MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(An Autonomous Institution – UGC, Govt.of India)

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(Affiliated to JNTUH, Hyderabad, Approved by AICTE –Accredited by NBA & NAAC-“A” Grade-ISO 9001:2015 Certified)

MATHEMATICS-III

B.Tech – II Year – I Semester

DEPARTMENT OF HUMANITIES AND SCIENCES

CONTENTS

(R18A0023)MATHEMATICS-III

Objectives: To learn

1. The expansion of a given function by Fourier series.
2. The Fourier sine and cosine transforms, properties, inverse transforms, and finite Fourier transforms.
3. Differentiation, integration of complex valued functions and evaluation of integrals using Cauchy's integral formula.
4. Taylor's series, Laurent's series expansions of complex functions and evaluation of integrals using residue theorem.
5. Transform a given function from z - plane to w – plane. Identify the transformations like translation, magnification, rotation, reflection, inversion, and Properties of bilinear transformations.

UNIT – I: Fourier series

Definition of periodic function, Fourier expansion of periodic functions in a given interval of length 2π, Fourier series of even and odd functions, Half-range Fourier sine and cosine expansions, Fourier series in an arbitrary interval.

UNIT – II: Fourier Transforms

Fourier integral theorem - Fourier sine and cosine integrals. Fourier transforms – Fourier sine and cosine transforms, properties. Inverse transforms and Finite Fourier transforms.

UNIT – III: Analytic functions

Complex functions and its representation on Argand plane, Concepts of limit, continuity, differentiability, Analyticity, and Cauchy-Riemann conditions, Harmonic functions – Milne – Thompson method. Line integral – Evaluation along a path and by indefinite integration – Cauchy's integral theorem (singly and multiply connected regions) – Cauchy's integral formula – Generalized integral formula.

UNIT – IV: Singularities and Residues

Radius of convergence, expansion of given function in Taylor's series and Laurent series. Singular point –Isolated singular point, pole of order m and essential singularity. Residues – Evaluation of residue by formula and by Laurent series. Residue theorem- Evaluation of improper integrals of the type

(a) $\int_{-\infty}^{\infty} f(x)dx$ (b) $\int_{c}^{c+2\pi} f(cos\Theta, sin\Theta)d\Theta$

UNIT – V: Conformal Mappings

Conformal mapping: Transformation of z-plane to w-plane by a function, conformal transformation. Standard transformations- Translation; Magnification and rotation; inversion and reflection, Transformations like e^z , log z, z^2, and Bilinear transformation. Properties of Bilinear transformation, determination of bilinear transformation when mappings of 3 points are given (cross ratio).

TEXT BOOKS:

i) Higher Engineering Mathematics by B.S. Grewal, Khanna Publishers.
ii) Higher Engineering Mathematics by B.V Ramana , Tata McGraw Hill.
iii)Advanced Engineering Mathematics by Kreyszig, John Wiley & Sons.

REFERENCES:

i) Complex Variables and Applications by James W Brown and Ruel Vance Churchill-Mc Graw Hill
ii)Mathematics-III by T K V Iyenger ,Dr B Krishna Gandhi, S Ranganatham and Dr MVSSN Prasad, S chand Publications.
iii) Advanced Engineering Mathematics by Michael Greenberg –Pearson publishers.

Course Outcomes: After going through this course the students will be able to

1. Find the expansion of a given function by Fourier series in the given interval.
2. Find Fourier sine, cosine transforms and inverse transformations.
3. Analyze the complex functions with reference to their analyticity and integration using Cauchy's integral theorem.
4. Find the Taylor's and Laurent series expansion of complex functions. Solution of improper integrals can be obtained by Cauchy's-Residue theorem.
5. Understand the conformal transformations of complex functions can be dealt with ease.

UNIT – I
FOURIER SERIES

Fourier series

Suppose that a given function $f(x)$ defined in $[-\pi,\pi]$ (or) $[0,2\pi]$ (or) in any other interval can be expressed as

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

The above series is known as the Fourier series for $f(x)$ and the constants $a_0,a_n,b_n\left(n=1,2,3-----\right)$ are called Fourier coefficients of $f(x)$

Periodic Function:-

A function $f(x)$ is said to be periodic with period $T>0$ if for all $x, f(x+T)=f(x)$,and T is the least of such values

Example:- (1) $\sin x=\sin(x+2\pi)=\sin(x+4\pi)=-----$ the function $\sin x$ is periodic with period 2π. There is no positive value T, $0<T<2\pi$ such that $\sin(x+T)=\sin x\,\forall\, x$

(2) $The\ period\ of\ \tan x\ is\ \pi$

(3) The period of sin$nnx\ is\ \frac{2\pi}{n}\quad i.e\ sinnx=sinn\left(\frac{2\pi}{n}+x\right)$

Euler's Formulae:-

The Fourier series for the function $f(x)$ in the interval $C\le x\le C+2\pi$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

Where $a_0=\frac{1}{\pi}\int_C^{C+2\pi}f(x)dx$

$a_n=\frac{1}{\pi}\int_C^{C+2\pi}f(x)\cos nx.\,dx$ and

$b_n=\frac{1}{\pi}\int_C^{C+2\pi}f(x)\sin nx.\,dx$

These values of a_0,a_n,b_n are known as Euler's formulae

Corollary:- If $f(x)$ is to be expanded as a Fourier series in the interval $0\le x\le 2\pi$, put $C=0$ then the formulae (1) reduces to

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx \qquad a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx.dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx.dx$$

Corollary 2:- If $f(x)$ is to expanded as a Fourier series in $[-\pi,\pi]$ put $c=-\pi$, the interval becomes $-\pi\le x\le\pi$ and the formulae (1) reduces to

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$$

Functions Having Points of Discontinuity :-

In Euler's formulae for a_0, a_n, b_n it was assumed that $f(x)$ is continuous. Instead a function may have a finite number of discontinuities. Even then such a function is expressible as a Fourier series

Let $f(x)$ be defined by

$$f(x) = \phi(x), c < x < x_0$$
$$= \phi(x), x_0 < x < c + 2\pi$$

Where x_0 is the point of discontinuity in $(c, c+2\pi)$ in such cases also we obtain the Fourier series for $f(x)$ in the usual way. The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\,dx + \int_{x_0}^{c+2\pi}\phi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\cos nx.dx + \int_{x_0}^{c+2\pi}\phi(x)\cos nx.dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_c^{x_0}\phi(x)\sin nx.dx + \int_{x_0}^{c+2\pi}\phi(x)\sin nx.dx\right]$$

Note :-

(i) $$\int_{-\pi}^{\pi}\cos mx \cos nx.\,dx = \begin{cases} 0 \; for\; m \neq n \\ \pi, for\; m = n > 0 \\ 2\pi, for\; m = n > 0 \end{cases}$$

(ii) $$\int_{-\pi}^{\pi}\sin mx \sin nx.\,dx = \begin{cases} 0 \; for\;\; m = n = 0 \\ \pi, for\; m \neq n\; > 0 \end{cases}$$

Problems:-

Fourier Series in $[-\pi, \pi]$

1. ***Express*** $f(x) = x$ ***as Fourier series in the interval*** $-\pi < x < \pi$

Sol : Let the function x be represented as a Fourier series

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0 \qquad (\because x \text{ is odd function})$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.\,dx$$

$$= 0\,(x\cos nx \text{ is odd function and } \cos nx \text{ is even function})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x)\sin nx.\,dx = = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\ \sin nx.\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{-\pi}^{\pi} x\ \sin nx.\,dx\right]\quad [\because x\sin nx \text{ is even function}]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right) - (0+0)\right]$$

$$(\because \sin n\pi = 0, \sin 0 = 0)$$

$$= -\frac{2}{n}\cos n\pi = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}\ \forall n = 1,2,3.....$$

Substituting the values of a_0, a_n, b_n in (1), We get

$$x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$$

$$= -\pi + 2\left[\sin x \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots.\right]$$

2. **Express $f(x) = x - \pi$ as Fourier series in the interval $-\pi < x < \pi$**

Sol:

Let the function $x - \pi$ be represented by the Fourier series

$$f(x) = x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \to (1)\qquad \text{Then}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-\pi)dx \quad = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx = \pi\int_{-\pi}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left[0 - \pi.2\int_0^{\pi} dx\right]\quad (\because x \text{ is odd function})$$

$$= \frac{1}{\pi}[-2\pi(x)_0^{\pi}] \qquad = -2(\pi - 0) = -2\pi$$

$$\text{and } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.\,dx \quad = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx.\,dx$$

$$= \frac{1}{x}\left[\int_{-\pi}^{\pi} x\cos nx.\,dx - \pi\int_{-\pi}^{\pi}\cos nx.\,dx\right] \qquad = \frac{1}{\pi}\left[0 - 2\pi\int_0^{\pi}\cos nx.dx\right]$$

$$(x\cos nx \text{ is odd function and } \cos nx \text{ is even function})$$

$$\therefore\ a_n = -2\int_0^{\pi} \cos nx.\,dx = -2\left(\frac{\sin nx}{n}\right)_0^{\pi}$$

$$= \frac{-2}{n}(\sin n\pi - \sin 0) = \frac{-2}{n}(0-0) = 0 \ \text{ for } n = 1,2,3\ldots\ldots\ldots.$$

$$\therefore\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx.\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\ \sin nx.\,dx - \pi\int_{-\pi}^{\pi}\sin nx.\,dx\right]$$

$$= \frac{1}{\pi}\left[2\int_{-\pi}^{\pi} x\ \sin nx.\,dx - \pi(0)\right] \quad [\because x\sin nx \text{ is even function}]$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right) - (0+0)\right] \ (\because \sin n\pi = 0)$$

$$= \frac{-2}{\pi}\cos n\pi = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}\ \forall n = 1,2,3.....$$

Substituting the values of a_0, a_n, b_n in (1), We get

$$\therefore\ x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$$

$$= -\pi + 2\left[\sin x\frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\ldots.\right]$$

3. Find the Fourier series to represent the function e^{-ax} *from* $-\pi \le x \le \pi$.

Deduce from this that $\frac{\pi}{\sinh\pi} = 2\left[\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} ------\right]$ Sol. Let the function e^{-ax} be represented by the Fourier series

$$e^{-ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \to (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}dx = \frac{1}{\pi}\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi} = \frac{-1}{a\pi}\left(e^{-a\pi} - e^{a\pi}\right) = \frac{e^{a\pi} - e^{-a\pi}}{a\pi}$$

Then

$$\therefore \frac{a_0}{2} = \left[\frac{e^{a\pi} - e^{-a\pi}}{2}\right]\frac{1}{a\pi} = \frac{\sinh a\pi}{a\pi}$$

$$\therefore\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\cos nx.\,dx = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{-ax}\cos bx.\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$\therefore a_n = \frac{1}{\pi}\left\{\frac{e^{-a\pi}}{a^2+n^2}(-a\cos n\pi + 0) - \frac{e^{-a\pi}}{a^2+n^2}(-a\cos n\pi + 0)\right\}$$

$$= \frac{a}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi})\cos n\pi = \frac{2a\cos n\pi\ \sinh a\pi}{\pi(a^2+n^2)}$$

$$= \frac{(-1)^n 2a\ \sinh a\pi}{\pi(a^2+n^2)}(\because \cos n\pi = (-1)^n)$$

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx.\,dx$

$$\left[\because \int e^{ax}\sin bx.\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$=\frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}(0-n\cos n\pi) - \frac{e^{a\pi}}{a^2+n^2}(0-n\cos n\pi)\right]$$

$$= \frac{n\cos n\pi(e^{a\pi}-e^{-a\pi})}{\pi(a^2+n^2)} = \frac{(-1)^n 2n\ \sinh a\pi}{\pi(a^2+n^2)}$$

Substituting the values of $\frac{a_0}{2}, a_n$ *and* b_n in (1) we get

$$e^{-a\pi} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2a\ \sinh a\pi}{\pi(a^2+n^2)}\cos nx + (-1)^n 2n\frac{\sinh a\pi}{\pi(a^2+n^2)}\sin nx\right]$$

$$= \frac{2\sinh a\pi}{a}\left\{\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \frac{a\cos 3x}{3^2+a^2} + \ldots\right)\left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2}\ldots\right)\right\} \text{——}(2)$$

Deduction: –

Putting $x=0$ and $a=1$ in (2), we get

$$1=\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}-\frac{1}{2}+\frac{1}{2^2+1}-\frac{1}{3^2+1}+\frac{1}{4^2+1}---\right] \Rightarrow \frac{\pi}{\sinh\pi} = 2\left(\frac{1}{2^2+1}-\frac{1}{3^2+1}+\frac{1}{4^2+1}----\right)$$

4. Find the Fourier Series of $f(x) = x + x^2, -\pi < x < \pi$ and hence deduce the series

i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+----=\frac{\pi^2}{6}$ **ii)** $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}+--=\frac{\pi^2}{12}$

Sol: Let $x + x2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

To find ao $= \frac{1}{\pi}\int_{-\Pi}^{\Pi}(x+x^2)\,dx = \frac{1}{\Pi}\left(\frac{x^2}{2}+\frac{x^3}{3}\right)_{\pi}^{-\pi} = \frac{2}{3}\pi^2$

To find an $= \frac{1}{\pi}\int_{-\Pi}^{\Pi}(x+x^2)\cos nx\ dx$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\Pi}^{\Pi}$$

$$= \frac{1}{\Pi n^2}\left[(1+2x)(\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\Pi n^2}\left[(1+2\pi)(\cos n\pi) - (1-2\pi)(\cos n\pi)\right]$$

$$= \frac{1}{\Pi n^2}(4\pi\ \cos n\pi) = \frac{4}{n^2}(-1)^n$$

To find bn $= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\ dx$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{-\cos nx}{n} - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\frac{-\cos n\Pi}{n} - 0 + 2\left(\frac{\cos n\Pi}{n^3}\right)\right] - \left[(\pi+\pi^2)\frac{-\cos n\pi}{n} - 0 + 2\left(\frac{\cos n\pi}{n^3}\right)\right] = -\frac{2}{n}(-1)^n$$

Substituting in (1), the required Fourier series is,

$$x + x^2 = \frac{\pi^2}{3} - 4(\text{Cos } x - \text{Cos}\frac{2x}{4} + \text{Cos}\frac{3x}{9} + ..) + 2(\text{Sin } x - \text{Sin}\frac{2x}{4} + \text{Sin}\frac{3x}{9} + \cdots)$$

5. Find the Fourier series of the periodic function defined as $f(x) = \begin{bmatrix} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{bmatrix}$ Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ------ = \frac{\pi^2}{8}$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \to (1)$ then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)dx + \int_0^{\pi} x dx\right] = \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} + \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{1}{\pi}\left[\frac{-\pi^2}{2}\right] = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx.dx + \int_0^{\pi} x\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[0 + \frac{1}{n^2}\cos n\pi - \frac{1}{\pi n^2}\right]$$

$$= \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{1}{\pi n^2}[(-1)^n_{-1}]$$

$$a_1 = \frac{-2}{1^2.\pi}, a_2 = 0, a_3 = \frac{-2}{3^2.\pi}, a_4 = 0, a_5 = \frac{-2}{5^2.\pi} ---$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx.dx + \int_0^{\pi} x\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-x\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1 - 2\cos n\pi)$$

$$b_1 = 3, b_2 = \frac{-1}{2}, b_3 = 1, b_4 = \frac{-1}{4} \text{ and so on}$$

Substituting the values of a_0, a_n *and* b_n in (1), we get

$$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ---\right) + \left(3\sin x - \frac{\sin 3x}{2} + \frac{3\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots\right) \ldots(2)$$

<u>**Deduction:-**</u>

Putting $x=0$ in (2), we obtain $f(0) = \frac{-\pi}{4} - \frac{2}{4}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots.\right)$ —— .... (3)

Now $f(x)$ is discontinuous at $x = 0$

$f(0-0) = -\pi$ *and* $f(0+0) = 0$

$f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = \frac{-\pi}{2}$

Now (3) becomes $\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + ---\right) \to \frac{1}{1^2} + \frac{1}{3^2} + {}^{1}/_{5^2} + \cdots = \frac{\pi^2}{8}$

6. Find the Fourier series of the periodic function defined as $f(x) = \begin{bmatrix} -\pi, & -\pi < x < 0 \\ \pi, & 0 < x < \pi \end{bmatrix}$

Sol. Let $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$ then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \quad = \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)dx + \int_0^{\pi}\pi dx\right]$$

$$= \frac{1}{\pi}\left[-\pi(x)_{-\pi}^{0} \; + \pi(x)_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \pi^2\right] = \frac{1}{\pi}[0] = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx \; = \frac{1}{\pi}\left[\int_{-\pi}^{o}(-\pi)\cos nx.dx + \int_0^{\pi}\pi\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} \; + \pi\left(\frac{\sin nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}(0) \quad (\because \sin 0 = 0, \sin n\pi = 0)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx.dx + \int_0^{\pi}\pi\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-\pi\frac{\cos nx}{n}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1 - \cos n\pi) - \frac{\pi}{n}(\cos n\pi - cos0)\right]$$

$$= \frac{1}{n}(2 - 2\cos n\pi) = \frac{1}{n}(2 - 2(-1)^n) = \begin{cases} 0 \; when \; n \; is \; even \\ \dfrac{4}{n} \; when \; n \; is \; odd \end{cases}$$

Substituting the values of a_0, a_n and b_n in (1), we get $f(x) = \sum_{n=1}^{\infty}\frac{4}{n}\sin(nx)$ *where n is odd*

$$f(x) = 4\left(\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) \pm - -\right)$$

Fourier Series for $f(x)$ in $[0, 2\pi]$

1. ***Obtain the Fourier series for the function*** $f(x) = e^x$ ***from*** $x = [0, 2\pi]$

Sol: Let $e^x = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$

Then $ao = \dfrac{1}{\pi}\int_0^{2\Pi} f(x)\,dx \; = \dfrac{1}{\pi}\int_0^{2\Pi} e^x \, dx = \dfrac{1}{\pi}(e^x)_0^{2\pi} = \dfrac{1}{\pi}(e^{2\Pi} - 1)$

and $an = \dfrac{1}{\pi}\int_0^{2\Pi} f(x)\cos nx\,dx \; = \dfrac{1}{\pi}\int_0^{2\Pi} e^x \cos nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi} = \frac{e^{2\pi} - 1}{\pi(1+n^2)}$$

$$Finally \quad bn = \frac{1}{\pi}\int_0^{2\Pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{2\Pi} e^x \sin nx\ dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi} = \frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}$$

$$Hence\ e^x = \frac{e^{2\pi}-1}{\pi} + \sum_{n=1}^{\infty}\frac{e^{2\pi}-1}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}\sin nx$$

$$= \frac{e^{2\pi}-1}{2\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} - \sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}\right]$$

This is the required Fourier series.

2. *Obtain the Fourier series to represent the function*

$f(x) = kx(\pi - x)$ *in* $0 < x < 2\pi$ *Where k is a constant.*

$Sol: -\ Given\ f(x) = kx(\pi - x)\ in\ 0 < x < 2\pi\ fourier\ series\ of\ the\ function\ f(x)$

$$f(x) = kx(\pi - x)\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\, sinnx ----------(1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\,dx = \frac{k}{\pi}\left[\pi\frac{x^2}{2} - \frac{x^3}{3}\right]_0^{2\pi} = -\frac{2\pi^2 k}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,cosnx\,dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\,cosnx\,dx$$

$$= \frac{k}{\pi}\left[(\pi x - x^2)\frac{\sin nx}{n} - (\pi - 2x)\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\left\{0 + \frac{-3\pi}{n^2}cos2n\pi + 0\right\} - \left\{0 + \frac{\pi}{n^2} + 0\right\}\right] = \frac{k}{\pi}\left(\frac{-4\pi}{n^2}\right) = -\frac{4k}{n^2}(n \neq 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\,sinnx\,dx = \frac{1}{\pi}\int_0^{2\pi} kx(\pi - x)\sin nx\,dx$$

$$= \frac{k}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{k}{\pi}\left[\left\{\frac{2\pi^2}{n} + 0 - \frac{2}{n^3}\right\} - \left\{0 + 0 - \frac{2}{n^3}\right\}\right] = \frac{2k\pi}{n}$$

put the values of a_0, a_n, b_n *in (1) we get*

$$f(x) = -\frac{\pi^2 k}{3} - 4k\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + 2k\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

3.. Find the fourier series expansion of the

function $\mathbf{f(x) = \frac{(\pi - x)^2}{4}}$ **in the interval** $\mathbf{0 < x < 2\pi}$

Sol:

$$\text{Given } f(x) = \frac{(\pi - x)^2}{4}\ 0 < x < 2\pi$$

fourier series of the function f(x) is given by

$$f(x)=\frac{(\pi-x)^2}{4}=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\,\sin nx--------(1)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}dx=\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{3}\right]_0^{2\pi}=\frac{\pi^2}{6}--(2)$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\cos nx\,dx$$

$$=\left[\frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-\{2(\pi-x)\}\left(-\frac{\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right]\right]_0^{2\pi}=\frac{1}{4\pi}\left[\frac{2\pi}{n^2}+\frac{2\pi}{n^2}\right]$$

$$=\frac{1}{n^2}-(3)$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\sin nx\,dx$$

$$=\left[\frac{1}{4\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)-\{2(\pi-x)\}\left(-\frac{\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]\right]_0^{2\pi}$$

$$=\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]=0\,;b_n=0------(4)$$

put the values of a_0,a_n,b_n in (1) we get

$$f(x)=\frac{(\pi-x)^2}{4}=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+---$$

4. Expand $f(x)=\begin{cases}1; & 0<x<\pi\\ 0 & \pi<x<2\pi\end{cases}$ as a Fourier Series.

Sol:- The Fourier series for the function in $(0,22\pi)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\,\sin nx\ldots\ldots..(1)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\left[\int_0^{\pi}1\,dx+\int_{\pi}^{2\pi}0\,dx\right]=\frac{1}{\pi}(x)_0^{\pi}=1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx.dx = \frac{1}{\pi}\left[\int_0^{\pi}(1)\cos nx.dx + \int_{\pi}^{2\pi}(0)\cos nx.dx\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin nx}{n}\right)_0^{\pi} = 0$$

$$= \frac{1}{\pi}(0) \quad (\because \sin 0 = 0, \sin n\pi = 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx.dx = \frac{1}{\pi}\left[\int_0^{\pi}(1)\sin nx.dx + \int_{\pi}^{2\pi} 0.\sin nx.dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(1)\sin nx.dx\right] = \frac{1}{\pi}\left(\frac{-\cos nx}{n}\right)_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - \cos 0) = -\frac{1}{n\pi}[(-1)^n - 1]$$

$$\therefore b_n = \begin{cases} 0 & when\ n\ is\ even \\ \frac{2}{n\pi} & when\ n\ is\ odd \end{cases}$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x) = \frac{1}{2} + \frac{2}{n\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin nx = \frac{1}{2} + \frac{2}{n\pi}\left(sinx + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ..\right)$$

5. Obtain Fourier series expansion of $f(x) = (\pi - x)^2$ **in** $0 < x < 2\pi$ **and deduce the value of** $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ---- = \frac{\pi^2}{6}$

Sol:-

Given $f(x) = (\pi - x)^2 \ 0 < x < 2\pi$

fourier series of the function f(x) is given by

$$f(x) = (\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx --------(1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 dx$$

$$\frac{1}{\pi}\int_0^{2\pi}[\pi^2 + x^2 - 2\pi x]\,dx = \frac{2\pi^2}{3} --(2)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2\cos nx\,dx$$

$$= \left[\frac{1}{\pi}\left[(\pi - x)^2\frac{\sin nx}{n} - \{2(\pi - x)\}\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{4}{n^2} - (3)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(\pi - x)^2 \sin nx\,dx$$

$$= \left[\frac{1}{\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - \{2(\pi - x)\}\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right) - \left(-\frac{\pi^2}{n} + \frac{2}{n^3}\right)\right] = 0\,; b_n = 0 ------(4)$$

put the values of a_0, a_n, b_n in (1) we get

$$f(x) = (\pi - x)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{\pi^2}{3} + \frac{4\cos x}{1^2} + \frac{4\cos 2x}{2^2} + \frac{4\cos 3x}{3^2} + ---$$

