transmitted per minute
c. Number of digital pulses received per second
d. Number of digital pulses received per minute
ANSWER: (a)Number of digital pulses transmitted per second

27) As the signalling rate increases, ________
a. Width of each pulse increases
b. Width of each pulse decreases
c. Width of each pulse remains unaffected
d. None of the above
ANSWER: (b)Width of each pulse decreases

28) Which phenomenon occurs due to an increase in the channel bandwidth during the transmission of narrow pulses in order to avoid any intervention of signal distortion?
a. Compression in time domain
b. Expansion in time domain
c. Compression in frequency domain
d. Expansion in frequency domain
ANSWER: (d)Expansion in frequency domain

29) Why are the negative & positive phase shifts introduced for positive & negative frequencies respectively in amplitude and phase spectra?
a. To change the symmetry of the phase spectrum
b. To maintain the symmetry of the phase spectrum
c. Both a & b
d. None of the above
ANSWER: (b) To maintain the symmetry of the phase spectrum

30) Duality Theorem / Property of Fourier Transform states that __________
a. Shape of signal in time domain & shape of spectrum can be interchangeable
b. Shape of signal in frequency domain & shape of spectrum can be interchangeable
c. Shape of signal in time domain & shape of spectrum can never be interchangeable
d. Shape of signal in time domain & shape of spectrum can never be interchangeable
ANSWER: (a) Shape of signal in time domain & shape of spectrum can be interchangeable

31) Which property of fourier transform gives rise to an additional phase shift of $-2\pi f t_d$ for the generated time delay in the communication system without affecting an amplitude spectrum?
a. Time Scaling
b. Linearity
c. Time Shifting
d. Duality
ANSWER: (c) Time Shifting

32) Which among the below assertions is precise in accordance to the effect of time scaling?
A : Inverse relationship exists between the time and frequency domain representation of signal
B : A signal must be necessarily limited in time as well as frequency domains

a. A is true & B is false
b. A is false & B is true
c. Both A & B are true
d. Both A & B are false
ANSWER: (a)A is true & B is false

33) Which is/are the mandatory condition/s to get satisfied by the transfer function for the purpose of distortionless transmission?
a. Amplitude Response should be constant for all frequencies
b. Phase should be linear with frequency passing through zero
c. Both a & b
d. None of the above
ANSWER: (c)Both a & b

34) A Laplace Transform exists when _______
A. The function is piece-wise continuous
B. The function is of exponential order
C. The function is piecewise discrete
D. The function is of differential order

a. A & B
b. C & D
c. A & D
d. B & C
ANSWER: (a) A & B

35) Where is the ROC defined or specified for the signals containing causal as well as anti-causal terms?
a. Greater than the largest pole
b. Less than the smallest pole
c. Between two poles
d. Cannot be defined
ANSWER: (c) Between two poles

36) What should be the value of laplace transform for the time-domain signal equation e^{-at} cos ωt.u(t)?
a. 1 / s + a with ROC $\sigma > -a$
b. $\omega / (s + a)^2 + \omega^2$ with ROC $\sigma > -a$
c. $s + a / (s + a)^2 + \omega^2$ with ROC $\sigma > -a$
d. $A\omega / s^2 + \omega^2$ with ROC $\sigma > 0$
ANSWER: (c) $s + a / (s + a)^2 + \omega^2$ with ROC $\sigma > -a$

37) According to the time-shifting property of Laplace Transform, shifting the signal in time domain corresponds to the ______
a. Multiplication by e^{-st0} in the time domain

b. Multiplication by e^{-st0} in the frequency domain
c. Multiplication by e^{st0} in the time domain
d. Multiplication by e^{st0} in the frequency domain
ANSWER: (b) Multiplication by e^{-st0} in the frequency domain

38) Which result is generated/ obtained by the addition of a step to a ramp function?
a. Step Function shifted by an amount equal to ramp
b. Ramp Function shifted by an amount equal to step
c. Ramp function of zero slope
d. Step function of zero slope
ANSWER: (b) Ramp Function shifted by an amount equal to step

39) Unilateral Laplace Transform is applicable for the determination of linear constant coefficient differential equations with ________

a. Zero initial condition
b. Non-zero initial condition
c. Zero final condition
d. Non-zero final condition
ANSWER: (b) Non-zero initial condition

40) What should be location of poles corresponding to ROC for bilateral Inverse Laplace Transform especially for determining the nature of time domain signal?

a. On L.H.S of ROC
b. On R.H.S of ROC
c. On both sides of ROC
d. None of the above
ANSWER: (c) On both sides of ROC

41) Generally, the convolution process associated with the Laplace Transform in time domain results into________

a. Simple multiplication in complex frequency domain
b. Simple division in complex frequency domain
c. Simple multiplication in complex time domain
d. Simple division in complex time domain
ANSWER: (a) Simple multiplication in complex frequency domain

42) An impulse response of the system at initially rest condition is basically a response to its input & hence also regarded as,

a. Black's function
b. Red's function
c. Green's function
d. None of the above
ANSWER: (c) Green's function

43) When is the system said to be causal as well as stable in accordance to pole/zero of ROC specified by system transfer function?

a. Only if all the poles of system transfer function lie in left-half of S-plane
b. Only if all the poles of system transfer function lie in right-half of S-plane
c. Only if all the poles of system transfer function lie at the centre of S-plane
d. None of the above
ANSWER: (a) Only if all the poles of system transfer function lie in left-half of S-plane

44) Correlogram is a graph of ______

a. Amplitude of one signal plotted against the amplitude of another signal
b. Frequency of one signal plotted against the frequency of another signal
c. Amplitude of one signal plotted against the frequency of another signal
d. Frequency of one signal plotted against the time period of another signal
ANSWER: (a) Amplitude of one signal plotted against the amplitude of another signal