Deduction:-

Putting $x = 0$ in the above equation we get

$$f(0) = (\pi - 0)^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} = \frac{\pi^2}{3} + \frac{4\cos 0}{1^2} + \frac{4\cos 0}{2^2} + \frac{4\cos 0}{3^2} + ---$$

$$\pi^2 = \frac{\pi^2}{3} + \frac{4}{1^2} + \frac{4}{2^2} + \frac{4}{3^2} + ---$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ---\right]$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ---- = \frac{\pi^2}{6}$$

Even and Odd Functions:-

A function $f(x)$ is said to be even *if* $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

Example:- $x^2, x^4 + x^2 + 1, e^x + e^{-x}$ are even functions

$x^3, x, \sin x, \operatorname{cosec} x$ are odd functions

Note1:-

1. Product of two even (or) two odd functions will be an even function
2. Product of an even function and an odd function will be an odd function

Note 2:- $\int_{-a}^{a} f(x)dx = 0$ when $f(x)$ is an odd function

$= 2\int_0^a f(x)\,dx$ when $f(x)$ is even function

Fourier series for even and odd functions

We know that a function $f(x)$ defined in $(-\pi, \pi)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx$

Case (i):- when $f(x)$ is an even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)dx$$

Since $\cos nx$ is an even function, $f(x)\cos nx$ is also an even function

Hence $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, . \, dx$

$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx \, . \, dx$

Since $\sin nx$ is an odd function, $f(x)\sin nx$ is an odd function

$$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx.dx = 0$$

$\therefore$ If a function $f(x)$ is even in $(-\pi,\pi)$, its Fourier series expansion contains only cosine terms

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx.dx, n = 0,1,2,-----$

Case 2:- when $f(x)$ is an odd function in $(-\pi,\pi)$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0$ Since $f(x)$ is odd

Since $\cos nx$ is an even function, $f(x)\cos nx$ is an odd function and hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx.dx = 0$$

Since $\sin nx$ is an odd function; $f(x)\sin nx$ is an even function

$\therefore b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx. \, dx$

$= \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx. \, dx$

Thus, if a function $f(x)$ defined in $(-\pi,\pi)$ is odd, its Fourier expansion contains only sine terms

$\therefore f(x) = \sum_{n=1}^{\pi} b_n \sin nx$ Where $b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx.dx$

Even and Odd Functions:-

Problems:-

1. Expand the function $f(x) = x^2$ as a Fourier series in $[-\pi,\pi]$, hence deduce that

(i) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+----=\frac{\pi^2}{12}$

Sol. Since $f(-x)=(-x)^2=x^2=f(x)$

$\therefore f(x)$ is an even function

Hence in its Fourier series expansion, the sine terms are absent

$$\therefore x^2=\frac{a_0}{2}\sum_{n=1}^{\infty} a_n\cos nx \ldots\ldots \text{———}(1)$$

Where $a_0=\frac{2}{\pi}\int_0^{\pi}x^2dx=\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi}=\frac{2\pi^2}{3}\ \ldots\ldots$ ———(2)

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx.dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx.dx$$

$$=\frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right)-2x\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$=\frac{2}{\pi}\left[0+2\pi\frac{\cos n\pi}{n^2}+2.0\right]=\frac{4\cos n\pi}{n^2}=\frac{4}{n^2}(-1)^n$ --------------(3)

Substituting the values of a_0 and a_n from (2) and (3) in (1) we get

$$x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx=\frac{\pi^2}{3}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$=\frac{\pi^2}{3}-4\left(\cos x-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-\frac{\cos 4x}{4^2}+---\right)\to(4)$$

Deduction:- Putting $x=0$ in (4), we get

$$0=\frac{\pi^2}{3}-4\left(1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+---\right)\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+---=\frac{\pi^2}{12}$$

2. Find the Fourier series to represent the function $f(x)=|\sin x|,-\pi<x<\pi$

Sol: Since $|\sin x|$ is an even function, $b_n=0\ \ for\ all\ n$

Let $f(x)=|\sin x|=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx\to(1)$

Where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin x|dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}(-\cos x)_0^{\pi}$

$$=\frac{-2}{\pi}(-1-1)=\frac{4}{\pi}$$

and $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx.dx=\frac{2}{\pi}\int_0^{\pi}\sin x.\cos nx\,dx$

$$=\frac{1}{\pi}\int_0^{\pi}[sin(1+n)x+sin(1-n)x]\,dx$$

$$=\frac{1}{\pi}\left[-\frac{cos(1+n)x}{1+n}-\frac{cos(1-n)x}{1-n}\right]_0^{\pi}(n\neq 1)$$

$$=-\frac{1}{\pi}\left[\frac{cos(1+n)\pi}{1+n}+\frac{cos(1-n)\pi}{1-n}-\frac{1}{1+n}-\frac{1}{1-n}\right]_0^{\pi}(n\neq 1)$$

$$=\frac{-1}{\pi}\left[\frac{(-1)^{n+1}-1}{1+n}+\frac{(-1)^{n+1}-1}{1-n}\right] \quad =\frac{-1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}-\left\{\frac{1}{1+n}+\frac{1}{1-n}\right\}\right]$$

$$=\frac{-1}{\pi}\left[(-1)^{n+1}\frac{2}{1-n^2}-\frac{2}{1-n^2}\right]=\frac{2}{\pi(n^2-1)}[(-1)^{n+1}-1]$$

$$=\frac{-2}{\pi(n^2-1)}[1+(-1)^n] \quad (n\neq 1)$$

$$\therefore a_n=\begin{cases}0 \ \ if\ n\ is\ odd\ and\ n\neq 1\\ \frac{-4}{\pi(n^2-1)} \ \ if\ n\ is\ even\end{cases}$$

For $n=1, a_1=\frac{2}{\pi}\int_0^\pi \sin x.cons\ x\,dx \quad =\frac{1}{\pi}\int_0^\pi \sin 2x\,dx$

$$=\frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^\pi \quad =\frac{-1}{2\pi}(\cos 2\pi-1)=0$$

Substituting the values of a_0, a_1 *and* a_n in (1) We get $|\sin x|=\frac{2}{\pi}+\sum_{n=2,4,--}^{\infty}\frac{-4}{\pi(n^2-1)}\cos nx$

$$=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=2,4---}^{\infty}\frac{\cos nx}{n^2-1}=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \quad \text{(Replace n by 2n)}$$

Hence $|\sin x|=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+----\right)$

3. *Show that for* $-\pi<x<\pi, \sin ax=\frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-......\right]$

(a **is not an integer**)

Sol: - As sin ax is an Odd function. It's Fourier series expansion will consist of sine terms only

$$\therefore \sin ax=\sum_{n=1}^{\infty}b_n\sin nx \quad \text{------------- (1)}$$

Where $b_n=\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^\pi \sin ax.\sin nx\,dx=\frac{2}{\pi}\int_0^\pi[\cos(a-n)x-\cos(a+n)x]dx$

$[\because 2\sin A\sin B=\cos(A-B)-\cos(A+B)]$

$$b_n=\frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n}-\frac{\sin(a+n)x}{a+n}\right]_0^\pi$$

$$=\frac{1}{\pi}\left[\frac{\sin a\pi\cos n\pi-\cos\pi\sin n\pi}{a-n}-\frac{\sin a\pi\cos n\pi+\cos a\pi\sin n\pi}{a+n}\right]$$

$$b_n=\frac{1}{\pi}\left[\frac{\sin a\pi.\cos n\pi}{a-n}-\frac{\sin a\pi.\cos n\pi}{a+n}\right][\because \sin n\pi=0]$$

$$=\frac{1}{\pi}\sin a\pi\cos n\pi\left(\frac{1}{a-n}-\frac{1}{a+n}\right)=\frac{1}{\pi}\sin a\pi\,(-1)^n\left(\frac{a+n-a+n}{a^2-n^2}\right)=\frac{(-1)^n 2n}{\pi(a^2-n^2)}\sin a\pi$$

Substituting these values in (1), we get

$$\sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{(a^2-n^2)}\sin nx = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(n^2-a^2)}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-......\right]$$

4. **Find the Fourier series to represent the function $f(x) = Sin\, x, -\pi < x < \pi$.**

Sol:- since sin x is an odd function $a_0 = a_n = 0$

Let $f(x) = \sum b_n \sin nx, where$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi}[\cos(1-n)\,x - \cos(1+n)\,x]\, dx$$

$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi}$ (n$\neq$ 1) $\quad = 0(n \neq 1)$

If n=1 $b_1 = \frac{2}{\pi}\int_0^{\pi} sin^2x\, dx = \frac{2}{\pi}\int_0^{\pi}\frac{1-\cos 2x}{2}\, dx = \frac{1}{\pi}\left(x - \frac{Sin2x}{2}\right)_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1 \therefore f(x) = b_1 sinx = sinx$

5. *Show that for* $-\pi < x < \pi$, $\sin k\,x = \frac{2\sin k\pi}{\pi}\left[\frac{\sin x}{1^2-k^2}-\frac{2\sin 2x}{2^2-k^2}+\frac{3\sin 3x}{3^2-k^2}-......\right]$

(k **is not an integer**)

Sol: - As sin kx is an Odd function.

It's Fourier series expansion will consist of sine terms only

$$\therefore \sin \mathrm{k}\, x = \sum_{n=1}^{\infty} b_n \sin nx$$

------------- (1)

Where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}\sin \mathrm{kx}.\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}[\cos(\mathrm{k}-n)\,\mathrm{x} - \cos(\mathrm{k+n})\,\mathrm{x}]dx$$

$[\because 2\sin A \sin B = \cos(\mathrm{A-B}) - \cos(\mathrm{A+B})]$

$$b_n = \frac{1}{\pi}\left[\frac{\sin(\mathrm{k}-n)x}{k-n} - \frac{\sin(\mathrm{k}+n)x}{k+n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin k\pi \cos \mathrm{n}\,\pi - \cos \pi \sin \mathrm{n}\,\pi}{k-n} - \frac{\sin k\,\pi \cos n\,\pi + \cos k\pi \sin \mathrm{n}\,\pi}{k+n}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{\sin k\pi.\cos \mathrm{n}\,\pi}{k-n} - \frac{\sin k\,\pi.\cos n\pi}{k+n}\right][\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\sin k\,\pi \cos n\pi\left(\frac{1}{k-n} - \frac{1}{k+n}\right) = \frac{1}{\pi}\sin \mathrm{k}\,\pi\, (-1)^n\left(\frac{k+n-k+n}{k^2-n^2}\right) = \frac{(-1)^n 2n}{\pi(\mathrm{k}^2-n^2)}\sin k\,\pi$$

Substituting these values in (1), we get

$$\sin kx = \frac{2\sin k\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{(k^2-n^2)}\sin nx = \frac{2\sin k\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(n^2-k^2)}\sin nx$$

$$= \frac{2\sin k\pi}{\pi}\left[\frac{\sin x}{1^2-k^2}-\frac{2\sin 2x}{2^2-k^2}+\frac{3\sin 3x}{3^2-k^2}-\ldots\ldots\right]$$

Half –Range Fourier Series:-

1) The sine series:-

$$f(x)=\sum_{n=1}^{\infty}b_n\sin nx \text{ where } b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx$$

2) The cosine series:-

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \text{ where } a_0=\frac{2}{\pi}\int_0^{\pi}f(x)dx \text{ and}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx.dx$$

Note:-

1) Suppose $f(x)=x$ in $[0,\pi]$.It can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$
2) If $f(x)=x^2$ in $[0,\pi]$ can have Fourier cosine series expansion as well as Fourier sine series expansion in $[0,\pi]$.

Half –Range Fourier Series:-

Problems:

1. **Find the half range sine series for . $f(x)=x(\pi-x)$, in $0<x<\pi$**

Deduce that $\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+----=\frac{\pi^3}{32}$

Sol. The Fourier sine series expansion of $f(x)$ in $(0,\pi)$ is $f(x)=x(\pi-x)=\sum_{n=1}^{\infty}b_n\sin nx$

Where $b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx.dx$; $b_n=\frac{2}{\pi}\int_0^{\pi}x(\pi-x)\sin nx.dx$

$$=\frac{2}{\pi}\int_0^{\pi}\left(\pi x-x^2\right)\sin nx.dx$$

$$=\frac{2}{\pi}\left[\left(\pi x-x^2\right)\left(\frac{-\cos nx}{n}\right)-(\pi-2x)\left(\frac{-\sin nx}{n^2}\right)+(-2)\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\frac{2}{n^3}(1-\cos n\pi)\right]=\frac{4}{n\pi^3}\left(1-(-1)^n\right)$$

$$b_n=\begin{cases}0, \text{ when } n \text{ is even}\\ \frac{8}{\pi n^3}, \text{ when } n \text{ is odd}\end{cases}$$

Hence

$$x(\pi - x) = \sum_{n=1,3,5\ldots} \frac{8}{\pi n^3} \sin nx \ \ (or) \quad x(\pi - x) = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + ---- \right) \rightarrow (1)$$

Deduction:-

Putting $x = \frac{\pi}{2}$ in (1), we get

$$\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(sin\frac{\pi}{2} + \frac{1}{3^3} sin\frac{3\pi}{2} + \frac{1}{5^3} sin\frac{5\pi}{2} + ---\right)$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\left[1 + \frac{1}{3^3} sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3} sin\left(2\pi + \frac{\pi}{2}\right) + \frac{1}{7^3} sin\left(3\pi + \frac{\pi}{2}\right) + ---\right]$$

(or) $\frac{\pi^2}{32} = 1 + \frac{1}{3^3} + \frac{1}{5^3} + \frac{1}{7^3} + ---$

2.**Find the half- range sine series for the function** $f(x) = \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} in\ (0, \pi)$

Sol. Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ —— (1)

$Then\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx.dx = \frac{2}{\pi}\int_0^{\pi} \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}}.\sin nx.dx$

$= \frac{2}{\pi(e^{a\pi} - e^{-a\pi})}\left[\int_0^{\pi} e^{ax} \sin nx.dx - \int_0^{\pi} e^{-ax} \sin nx.dx\right]$

$$= \frac{2}{\pi(e^{a\pi} - e^{-a\pi})}\left[\left[\frac{e^{ax}}{a^2 + n^2}(a \sin nx - n \cos nx)\right]_0^{\pi} - \left[\frac{e^{-ax}}{a^2 + n^2}(-a \sin nx - n \cos nx)\right]_0^{\pi}\right]$$

$$= \frac{2}{\pi(e^{a\pi} - e^{-a\pi})}\left[\frac{-e^{a\pi}}{a^2 + n^2} n(-1)^n + \frac{n}{a^2 + n^2} + \frac{-e^{a\pi}}{a^2 + n^2} n(-1)^n - \frac{n}{a^2 + n^2}\right]$$

$$= \frac{2n(-1)^n}{\pi(e^{a\pi} - e^{-a\pi})}\left[\frac{e^{-ax} - e^{ax}}{n^2 + a^2}\right] = \frac{2n(-1)^{n+1}}{\pi(n^2 + a^2)} -- —— (2)$$

Substituting (2) in (1), we get

$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{a^2 + n^2} \sin nx = \frac{2}{\pi}\left[\frac{\sin x}{a^2 + 1^2} - \frac{2 \sin 2x}{a^2 + 2^2} + \frac{3 \sin 3x}{a^2 + 3^2} - ----\right]$

Fourier series of $f(x)$ defined in $[c, c + 2l]$

It can be seen that role played by the functions

$1, \cos x, \cos 2x, \cos 3x, ...\sin x, \sin 2x.........$

In expanding a function $f(x)$ defined in $[c, c + 2\pi]$ as a Fourier series, will be played by

$$1, \cos\left(\frac{\pi x}{e}\right), \cos\left(\frac{2\pi x}{e}\right), \cos\left(\frac{3\pi x}{e}\right), ...$$

$$\sin\left(\frac{\pi x}{e}\right), \sin\left(\frac{2\pi x}{e}\right), \sin\left(\frac{3\pi x}{e}\right), ...$$

In expanding a function $f(x)$ defined in $[c, c + 2l]$

$$(i) \int_c^{c+2l} sin\left(\frac{m\pi x}{l}\right).cos\left(\frac{n\pi x}{l}\right) dx = 0$$

$$(ii)\ \int_c^{c+2l} sin\left(\frac{m\pi x}{l}\right).sin\left(\frac{n\pi x}{l}\right)dx = \begin{cases} 0, & if\ m \neq n \\ l, & if\ m = n \neq 0 \\ 0, & if\ m = n = 0 \end{cases}$$

$$(iii)\ \int_c^{c+2l} cos\left(\frac{m\pi x}{l}\right).cos\left(\frac{n\pi x}{l}\right)dx = \begin{cases} 0, & if\ m \neq n \\ l, & if\ m = n \neq 0 \\ 2l, & if\ m = n = 0 \end{cases}$$

[It can be verified directly that, when m, n are integers]

Fourier series of $f(x)$ defined in $[0, 2l]$:

Let $f(x)$ be defined in $[0,2l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\, cos\frac{n\pi x}{l} + b_n\, sin\frac{n\pi x}{l}\right] \to (1)$

Where $a_n = \frac{1}{l}\int_0^{2l} f(x)cos\frac{n\pi x}{l}dx \to (2)$

$.and\ b_n = \frac{1}{l}\int_0^{2l} f(x)sin\frac{n\pi x}{l}dx \to (3)$

Fourier Series Of $f(x)$ Defined In $[-l, l]$:

Let $f(x)$ be defined in $[-l,l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$where\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

Fourier series for even and odd functions in $[-l, l]$:-

Let $f(x)$ be defined in $[-l,l]$. If $f(x)$ is even $f(x)\cos\frac{n\pi x}{l}$ is also even

$$\therefore a_n = \frac{1}{l}\int_{-l}^{l} f(x)cos\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^{l} f(x)cos\frac{n\pi x}{l}dx$$

And $f(x)sin\frac{n\pi x}{l}\ is\ odd$

$$\therefore b_n = \frac{1}{l}\int_{-l}^{l} f(x)sin\frac{n\pi x}{l}dx = 0\forall n$$

Hence if $f(x)$ is defined in $[-l,l]$ and is even its Fourier series expansion is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

If $f(x)$ is defined in $[-l,l]$and its odd its Fourier series expansion is given by

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\quad where\ b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

<u>Note:-</u> In the above discussion if we put $2l=2\pi, l=\pi$ we get the discussion regarding the intervals $[0,2\pi]$ and $[-\pi,\pi]$ as special cases

<u>Fourier series of $f(x)$ defined in $[c, c+2l]$</u>

<u>Problems:</u>-

1.Express $f(x)=x^2$ as a Fourier series in $[-l,l]$

Sol: Since $f(-x)=(-x)^2=x^2=f(x)$

Therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l,l]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\ where\ a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$$

Hence $a_0=\frac{2}{l}\int_0^l x^2dx=\frac{2}{l}\left(\frac{x^3}{3}\right)_0^l=\frac{2l^2}{3}$

$$Also\ a_n=\frac{2}{l}\int_0^l f(x)cos\frac{n\pi x}{l}dx$$

$$=\frac{2}{l}\left[x^2\left[\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]-2x\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)+2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$=\frac{2}{l}\left[2x\frac{cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right]_0^l$$

(Since the first and last terms vanish at both upper and lower limits)

$$\therefore a_n = \frac{2}{l}\left[2l\frac{\cos n\pi}{n^2\pi^2/l^2}\right] = \frac{4l^2\cos n\pi}{n^2\pi^2} = \frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - - - - -\right]$$

2. **Obtain Fourier series for f(x) = x^3 in [-1, 1].**

Sol: The given function is x^3 which is odd

a_0=, a_n=0, b_n= $b_n = \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^1 x^3 \sin n\pi x\ dx$

$$=2\left[-x^3\frac{\cos n\pi x}{n\pi} + 3x^2\sin\frac{n\pi x}{n^2\pi^2} + 6x\frac{\cos n\pi x}{n^3\pi^3} - 6\sin\frac{n\pi x}{n^4\pi^4}\right]_0^1$$

$$= 2\left[\frac{-(-1)^n}{n\pi} + \frac{6(-1)^n}{n^3\pi^3}\right]$$

$\therefore$ f(x) $=2[\left(\frac{1}{\pi} - \frac{6}{\pi^3}\right)\sin x + \left(-\frac{1}{2\pi} + \frac{6}{2^3\pi^3}\right)\sin 2\pi x + \left(\frac{1}{3\pi} - \frac{6}{3^3\pi^3}\right)\sin 3\pi x + \left(-\frac{1}{4\pi} + \frac{6}{4^3\pi^3}\right)\sin 4\pi x]$

3. Find a Fourier series with period 3 to represent $f(x) = x + x^2\ \ in\ (0,3)$

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right) \to (1)$

Here $2l = 3,\quad l = 3/2$ Hence (1) becomes

$$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right) \to (2)$$

$$Where\ \ a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}\int_0^3 (x + x^2)\,dx = \frac{2}{3}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_0^3 = 9$$

$$and\ \ a_n = \frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{3}\int_0^3 (x + x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

Integrating by parts, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2} - \frac{9}{4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{4n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$$

Finally $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}dx \quad = \frac{2}{3}\int_0^3 (x + x^2)\sin\left(\frac{2n\pi x}{3}\right)dx = \frac{-12}{n\pi}$

Substituting the values of a's and b's in (2) we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Half- Range Expansion of $f(x)$ in $[0,l]$:-

Some times we will be interested in finding the expansion of $f(x)$ defined in $[0,l]$ in terms of sines only (or) in terms of cosines only. Suppose we want the expansion of $f(x)$ in terms of sine series only. Define $f_1(x)=f(x)$ in $[0,l]$ and $f_1(x)=-f_1(x)\forall n$ with $f_1[2l+x]=f_1(x)$, $f_1(x)$ is an odd function in $[-l,l]$. Hence its Fourier series expansion is given by

$$f_1(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}dx$$

$$where\ b_n=\frac{2}{l}\int_0^1 f_1(x)dx$$

The above expansion is valid for $x\ in\ [-l,l]$ in particular for $x\ in\ [0,l]$,

$f_1(x)=f(x)$ and $f_1(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}dx$ where $b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

This expansion is called the half- range sine series expansion of $f(x)$ in $[0,l]$. If we want the half – range expansion of $f(x)$ in $[0,l]$, only in terms of cosines, define $f_1(x)=f(x)$ in $[0,l]$ and $f_1(-x)=f_1(x)$ for all x with

$f_1(x+2l)=f_1(x)$.

Then $f_1(x)$ is even in $[-l,l]$ and hence its Fourier series expansion is given by

$$f_1(x)=\frac{a_0}{2}+\sum a_n\cos\frac{n\pi x}{l}$$

$$where\ a_n=\frac{2}{l}\int_0^l f_1(x)\cos\frac{n\pi x}{l}dx$$

The expansion is valid in $[-l,l]$ and hence in particular on $[0,l]$,

$f_1(x)=f(x)$ hence in $[0,l]$

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

Where $a_n=\frac{2}{l}\int_0^1 f(x)\cos\frac{n\pi x}{l}dx$

1. The half range sine series expansion of $f(x)=\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}$ in $(0,l)$is given by

   Where $b_n=\frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}dx$

2. The half range cosine series expansion of $f(x)\ in\ [0,l]$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$where\ b_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

Problems:-

1. **Find the half- range sine series of** $f(x) = 1\ in\ [0,l]$

Sol: The Fourier sine series of $f(x)$ $in\ [0,l]$ is given by $f(x) = 1 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$Here\ b_n = \frac{2}{1}\int_0^l f(x) sin\frac{n\pi x}{1}dx = \frac{2}{1}\int_0^l 1.sin\frac{n\pi x}{1}dx$$

$$= \frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l = \frac{2}{n\pi}\left[-\cos\frac{n\pi x}{l}\right]_0^l = \frac{2}{n\pi}\left(-\cos n\pi + 1\right) = \frac{2}{n\pi}\left[(-1)^{n+1} + 1\right]$$

$$\therefore b_n = \begin{cases} 0 & \text{when n is even} \\ \frac{4}{n\pi} & \text{when n is odd} \end{cases}$$

Hence the required Fourier series is $f(x) = \sum_{n=1,3,5---}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi x}{l}$

$$i.e\ 1 = \frac{4}{\pi}\left(\sin\frac{n\pi}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + {}^1/_5 \sin\frac{5\pi x}{l}\ \ldots\ldots..\right)$$

2. **Find the half – range cosine series expansion of** $f(x) = \sin\left(\frac{\pi x}{l}\right)$ **in the range** $0 < x < l$

Sol : The half-range Fourier Cosine Series is given by

$$f(x) = sin\left(\frac{\pi x}{l}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}\ \ldots\ldots \quad \text{——}(1)$$

Where $a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l sin\frac{\pi x}{l}dx = \frac{2}{l}\left[\frac{-\cos\pi x/l}{\pi/l}\right]_0^l = \frac{-2}{\pi}(\cos\pi - 1) = \frac{4}{\pi}$

and $a_n = \frac{2}{l}\int_0^l f(x) cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l sin\left(\frac{\pi x}{l}\right) cos\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{1}{l}\int_0^l \left[\frac{sin(n+1)\pi x}{l} - \frac{sin(n-1)\pi x}{l}\right]dx$$

$$= \frac{1}{l}\left[-\frac{-\frac{\cos(n+1)\pi x}{l}}{(n+1)\pi/l} + \frac{\cos(n-1)\pi x/l}{(n+1)\pi/l}\right]_0^l \quad (n \neq 1)$$

$$= \frac{1}{l}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] \quad (n \neq 1)$$

When n is odd $a_n = \frac{1}{\pi}\left[\frac{-1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = 0$

When n is even $a_n = \frac{1}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$

$= \frac{-4}{\pi(n+1)(n-1)} \quad (n \neq 1)$

If $n = 1, a_1 = \frac{1}{l}\int_0^l 2\sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{\pi x}{l}\right)dx = \frac{1}{l}\int_0^l \sin\left(\frac{2\pi x}{l}\right)dx$

$= \frac{1}{l}\cdot\frac{1}{2\pi}\left[-cos\left(\frac{2\pi x}{l}\right)\right]_0^l = \frac{-1}{2\pi}(\cos 2\pi - cos 0) = {}^{-1}/_{2\pi}(1-1) = 0$

from equation(1) we have∴ $\sin\left(\frac{\pi x}{l}\right) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3} + \frac{\cos(4\pi x/l)}{3.5} + ---\right]$

3. Obtain the half range cosine series for f(x) = x – x², 0$\leq x \leq 1$.

Sol:The half range cosine series for f(x) in $0 \leq x \leq 1$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

Where $a_0 = \frac{2}{1}\int_0^1 f(x)\,dx = 2\int_0^1 (x - x^2)dx = 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

$a_n = 2\int_0^1 (x - x^2)\cos n\pi x\, dx$

$= 2\left[(x - x^2)\frac{\cos n\pi x}{n\pi} + (1-2x)\frac{\cos n\pi x}{n\pi^2}\right]_0^1 = 2\left[(-1)\frac{\cos n\pi x}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = 2\left[\frac{(-1)^{n+1} - 1}{n^2\pi^2}\right]$

∴ The cosine series of f(x) is given by,

$f(x) = \frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1} - 1}{n^2}\right\}\cos n\pi x \quad = \frac{1}{6} - \frac{4}{\pi^2}\left\{\frac{\cos 2\pi x}{2^2} + \frac{\cos 4\pi x}{4^2} + ----\right\}$

4. Obtain the half range sine series for e^x in 0<x<1.

Sol: The sine series is $\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

Where bn $= \frac{2}{l}\int_0^1 f(x)\sin\frac{n\pi x}{l}\,dx$

$= 2\int_0^1 e^x \sin n\pi x\, dx = \left[\frac{2e^x}{(1+n^2\pi^2)}[\sin n\pi x - n\pi x \cos n\pi x]\right]_0^1$

$= \frac{2}{(1+n^2\pi^2)}[-n\pi e\cos n\pi + n\pi] = \frac{2}{(1+n^2\pi^2)}[1 - e(-1)^n]$

$\therefore e^x = 2\pi\left[\frac{(1+e)}{1+\pi^2}\sin\pi x + \frac{2(1-e)}{1+4\pi}\sin 2\pi x + \frac{3(1+e)}{1+9\pi}\sin 3\pi x + ---\right]$

UNIT-II

FOURIER TRANSFORMS

Dirichlet's condition :

A function f(x) is said to satisfy dirichlets conditions in the interval (a,b) if

(i) f(x) defined and is single valued function except possibly at a finite number of points in the interval (a,b) and

(ii) f(x) and $f^1(x)$ are piecewise continuous in(a,b)

Fourier Integral Theorem:

If $f(x)$ is defined in $(-l,l)$ and satisfies Dirichlet's condition, then

$$f(x)=\frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dt.d\lambda$$

Fourier Sine Integral:

The Fourier sine integral for f(x) is given by

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t dt.d\lambda$$

Fourier Cosine Integral:

The Fourier cosine integral for f(x) is given by

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\cos\lambda x\int_0^{\infty} f(t).\cos\lambda t.dtd\lambda$$

1. Using Fourier integral show that

$$e^{-ax}-e^{-bx}=\frac{2\left(b^2-a^2\right)}{\pi}\int_0^{\infty}\frac{\lambda\sin\lambda x}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}d\lambda, a>0, b>0$$

Sol. Since the integral on R.H.S contains sine term use Fourier sine integral formula.

We know that the F.S.I for f(x) is given by.

$$f(x)=\frac{2}{\pi}\int_0^{\infty}\sin\lambda x\int_0^{\infty} f(t)\sin\lambda t.dt.d\lambda \ldots\ldots\ldots\ldots\ldots.. (1)$$

Here $f(x)=e^{-ax}-e^{-bx}$; $\quad\therefore f(t)=e^{-at}-e^{-bt}$

$$\therefore e^{-ax}-e^{-bx}=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty\left(e^{-at}-e^{-bt}\right)\sin\lambda t\,dt\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty e^{-at}\sin\lambda t\,dt-\int_0^\infty e^{-bt}\sin\lambda t dt\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{e^{-at}}{a^2+\lambda^2}\left(-a\sin\lambda t-\lambda\cos\lambda t\right)-\frac{e^{-bt}}{b^2+\lambda^2}\left(-b\sin\lambda t-\lambda\cos\lambda t\right)\right]_0^\infty d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{\lambda^2+a^2}-\frac{\lambda}{\lambda^2+b^2}\right]d\lambda=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\lambda.\left[\frac{1}{\lambda^2+a^2}-\frac{1}{\lambda^2+b^2}\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\frac{\lambda\left(b^2-a^2\right)}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}d\lambda$$

$$\therefore e^{-ax}-e^{-bx}=\frac{2\left(b^2-a^2\right)}{\pi}\int_0^\infty \frac{\lambda.\sin\lambda x}{\left(\lambda^2+a^2\right)\left(\lambda^2+b^2\right)}d\lambda$$

Hence proved

2. Using Fourier Integral, show that $\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}.\sin\lambda x\,d\lambda=\begin{cases}\frac{\pi}{2} if\ 0<x<\pi\\ 0, if\ x>\pi\end{cases}$

Sol. Since the integral on R.H.S. contains the sine term we use Fourier Sine Integral formula. The Fourier Sine Integral for f(x) is given by.

$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\int_0^\infty f(t).\sin\lambda t\,dt.d\lambda$ --------------------- (1)

Let $f(x)=\begin{cases}\frac{\pi}{2}, & 0<x<\pi\\ 0, & x>\pi\end{cases}$ ----------------------------(2)

Using (2) in (1), we get

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\left[\int_0^\pi f(t)\sin\lambda t\,dt+\int_\pi^\infty f(t)\sin\lambda t\,dt\right]d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x.\left[\int_0^\pi \frac{\pi}{2}.\sin\lambda t\,dt\right]d\lambda \quad =\frac{2}{\pi}\int_0^\infty \sin\lambda x.\left[\frac{\pi}{2}\left(\frac{-1}{\lambda}\right).\cos\lambda t\right]_0^\pi d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{-\pi}{2\lambda}\left(\cos\lambda\pi-1\right)\right]d\lambda=\frac{\pi}{2}\times\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{1-\cos\lambda\pi}{\lambda}\right]d\lambda$$

$$\therefore f(x)=\int_0^\infty \frac{\left(1-\cos\lambda\pi\right)}{\lambda}\sin\lambda x\,d\lambda \ or \int_0^\infty \frac{\left(1-\cos\lambda\pi\right)}{\lambda}.\sin\lambda x\,d\lambda=\begin{cases}\frac{\pi}{2}, & 0<x<\pi\\ 0, & x>\pi\end{cases}$$

3. Express $f(x) = \begin{cases} 1 \; for \; 0 \leq x \leq \pi \\ 0 \; for \quad x > \pi \end{cases}$ as a Fourier cosine integral and hence evaluate $\int_0^\infty \frac{\cos \lambda x \sin \lambda \pi}{\lambda}\, d\lambda$

Sol:- Fourier cosine integral of f(x) is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \int_0^\infty f(t)\cos \lambda t \, dt \; d\lambda$$

$$f(x) = \begin{cases} 1 \; for \; 0 \leq x \leq \pi \\ 0 \; for \quad x > \pi \end{cases}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos \lambda x \left[\int_0^\pi \cos \lambda t \, dt\right] d\lambda = \frac{2}{\pi}\int_0^\infty \cos \lambda x \left[\frac{\sin \lambda t}{\lambda}\right]_0^\pi d\lambda = \frac{2}{\pi}\int_0^\infty \frac{\cos \lambda x \sin \lambda \pi}{\lambda}\, d\lambda$$

$$\therefore \int_0^\infty \frac{\cos \lambda x \sin \lambda \pi}{\lambda}\, d\lambda = \frac{\pi}{2} f(x) = \begin{cases} \frac{\pi}{2} \;\; for \quad 0 \leq x \leq \pi \\ 0 \quad for \qquad x > \pi \end{cases}$$

At $x = \pi$ which is a point of discontinuity of f(x) ,the value of the above integral

$$\int_0^\infty \frac{\cos \lambda x \sin \lambda \pi}{\lambda}\, d\lambda = \frac{\pi}{2}\left(\frac{1+0}{2}\right) = \frac{\pi}{4}$$

<u>FOURIER TRANSFORM OR COMPLEX FOURIER TRANSFORM</u>

<u>The Infinite Fourier Transform of f(x)</u> :

The Fourier transform of a function f(x) is given by.

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} f(x).e^{ipx}\, dx$$

The inverse Fourier transform of F(p) is given by.

$$f(x) = F^{-1}\{F(p)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(p).e^{-ipx}\, dp$$

<u>Fourier sine Transform</u>:

The Fourier sine Transform of a function f(x) is given by

$$F_s\{f(x)\} = F_s(p) = \int_0^\infty f(x).\sin px\, dx$$

The inverse Fourier sine Transform of F_s (p) is given by

$$f(x) = F_s^{-1}\{F_s(p)\} = \frac{2}{\pi}\int_0^\infty F_s(p).\sin px\, dp$$

Fourier cosine Transform:

The Fourier cosine Transform of a function f(x) is given by

$$F_c\left\{f(x)\right\} = F_c(p) = \int_0^\infty f(x).\cos px\,dx$$

The inverse Fourier cosine Transform of $F_c(p)$ is given by

$$f(x) = F_c^{-1}\{F_c(p)\} = \frac{2}{\pi}\int_0^\infty F_c(p).\cos px\,dp$$

Problems:

1. Find the Fourier transform of f(x) defined by $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$

Sol. We have $F\left\{f(x)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx} f(x)dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-a} e^{ipx} f(x)dx = \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a} e^{ipx} f(x)\,dx + \int_{a}^{\infty} e^{ipx} f(x)\,dx\right]$$

$$F\left\{f(x)\right\} = \frac{1}{\sqrt{2\pi}}\left[x^2\frac{e^{ipx}}{ip} + \frac{2}{p^2}xe^{ipx} + \frac{2i}{p^3}e^{ipx}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{a^2}{ip}\left(e^{ipa} - e^{-ipa}\right) + \frac{2a}{p^2}\left(e^{ipa} + e^{-ipa}\right) + \frac{2i}{p^3}\left(e^{ipa} - e^{-ipa}\right)\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2a^2\sin ap}{p} + \frac{4a}{p^2}\cos ax - \frac{4}{p^3}\sin ap\right]$$

2. Find the Fourier transform of f(x) defined by $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ *and hence evaluate*

$$\int_0^\infty \frac{\sin p}{p}dp.and \int_{-\infty}^{\infty} \frac{\sin ap.\cos px}{p}dp$$

Sol. We have $F\left\{f(x)\right\} = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$

$$= \int_{-\infty}^{-a} e^{ipx} f(x)dx + \int_{-a}^{a} e^{ipx} f(x)\,dx + \int_{a}^{\infty} e^{ipx} f(x)\,dx = \int_{-a}^{+a}(1)e^{ipx}dx$$

$$=\left[\frac{e^{ipx}}{ip}\right]_{-a}^{+a}=\frac{e^{ipa}-e^{-ipa}}{ip}=\frac{2}{p}.\frac{e^{ipa}-e^{-ipa}}{2i}=\frac{2\sin pa}{p}=F\{f(x)\}=\frac{2\sin pa}{p}=F(p)$$

We know that $F(p)=\int_{-\infty}^{\infty}e^{ipx}f(x)dx$

Then by the inversion formula,

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ipx}.F(p)dp=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ipx}.\frac{2\sin pa}{p}dp$$

$$=\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\frac{2\sin ap}{p}.(\cos px-i\sin px)dp\right]=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap}{p}\cos px\ dp-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap}{p}\sin px\ dp$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p}dp$$ [Since the second integral is an odd]

$$or\ \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p}dp=\begin{cases}1, & |x|<a\\ 0, & |x|>a\end{cases}$$

$$\therefore\int_{-\infty}^{\infty}\frac{\sin ap\cos px}{p}dp=\begin{cases}\pi, & |x|<a\\ 0, & |x|>a\end{cases}$$

If x=0 and a=1, then

$$\int_{-\infty}^{\infty}\frac{\sin p}{p}dp=\pi\ or\ 2\int_{0}^{\infty}\frac{\sin p}{p}dp=\pi\ or\int_{0}^{\infty}\frac{\sin p}{p}dp=\frac{\pi}{2}$$

Note: $\int_{0}^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$

3. Find the Fourier transform of f(x) defined by $f(x)=\begin{cases}1-x^2, & |x|\le 1\\ 0, & |x|>1\end{cases}$

Hence evaluate $(i)\int_{0}^{\infty}\frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}dx$ $(ii)\int_{0}^{\infty}\frac{x\cos x-\sin x}{x^3}dx$

Sol. We have $F\{f(x)\}=\int_{-\infty}^{\infty}e^{ipx}f(x)dx=\int_{-\infty}^{-1}e^{ipx}f(x)dx+\int_{-1}^{1}e^{ipx}f(x)dx+\int_{1}^{\infty}e^{ipx}f(x)dx$

$$=\int_{-1}^{1}(1-x^2).e^{ipx}dx=\left\{\left[\frac{(1-x^2)}{ip}-\frac{(-2x)}{i^2p^2}+\frac{(-2)}{i^3p^3}\right]e^{ipx}\right\}_{x=-1}^{1}=\left(\frac{-2}{p^2}+\frac{2}{ip^3}\right)e^{ip}-\left(\frac{2}{p^2}+\frac{2}{ip^3}\right)e^{-ip}$$

$$=\frac{-2}{p^2}\left(e^{ip}+e^{-ip}\right)+\frac{2}{ip^3}\left(e^{ip}-e^{-ip}\right)=\frac{-4}{p^2}\left(\frac{e^{ip}+e^{-ip}}{2}\right)+\frac{4i}{ip^3}\left(\frac{e^{ip}-e^{-ip}}{2i}\right)=\frac{-4}{p^2}\cos p+\frac{4}{p^3}\sin p$$

$$= \frac{4}{p^3}\left(\sin p - p\cos p\right) = F\left(p\right)$$

Second Part: By inversion formula, we have

i. $f\left(x\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}.F\left(p\right)dp$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}.\frac{4\left(\sin p - p\cos p\right)}{p^3}dp = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Putting $x = \frac{1}{2}$ *in* (1), *we get*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i p/2}.\frac{4\left(\sin p - p\cos p\right)}{p^3}dp = \begin{cases} 1-\frac{1}{4} = \frac{3}{4} \\ 0 \end{cases}$$

$$or \int_{-\infty}^{\infty} \frac{1}{p^3}\left(p\cos p - \sin p\right)e^{-i p/2}dp = \frac{-3\pi}{8}$$

$$or \int_{-\infty}^{\infty} \frac{1}{p^3}\left(p\cos p - \sin p\right)\left(\cos p/2 - i\sin p/2\right)dp = \frac{-3\pi}{8}$$

$$or \int_{-\infty}^{\infty} \frac{p\cos p - \sin p}{p^3}.\cos p/2\, dp = \frac{-3\pi}{8} \left(Equating\ real\ parts\right)$$

$$or\ 2\int_{0}^{\infty} \frac{p\cos p - \sin p}{p^3}.\cos p/2\, dp = \frac{-3\pi}{8}\ [\text{since integral is even}]$$

$$or \int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = \frac{-3\pi}{16}$$

ii. Putting $x = 0$ *in* (1), *we get*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{4}{p^3}\left(\sin p - p\cos p\right)dp = 1 \quad or \int_{-\infty}^{\infty}\frac{\sin p - p\cos p}{p^3}dp = \frac{\pi}{2}$$

$$or\ 2\int_{0}^{\infty}\frac{\sin p - p\cos p}{p^3}dp = \frac{\pi}{2}[\because Integral\ is\ even]\ or \int_{0}^{\infty}\frac{p\cos p - \sin p}{p^3}dp = -\frac{\pi}{4}$$

$$or \int_{0}^{\infty}\frac{x\cos x - \sin x}{x^3}dx = -\frac{\pi}{4}$$

4. **Find the Fourier Transform of f(x)=** $\begin{cases} 0 & if\ x \le a \\ 1\ if\ a < x \le b \\ 0 & if\ x \ge b \end{cases}$

By definition $F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x).e^{ipx}dx = \frac{1}{\sqrt{2\pi}}\int_a^b e^{ipx}dx = \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ipx}}{ip}\right]_a^b$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ibx} - e^{iax}}{ip}\right]$$

5. Find the Fourier Transform of f(x) defined by $f(x) = e^{\frac{-x^2}{2}}, -\infty < x < \infty$ **or,**

Show that the Fourier Transform of $e^{\frac{-x^2}{2}}$ **is reciprocal.**

Sol. We have $F\{f(x)\} = \int_{-\infty}^{\infty} f(x).e^{ipx}dx = \int_{-\infty}^{\infty} e^{\frac{-x^2}{2}}.e^{ipx}dx$

$$= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-ip)^2} e^{-p^2/2}dx = e^{-p^2/2}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-ip)^2}dx$$

Put $\frac{1}{\sqrt{2}}(x-ip) = t\ so\ that\ \frac{1}{2}(x-ip)^2 = t^2\ and\ dx = \sqrt{2}dt$

$$\therefore F\{f(x)\} = e^{-p^2/2}\int_{-\infty}^{\infty} e^{-t^2}\sqrt{2}dt = \sqrt{2}.e^{-p^2/2}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$= \sqrt{2}.e^{-p^2/2}\sqrt{\pi}\left[\because \int_{-\infty}^{\infty} e^{-t^2}dt = \sqrt{\pi}\right] = \sqrt{2\pi}.e^{-p^2/2}$$

6. Find the Fourier Transform of f(x) defined by

$$f(x) = \begin{cases} e^{iqx}, & \alpha < x < \beta \\ 0, & x < \alpha\ and\ x > \beta \end{cases}\ or\ f(x) = \begin{cases} e^{ikx}, & a < x < b \\ 0, & x < a\ and\ x > b \end{cases}$$

Sol. We have $F\{f(x)\} = \int_{-\infty}^{\infty} e^{ipx} f(x)dx$

$$= \int_{-\infty}^{\alpha} e^{ipx} f(x)dx + \int_{\alpha}^{\beta} e^{ipx} f(x)dx + \int_{\beta}^{\infty} e^{ipx} f(x)dx$$

$$= \int_{\alpha}^{\beta} e^{ipx}.e^{iqx} dx = \int_{\alpha}^{\beta} e^{i(p+q)x} dx = \frac{1}{i(p+q)}\left[e^{i(p+q)x}\right]_{\alpha}^{\beta} = \frac{e^{i(p+q)\alpha} - e^{i(p+q)\beta}}{i(p+q)} = F(p)$$

The finite Fourier sine and cosine Transforms:

- The finite Fourier sine transform of f(x) when 0<x<l, is defined as

$$F_s\{f(x)\} = \int_0^l f(x).\sin\left(\frac{n\pi x}{l}\right)dx = F_s(n)$$

Where n is an integer.

The inverse Fourier sine transform of F_s (n) is given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_s(n).\sin\frac{n\pi x}{l}$$

- The finite Fourier cosine transform of f(x), when 0<x<l, is given by

$$F_c\{f(x)\} = \int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx = F_c(n)$$

Where n is an integer.

The inverse Fourier sine transform of F_c (n) is given by

$$f(x) = \frac{2}{l}\sum_{n=1}^{\infty} F_c(n).\cos\left(\frac{n\pi x}{l}\right)$$

$$f(x) = \frac{1}{l}F_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} F_c(n).\cos\left(\frac{n\pi x}{l}\right)$$

PROBLEMS RELATED TO INFINITE FOURIER

SINE AND COSINE TRANSFORMS:

1. **Find the Fourier cosine transform of the function f(x) defined by** $f(x) = \begin{cases} \cos x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Sol. We have $F_c\{f(x)\} = \int_0^{\infty} f(x).\cos px\,dx = \int_0^a f(x)\cos px\,dx + \int_a^{\infty} f(x)\cos px\,dx$

$$= \int_0^a \cos x.\cos px\,dx = \frac{1}{2}\int_0^a 2\cos px.\cos x dx = \frac{1}{2}\int_0^a\left[\cos(p-1)x + \cos(p+1)x\right]dx$$

$$= \frac{1}{2}\left[\frac{1}{(p-1)}\sin(p-1)x + \frac{1}{(p+1)}\sin(p+1)x\right]_0^a = \frac{1}{2}\left[\frac{\sin(p-1)a}{(p-1)} + \frac{\sin(p+1)a}{p+1}\right]$$

2. **Find the Fourier sine transform of f(x) defined by** $f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x \geq a \end{cases}$

Sol. We have $F_s\{f(x)\}=\int_0^\infty f(x).\sin px\,dx$

$$=\int_0^a f(x).\sin px\,dx+\int_a^\infty f(x)\sin px\,dx=\int_0^a \sin x\sin px dx=\frac{1}{2}\int_0^a 2\sin x.\sin px\,dx$$

$$=\frac{1}{2}\int_0^a\left[\cos(1-p)x-\cos(1+p)x\right]dx=\frac{1}{2}\left[\frac{\sin(1-p)x}{1-p}-\frac{\sin(1+p)x}{1+p}\right]_0^a$$

$$=\frac{1}{2}\left[\frac{\sin(1-p)a}{1-p}-\frac{\sin(1+p)a}{1+p}\right]$$

3. Find the Fourier cosine transform of $2e^{-3x}+3e^{-2x}$

We have $F_c\{f(x)\}=\int_0^\infty f(x)\cos px\,dx=\int_0^\infty\left(2e^{-3x}+3e^{-2x}\right)\cos px\,dx$

$$=2\int_0^\infty e^{-3x}\cos px\,dx+3\int_0^\infty e^{-2x}\cos px\,dx$$

$$=2\left[\frac{e^{-3x}}{9+p^2}(-3\cos px+p\sin px)\right]_0^\infty+3\left[\frac{e^{-2x}}{4+p^2}(-2\cos px+p\sin px)\right]_0^\infty$$

$$=-2\times\frac{1}{9+p^2}\times(-3)-3\frac{1}{4+p^2}\times(-2)=\frac{6}{p^2+25}+\frac{6}{p^2+4}$$

4. Find Fourier cosine and sine transforms of $e^{-ax}, a>0$ **and hence deduce the inversion formula (or) deduce the integrals** $i.\ \int_0^\infty\frac{\cos px}{a^2+p^2}dp\quad ii.\ \int_0^\infty\frac{p\sin px}{a^2+p^2}dp$

Sol. Let $f(x)=e^{-ax}$

We have $F_c\{f(x)\}=\int_0^\infty f(x)\cos px\,dx$

$$=\int_0^\infty e^{-ax}.\cos px\,dx=\left[\frac{e^{-ax}}{a^2+p^2}(-a\cos px+p\sin px)\right]_0^\infty$$

$$=-\frac{1}{a^2+p^2}(-a(1)+p(0))=\frac{a}{a^2+p^2}=F_c(p)\text{ and }F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx$$

$$=\int_0^\infty e^{-ax}.\sin px\,dx=\left[\frac{e^{-ax}}{a^2+p^2}(-a\sin px-p\cos px)\right]_0^\infty=-\frac{1}{a^2+p^2}(-a(0)-p(1))=\frac{p}{a^2+p^2}=F_s(p)$$

Deduction: i. Now by the inverse Fourier cosine Transform, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty F_c(p).\cos px\,dp$$

$$\therefore e^{-ax}=\frac{2}{\pi}\int_0^\infty \frac{a}{p^2+a^2}\cos px\,dp=\frac{2a}{\pi}\int_0^\infty \frac{\cos px}{a^2+p^2}dp \text{ or } \int_0^\infty \frac{\cos px}{a^2+p^2}dp=\frac{\pi}{2a}e^{-ax}$$

ii. Now by the inverse Fourier sine transform, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(p).\sin px\,dp. \quad \therefore e^{-ax}=\frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$$

$$\text{or } \int_0^\infty \frac{p\sin px}{a^2+p^2}dp=\frac{\pi}{2}e^{-ax}$$

5. Find the Fourier sine and cosine transform of $2e^{-5x}+5e^{-2x}$

Sol. Let $f(x)=2e^{-5x}+5e^{-2x}$

i. The Fourier sine transform of f(x) is given by

$$F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx=\int_0^\infty \left(2e^{-5x}+5e^{-2x}\right)\sin px\,dx$$

$$=2.\int_0^\infty e^{-5x}.\sin px\,dx+5\int_0^\infty e^{-2x}.\sin px\,dx$$

$$=2.\left[\frac{e^{-5x}}{25+p^2}(-5\sin px-p\cos px)\right]_0^\infty+5\left[\frac{e^{-2x}}{4+p^2}(-2\sin px-p\cos px)\right]_0^\infty$$

$$=-2.\times\frac{1}{25+p^2}(-p)-5\times\frac{1}{4+p^2}(-p)=\frac{2p}{p^2+25}+\frac{5p}{p^2+4}$$

ii. We have $F_c\{f(x)\}=\int_0^\infty f(x)\cos px\,dx=\int_0^\infty \left(2e^{-5x}+5e^{-2x}\right)\cos px\,dx$

$$=2\int_0^\infty e^{-5x}\cos px\,dx+5\int_0^\infty e^{-2x}\cos px\,dx$$

$$=2\left[\frac{e^{-5x}}{25+p^2}(-5\cos px+p\sin px)\right]_0^\infty+5\left[\frac{e^{-2x}}{4+p^2}(-2\cos px+p\sin px)\right]_0^\infty$$

$$=-2\times\frac{1}{25+p^2}\times(-5)-5\frac{1}{4+p^2}\times(-2)=\frac{10}{p^2+25}+\frac{10}{p^2+4}$$

6. Find the Fourier sine Transform of $e^{-|x|}$ and hence evaluate $\int_0^\infty \frac{x\sin mx}{1+x^2}dx$

Sol. Let $f(x)=e^{-|x|}$

We have

$$F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx=\int_0^\infty e^{-|x|}\sin px\,dx$$

$$=\int_0^\infty e^{-x}\sin px\,dx \quad [\because |x|=x \text{ in } (0,\infty)]$$

$$=\left[\frac{e^{-x}}{1+p^2}(-\sin px-p\cos px)\right]_0^\infty=-\frac{1}{1+p^2}(-p)=\frac{p}{1+p^2}=F_s(p)$$

Now by the inverse Fourier sine transform, we have

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(p)\sin px\,dp \;\therefore e^{-|x|}=\frac{2}{\pi}\int_0^\infty \frac{p}{1+p^2}\sin px\,dp$$

Chang x to m on both sides

$$e^{-|m|}=\frac{2}{\pi}\int_0^\infty \frac{p\sin pm}{1+p^2}dp=\frac{2}{\pi}\int_0^\infty \frac{x\sin mx}{1+x^2}dx, \text{where } p \text{ is replaced by } x$$

$$\therefore \int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-|m|}$$

7. Show that the Fourier sine transform of

$$f(x)=\begin{cases} x, & \text{for } 0<x<1 \\ 2-x, & \text{for } 1<x<2 \\ 0, & \text{for } x>2 \end{cases} \quad is \quad \frac{2\sin p.(1-\cos p)}{p^2}$$

Sol. By definition, $F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx$

$$=\int_0^1 f(x).\sin px\,dx+\int_1^2 f(x)\sin px\,dx+\int_2^\infty f(x)\sin px\,dx$$

$$=\int_0^1 x.\sin px\,dx+\int_1^2 (2-x).\sin px\,dx$$

$$=\left[-\frac{x}{p}\cos px+\frac{1}{p^2}\sin px\right]_0^1+\left[\frac{-(2-x)}{p}\cos px+\frac{(-1)}{p^2}\sin px\right]_1^2$$

$$=\frac{-\cos p}{p}+\frac{1}{p^2}\sin p-\frac{1}{p^2}\sin 2p+\frac{\cos p}{p}+\frac{1}{p^2}\sin p$$

$$=\frac{2\sin p-\sin 2p}{p^2}=\frac{2\sin p-2\sin p\cos p}{p^2}=\frac{2\sin p\left(1-\cos p\right)}{p^2}$$

8. Find the Fourier cosine transform of $f\left(x\right)$ defined by $f\left(x\right)=\begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

By definition, $F_c\left\{f\left(x\right)\right\}=\int_0^\infty f\left(x\right)\cos px\,dx$

$$=\int_0^1 f\left(x\right).\cos px\,dx+\int_1^2 f\left(x\right)\cos px\,dx+\int_2^\infty f\left(x\right)\cos px\,dx$$

$$=\int_0^1 x.\cos px\,dx+\int_1^2\left(2-x\right).\cos px\,dx$$

$$=\left[\frac{x}{p}\sin px+\frac{1}{p^2}\cos px\right]_0^1+\left[\frac{\left(2-x\right)}{p}\sin px+\frac{\left(-1\right)}{p^2}\cos px\right]_1^2$$

$$=\frac{\sin p}{p}+\frac{\cos p}{p^2}-\frac{1}{p^2}\cos 2p+\frac{\sin p}{p}+\frac{1}{p^2}\cos p=2\frac{\sin p}{p}+2\frac{\cos p}{p^2}-\frac{1}{p^2}\cos 2p=\frac{2p^2\sin p+2\cos p-\cos 2p}{p^2}$$

9. Find the inverse Fourier cosine transform $f\left(x\right)$ of

$$F_c\left(p\right)=\begin{cases}\frac{1}{2a}\left(a-\frac{p}{2}\right), & when \quad p<2a \\ 0, & when \quad p\geq 2a\end{cases}$$

Sol. From the inverse Fourier cosine transform, we have

$$f\left(x\right)=\frac{2}{\pi}\int_0^\infty F_c\left(p\right).\cos px\,dp$$

$$=\frac{2}{\pi}\left[\int_0^{2a}\frac{1}{2a}\left(a-\frac{p}{2}\right)\cos px\,dp+\int_{2a}^\infty 0.\cos px\,dp\right]$$

$$=\frac{2}{\pi}\times\frac{1}{2a}\int_0^{2a}\left(a-\frac{p}{2}\right)\cos px\,dp=\frac{1}{a\pi}.\left[\frac{a-{}^{p}\!/_{2}}{x}.\sin px-\frac{1}{2x^2}\cos px\right]_{p=0}^{2a}$$

$$=\frac{1}{a\pi}\left[0-\frac{1}{2x^2}\cos 2ax+\frac{1}{2x^2}\right]=\frac{1}{2a\pi x^2}\left(1-\cos 2ax\right)=2\frac{\sin^2 ax}{2a\pi x^2}=\frac{\sin^2 ax}{a\pi x^2}$$

10. Find the Fourier cosine transform of $(a)\quad e^{-ax}\cos ax \quad (b)\quad e^{-ax}\sin ax$

Sol. (a). Let $f(x)=e^{-ax}\cos ax.$ *Then*

$$F_c\{f(x)\}=\int_0^\infty f(x).\cos px\,dx$$

$$=\int_0^\infty a^{-ax}\cos ax\cos px\,dx=\frac{1}{2}\int_0^\infty e^{-ax}.2\cos px.\cos ax\,dx$$

$$=\frac{1}{2}\int_0^\infty e^{-ax}\left[\cos(p+a)x+\cos(p-a)x\right]dx$$

$$=\frac{1}{2}\left[\int_0^\infty e^{-ax}\cos(p+a)x+\int_0^\infty e^{-ax}\cos(p-a)x\,dx\right]$$

$$=\frac{1}{2}\left[\left[\frac{e^{-ax}}{a^2+(p+a)^2}\left(-a\cos(p+a)x+(p+a)\sin(p+a)x\right)\right]_0^\infty+\left[\frac{e^{-ax}}{a^2+(p-a)^2}\left(-a\cos(p-a)x+(p-a)\sin(p-a)x\right)\right]_0^\infty\right]$$

$$=\frac{1}{2}\left[-\frac{1}{a^2+(p+a)^2}(-a.1)-\frac{1}{a^2+(p-a)^2}(-a)1\right]$$

$$=\frac{1}{2}\left[\frac{a}{a^2+(p+a)^2}+\frac{a}{a^2+(p-a)^2}\right]=\frac{a}{2}\left[\frac{a^2+(p-a)^2+a^2+(p+a)^2}{\left[a^2+(p+a)^2\right].\left[a^2+(p-a)^2\right]}\right]$$

$$=\frac{a}{2}\times\frac{2a^2+2\left(a^2+p^2\right)}{\left[a^2+(p+a)^2\right].\left[a^2+(p-a)^2\right]}=\frac{a\left(2a^2+p^2\right)}{\left(a^2+(p+a)^2\right).\left(a^2+(p-a)^2\right)}$$

b. Let $f(x)=e^{-ax}\sin ax$ *Then*

$$F_c\{f(x)\}=\int_0^\infty f(x)\cos px\,dx$$

$$=\int_0^\infty e^{-ax}.\sin ax\cos px\,dx=\frac{1}{2}\int_0^\infty e^{-ax}\left(2\cos px\sin ax\right)dx$$

$$=\frac{1}{2}\int_0^\infty e^{-ax}\left[\sin(p+a)x-\sin(p-a)x\right]dx$$

$$=\frac{1}{2}\left\{\left[\frac{e^{-ax}}{a^2+(p+a)^2}\left(-a\sin(p+a)x-(p+a)\cos(p+a)x\right)\right]_0^\infty-\left[\frac{e^{-ax}}{a^2+(p-a)^2}\left(-a\sin(p-a)x-(p-a)\cos(p-a)x\right)\right]_0^\infty\right\}$$

$$=\frac{1}{2}\left[\frac{p+a}{a^2+(p+a)^2}-\frac{(p-a)}{a^2+(p-a)^2}\right]=\frac{1}{2}\left(\frac{p}{p^2+(p+a)^2}-\frac{p}{p^2+(p-a)^2}\right)+\frac{1}{2}\left(\frac{a}{p^2+(p+a)^2}+\frac{a}{a^2+(p-a)^2}\right)$$

$$=\frac{p}{2}\times\frac{(-4ap)}{\left(p^2+(p+a)^2\right).\left(p^2+(p-a)^2\right)}+\frac{a}{2}\times\frac{2\left(p^2+2a^2\right)}{\left(p^2+(p+a)^2\right)\left(p^2+(p-a)^2\right)}$$

$$=\frac{-2ap^2}{\left(p^2+(p+a)^2\right).\left(p^2+(p-a)^2\right)}+\frac{a\left(p^2+2a^2\right)}{\left(p^2+(p+a)^2\right).\left(p^2+(p-a)^2\right)}$$

Note: (i) $F_s\{x.f(x)\}=-\frac{d}{dp}\{F_c(p)\}$

(ii) $F_c\{x\,f(x)\}=\frac{d}{dp}\{F_s(p)\}$

11. Find the fourier sine transform of $\frac{1}{x}$

Sol:the fourier sine transform of the given function f(x)= $\frac{1}{x}$

$F_s\{f(x)\}=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x).\sin px\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{x}.\sin px\,dx$ put px=t dx=$\frac{dt}{p}$ $=\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{\frac{t}{p}}.\frac{dt}{p}$

$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin t}{t}.dt = \sqrt{\frac{2}{\pi}}.\frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$ $\left(since \int_0^\infty \frac{\sin t}{t}.dt = \frac{\pi}{2}\right)$

12. Find the Fourier sine and cosine transform of xe^{-ax}

Sol. Let $f(x)=e^{-ax}$

Fourier sine Transform:

We know that $F_s\{x\,f(x)\}=\frac{-d}{dp}\{F_c(p)\}=\frac{-d}{dp}\{F_c\{f(x)\}\}$

$$\therefore F_s\{x.e^{-ax}\}=\frac{-d}{dp}\left[F_c\{e^{-ax}\}\right]=\frac{-d}{dp}\left(\frac{a}{p^2+a^2}\right)=(-a)\left(-\frac{1}{\left(p^2+a^2\right)^2}\right).2p=\frac{2ap}{\left(p^2+a^2\right)^2}$$

Fourier cosine Transform:

We know that $F_c\{x.f(x)\}=\frac{d}{dp}\left[F_s(p)\right]=\frac{d}{dp}\left[F_s\{f(x)\}\right]$

$$\therefore F_c\{x.e^{-ax}\}=\frac{d}{dp}\left[F_s\{e^{-ax}\}\right]=\frac{d}{dp}\left(\frac{p}{p^2+a^2}\right)=\frac{\left(p^2+a^2\right).1-p.(2p)}{\left(p^2+a^2\right)^2}=\frac{a^2-p^2}{\left(p^2+a^2\right)^2}$$

13. Find the Fourier sine transform of $\frac{x}{a^2+x^2}$ and Fourier cosine transform of $\frac{1}{a^2+x^2}$

Sol. Fourier sine transforms:

We have $F_s\left\{e^{-ax}\right\}=\frac{p}{a^2+p^2}=F_s\left(p\right)$

The inverse Fourier sine transforms of e^{-ax} is $e^{-ax}=\frac{2}{\pi}\int_0^\infty F_s\left(p\right).\sin px\,dp=\frac{2}{\pi}\int_0^\infty \frac{p}{a^2+p^2}\sin px\,dp$

$$or\ \int_0^\infty \frac{p\sin px}{a^2+p^2}dp=\frac{\pi}{2}e^{-ax}$$

Changing p to x and x to p, we get

$$\int_0^\infty \frac{x}{a^2+x^2}\sin xp\,dx=\frac{\pi}{2}e^{-ap}$$

Hence $F_s\left\{\frac{x}{a^2+x^2}\right\}=\frac{\pi}{2}e^{-ap}$

Fourier cosine Transform:

We have $F_c\left\{e^{-ax}\right\}=\frac{a}{p^2+a^2}=F_c\left(p\right)$

The inverse Fourier cosine transform of e^{-ax} is

$$e^{-ax}=\frac{2}{\pi}\int_0^\infty F_c\left(p\right).\cos px\,dp=\frac{2}{\pi}\int_0^\infty \frac{a}{p^2+a^2}\cos px\,dp\ or\int_0^\infty \frac{1}{p^2+a^2}\cos px\,dp=\frac{\pi}{2a}e^{-ax}$$

Changing p to x and x to p

$$\int_0^\infty \frac{1}{x^2+a^2}\cos xp.dx=\frac{\pi}{2a}.e^{-ap}$$

Hence $F_c\left\{\frac{1}{x^2+a^2}\right\}=\frac{\pi}{2a}.e^{-ap}$

14. Find the Fourier sine and cosine transform of $f\left(x\right)=\frac{e^{-ax}}{x}$ and deduce that

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin sx\,dx=Tan^{-1}\left(\frac{s}{a}\right)-Tan^{-1}\left(\frac{s}{b}\right)$$

Sol. **Fourier Sine Transforms:**

We have $F_s\left\{f(x)\right\}=\int_0^\infty f(x).\sin px\,dx$

$$=\int_0^\infty \frac{e^{-ax}}{x}.\sin px\,dx$$

$$\therefore F_s\left\{f(x)\right\}=\int_0^\infty \frac{e^{-ax}}{x}.\sin px\,dx$$

Differentiation w.r.t 'p', we get

$$\frac{d}{dp}\left[F_s\left\{f(x)\right\}\right]=\frac{d}{dp}\left[\int_0^\infty \frac{e^{-ax}}{x}\sin px\,dx\right]$$

$$=\int_0^\infty \frac{\partial}{\partial p}\left(\frac{e^{-ax}}{x}.\sin px\right)dx=\int_0^\infty \frac{e^{-ax}}{x}.x.\cos px\,dx$$

$$=\int_0^\infty e^{-ax}.\cos px\,dx=\left[\frac{e^{-ax}}{a^2+p^2}\left(-a\cos px+p\sin px\right)\right]_0^\infty$$

$$\frac{d}{dp}\left[F_s\left\{f(x)\right\}\right]=\frac{a}{p^2+a^2}$$

Integrating w.r.t. p

$$F_s\left\{f(x)\right\}=\int_0^\infty \frac{a}{p^2+a^2}dp=Tan^{-1}\left(\frac{p}{a}\right)+c$$

If p=0 then $F_s\left\{f(x)\right\}=0\,and\;c=0$

$$\therefore F_s\left\{f(x)\right\}=Tan^{-1}\left(p/a\right)\;if\;\;p>0$$

$$or\;\;F_s\left\{\frac{e^{-ax}}{x}\right\}=Tan^{-1}\left(p/a\right).if\;\;p>0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Deduction: We know that the Fourier sine transform of f(x) is given by

$$F_s\left\{f(x)\right\}=\int_0^\infty f(x)\sin px\,dx \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Suppose let $f(x)=\dfrac{e^{-ax}-e^{-bx}}{x}$ ……………….(3)

Using (3) in (2), we get

$$\int_0^\infty \frac{e^{-ax}-e^{-bx}}{x}\sin px\,dx=\int_0^\infty \frac{e^{-ax}}{x}.\sin px\,dx-\int_0^\infty \frac{e^{-bx}}{x}.\sin px\,dx$$

$$= F_s\left\{\frac{e^{-ax}}{x}\right\} - F_s\left\{\frac{e^{-bx}}{x}\right\} = Tan^{-1}\left(p/a\right) - Tan^{-1}\left(p/b\right) [\text{using}\,(1)]$$

$$or \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x} \sin sx\, dx = Tan^{-1}\left(s/a\right) - Tan^{-1}\left(s/b\right)$$

Fourier Cosine Transform:

We have $F_c\left\{e^{-ax}\right\} = \dfrac{a}{p^2 + a^2} = F_c(p)$

The inverse Fourier cosine transform of e^{-ax} *is*

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty F_c(p).\cos px\, dp = \frac{2}{\pi}\int_0^\infty \frac{a}{p^2+a^2}\cos px\, dp \quad or \int_0^\infty \frac{1}{p^2+a^2}\cos px\, dp = \frac{\pi}{2a}e^{-ax}$$

Changing p to x and x to p

$$\int_0^\infty \frac{1}{x^2+a^2}\cos xp.dx = \frac{\pi}{2a}.e^{-ap}$$

Hence $F_c\left\{\dfrac{1}{x^2+a^2}\right\} = \dfrac{\pi}{2a}.e^{-ap}$

15. Find the finite Fourier sine & cosine transform of f(x),

defined by f(x)=2x, where $o < x < 2\pi$

Sol. We have $F_s\left\{f(x)\right\} = \int_0^l f(x).\sin\dfrac{(n\pi x)}{l}dx$

$$= \int_0^{2\pi} 2x.\sin\left(\frac{nx}{2}\right)dx = 2\left[-\frac{2}{n}.x.\cos\frac{nx}{2} + \frac{4}{n^2}\sin\frac{nx}{2}\right]_0^{2\pi} = 2\left[-\frac{4\pi}{n}\cos n\pi + \frac{4}{n^2}\sin n\pi\right] = \frac{-8\pi}{n}\cos n\pi$$

$$= \frac{8\pi}{n}(-1)^{n+1} = F_s(n)$$

$$Also\, F_c\left\{f(x)\right\} = \int_0^l f(x).\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \int_0^{2\pi} 2x.\cos\left(\frac{nx}{2}\right)dx = 2\left[\frac{2}{n}.x.\sin\frac{nx}{2} + \frac{4}{n^2}\cos\frac{nx}{2}\right]_0^{2\pi}$$

$$=2\left[\frac{4}{n}\sin n\pi+\frac{4}{n^2}\cos n\pi-\frac{4}{n^2}\right]=2\times\frac{4}{n^2}\left(\cos n\pi-1\right)=\frac{8}{n^2}\left[\left(-1\right)^n-1\right]=F_c\left(n\right)$$

16. Find the finite Fourier sine transform of f(x), defined by f(x) =2x, where $o<x<4$

Sol:-The finite fourier sine transform of f(x) in 0<x<l

$$F_s\left\{f\left(x\right)\right\}=\int_0^l f\left(x\right).\sin\frac{\left(n\pi x\right)}{l}dx$$ Here f(x) = 2x and $l=4$

$$=\int_0^4 2x.\sin\left(\frac{n\pi x}{4}\right)dx=\left[2x\left(-\frac{4}{n\pi}\right)\cos\frac{n\pi x}{4}\right]_0^4+\int_0^4 2\left(\frac{4}{n\pi}\right)\cos\frac{n\pi x}{4}dx$$

$$=\left[-\frac{8}{n\pi}x\cos\frac{n\pi x}{4}\right]_0^4+\frac{32}{n^2\pi^2}\left[\sin\frac{n\pi x}{4}\right]_0^4=-\frac{32}{n\pi}(\cos n\pi-0)+0=-\frac{32}{n\pi}(-1)^n$$

17. Find the inverse finite sine transform f(x) if $F_s\left(n\right)=\dfrac{1-\cos n\pi}{n^2\pi^2}\,where\,0<x<\pi$

Sol. From the inverse finite sine transform, we have

$$f\left(x\right)=\frac{2}{l}\sum_{n=1}^{\infty}F_s\left(n\right).\sin\left(\frac{n\pi x}{l}\right)=\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2\pi^2}\right)\sin nx=\frac{2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\pi}{n^2}\right)\sin nx$$

18. Find the inverse finite cosine transform f(x), if

$$F_c\left(n\right)=\frac{\cos\left(\frac{2n\pi}{3}\right)}{\left(2n+1\right)^2},where\,0<x<4$$

Sol. From the inverse finite cosine transform, we have

$$f\left(x\right)=\frac{1}{l}F_c\left(0\right)+\frac{2}{l}\sum_{n=1}^{\infty}F_c\left(n\right)\cos\left(\frac{n\pi x}{l}\right)$$

$$=\frac{1}{4}.1+\frac{2}{4}\sum_{n=1}^{\infty}F_c\left(n\right).\cos\left(\frac{n\pi x}{l}\right)=\frac{1}{4}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{2n\pi}{3}\right)}{\left(2n+1\right)^2}\cos\left(\frac{n\pi x}{4}\right)$$

UNIT – III

ANALYTIC FUNCTIONS

Introduction: Complex analysis is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, in physics, thermodynamics, and also in engineering fields such as aerospace, mechanical and electrical engineering. Complex analysis is widely applicable to two dimensional problems in physics. In this unit we discuss about limit, differentiation and continuity of complex function and analyticity of a function and also complex integration.

We are familiar with the concepts of limit, continuity,differentiation and integration of function of real variable. Similar concepts can be defined with reference to complex variables also and their study constitutes "Complex analysis". A basic understanding of complex variable theory will be useful in diverse branches of science and engineering.

Definitions:

Complex number: A number which is in the form of $z = x + iy$ where $x, y \in R$ and $i^2 = -1$ is called complex number. Here x is real part and y is imaginary part of z.

(or)

A complex number z is defined as the ordered pair (x, y) of real numbers. i.e., $z = (x, y)$

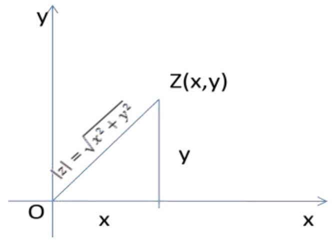
Set of complex numbers: The complex number set is denoted by $\mathbb{C}$ and $\mathbb{C} = \{z \,/\, z = x+iy,\ x,y \epsilon \mathbb{R},\ i^2 = -1\}$

$\mathbb{C} = \{(x,y),\ x,y \epsilon \mathbb{R},\ i^2 = -1\}$

Argand plane: We have seen that complex numbers are represented by points $(x, y) \epsilon\ \mathbb{R}^2$ and conversely. After this representation $\mathbb{R}^2$ is called the Argand plane where $(x, y) = x + iy$. After this representation the x and y axes are called real and imaginary axes.

Modulus of a complex number: The modulus or absolute value of complex number z is denoted by $|z|$ and it is defined as its distance from the origin.

i.e.,$|z| = \sqrt{x^2 + y^2}$



Now $x \leq |x| \leq \sqrt{x^2+y^2}$, $y \leq |y| \leq \sqrt{x^2+y^2}$

i.e., Rez$\leq |z|$ i.e.,Img z $\leq |z|$

Conjugate of a complex number: The conjugate of a complex number$z = x + iy$is denoted by $\bar{z}$ and it is defined as the mirror image of z in the real axis.

i.e., $\bar{z}$ = x – iy [i.e., $\bar{z}$ = (x,-y)]

Properties of conjugate:

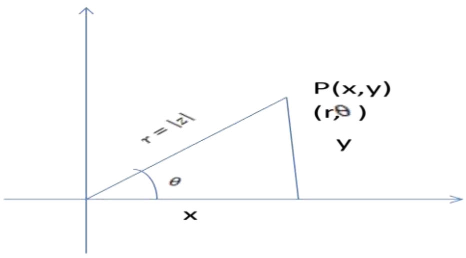
- $\bar{\bar{z}}$ = z, $\forall\ z\epsilon\mathbb{C}$
- $\bar{z} = z \iff z\ is\ real$
- $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$
- $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
- z +$\bar{z}$ = 2 Rez $\Rightarrow$ Rez = $\frac{z+\bar{z}}{2}$
- z - $\bar{z}$ = 2 Imgz $\Rightarrow$ Imgz = $\frac{z-\bar{z}}{2i}$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$, provided $z_2 \neq 0$

Properties of modulus:

- $|z| \geq 0$ i.e., $|z|$ is always non-negative
- $|z| = |\bar{z}| = |-z| = |\overline{-z}|$ also Rez $\leq |z|$, Imgz $\leq |z|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, where $z_2 \neq 0$
- $|z|^2 = z\,\bar{z}$
- $|z_1 + z_2| \leq |z_1| + |z_2|$
- $\big||z_1| - |z_2|\big| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

The Polar form or Exponential form of complex number:

Let $z = x + iy$ or $z = (x, y)$ be complex number



Here $\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\therefore z = x + iy = r\cos\theta + i\,r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$Z = re^{i\theta}$, which is a complex number in polar form

Here $r = |z|$ and $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

(r, θ) are called polar coordinates of a point P

- Here θ is called the argument or amplitude of z and denoted by $arg(z)$ or $amp(z)$

i.e.,$arg z = tan^{-1}\frac{y}{x}$

- The Specific value of $arg z$, satisfying $-\pi < arg z < \pi$ is called the principle value of $arg z$
- For any two complex numbers z_1, z_2 we have

$$arg(z_1.z_2) = arg z_1 + arg\, z_2$$

$$arg\left(\frac{z_1}{z_2}\right) = \frac{arg z_1}{arg z_2}$$

- $|z - z_0| = r$ represents a circle with centre at z_0 and radius r

Let $z = (x, y)$ and $z_0 = (a, b)$

$|z - z_0| = \sqrt{(x-a)^2 + (y-b)^2} = r$

$$\Rightarrow (x-a)^2 + (y-b)^2 = r^2$$

- $|z - z_0| = r \Leftrightarrow z - z_0 = \mathrm{r}e^{i\theta}, 0 \leq \theta \leq 2\pi$

  $\Leftrightarrow z = z_0 + \mathrm{r}e^{i\theta}, 0 \leq \theta \leq 2\pi$
- $|z| = r$ represents a circle with centre at origin and radius r
- $|z| = r \Leftrightarrow \mathrm{Z} = \mathrm{r}e^{i\theta}, 0 \leq \theta \leq 2\pi$

Neighbourhood (or) δ – Disc around $= z_0$:

Let $z_0 \epsilon \mathbb{C}$ and $\delta > 0$

$N_\delta(z_0) = \{z \epsilon \mathbb{C} | z - z_0| < \delta\}$ is called the δ – neighbourhood of z_0

Deleted δ – neighbourhood of z_0:

$\boldsymbol{N_\delta}^*(\boldsymbol{z_0}) = N_\delta(z_0) - \{\boldsymbol{z_0}\}$

$= \{z \epsilon \mathbb{C} / 0 < |z - z_0| < \delta\}$

It is known as deleted δ – neighbourhood of z_0.

Pathwise connected: A non-empty subset 'S' of $\mathbb{C}$ is said to be pathwise connected or arcwise connected, if every pair of points in 'S' can joined by a polygond arc which is entirely in 'S

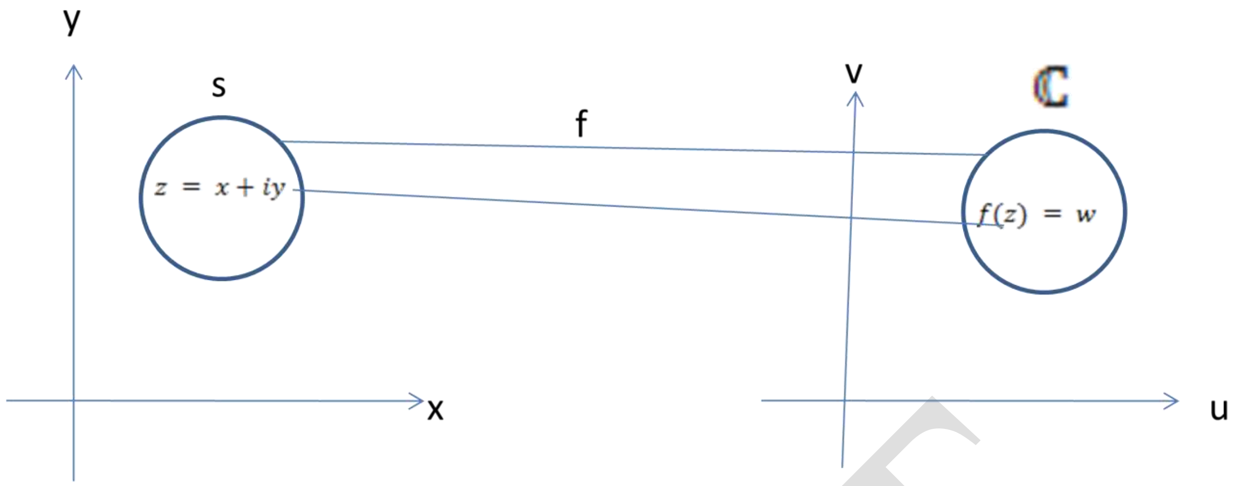
i.e., for each pair of points in'S' there exists a path joining than which entirely lies inside 'S'.

Domain:A non-empty open connected set in $\mathbb{C}$ is said to be a domain.

Function of a complex variable: Let 'S' be a non-empty subset of the argandplane $\mathbb{C}$. A function $f: S \to \mathbb{C}$ is a rule which assigns a unique value $f(z) \epsilon \mathbb{C}$ for each $z \epsilon$ S, then we write $f(z) = w, z \epsilon S$ and we say that 'f' is a complex valued function at complex variable z.

(or)

Let S $\subseteq \mathbb{C}$, a rule f:S$\to \mathbb{C}$ is called complex function if for every $z \epsilon$ S, there exist a unique image $f(z) \epsilon \mathbb{C}$, we write it as $f(z) = w$, for $z \epsilon$ S



Range: The set $\{f(z) \,/ z \in S\}$ is called the range of 'f'

$f(z)$can be written as $w = f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$

Here $u(x, y), v(x, y)$ are real valued functions of x, y

Definition of limit of a complex function: Let $f(z)$ be a complex function, a complex number $l\epsilon\mathbb{C}$ is said to be a limit of a function $f(z)$ as z tends to z_0. If for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$

Symbolically we write $\lim_{z \to z_0} f(z) = l$

Continuity of complex function:A function $f(z)$ is said to be continuous at $z = z_0$

If $\lim_{z \to z_0} f(z) = f(z_0)$

Derivative of $f(z)$: Let $f(z)$ be a given function defined on a nbd of z_0 then $f(z)$ is said to be differentiable at z_0 if $\lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ exists and it is denoted by $f'(z_0)$

i.e.,$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$

Taking $z - z_0 = \Delta z$

$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z)-f(z_0)}{z-z_0}$

Analytic function:A function $f(z)$ is said to be analytic at a point z_0, if $f(z)$ is differentiable at every point z in the ϵ - neighbourhood of z_0.

i.e., $f'(z)$ exist for all z such that $|z - z_0| < \epsilon$, where $\epsilon > 0$ then $f(z)$ is said to be analytic at z_0.

Note: $f(z)$ is analytic at z_0 means

(i) $f'(z_0)$ exists

(ii) $f'(z)$ exist at every point z in a neighbourhood of z_0.

Definition: Let D be a domain of complex numbers, if $f(z)$ is analytic at every $z \epsilon D$, then $f(z)$ is said to be analytic in the domain D.

Definition: If $f(z)$ is analytic at every point z on the complex plane then $f(z)$ is said to be an entire function.

Properties of analytic function:

- If $f(z)$ and $g(z)$ are analytic then $f \pm g, f.g, \frac{f}{g} (g \neq 0)$ are also analytic function.
- Analytic function of an analytic function is analytic
- An entire function of an entire function is entire
- Derivative of an analytic function is itself analytic

Cauchy – Riemann (C-R) Equations:

C-R equations are used to test the analyticity of a complex function.

Statement: The necessary and sufficient condition for the derivative of the function $f(z) = u(x,y) + iv(x,y)$ to exist for all values of z in domain $\mathbb{R}$ are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in $\mathbb{R}$

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

These two are called C-R equations.

Note: The converse of above theorem is need not be true.

i.e., even though C-R equations are satisfied by $f(z)$ but $f(z)$ may not be differentiable.

Eg: $f(z) = \sqrt{|xy|}$ satisfies C-R equations at (0,0) but it is not differential at (0,0)

Laplace operator: The Laplace operator is denoted by ∇^2 and defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\Rightarrow \nabla^2 \emptyset = \frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2}$$

Result: If $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then u and v satisfy laplace equation.

i.e.,$\nabla^2 u = 0$ *and* $\nabla^2 v = 0$

i.e., $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

and$u$ and v have continuous second order partial derivatives in D.

Harmonic function: The function which satisfy the Laplace equation is called harmonic function.

i.e., funtion $\emptyset$ is said to be Harmonic if $\nabla^2 \emptyset = 0$

i.e.,$\frac{\partial^2 \emptyset}{\partial x^2} + \frac{\partial^2 \emptyset}{\partial y^2} = 0$

Note: If$f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, the u and v satisfy the Laplacce equation

i.e., $\nabla^2 u = 0$ *and* $\nabla^2 v = 0$ and we have continuous second order partial derivatives in D.

Conjugate Harmonic function: Two harmonic funtions$u$ and v are said to be harmonic conjugate to each other if

(i) u and v satisfy the C-R equations

(ii) u and v are real and imaginary parts of analytic function $f(z)$

i.e.,$f(z) = u + iv$

Polar form of C-R equations: If$f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ and $f(z)$ is derivable at $z_0 = r_0 e^{i\theta_0}$ then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Problems:

1. Show that $f(z) = xy + iy$ is everywhere continuous but it is not analytic.

Sol. To prove f is continuous it is enough to prove that $\lim_{z\to z_0} f(z) = f(z_0)$

Let z_0 is any point in the domain

Now $\lim_{z\to z_0} f(z) = \lim_{Z\to z_0} x_0y_0 + iy_0$

Now f(z_0) = $x_0y_0 + iy_0$

$\therefore \lim_{z\to z_0} f(z) =$f($z_0$)

Therefore f is continuous every where

Verification of Analyticity of $f(z)$:

Given $f(z) = xy + iy = u + iv$

$\Longrightarrow u = xy, v = y$

Now $\frac{\partial u}{\partial x}$ = y,$\frac{\partial u}{\partial y} = x$,$\frac{\partial v}{\partial x} = 0$,$\frac{\partial v}{\partial y} = 1$

Clearly $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

Here $f(z)$ is not satisfying the C-R equations

Therefore $f(z)$ is not analytic.

2.Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane?

Sol. Given$f(z) = z + 2\bar{z} = (x + iy) + 2(x - iy) = 3x - iy$

But $f(z) = u + iv$

Therefore $u = 3x$ and $v = -y$

$\frac{\partial u}{\partial x}$ = 3,$\frac{\partial u}{\partial y} = 0$,$\frac{\partial v}{\partial x} = 0$,$\frac{\partial v}{\partial y} = -1$

Therefore $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

C-R equations are not satisfied.

Therefore $f(z)$ is not analytic anywhere.

3.Prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Real\ f(z)|^2 = 2|f'(z)|^2$ where $w = f(z)$ is analytic?

Sol: Given $f(z)$ is analytic

$f(z) = u + iv$

Real part of $f(z) = u$

$|Real\ f(z)| = |u| = u \Longrightarrow |Real\ f(z)|^2 = u^2$

Now L.H.S = $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|Real\ f(z)|^2$

$= \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) u^2$

$= \frac{\partial^2 u^2}{\partial x^2}+\frac{\partial^2 u^2}{\partial y^2}$ ....(1)

Now $\frac{\partial}{\partial x}(u^2) = 2u\frac{\partial u}{\partial x}$

$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(u^2)\right] = \frac{\partial}{\partial x}\left[2u\,\frac{\partial u}{\partial x}\right] = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}\right]$ ....(2)

$\frac{\partial^2}{\partial y^2}(u^2) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y}(u^2)\right] = \frac{\partial}{\partial y}\left[2u\,\frac{\partial u}{\partial y}\right] = 2\left[\left(\frac{\partial u}{\partial y}\right)^2 + u\frac{\partial^2 u}{\partial y^2}\right]$ ....(3)

Substitute equation (2) and (3) in (1)

Then L.H.S = $2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

Since $f(z) = u + iv$ is analytic

$u$is a real part of analytic function $f(z)$

Therefore u is Harmonic function

i.e., u satisfies Laplace equation$\Longrightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$

Therefore L.H.S = $2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

Now R.H.S = $2|f'(z)|^2$

And $f(z) = u + iv \Longrightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

Since $f(z)$ is analytic $\Longrightarrow$ it will satisfy C-R equations

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Therefore $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$$\Longrightarrow |f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$$

$$\Longrightarrow |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Therefore R.H.S = $2\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$

Therefore L.H.S = R.H.S

4. Show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$, where $f(z)$ is an analytic function?

Sol: Let $z = x + iy, \bar{z} = x - iy$

We know that $z + \bar{z} = 2x \Longrightarrow x = \frac{z+\bar{z}}{2}$

$$z - \bar{z} = 2iy \Longrightarrow y = \frac{z-\bar{z}}{2i} = -\frac{i}{2}(z - \bar{z})$$

Let $f = f(x,y) \Longrightarrow f(z, \bar{z})$

Now $\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial z}\right) + \frac{\partial f}{\partial y}\left(\frac{\partial y}{\partial z}\right) = \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{-i}{2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)f$

$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{i}{2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)f$

$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f$

$\Longrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ …………….(1)

Hence $(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})log|f'(z)| = 4\frac{\partial^2}{\partial z\partial \bar{z}}\ log|f'(z)|$ [from equation (1)]

$= 4\frac{\partial^2}{\partial z\partial \bar{z}}\ .\frac{1}{2}.\ log|f'(z)|^2$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}log\left(f'(z)\overline{f'(z)}\right)$$

$$= 2\frac{\partial^2}{\partial z\partial \bar{z}}[\log f'(z)+\log f'(\bar{z})]$$

$$= 2\left[\frac{\partial}{\partial z}\frac{f''(\bar{z})}{f'(\bar{z})}+\frac{\partial}{\partial \bar{z}}\frac{f''(z)}{f'(z)}\right]$$

$$= 2\ (0+0)\ = 0$$

5. Show that the function $u(x,y) = x^3-3xy^2$ is harmonic and find its harmonic conjugate $v(x,y)$ and the analytic function $f(z) = u+iv$?

Sol: Given $u(x,y) = x^3-3xy^2$

$\frac{\partial u}{\partial x} = 3x^2-3y^2$ and $\frac{\partial u}{\partial y} = -6xy$

$\frac{\partial^2 u}{\partial x^2} = 6x$ and $\frac{\partial^2 u}{\partial y^2} = -6x$

$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$

Therefore u is Harmonic function.

Milne – Thomson's method: Given $u(x,y) = x^3-3xy^2 \Longrightarrow \frac{\partial u}{\partial x} = 3x^2-3y^2$ and

$$\frac{\partial u}{\partial y} = -6xy$$

Let $v(x,y)$ be the harmonic conjugate of u

Let $f(z) = u+iv$

Differentiate with respect to x

$$f'(z) = \frac{\partial u}{\partial x}+i\ \frac{\partial v}{\partial x}$$

$= \frac{\partial u}{\partial x} - i\ \frac{\partial u}{\partial y}$ (from C-R equations , we have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$)

$= (3x^2 - 3y^2) - i\,(-6xy)$

$= (3x^2 - 3y^2) + i\,(6xy)$

Now replace x by z and y by 0

$f'(z) = 3z^2$

Integrate on both sides,

$f(z) = z^3 + c$

$= (x + iy)^3 + c$

$= x^3 - iy^3 + 3x^2(iy) - 3xy^2 + c$

$f(z) = (x^3 - 3xy^2) + i\,(3x^2y - y^3) + c$

$f(z) = u + iv$

Therefore $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

Hence v is the Harmonic conjugate of u.

Constuction of analytic function whose real (or) imaginary part is known:

Let $u(x,y)$ be a harmonic function then there exists a harmonic conjugate $v(x,y)$ and $u(x,y)$ such that $f(z) = u + iv$ is analytic

Problems:

1.Find most general analytic (regular) function whose real part is $u = e^x[(x^2 - y^2)\cos y - 2xy\sin y]$

Sol: Let $f(z) = u + iv$ be analytic function

Differentiate with respect to x,

$f'(z) = \frac{\partial u}{\partial x} + i\ \frac{\partial v}{\partial x}$

$= \frac{\partial u}{\partial x} - i\ \frac{\partial u}{\partial y}$ (from C-R equations , *we have* $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$)

$$\frac{\partial u}{\partial x} = e^x[(x^2 - y^2)\cos y - 2xy\sin y] + e^x[2x\cos y - 2y\sin y]$$

$$\frac{\partial u}{\partial y} = e^x[-2y\cos y + (x^2 - y^2)(-\sin y) - 2x\sin y - 2xy\cos y]$$

$$\therefore \quad f'(z) = e^x[(x^2 - y^2)\cos y - 2xy\sin y + 2x\cos y - 2y\sin y]$$
$$- \; i\, e^x[-2y\cos y + (y^2 - x^2)\sin y - 2x\sin y - 2xy\cos y]$$

By Milne's Thomson method, replace x by z and y by 0

Hence $\quad f'(z) = e^z[z^2 + 2z]$

Now integrate on both sides,

$f(z) \; = e^z z^2 + \; c$

$$= e^{x+iy}(x + iy)^2 = e^x e^{iy}[(x^2 - y^2) + i2xy]$$

$$= e^x(\cos y + i\sin y)[(x^2 - y^2) + i2xy]$$

$$= e^x[(x^2 - y^2)\cos y - 2xy\sin y] + ie^x[(x^2 - y^2)\sin y + 2xy\cos y]$$

$f(z) \; = \; u + iv$

Where $u = e^x[(x^2 - y^2)\cos y - 2xy\sin y]$ and $v = e^x[(x^2 - y^2)\sin y + 2xy\cos y]$

Therefore v is harmonic conjugate of u

2.Find the analytic function $f(z) \; = \; u + iv$ if $u \; = \; a(1 + \cos\theta)$?

Sol: Given $u \; = \; a(1 + \cos\theta)$

Differentiate with respect to $\theta \; and \; r$, we get

$\frac{\partial u}{\partial \theta} = \; u_\theta \; = -a\sin\theta, \frac{\partial u}{\partial r} = \; \; u_r \; = 0$

The Cauchy-Riemann equations in polar coordinates are $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$$\Longrightarrow r\frac{\partial v}{\partial r} = \; -\frac{\partial u}{\partial \theta} = \; a\sin\theta$$

Therefore $\frac{\partial v}{\partial r} \; = \frac{1}{r}(a\sin\theta)$

Integrating with respect to r,

$v(r,\theta) = a\sin\theta . \log r + c(\theta)$ ....(1)

Differentiating (1) w.r.t. 'θ', we get

$\frac{\partial v}{\partial\theta} = a\cos\theta\,.\log r + \frac{dc}{d\theta} = r\frac{\partial u}{\partial r} = r.0 \Longrightarrow \frac{dc}{d\theta} = -a\cos\theta\,.\log r$

Again integrating, we get

$c(\theta) = a\sin\theta\log r + c_1$, Where c_1 is a constant.

Substituting $c(\theta)$ in equation (1), we get

$$v(r,\theta) = a\sin\theta . \log r + a\sin\theta\log r + c_1 = 2a\sin\theta\log r + c_1$$

Therefore $f(z) = u + iv = a(1 + \cos\theta + 2\sin\theta\log r) + c_1$

3.If $f(z) = u + iv$ is an analytic function of z and if $u - v = e^x(\cos y - \sin y)$ then find $f(z)$ in terms of z?

Sol: Given $u - v = e^x(\cos y - \sin y)$ ....(1)

Differentiate equation (1) partially w.r.to x

$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$ ....(2)

Again differentiate equation (1) partially w.r.to y

$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x(-\cos y - \sin y) = -e^x(\cos y + \sin y)$ ....(3)

Since $f(z)$ is analytic

Therefore it satisfies C-R equations

i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

equation (3) $\Longrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = e^x(\cos y + \sin y)$ ....(4)

equation (2) + equation (4) $\Longrightarrow \frac{\partial u}{\partial x} = e^x\cos y$

equation (4) – equation (2) $\Longrightarrow \frac{\partial v}{\partial x} = e^x\sin y$

Now $f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y$

$= e^x\,(\cos y + i \sin y) = e^x e^{iy} = e^{x+iy} = e^z$ by integrating we get $f(z) = e^z + c$

COMPLEX INTEGRATION

Introduction: Here we discuss the idea of line integral of a complex valued function $f(z)$ of a complex variable z in a simple way. It is intresting to note that some definite integrals involving real variables can be evaluated simply using the integral theory of complex variables and also we discuss Cauchy's integral theorem and their applications.

Piecewise continuous : real valued function 'f' is said to be piecewise continuous on $[a, b]$, if $[a, b]$ can be divided into a finite number of subintervals in which the function is continuous.

Continuous Arc: A set of points (x, y) , $x = x(t), y = y(t)(a \leq t \leq b)$where $x(t)$, $y(t)$continuous functions of the real variable are't' is called a continuous arc.

Path:A continuous complex valued function $'\gamma'$ defined on $[a, b]$ is called a path (or) arc in the argand plane

Where $\gamma(t) = x(t) + i\,y(t), a \leq t \leq b$

Note:A path is closed if $\gamma(a) = \gamma(b)$

Simple Path (Zordan Arc):A path is said to be simple if it does not intersect itself

i.e., $\gamma(t_1) \neq \gamma(t_2)$ for any $t_1, t_2 \epsilon (a, b)$

Smooth Path: The path $\gamma(t) = x(t) + i\,y(t)$, $t\epsilon(a, b)$ is said to be smooth, if $x'(t), y'(t)$ are continuous and donot vanish simultaneously for any value of 't'.

Piecewise smooth:A path γ is said to be piecewise smooth if thereexists a partition 'P' of $[a, b]$ thereexists $a = t_1 < t_2 < \cdots \ldots \ldots < t_{n-1} < t_n = b$ and γ is smooth on each subinterval $[t_{i-1}, t_i], 1 \leq i \leq n$.

Note: For a piecewise smooth $\gamma'(t)$ exist at $t_{0,} t_{1,\ldots\ldots\ldots}\ t_n$also at $t_{0,} t_{1,\ldots\ldots\ldots}\ t_n$ the right and left derivative exist but may not be equal at these points, we define $\gamma(t_i)$ = 0, $1 \leq i \leq n$

Contour: A piecewise smooth curve is called contour. If a contour is closed and does not intersect itself, it is called a closed contour.

Note: The length of the contour is sum of lengths of the smooth arcs constituting the contour.

Contour integration: Let $f(z)$ be a piecewise continuous function defined on a contour $\gamma(t) = x(t) + i\,y(t),\ a \leq t \leq b$ then the integral of $f(z)$ along $\gamma(t)$ is define by $\int_\gamma f(z)dz = \int_a^b f[\gamma(t)].\gamma'(t)dt$

This integral is called a contour (or) complex integral

Note: Re$\int_\gamma f(z)dz \neq \int_\gamma Re\, f(z)dz$

Line integral: Let $f(z)$ be a function of complex variable defined in a domain D. Let C be an arc in the domain joining from $z = \alpha$to $z = \beta$. Let C be defined by $x = x(t),\ y = y(t),\quad a \leq t \leq b$

Where $\alpha = x(a) + iy(a)$and $\beta = x(b) + iy(b)$.

Let $x(t), y(t)$ be having continuous first order derivatives in $[a, b]$. We define

$$\oint f(z)dz = \int_a^b f[x(t) + iy(t)][x(t) + iy(t)]dt$$

Problems:

1. **Evaluate $\int(2y + x^2)dx + (3x - y)dy$ along the parabola $x = 2t, y = t^2 + 3$ joining $(0, 3)$ and $(2, 4)$.**

Sol: At $x = 0, y = 3, t = 0\ and\ at\ x = 2, y = 4, t = 1$

Substituting for x and y in terms of t, we get

$$I = \int_{t=0}^{1}[2(t^2 + 3) + 4t^2]\,2dt + \int_{t=0}^{1}[6t - t^2 - 3]2tdt$$

$$= \int_0^1 (24t^2 - 2t^3 - 6t + 12)dt$$

$$= \left[\frac{24t^3}{3} - \frac{2t^4}{4} - \frac{6t^2}{2} + 12t\right]_0^1 = 8 + 12 - \frac{1}{2} - 3 = \frac{33}{2}.$$

2. **Evaluate $\oint(x + y)dx + x^2ydy$ along $y = 3x$ between $(0, 0)$ and $(3, a)$?**

Sol: Let I denote the given integral

Since $y = 3x \Longrightarrow dy = 3dx$

Substituting for y and dy in terms of x, we have

$$I = \int_0^3 (x+3x)dx + x^2(3x)(3dx) = \int_0^3 (4x+9x^3)dx = \left(4.\frac{x^2}{2} + 9.\frac{x^4}{4}\right)_0^3$$

$$= 2(9) + \frac{9}{4}(81)$$

$$= 18 + \frac{729}{4} = \frac{801}{4}$$

3. Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the paths $(i) y = x$ $(ii) y = x^2$

Sol: (i) Along OB whose equation is $y = x \Longrightarrow dy = dx$ and x varies from 0 to 1

Therefore $\int_0^{1+i}(x^2 - iy)dz = \int_{(0,0)}^{(1,1)}(x^2 - iy)(dx + idy)$

Therefore $\int_{OB}(x^2 - iy)dz = \int_{x=0}^{1}(x^2 - ix)(dx + idx)$

$$= (1+i)\int_0^1 (x^2 - ix)dx = (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$= (1+i) \left[\frac{1}{3} - \frac{i}{2}\right]$

(ii)Along the parabola whose equation is $y = x^2 \Longrightarrow dy = 2xdx$

Now $\int_0^{1+i}(x^2 - iy)dz = \int_{(0,0)}^{(1,1)}(x^2 - iy)(dx + idy)$

Therefore $\int_{Oc}(x^2 - iy)dz = \int_{x=0}^{1}(x^2 - ix^2)(dx + i2xdx)$

$$= (1-i)\int_{x=o}^{1} x^2(1+2ix)dx$$

$$= (1-i)\int_{x=o}^{1} (x^2 + 2ix^3)dx$$

$$= (1-i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right]_0^1 = (1-i)\left[\frac{1}{3} + \frac{i}{2}\right]$$

4. Evaluate $\int_{1-i}^{2+i}(2x+1+iy)dz$ along the straight line joining $(1,-i)$ and $(2,i)$?

Sol: We have $z = x + iy \Rightarrow dz = dx + idy$

Equation of the line joining the two points $(1,-1)$ and $(2,1)$ is

$$y + 1 = \frac{1-(-1)}{2-1}(x-1)$$

i.e., $y + 1 = 2(x+1)$ or $y = 2x - 3$

Therefore $z = x + iy = x + i(2x-3) = (1+2i)x - 3i$

$\Rightarrow dz = (1+2i)dx$

Also x varies from 1to 2.

Hence $\int_{1-i}^{2+i}(2x+1+iy)dz = \int_1^2[2x+1+i(2x-3)](1+2i)dx$

$$= (1+2i)\int_1^2 [2(1+i)x + (1-3i)dx]$$

$$= (1+2i)[(1+i)x^2 + (1-3i)x]_1^2$$

$$= (1+2i)[(1+i)4 + (1-3i)2 - (1+i) - (1-3i)]$$

$$= (1+2i)(4) = 4 + 8i$$

The Cauchy-Goursat Theorem: If a function $f(z)$ is analytic at all points interior to and on a simple closed curve C, then $\oint f(z)dz = 0$.

This is called Cauchy-Goursat theorem.

Cauchy's (Integral) Theorem: Let $f(z) = u(x,y) + iv(x,y)$ be analytic on and within a simple closed contour c and let $f'(z)$ be continuous there. Then

$$\oint f(z)dz = 0.$$

Proof: We have $f(z) = u(x,y) + iv(x,y)$ and $z = x + iy \Rightarrow dz = dx + idy$

Therefore $f(z)dz = (u + iv)(dx + idy) = (udx - vdy) + i(vdx + udy)$

Hence $\oint f(z)dz = \oint(udx - vdy) + i\oint(vdx + udy)$

$$= \iint_R \left[-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dxdy + i \iint_R \left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] dxdy \ldots\ldots\ldots (1)$$

Since $f'(z)$ is continuous, the four partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous in the region R enclosed by C. Hence we can apply Green's theorem.

Using Green's theorem in plane, assuming that R is the region bounded by C.

It is given that $f(z) = u + iv$ is analytic on and within c.

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ……………………………(2)

Therefore Using (2) in (1), we have

$$\oint f(z)dz = \iint_R 0dxdy + i \iint_R 0dxdy = 0$$

Hence the theorem follows.

Simple connected domain: A domain D is said to be simply connected if every simple closed curve that is in D can be shrink to a point without leaving the domain.

(or)

A simply connected domain is a domain without holes

Note: Every disc is simple connected domain

Eg: $A = \{z\epsilon\mathbb{C}/|z| < 1\}$, Disc with centre$(0,0)$ and radius $r = 1$

Multiply connected domain: A domain D is said to be multiply connected if it is not simply connected.

(or)

A multiply connected domain is a domain with holes.

Eg: The region between two concentric circles is a multiply connected

T = $\{z\epsilon\mathbb{C}/1 < |z| < 2\}$

Cauchy-Goursat Theorem For A Multiply Connected Region:

Statement: Let c denote a closed contour and $c_1, c_2, c_3, \dots \dots \dots c_k$ be a finite number of closed contours interior to c such that the interiors of the $c_j's$ do not have any points in common.

Let R be the region consisting of points on and within c except the interior points of c_j. If B denotes the positively oriented boundary of the region R, then

$\int_B f(z)dz = 0$, where $f(z)$ is analytic in the region R.

Result: The above theorem can also be stated as

If 'c' is a simple closed contour and $c_1, c_2, c_3, \dots \dots \dots c_n$ are closed contours within c and if $f(z)$ is analytic within c but on and outside the $c_i's$ then

$$\int_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz + \cdots \dots \dots \dots \dots . \int_{c_n} f(z)dz$$

Where the integrals are all taken in the anticlockwise sense around the curves.

Result: Let 'c' be a simple closed curve. Let $f(z)$ be analytic on and within 'c' everywhere except at $z = a$

$$\int_c f(z)dz = \int_{c_1} f(z)dz$$

Cauchy's Integral Formula:

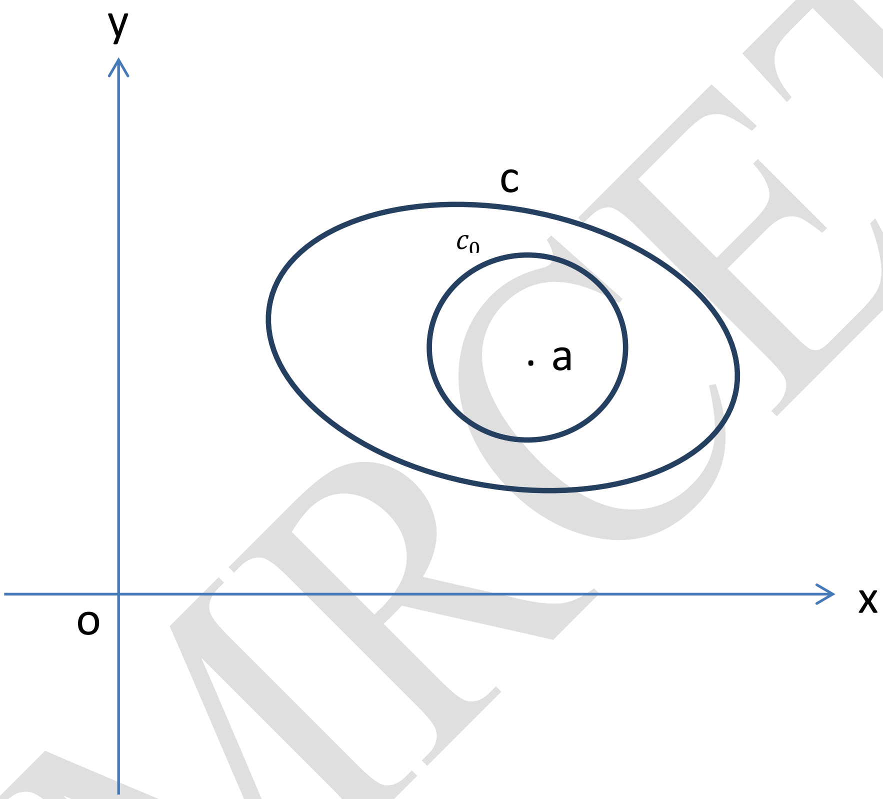
Statement: Let $f(z)$ be an analytic function everywhere on and within a closed contour c. If $z = a$ is any point within c, then

$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)}dz$

Where the integral is taken in the positive sense around c.

Proof: Let $f(z)$ be analytic within a closed contour. Let $z = a$ be within c. Choose a suitably small positive number r_0 and describe a circle c_0 with centre at a and radius r_0 so that this circle c_0 is entirely within c.Then $\frac{f(z)}{z-a}$ is analytic within c except at $z = a$.

Therefore $\frac{f(z)}{z-a}$ is analytic in the region between $c$and c_0.



Therefore by generalization to cauchy'sthorem, we get

$$\int_c \frac{f(z)}{(z-a)} dz = \int_{c_0} \frac{f(z)}{(z-a)} dz$$

$$= \int_{c_0} \frac{[f(z) - f(a)] + f(a)}{z-a}\, dz$$

$$= f(a) \int_c \frac{dz}{z-a} + \int_c \frac{f(z) - f(a)}{z-a} dz \ldots\ldots\ldots..(1)$$

Where the integrals around c_0 are all taken in the positive sense, on c_0: $z - a = r_0 e^{i\theta}$ and $dz = ir_0 e^{i\theta} d\theta$.

Hence, $\int_{c_0} \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{ir_0 e^{i\theta}}{r_0 e^{i\theta}} d\theta = i\int_0^{2\pi} d\theta = 2\pi i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2)$

For every positive r_0.

Also $f(z)$ is continuous at a. Hence, to each $\epsilon > 0$, there corresponds a positive δ such that

$$|f(z) - f(a)| < \epsilon \text{ } whenever \text{ } |z - a| < \delta.$$

Let us take $r_0 = \delta$. Then c_0 is $|z - z_0| = \delta$.

Hence, $\left|\int_{c_0} \frac{f(z)-f(a)}{z-a} dz\right| \le \int_{c_0} \frac{|f(z)-f(a)|}{|z-a|} |dz| < \frac{\epsilon}{\delta}\int_{c_0} |dz|$

$$< \frac{\epsilon}{\delta}(2\pi\delta) \langle \int_{c_0} |dz| = perimeter\ of\ the\ circle\ c_0 \rangle$$

$< 2\pi\epsilon$

Hence, the second integral on the R.H.S of (1) can be made arbitrarily small by taking r_0 sufficiently small. Thus,

$$\int_c \frac{f(z)}{(z-a)} dz = 2\pi i\, f(a) + \int_{c_0} \frac{f(z) - f(a)}{z - a} dz$$

L.H.S and the first term on the R.H.S are independent of r_0 and the second integral on the R.H.S can be made arbitrarily small. Further the second integral must also be independent of r_0.

Hence, it must be 0. Thus,

$$\int_c \frac{f(z)}{(z-a)} dz = 2\pi i\, f(a)$$

i.e., $(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)} dz.$

Hence the theorem follows.

Generalization Of Cauchy's Integral Formula:

Statement: If $f(z)$ is analytic on and within a simple closed curve c and if a is any point within c, then

$$f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz$$

Morera's Theorem:

If a function f is continuous throughout a simply connected domain D and if $\int_c f(z)dz = 0$ for every closed contour $c$in D, the $f(z)$ is analytic in D.

Problems:

1.Evaluate $\int_c \frac{z^2-z+1}{z-1}dz$ where $C: |z| = \frac{1}{2}$ taken in anticlockwise sense.

Sol: Let $f(z) = \frac{z^2-z+1}{z-1}$

Since $z = 1$ is outside c, $f(z)$ is analytic inside c.

By Cauchy's theorem, $\int_c f(z)dz = 0$.

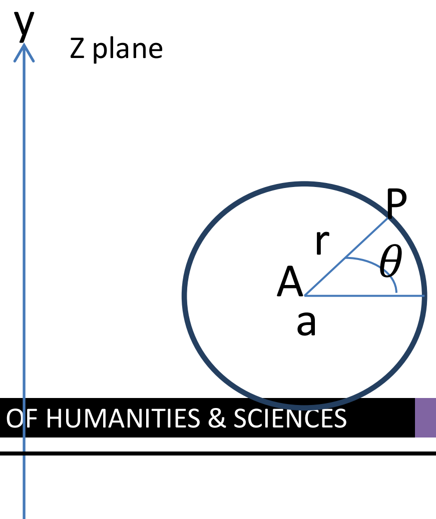
2.Prove that $\int_c \frac{1}{z-a}dz = 2\pi i$, where C is $|z-a| = r$.

Sol: Let A be the fixed complex number 'a' and P a variable point z on the circle.

Then $AP = z - a$. Let AP make an angle θ with x-axis. Then $AP = re^{i\theta}$.

Therefore $z - a = re^{i\theta}$

This is the parametric equation to the circle C and θ varies from 0 to 2π, r being constant.



Hence $\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{rie^{i\theta}}{re^{i\theta}} d\theta$

$= \int_0^{2\pi} id\theta$

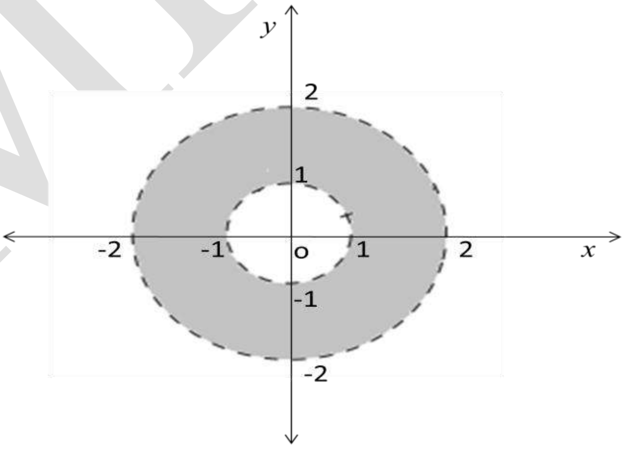
$= i(\theta)_0^{2\pi}$

$= 2\pi i.$

3.Consider the region $1 \le |z| \le 2$. If B is the positively oriented boundary of this region then show that $\int_B \frac{dz}{z^2(z^2+16)} = 0$.

Sol: Given $f(z) = \frac{1}{z^2(z^2+16)}$

$|z| = 1$ *and* $|z| = 2$are two circles with centre at (0,0) and radii equal to 1 and 2 respectively



The singular points of $f(z)$ are obtained by equating $z^2(z^2 + 16) = 0$

$$\Rightarrow z = 0\ (or) z^2 + 16 = 0$$

$$\Rightarrow z = 0\ (or) z = \pm 4i$$

$z = 0, 4i, -4i$are called singular points, which are outside of the region.

By Cauchy's integral therorem,

$\int_B \frac{dz}{z^2(z^2+16)} = 0.$

4.If B is the positively oriented boundary of the region between the circle $|z| = 4$ and the square with sides along the lines $x = \pm 1$ and $y = \pm 1$, then evaluate $\int_B \frac{z+2}{\sin\left(\frac{z}{2}\right)} dz$?

Sol: Let $f(z) = \frac{z+2}{\sin\left(\frac{z}{2}\right)}$

The given region is between $|z| = 4$ and the square $= \pm 1$ and $y = \pm 1,$

$|z| = 4$is the circle with centre $(0,0)$ and $r = 4$

The singular points of $f(z)$ are given by $\sin\left(\frac{z}{2}\right) = 0$

$\Rightarrow \frac{z}{2} = n\pi$, n is an integer

i.e., $z = 2n\pi$

$z = 0, \pm 2\pi, \pm 4\pi, \ldots \ldots \ldots \ldots$

Which are called singular points.

Here $z = 0$ lies inside of the square and all remaining points lies outside of the circle.

Therefore $f(z)$ is analytic within B.

By Cauchy's theorem,

$$\int_B \frac{dz}{z^2(z^2+16)} = 0$$

5.Evaluate$\int_c \frac{z^2+4}{z-3} dz$ where c is $(a)|z| = 5$ $(b)|z| = 2$ taken in anticlockwise?

Sol: $(a)|z| = 5$ is the circle with centre at $(0,0)$ and radius 5 units.

Given function is analytic everywhere except at $z = 3$ and it lies inside C.

$$\int_c \frac{z^2+4}{z-3} dz = \int_c \frac{f(z)}{z-a} dz$$

Where $(z) = z^2 + 4$, $a = 3$ and c is $|z| = 5$ taken in anticlockwise sense.

Using Cauchy's integral formula

$$\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a) = 2\pi i [z^2 + 4]_{z=a=3}$$

$$= 2\pi i(9 + 4) = 26\pi i$$

$(b)|z| = 2$is the circle with centre at $(0,0)$ and radius equal to 2. The point $z = 3$ is outside this curve.

Therefore the function $\frac{z^2+4}{z-3}$ is analytic on and within $c: |z| = 2$.

Hence by Cauchy's theorem $\int_c \frac{z^2+4}{z-3} dz = 0$

6.Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$.

Sol: Given $f(z) = e^{2z}$

$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$using partial fractions.

Therefore $\int_c \frac{e^{2z}}{(z-1)(z-2)} dz = \int_c \frac{e^{2z}}{z-2} dz - \int_c \frac{e^{2z}}{z-1} dz$

The points $z = 1,2$ lies inside c.

Because e^{2z} is analytic everywhere, according to Cauchy's integral formula,

$$\int_c \frac{e^{2z}}{z-2} dz - \int_c \frac{e^{2z}}{z-1} dz = [2\pi i e^{2z}]_{z=2} - [2\pi i e^{2z}]_{z=1} = 2\pi i[e^4 - e^2]$$

7.Use Cauchy's integral formula to evaluate $\int_c \frac{e^z}{(z^2+\pi^2)^2} dz$ where C is the circle $|z| = 4$.

Sol: $\frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z+\pi i)^2(z-\pi i)^2}$

$f(z) = e^z$is analytic within the circle $|z| = 4$ and the two singular points $z = \pm\pi i$ lies inside C.

Let $\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z+\pi i)^2(z-\pi i)^2}$

$= \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$

Solving for A, B, C and D, we get

$A = \frac{7}{2\pi^3 i}, B = \frac{-1}{4\pi^2}, C = \frac{-7}{2\pi^3 i}, D = \frac{-1}{4\pi^2}$

$$\int_c \frac{e^z}{(z^2+\pi^2)^2} dz = \frac{7}{2\pi^3 i}\int_c \frac{e^z}{(z+\pi i)} dz$$

$$-\frac{1}{4\pi^2}\int_c \frac{e^z}{(z+\pi i)^2} dz - \frac{7}{2\pi^3 i}\int_c \frac{e^z}{(z-\pi i)} dz - \frac{1}{4\pi^2}\int_c \frac{e^z}{(z-\pi i)^2} dz$$

Therefore by Cauchy's integral formula,

$$\int_c \frac{e^z}{(z^2+\pi^2)^2} dz = \frac{7}{2\pi^3 i} 2\pi i f(-\pi i) - \frac{1}{4\pi^2} 2\pi i f'(-\pi i) - \frac{7}{2\pi^3 i} 2\pi i f(\pi i)$$

$$-\frac{1}{4\pi^2} 2\pi i f'(\pi i)$$

$$= \frac{7}{\pi^2} e^{-i\pi} - \frac{i}{2\pi} e^{-i\pi} - \frac{7}{\pi^2} e^{i\pi} - \frac{i}{2\pi} e^{i\pi} = \frac{i}{\pi}$$

8.Find$f(2)$ and $f(3)$ if $f(a) = \int_c \frac{(2z^2-z-2)}{z-a} dz$ where C is the circle $|z| = 2.5$ using Cachy's integral formula?

Sol: Given $f(a) = \int_c \frac{(2z^2-z-2)}{z-a} dz$

$(i) a = 2$lies inside the circle $C: |z| = 2.5$

Let $\emptyset(z) = 2z^2 - z - 2$

By Cauchy's integral formula, $\emptyset(a) = \frac{1}{2\pi i}\int_c \frac{\emptyset(z)}{z-a} dz$

$$\Rightarrow 2\pi i \emptyset(a) = \int_c \frac{\emptyset(z)}{z-a} dz = f(a)$$

$\Rightarrow f(a) = 2\pi i \emptyset(a) = 2\pi i(2a^2 - a - 2)$

Therefore $f(2) = 2\pi i(8 - 2 - 2) = 8\pi i$

(ii)Taking $a = 3$, we get, $f(3) = \int_c \frac{(2z^2-z-2)}{z-3} dz$

Now, the point $z = 3$ lies outside C. Hence the integrand is analytic within and on C.

Therefore by Cauchy's theorem, $f(3) = \int_c \frac{(2z^2-z-2)}{z-3} dz = 0.$

9. Evaluate using Cauchy's theorem $\int_c \frac{z^3 e^{-z}}{(z-1)^3} dz$ where C is $|z-1| = \frac{1}{2}$.

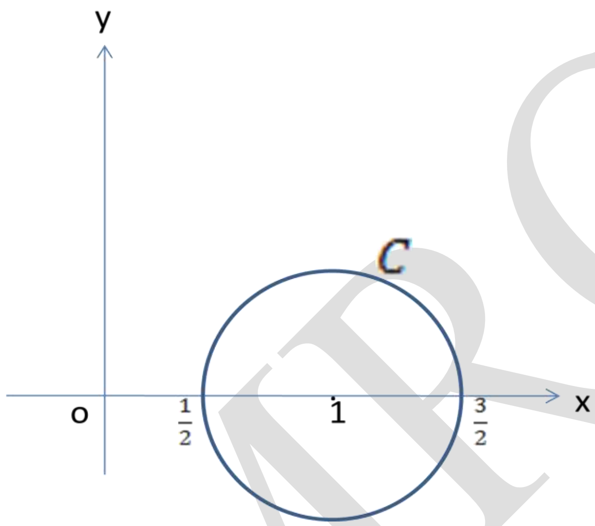
Sol: Given curve is $|z-1| = \frac{1}{2}$.

This is clearly a circle C with centre at 1 and radius 0.5 units.

The integrand has only one singular point at $z = 1$ and it lies inside C.

Consider the function $f(z) = z^3 e^{-z}$

This function is analytic at all points inside C.



Hence by Cauchy's integral formula,

$$f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$$

In this, take $a = 1$ and $n = 2$.

Then

$$f''(1) = \frac{2!}{2\pi i} \int_c \frac{z^3 e^{-z}}{(z-1)^3} dz$$

$$\therefore \int_c \frac{z^3 e^{-z}}{(z-1)^3} dz = \pi i f''(1)$$

$$= \pi i \left\{ \frac{d^2}{dz^2} [z^3 e^{-z}] \right\}_{z=1}$$

$$= \pi i \left\{ \frac{d}{dz} [3z^2 e^{-z} - z^3 e^{-z}] \right\}_{z=1}$$

$$= \pi i [6ze^{-z} - 3z^2 e^{-z} - (3z^2 e^{-z} - z^3 e^{-z})]_{z=1}$$

$$= \pi i [z^3 e^{-z} - 6z^2 e^{-z} + 6ze^{-z}]_{z=1}$$

$$= \pi i [e^{-1} - 6e^{-1} + 6e^{-1}] = \pi i e^{-1}$$

10. Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where c is the circle $|z| = 3$ using Cauchy's integral formula.

Sol: $f(z) = \sin \pi z^2 + \cos \pi z^2$ is analytic within the circle $|z| = 3$ and the singular points $a = 1,2$ lie inside c.

$$\therefore \int_c \frac{f(z)}{(z-1)(z-2)} dz = \int_c \left[\frac{1}{z-2} - \frac{1}{z-1} \right] f(z) dz = \int_c \frac{f(z)}{z-2} dz - \int_c \frac{f(z)}{z-2} dz$$

$= 2\pi i f(2) - 2\pi i f(1)$ (using Cauchy's integral formula)

$$= 2\pi i [(\sin 4\pi + \cos 4\pi) - (\sin \pi + \cos \pi)]$$

$$= 2\pi i [1 - (-1)] = 4\pi i$$

i.e., $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 4\pi i$

Assignment questions:

1.Find whether $f(z) = \frac{x - iy}{x^2 + y^2}$ is analytic or not.

2.Show that the real and imaginary parts of the function $w = \log z$ satisfy the C-R equations when z is not zero.

3.Prove that the function $f(z)$ defined by

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, (z \neq 0) \\ 0, (z = 0) \end{cases}$$

Is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

4.Find k such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic and find its conjugate.

5.Evaluate $\int_c (x-2y)dx + (y^2 - x^2)\,dy$ where C is the boundary of the first quadrant of the circle $x^2 + y^2 = 4$.

6.Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ if c is the square with the vertices at $1 \pm i, -1 \pm i$.

7.Evaluate $\int_c \frac{z^3 - \sin 3z}{\left(z-\frac{\pi}{2}\right)^3} dz$ with $C: |z| = 2$ using Cauchy's integral formula.

8.Evaluate $\int_c \frac{\log z}{(z-1)^3} dz$ where $C: |z-1| = \frac{1}{2}$ using Cauchy's integral formula.

9.Using Cauchy's integral formula, evaluate $\int_c \frac{z^4}{(z+1)(z-i)^2} dz$ where C is the ellipse $9x^2 + 4y^2 = 36$.

10.Evaluate $\int_c \frac{dz}{(z^2+4)^2}$ where $C: |z-i| = 2$.

11.Evaluate $\int_c \frac{e^z}{z(z+1)} dz$ where $C: |z-1| = 3$.

12.Evaluate $\int_c \frac{z^2-z+1}{z-1} dz$ where c is (i) $|z| = 1 (ii) |z| = \frac{1}{2}$ taken in anticlockwise sense.

UNIT –IV

Singularities and Residues

Introduction: In this unit, we discuss the method of expanding a given function about a point 'a' in powers of '$z-a$', as we proceed, we recognize that this theory enables us in evaluating certain real & complex integrals easily. Here we discuss Taylor's series & Laurent series expansion of $f(z)$about point 'a'.

In this unit we also discuss about Residue Theorem which is useful to evaluate certain real integrals.

Sequence: A sequence $\{Z_n\}$ is a function from $N \longrightarrow C$ i.e., $Z_n : N \longrightarrow C$

Series: Let $\{Z_n\}_{n=1}^{\infty}$be a sequence, the n[th] partial sum of sequence is called series and it is denoted by $\sum_{n=1}^{\infty} Z_n$

Power Series: Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of complex no's the series $\sum_{n=1}^{\infty} a_n (z-z_0)^n$ is called a power series of z_0.

- The Series $\sum_{n=1}^{\infty} a_n z^n$ is a power series about the origin.
- If a series $\sum_{k=0}^{\infty} a_k$ converges at every point of circle 'C' & diverges at every point outside the circle 'C', then such a Circle 'C' is said to be circle of convergence of the series $\sum_{k=0}^{\infty} a_k$. The Radius R of the Circle 'C' called the radius of convergence of the series $\sum_{k=0}^{\infty} a_k$.
- The formula to find radius of convergence (R) is $\frac{1}{R} = Lt_{n\to\infty} Sup\left|\frac{a_{n+1}}{a_n}\right|$ (or) $\frac{1}{R} = Lt_{n\to\infty} Sup|a_n|^{1/n}$.

1. Find the circle of convergence of the series $\sum_{n=1}^{\infty} (\log z)^n z^n$

Sol. We have $\sum_{n=1}^{\infty} (\log z)^n z^n = \sum_{n=1}^{\infty} a_n z^n$

on comparing $a_n = (\log z)^n$

we know that $\frac{1}{R} = Lt_{n\to\infty} Sup|a_n|^{1/n}$

$= Lt_{n\to\infty} Sup|(\log z)^n|^{1/n}$

$\frac{1}{R} = \infty$

$R = 0$

Radius of Convergence =0

i.e., Circle with zero radius.

Hence the circle of convergence is $|z| = 0$

2. Find the circle of convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$

Sol. We have $a_n=\frac{(-1)^{n-1}}{(2n-1)!}$ $\quad a_{n+1}=\frac{(-1)^n}{(2n+1)!}$

$$\frac{1}{R}=Lt_{n\to\infty}Sup\left|\frac{a_{n+1}}{a_n}\right|$$

$$=Lt_{n\to\infty}Sup\left|\frac{\cancel{(2n-1)!}}{(2n+1)(2n)\cancel{(2n-1)!}}\right|$$

$$=Lt_{n\to\infty}\left|\frac{1}{(2n+1)(2n)}\right|$$

$$=0$$

$\therefore R\to\infty$, Circle with ∞ radius

$\therefore$The given series is convergent everywhere in the complex plane.

Taylor's Theorem:

Let $f(z)$ be analytic at all points within a circle C with center at *'a'* & radius r. then at each point *'z'* within *'C'*.

$$f(z)=f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\frac{f'''(a)}{3!}(z-a)^3+\ \dots\dots\ \_(1)$$

i.e., the series on the right hand side in (1) converges to *f(z)* whenever $|z-a|<r$

- The expansion in (1) on the R.H.S is called the Taylor's series expansion of $f(z)$ in power of $(z-a)$ (or) Taylor's series expansion of $f(z)$ about z =a (around z = a)

Maclaurin's Series:

Taylor's series expansion about a=0 is called Macluarin's Series i.e.,

$$f(z)=f(0)+f'(0)(z)+\frac{f''(0)}{2!}(z)^2+\frac{f'''(0)}{3!}(z)^3+\ \dots\dots\ \_(2)$$

which is called Maclaurin's Theorem.

Note: Suppose we want Taylor's Series expansion of $f(z)$ around $z=a$. Then $f(z)$ must be analytic at $z=a$ & within circle C: $|z-a|=R$, where R is as large as possible.

<u>Expansion of some standard functions:</u>

1. $e^z=1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\dots\dots\dots\dots=\sum_{n=1}^{\infty}\frac{z^n}{n!}\ \forall\ z$ i.e., $|z|<\infty$
2. $sinz=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\ \dots\dots\dots$
3. $cosz=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\ \dots\dots\dots$
4. $sinhz=\sum_{n=1}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$

<u>Important Note</u>: To obtain Taylor's series expansion of $f(z)$ around about $z=a$, then put $z-a=0$. Then

$$f(z) = f(w+a) = \phi(w) \qquad \text{(say)}$$

now write the Maclaurin's series expansion of $\phi(w)$.

Finally substitute$w = z - a$, then we get required Taylor's Series.

Problems on Taylor's Series Expansion of $f(z)$:

1. Expand e^z as Taylor's series about $z = 1$

Sol: Given $f(z) = e^z, \quad z = 1$

Let $z - 1 = w \Rightarrow z = 1 + w$

Now, write Maclaurin's series for $\phi(w)$

i.e., $\phi(w) = \phi(0) + \phi'(0)(w) + \frac{\phi''(0)}{2!}(w)^2 + \frac{\phi'''(0)}{3!}(w)^3 + \dots\dots\dots\dots$

$$\phi(w) = e.e^w \quad \phi'(w) = e.e^w \quad \phi''(w) = e.e^w$$

$$\phi(0) = e \qquad \phi'(0) = \text{e} \qquad \phi''(0) = e$$

$$\therefore \phi(w) = e + ew + \frac{w^2}{2!}e + \dots\dots\dots\dots$$

$$\phi(w) = e[1 + w + \frac{w^2}{2!} + \dots\dots\dots\dots]$$

Now replace w by $z - 1$

$$\phi(z-1) = e[1 + (z-1) + \frac{(z-1)^2}{2!} + \dots\dots\dots\dots]$$

which is the Taylor's series of $f(z) = e^z$ about $z = 1$.

2. Find Taylors series of $f(z) = \frac{1}{(1+z)^2}$ about $z = -i$

Sol: We know that Taylor's Theorem for $f(z)$ is

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \dots\dots \quad \text{\_(1)}$$

$put\ a = -\text{i}$

$$f(z) = f(-i) + f'(-i)(z+i) + \frac{f''(-i)}{2!}(z+i)^2 + \frac{f'''(-i)}{3!}(z+i)^3 +$$

$$f(z) = \frac{1}{(1+z)^2} \Rightarrow f(-i) = \frac{i}{2}$$

$$f'(z) = \frac{-2}{(1+z)^3} \Rightarrow f'(-i) = \frac{-1\,.2!}{(1-i)^3}$$

$$f''(z) = \frac{6}{(1+z)^4} \Rightarrow f''(-i) = \frac{3!}{(1-i)^4}$$

Sub. All above in (1) then

$$f(z) = \frac{i}{2} + \frac{-1\,.2!}{(1-i)^3}(z+i) + \frac{3!}{(1-i)^4}(z+i)^2 + \dots\dots\dots\dots\dots\dots\dots$$

3. Expand $\frac{z}{(z+1)(z-2)}$ about $z = 1$ (or)

Write the Taylor's series expansion of $\frac{z}{(z+1)(z-2)}$ about $z = 1$

Sol: Given $f(z) = \frac{z}{(z+1)(z-2)}$ & a=1

$$\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} \text{ (by partial fractions)}$$

$$\frac{z}{(z+1)\cancel{(z-2)}} = \frac{A(z-2)+B(Z+1)}{(z+1)\cancel{(z-2)}} \Rightarrow z = A(z-2) + B(Z+1)$$

on solving it $A = 1/3\,, B = \ 2/3$

$$\frac{z}{(z+1)(z-2)} = \frac{2}{3(z+1)} + \frac{1}{3(z-2)}$$

$$\therefore f(z) = \frac{2}{3(z+1)} + \frac{1}{3(z-2)}$$

Now let $z - 1 = w \ \Rightarrow z = 1 + w$

$$= \frac{2}{3(w+2)} + \frac{1}{3(w-1)}$$

$$= \frac{1}{3}\left[1 + \frac{w}{2}\right]^{-1} - \frac{1}{3}[1 + w]^{-1}$$

$$= \frac{1}{3}\left[1 - \frac{w}{2} + \frac{w^2}{4} - \frac{w^3}{8} + \ \ldots\ldots\ldots..\right] - \frac{1}{3}[1 + w + w^2 + w^3 + \cdots]$$

$$\left(\text{ if } \left|\frac{w}{2}\right| < 1 \ \Rightarrow |w| < 1\,;\ |w| < 2 \ \Rightarrow |w| < 1\right)$$

$$f(z) = \frac{1}{3}\left[1 - \frac{z-1}{2} + \frac{(z-1)^2}{4} - \frac{(z-1)^3}{8} + \ \ldots..\right] - \frac{1}{3}[1 + (z-1) + (z-1)^2 + \ \ldots..]$$

i.e., this series is valid in the region $|z - 1| < 1$

Assignment Questions:

1. Find the Taylor's series for $\frac{z}{z+2}$ above $z = 1$. Also find the region of convergence.
2. Expand $\log z$ by Taylor's Series about $z = 1$
3. Obtain the expansion of $\frac{1}{(z-1)(z-3)}$ in a Taylor's series in power of $(Z - 4)$ and determine the region of convergence.
4. Expand $f(z) = \frac{1}{z^2-z-6}$ about $(i)\ z = -1\ (ii)\ z = 1$
5. Find the Taylor's series expansion of $f(z) = \frac{2z^3+1}{z^2+z}$ about point $(i)\ z = -i\ (ii)\ z = 1$

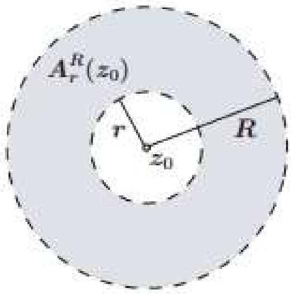
Laurent's series Expansion: we have seen under Taylors series that if $f(z)$ is analytic at $z = a$, we can have a series expansion of $f\ (z)$ in non-negative powers of $(z - a)$ which is valid in a region given by $|z - a| < R$ for suitable R.

Laurent's theorem gives a procedure to expand a given function in powers of $(z - a)$. The series expansion may have positive as well as negative powers.

Laurent's Theorem:

Let C_1 and C_2 be two circular given by $|z' - z_0| = r$ and $|z' - z_0| < R$ respectively where $r < R$.

Let $f(z)$ be analytic on C_1 and C_2 throughout the region between the two circles. Let Z be any point in the ring shaped region between the two circles C_1 and C_2.



then

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

which is called Laurent's series expansion of f(z) about z=z_0.

where $a_n = \frac{1}{2\pi i} \oint_{c_1} \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$

and $b_n = \frac{1}{2\pi i} \oint_{c_2} \frac{f(z')}{(z'-z_0)^{-n+1}}\, dz'$

where integrals are taken around C_1 and C_2 in the anti clockwise direction.

Problems:

1. **Find Laurent's series for $f(z) = \frac{1}{z^2(1-z)}$ & Find the region of convergence (or) Find two Laurent's series expansion in powers of z for $f(z) = \frac{1}{z^2(1-z)}$ & specify the regions in which these expansions are valid.**

Sol: Given $f(z) = \frac{1}{z^2(1-z)}$

The singular points are z=0 and z=1

Now $f(z) = \frac{1}{z^2(1-z)} = \frac{1}{z^2}(1 - z)^{-1}$

$= \frac{1}{z^2}[1+z+z^2+ \ldots\ldots\ldots]$ valid only if $z \neq 0$ & $|z| < 1$

$= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \ldots\ldots\ldots$ is valid only if $0 < |z| < 1$

$= \sum_{n=0}^{\infty} z^{n-2}$ if $0 < |z| < 1$

which is one Laurents series expansion in powers of Z.

$$f(z) = \frac{1}{z^2(1-z)} = \frac{-1}{z^2(z-1)}$$

$$= \frac{-1}{z^2.z\,(1-\frac{1}{z})} = \frac{-1}{z^3(1-\frac{1}{z})} = \frac{-1}{z^3}\left(1 - \frac{1}{z}\right)^{-1}$$

$= -\left(\frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \ \dots\dots\dots\dots\right)$ if $|z| > 1$

$= - \sum_{n=0}^{\infty} z^{-n-3}$ if $|z| > 1$

$= - \sum_{n=0}^{\infty} (z-0)^{-n-3}$ if $|z| > 1$

Only principal part analytic part is not there

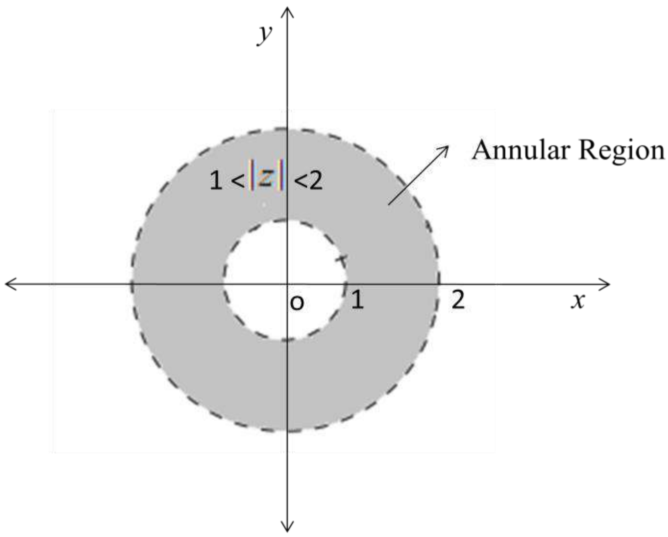
This is the another Laurent's series expansion in powers of z.

2. Expand $f(z) = \frac{1}{z^2-3z+2}$ in the region (i) $1 < |z| < 2$ (ii) $0 < |z-1| < 1$

Sol: $f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

A=-1, B=1

$\therefore f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$



The singular points of f(z) are Z=1,2

(i) Consider $1 < |z| < 2$

i.e., $1 < |z|$, $|z| < 2$

$\left|\frac{1}{z}\right| < 1$, $\left|\frac{z}{2}\right| < 1$

$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$

$= \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}$

$= \frac{1}{-2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$

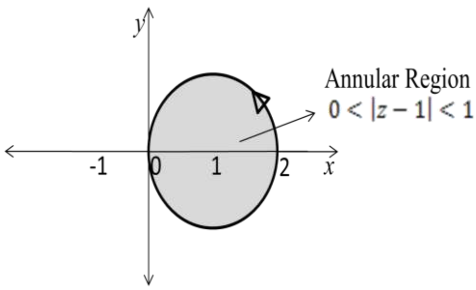
$= \frac{1}{-2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3 + \ \dots\dots\right) - \frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2 + \ \dots\dots\dots\right)$

valid only if $\left|\frac{1}{z}\right| < 1$, $\left|\frac{z}{2}\right| < 1$

$= \frac{1}{-2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1}$ if $1 < |z| < 2$

This is the Laurent's series expansion of $f(z)$ about z=0 (or) in powers of Z in the region $1 < |z| < 2$

(ii) Consider $0 < |z-1| < 1$



We have $f(z) = \frac{-1}{z-1} + \frac{1}{z-2}$

The function $f(z)$ is analytic

in the ring shaped region $0 < |z-1| < 1$

$f(z) = \frac{-1}{z-1} + \frac{1}{(z-1)-1}$

$= \frac{-1}{z-1} - (1-(z-1))^{-1}$

$= \frac{-1}{z-1} - (1-(z-1)+(z-1)^2 + \ \dots\dots\dots)$

$$= -(1-z)^{-1} - \sum_{n=0}^{\infty}(z-1)^n$$

Principal part + Analytic part

This is the Laurent's series expansion of $f(z)$ about $z = 1$ (or) in powers of $(z-1)$ in the region $0 < |z-1| < 1$

3. Expand $\frac{1}{(z^2+1)(z^2+2)}$ in positive & negative powers of z if $1 < |z| < \sqrt{2}$

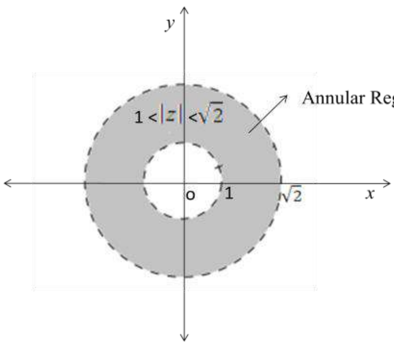
Sol. Given $f(z) = \frac{1}{(z^2+1)(z^2+2)} = \frac{1}{(z^2+1)} - \frac{1}{(z^2+2)}$

Given region is $1 < |z| < \sqrt{2}$

i.e., $1 < |z|$, $|z| < \sqrt{2}$

$\left|\frac{1}{z}\right| < 1$, $\left|\frac{z}{\sqrt{2}}\right| < 1$

$\left|\frac{1}{z^2}\right| < 1$, $\left|\frac{z^2}{2}\right| < 1$



$$f(z) = \frac{1}{(z^2+1)} - \frac{1}{(z^2+2)}$$

$$= \frac{1}{z^2\left(1+\frac{1}{z^2}\right)} - \frac{1}{2\left(1+\frac{z^2}{2}\right)}$$

$$= \frac{1}{z^2}\left(1+\frac{1}{z^2}\right)^{-1} - \frac{1}{2}\left(1+\frac{z^2}{2}\right)^{-1}$$

$$= \frac{1}{z^2}\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \ldots\ldots\ldots.\right] - \frac{1}{2}\left[1 - \frac{z^2}{2} + \frac{z^4}{2^2} - \frac{z^6}{2^3} + \ldots\ldots\ldots\ldots..\right]$$

$$f(z) = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z}\right)^{2n+2} + \sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^{2n}}{2^{n+1}}$$

Principal part of Laurent's series   Analytic part of Laurent's series

Assignment Problems:

1. Obtain all the Laurent's series of the function $\frac{7z-2}{(z+1)z(z-2)}$ about $z = -1$
2. Expand $\frac{1}{z(z^2-3z+2)}$ in the region (a) $1 \le |z| \le 2$ (b) $0 \le |z| \le 1$ (c) $|z| \ge 2$
3. Find the Laurent's series expansion of the function $f(z) = \frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ in the region $3 \le |z+2| \le 5$

Contour Integration

We have studied the functions which are analytic in a given region. But there are several functions which are not analytic at certain points of its domain. Such exceptional points are called the 'singularities' of the function & a type of a singular point is called a 'Pole'. Now we study above different types of singularities & finding residues of a function at a pole. Also we prove Residue theorem which is useful to evaluate certain real integrals.

Definition:

Zero (or) root of analytic function: It is a value of Z such that $f(z) = 0$ (or) A point *'a'* is called a zero of an analytic function $f(z)$ if $f(a) = 0$.

Ex: $f(z) = z - 1$, here $f(1) = 0 \therefore$ '1' is called zero (or) root of $f(z)$

Zero of n^th order : Let $f(z)$ be analytic function, if the root 'a' of $f(z)$ repeated 'n' times then 'a' is called root (or) zero of the nth order. & we write it as $f(z) = (z-a)^m \phi(z)$ where $\phi(z) \neq 0$.

Examples:

1. $f(z) = (z-1)^3$, $f(1) = 0$, Hence '1' is called zero of 3^rd order.
2. $f(z) = \frac{1}{1-z}$, then $f(\infty) = 0$, Hence '∞' is called zero of order 1, it is a simple pole.
3. $f(z) = sinz$, the zeros of f(z) are z=0, $\pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi \ldots \ldots \ldots$..
4. $f(z) = e^{tanz}$ has no zeros ($\because e^z \neq 0$)

Singular Point: A singular point of a function $f(z)$ is the point at which the function $f(z)$ is not analytic.

(or)

A point 'a' is said to be a singularity of $f(z)$ if $f(z)$ is not analytic at 'a'

Singularities are classical into two types:

(i) Isolated Singularity

(ii) Non- isolated singularity

Isolated singularity: A point $z = a$ is called an isolated singularity of an analytic function $f(z)$ if (i) $f(z)$ is not analytic at 'a'

(ii) $f(z)$ is analytic in the deleted neighborhood of $z = a$

Ex.1. $\boldsymbol{f(z) = \frac{1}{z-1}}$

Here $z = 1$ is a singularity of $f(z)$

Further $z = 1$ is a isolated singularity of $f(z)$ since $f(z)$ is analytic in the deleted neighborhood of $z = 1$.

Ex. 2. $f(z) = \frac{1}{(z-1)(z-2)}$

Here $z = 1, 2$ are singularities of $f(z)$

Further $z = 1,2$ are isolated singularity of $f(z)$ since $f(z)$ is analytic in the deleted neighborhood of $z = 1,2$.

Ex.3. $f(z) = \frac{e^z}{z^2+1}$

Here $z = \pm i$ are two isolated singular points of $f(z)$

Ex.4. $f(z) = \frac{2}{\sin z}$

The isolated singular points are z = $\pm\pi, \pm 2\pi, \pm 3\pi, \pm 4\pi \ldots\ldots\ldots..$

Non-Isolated Singularity: A Singularity which is not isolated is called a non isolated singularity.

i.e., A singularity 'a' of $f(z)$ is said to be a non-isolated singularity if every neighborhood of 'a' contains a singularity other than'a'.

Ex. $f(z) = \frac{1}{\sin\left(\frac{1}{z}\right)}$

$\sin\left(\frac{1}{z}\right) = 0 \Rightarrow \frac{1}{z} = \pm n\pi \Rightarrow$z=$\frac{1}{n\pi}$, n=$\pm 1, \pm 2, \pm 3, \pm 4 \ldots\ldots\ldots..$

The singularities of $f(z)$ are $\frac{1}{n\pi}$, n=$\pm 1, \pm 2, \pm 3, \pm 4 \ldots\ldots\ldots..$

It may be noted that $Lt_{n\to\infty} \frac{1}{n\pi} = 0$

i.e., z=0 is the limit sequence of singularity.

∴ Every neighborhood of '0' contains a singularity $\frac{1}{n\pi}$ for sufficiently large 'n'

∴ z=0 is a non- isolated singularity.

Note: If $z = a$ is an isolated singularity of $f(z)$, then f(z) is analytic in deleted neighborhood say $0 < |z-a| < R$, R > 0

∴ $f(z)$ has Laurent's expansion which is valid in the annulus $0 < |z-a| < R$

We know that the Laurent's series expansion of $f(z)$ is

$f(z) = \sum_{n=1}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ valid in $0 < |z-a| < R$

In this expansion $\sum_{n=1}^{\infty} a_n(z-a)^n$ is called the analytic part and $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is called the Principal part of the expansion.

1. **Removable Singularity**: If the principle part of the Laurent's expansion of $f(z)$ around the singular point $z = a$contains no terms. Then singularity is said to be a 'Removable Singularity" of $f(z)$.

   In this case $f(z)$= $\sum_{n=1}^{\infty} a_n(z-a)^n$

   In this case the singularity can be removed by appropriately defining the function $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = 0$, such a singularity is called removable singularity.

Note: If $Lt_{z\to a} f(z) = finite$ then $z = a$ is a removable singularity.

Ex.1: If $f(z) = \frac{1-Cos\, z}{z}$

Hence $z = 0$ is isolated singularity of $f(z)$

$$Lt_{z\to 0} f(z) = Lt_{z\to 0} \frac{1-Cos\, z}{z} \quad \left(\frac{0}{0}\ form\right)$$

$$= Lt_{z\to 0} \frac{\sin z}{1} \quad (L\ hospitals\ Rule)$$

= 0 (finite)

$\therefore z = 0$ is called removable singularity of $f(z)$

Ex.2: If $f(z) = \frac{Sinz}{z}$

z=0 is removable singularity

2. **Pole:** If the principal part of Laurent's series expansion of $f(z)$ around singular point $z = a$. Then $z = a$ is called a pole.

   - If $b_m \neq 0$ & $b_k = 0$ for $k = m+1, m+2, \ldots \ldots \ldots \ldots$

   Then $z = a$ is called a ple of order 'm'

   - A pole of order 1 is called a simple pole.

Ex: $\boldsymbol{f(z) = \frac{z^2}{(z-1)(z+2)^2}}$

Here, $z = 1 , -2$ are isolated singular points

Hence $z = 1$ is a simple pole

$z = -2$ is a pole of order 2

Essential Singularity: If the principle part of the Laurent's series expansion of $f(z)$around $z = a$ (Singular point) contains infinitely many terms then $z = a$ is called an Essential singularity of $f(z)$.

Example for Removable singularity, pole, Essential singularity:

Ex 1: $\mathbf{f(z)} = \frac{\mathbf{z^2-2z+3}}{\mathbf{z-2}} = \frac{z(z-2)+3}{z-2} = z + \frac{3}{z-2}$

Hence $z = 2$ is a singular point & it is Isolated

$f(z) = z + 3(z-2)^{-1}$

which is Laurent's series expansion of f(z) around $z = 2$. It contains only one –ve power of order one.

$\therefore z = 2$ is called a simple pole.

Ex 2: $\boldsymbol{f(z) = e^{1/z} = \frac{1}{e^{-1/z}}}$

The singular point are given by $e^{-1/z} = 0$

$\Rightarrow \frac{1}{z} = \infty$

$\Rightarrow z = 0$

$z = 0$ is the Singular point of $f(z)$ & it is Isolated.

Now $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \dots\dots\dots\dots$ if $0 < |z| < \infty$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}(z-0)^{-n}$$

which is Laurent's Series expansion of $f(z)$ above $z = 0$ & It contains infinitely many –ve powers of $(z-0)$ (principle part contains Infinite no. of terms)

$\therefore z = 0$ is called Essential Singularity of $f(z)$.

Singularity at Infinity: Let the function is f(z), to find the singularitites of f(z) at z=∞ then put $z = \frac{1}{t}$ in $f(z)$.

Then $f(z) = f\left(\frac{1}{t}\right) = F(t)$ [say]

Now the singularity of $F(t)$ at $t = 0$ is the singularity of $F(z)$ at $z = \infty$

Laurent's Theorem:

Let C_1 and C_2 be two circular given by $|z' - a| = r_1$ and $|z' - a| < r_2$ respectively where $r_2 < r_1$.

Let f(z) be analytic on C_1 and C_2 throughout the region between the two circles. Let Z be any point in the ring shaped region between the two circles C_1 and C_2 . then

$f(z) = \sum_{n=1}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ which is called Laurent's series expansion of f(z) about z=a.

where $a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{(z'-a)^{n+1}}\, dz'$ and $b_n = \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')}{(z'-a)^{-n+1}}\, dz'$

where the integrals are taken around C_1 and C_2 in the anti clockwise direction.

Residue at a pole: Let $z = a$ be the pole of a function $f(z)$ then residue of $f(z)$ at $z = a$ is denoted by $Res_{z=a}[f(z)]$ and it is defined as the coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion i.e., $b_1$is the residue

i.e., $b_1 = \frac{1}{2\pi i}\int_c f(z)$

$\int_c f(z) = 2\pi i \times b_1$=$2\pi i \times Res_{z=a}[f(z)]$

- if $z = a$ is the simple pole of $f(z)$

  then $Res_{z=a}[f(z)] = Lt_{z\to a}(z-a)f(z)$

- if $z = a$ is the pole of order 'm' of $f(z)$

  then $Res_{z=a}[f(z)] = \frac{1}{(m-1)!} Lt_{z\to a}\left[\frac{d^m}{dz^{m-1}}(z-a)^m . f(z)\right]$

Cauchy's Residue Theorem

Statement: Let C be any positively oriented simple closed contour. Let f(z) is analytic on & with in 'C' except at a finite number of poles $z_1, z_2, \dots \dots z_n$ within 'c' and $R_1, R_2, \dots \dots R_n$ be the residue of $f(z)$ at these poles, then $\int_c f(z)\,dz = 2\pi i\,[R_1 + R_2 + \dots \dots + R_n]$

(or)

$$\int_c f(z)\,dz = 2\pi i[\,sum\ of\ the\ residues\ at\ the\ poles\ with\ in\ C]$$

Proof: Let $c_1, c_2, \dots \dots c_n$ be the circles with center at $z_1, z_2, \dots \dots z_n$ respectively

The raddi so small therefore all circle $c_1, c_2, \dots \dots c_n$ are entirely lie in C

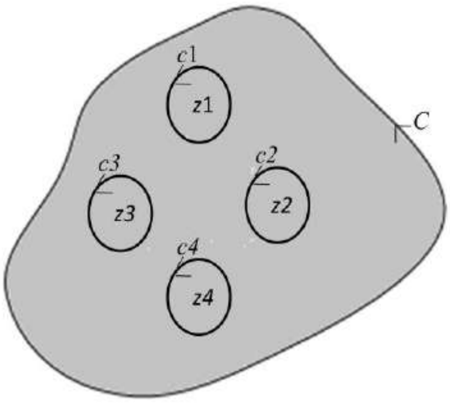
and They do not overlap.

Now f(z) is analytic within the region enclosed by the curve 'c' between these circles.

∴ By Cauchy's theorem for multiply connected regions we have

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \cdots \dots \dots \dots .. + \int_{c_2} f(z)\,dz \qquad \text{———— (1)}$$

But by definition we have



$$\frac{1}{2\pi i}\int_{c_1} f(z)\,dz = Res_{z=z_1}[f(z)]$$

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$$\frac{1}{2\pi i}\int_{c_n} f(z)\,dz = Res_{z=z_n}[f(z)]$$

$$\int_c f(z)\,dz = 2\pi i Res_{z=z_1}[f(z)] + 2\pi i Res_{z=z_2}[f(z)] + \dots\dots\dots\dots + 2\pi i Res_{z=z_n}[f(z)]$$

$$= \ 2\pi i\ \{Res_{z=z_1}[f(z)] + Res_{z=z_2}[f(z)] + \dots \dots \dots \dots .. + Res_{z=z_n}[f(z)]\}$$

$$= \ 2\pi i\,[R_1 + R_2 + \dots \dots + R_n]$$

$$= 2\pi i[\,sum\ of\ the\ residues\ at\ the\ poles\ with\ in\ C]$$

Hence Proved

Problems related to poles & Residues:

1. Expand $f(z)=\frac{e^z}{(z-1)^2}$ as a Laurent's series about $z=1$ & hence find the residue at that point.

Sol: Given $f(z)=\frac{e^z}{(z-1)^2}$ & $z=1$

It is required to find Laurent's series expansion around $z=1$

(i.e., in powers of $(z-1)$)

$$f(z)=(z-1)^{-2}e^{(z-1)+1}=(z-1)^{-2}e^{(z-1)}.e$$

$$=e.(z-1)^{-2}\left[1+(z-1)+\frac{(z-1)^2}{2!}+\ldots\ldots\ldots\ldots\right]$$

$$=\frac{e}{(z-1)^2}\left[1+(z-1)+\frac{(z-1)^2}{2!}+\ldots\ldots\ldots\ldots\right]$$

$$=\frac{e}{(z-1)^2}+\frac{e}{(z-1)}+\frac{e}{2!}+\frac{e(z-1)}{9}+\ldots\ldots\ldots\ldots$$

$$=\left[\frac{e}{2!}+\frac{e(z-1)}{9}+\ldots\ldots\ldots\ldots..\right]+\left[\frac{e}{(z-1)}+\frac{e}{(z-1)^2}+\cdots\ldots\ldots\ldots\ldots\right]$$

$+ve$ powers of $(z-1)$ Analytical part

$-ve$ powers of $(z-1)$ Principle part

Given $f(z)=\frac{e^z}{(z-1)^2}$, $z=1$ is a pole order 2

& Residue of $f(z)$ at $z=1$ is coefficient of $\frac{1}{(z-1)}$ in Laurent's series expansion

i.e., $Res_{z=1}[f(z)]=e$

2. Find the poles of the function (i) $\frac{z}{\cos z}$ (ii) $\cot z$ (iii) $\frac{z}{z^2-3z+2}$

Sol. (i) $f(z)=\frac{z}{\cos z}$

Poles of $f(z)$ are given by denominator = 0

i.e., $\cos z=0$

i.e., $=(2n+1)\frac{\pi}{2}$, $n=0\pm 1,\pm 2\ldots\ldots..$

$\therefore$ The poles are $z=\pm\frac{\pi}{2},\pm\frac{3\pi}{2},\ldots..$,which are poles of order 1(simple poles).

(ii) $f(z) = \cot z$

$f(z) = \cot z = \frac{cosz}{sinz}$

Poles are given by $sinz = 0$

i.e., $z = n\pi$ where $n = 0 \pm 1, \pm 2 \ldots \ldots$

$\therefore$ The poles are $z = 0, \pm\pi, \pm 2\pi, , \pm 3\pi$ ………., which are poles of order 1(simple poles).

(iii) $f(z) = \frac{z}{z^2-3z+2}$

Poles are given by $z^2 - 3z + 2 = 0$

$z = 1,2$ are called poles, which are simple poles.

3. Find the poles of the function $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$ and residues at the poles.

Sol: Given $f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}$

The poles of $f(z)$ are given by $(z-1)^4(z-2)(z-3) = 0$

$\Rightarrow z = 1, 2, 3$

$here\ z = 1$ is a pole of order 4, $z = 2, 3$ are poles of order 1.

i) Residue at pole $z = 2$

w.k.t If $z = a$ is apole of order 1 then

$Res_{z=a}[f(z)] = Lt_{z\to a}(z-a)f(z)$

$Res_{z=2}[f(z)] = Lt_{z\to 2}(z-2)f(z) = Lt_{z\to 2}\cancel{(z-2)}\frac{z^3}{(z-1)^4\cancel{(z-2)}(z-3)} = \frac{8}{1(-1)} = -8$

ii) Residue at pole $z = 3$

$Res_{z=3}[f(z)] = Lt_{z\to 3}(z-3)f(z) = Lt_{z\to 3}\cancel{(z-3)}\frac{z^3}{(z-1)^4(z-2)\cancel{(z-3)}} = \frac{27}{16.1} = \frac{27}{16}$

iii) Residue at pole $z = 1$

Here z=1 is apole of order '4'

w.k.t if $z = a$ is pole of order 'm' then

then $Res_{z=a}[f(z)] = \frac{1}{(m-1)!} Lt_{z\to a}\left[\frac{d^m}{dz^{m-1}}(z-a)^m . f(z)\right]$

here $m = 4, a = 1$

$$Res_{z=1}[f(z)] = \frac{1}{3!} Lt_{z\to 1}\left[\frac{d^3}{dz^3}\cancel{(z-a)^4}.\frac{z^3}{\cancel{(z-1)^4}(z-2)(z-3)}\right]$$

$$Res_{z=1}[f(z)] = \frac{1}{6} Lt_{z\to 1}\left[\frac{d^3}{dz^3}\cdot\frac{\boldsymbol{z^3}}{(\boldsymbol{z-2})(\boldsymbol{z-3})}\right] \text{ ——— } (1)$$

Let us find out $\frac{d^3}{dz^3}\left[\frac{\boldsymbol{z^3}}{(\boldsymbol{z-2})(\boldsymbol{z-3})}\right]$

$$\frac{z^3}{(z-2)(z-3)} = Az + B + \frac{C}{z-2} + \frac{D}{z-3}$$

Hence $A = 1, B = 5, C = -8, D = 27$

$$\frac{z^3}{(z-2)(z-3)} = z + 5 - \frac{8}{z-2} + \frac{27}{z-3}$$

By solving $\frac{d^3}{dz^3}\left[\frac{\boldsymbol{z^3}}{(\boldsymbol{z-2})(\boldsymbol{z-3})}\right] = \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$ ——— (2)

Sub. (2) in (1)

$$Res_{z=1}[f(z)] = \frac{1}{6} Lt_{z\to 1} \frac{48}{(z-2)^4} - \frac{162}{(z-3)^4}$$

$$= \frac{1}{6}\left[48 - \frac{162}{16}\right]$$

$$Res_{z=1}[f(z)] = \frac{101}{16}$$

4. Find the Residues of $\boldsymbol{f(z) = \frac{1}{z(e^z-1)}}$

Sol. Given $f(z) = \frac{1}{z(e^z-1)}$

The poles of $f(z)$ are given by $z(e^z - 1) = 0$

$$z = 0 \text{ or } e^z - 1 = 0$$

$$e^z = 1 \Rightarrow e^z = e^{2n\pi i}, n = 0, \pm 1, \pm 2 \ldots\ldots$$

$$z = 2n\pi i$$

$\therefore$ The poles are $= 0, 2n\pi i$, $n = 0, \pm 1, \pm 2 \ldots\ldots$

When $n = 0$ then $z = 0,0$

$\therefore z = 0$ is apole of order 2

$$f(z) = \frac{1}{z(e^z-1)} = \frac{1}{z\left[\left(\cancel{1}+z+\frac{z^2}{2}+\frac{z^3}{3!}+\ldots\ldots\ldots\right)-\cancel{1}\right]}$$

$$= \frac{1}{z\times z\left[1+\frac{z}{2}+\frac{z^2}{3!}+\cdots\ldots\ldots\right]}$$

$$= \frac{1}{z^2}\left[1 + \left(\frac{z}{2} + \frac{z^2}{3!} + \cdots \ldots\ldots\ldots\ldots\ldots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1 - \left(\frac{z}{2} + \frac{z^2}{3!} + \cdots \ldots\ldots\ldots\ldots\ldots\right) + \left(\frac{z}{2} + \frac{z^2}{3!} + \cdots \ldots\ldots\ldots\right)^2 \ldots\ldots\ldots\right]$$

$$= \frac{1}{z^2}\left[1 - \frac{z}{2!} + \left(\frac{1}{4} - \frac{1}{6}\right)z^2 + \left(-\frac{1}{24} + \frac{1}{6} - \frac{1}{8}\right)z^3 + \ldots\ldots\right]$$

$$f(z) = \frac{1}{z^2} - \frac{1}{2z} + \frac{1}{12} + \frac{1}{360}z^2 + \ldots\ldots\ldots\ldots$$

Which is a Laurent's series Expansion of $f(z)$ in powers of z.

$$\therefore Res_{z=0}[f(z)] = Coeffiecient\ of\ \frac{1}{z} = -\frac{1}{2}$$

Assignment Questions:

Find the poles & the corresponding residues of

(1) $f(z) = \frac{e^z}{(1+z)^2}$

(2) $f(z) = \frac{z^2}{z^4-1}$

(3) $f(z) = \frac{z^2+2z}{(z+1)^2(z^2+4)}$

(4) $f(z) = \frac{ze^z}{(z-1)^3}$

(5) $f(z) = \frac{z^2}{(z+1)^2(z+2)}$

Problems related to evaluation of integrals using residue theorem:

1.Evaluate $\oint \frac{4-3Z}{Z(Z-1)(Z-2)}\, dz$ where 'C' is the circle $|z| = 3/2$ using residue theorem.

Sol: let $f(z) = \frac{4-3Z}{Z(Z-1)(Z-2)}$

The poles of f(z) are given by $z(z-1)(z-2) = 0 \Rightarrow z = 0,1,2$

$z = 0,1,2$ are the poles of order 1.

The given curve c is $|z| = \frac{3}{2} \Rightarrow |z-0| = \frac{3}{2}$

$$\Rightarrow |x + iy - 0| = 3/2$$

$$\Rightarrow |(x-0) + iy| = 3/2$$

$$\Rightarrow \sqrt{(x-0)^2 + y^2} = 3/2$$

$$\Rightarrow (x-0)^2 + (y-0)^2 = 1.5$$

which is a circle with center $(0,0)$ & $r = 1.5$

The poles $z = 0,1$ are only lies inside the curve $'c'$

We required to find the residues at the poles $z = 0, 1$

Residue of $f(z)$ at $z = 0$:

w.k.t $\quad Res\, f(z)_{at\ z=a} = Lt_{z\to a}(z-a)f(z)$

$R_1 = Res\, f(z)_{at\ z=0} = Lt_{z\to 0}(z-0)\frac{4-3Z}{Z(Z-1)(Z-2)} = 4/2 = 2 \Rightarrow \quad R_1 = 2$

Residue of $f(z)$ at $z=1$:

$R_2 = Res\, f(z)_{at\ z=1} = Lt_{z\to 1}(z-1)\frac{4-3Z}{Z(Z-1)(Z-2)} = 1/1(-1) = -1 \Rightarrow R_2 = -1$

∴ By Cauchy Residue theorem:

$$\oint \frac{4-3Z}{Z(Z-1)(Z-2)}dz = 2\pi i(R_1 + R_2)$$

$$= 2\pi i(2-1)$$

$$= 2\pi i$$

Note: $\int f(z)dz = 2\pi i(\text{sum of residues})$

2. Obtain the Laurent's Series for the function $f(z) = \frac{1}{z^2\, sinhz}$ & evaluate $\int \frac{dz}{z^2\, sinhz}$ where 'C' is the circle $|z-1| = 2$

Sol : Given $f(z) = \frac{1}{z^2\, sinhz}$

$$= \frac{1}{z^2\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\ldots\right)} \qquad \left(since\ sinhz = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)$$

$$= \frac{1}{z^3\left[1+\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)\right]}$$

$$= \frac{1}{z^3}\left[1+\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)\right]^{-1}$$

$$= \frac{1}{z^3}\left[1-\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)+\left(\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)^2\ldots.\right]$$

$$[since\ (1+x)^{-1} = 1 - x + x^2 + x^3 \ldots]$$

$$= \frac{1}{z^3}\left[1-\frac{z^2}{6}-\frac{z^4}{120}+\frac{z^4}{36}\ldots.\right]$$

$$= \frac{1}{z^3}\left[1-\frac{z^2}{6}+\left(\frac{1}{36}-\frac{1}{120}\right)z^4\ldots.\right]$$

$f(z) = \frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z^4$ ... is called L.S exp of f(z) about 0

The highest power of (z-0) is 3

Therefore $z = 0$ is a pole of circle 3

The given circle c is $|z-1| = 2$; $|x + iy - 1| = 2$;

$|x - 1 + iy| = 2$; $\sqrt{(x-1)^2 + y^2}$=2 at (1,0) $r = 2$

The pole $z = 0$ lies inside c

$$R_1 = Res\, f(z)_{at\ z=0} = coefficient\ \ of\ \frac{1}{z} in\ L.S\ exp = -1/6$$

By residue theorem $\int f(z)dz = 2\pi i(sum\ of\ residues\ )$

$$\int \frac{dz}{z^2\ sinhz} = 2\pi i(R_1) = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}$$

3. Evaluate $\int \frac{dz}{sinhz}$, where c is the circle $|z| = 4$ using residue theorem .

Sol: Given $f(z) = \frac{1}{sinhz}$

The poles of $f(z)$ are given by $sinhz = 0$

$Z = \pm n\pi i$, $n = 0, \pm 1, \pm 2, ...$

$Z = 0, \pi i, -\pi i, 2\pi i, -2\pi i$ .....

Which are the poles of order 1

$[(0,0)\ (0,\pi), (0,-\pi), (0,2\pi), (0,-2\pi) \dots.]$

The given curve 'C' is |z| = 4 which is a circle with center (0,0) & $radius\ \ r = 4$

Here the only poles lies inside the curve "c" are z=0, πi, -πi,

<u>Residue at *z=0*:</u>

$$R_1 = Res\, f(z)_{at\ z=0} = lt_{\ z\to 0}(z-0)f(z)$$

$$= lt_{\ z\to 0}\ z.\frac{1}{sinhz}$$

$= \frac{0}{0}$ is indeterminant form

$= lt_{\ z\to 0} \ \frac{1}{coshz}$ (L-hospital rule)

$= \frac{1}{cos0} = \frac{1}{1}$

$R_1 = 1$

<u>Residue at $z = \pi i$</u>

$$R_2 = \ Res\ f(z)_{at\ \ z=\pi i} = lt_{\ z\to \pi i}(z-\pi i)f(z)$$

$= lt_{\ z\to\pi i}(z-\pi i).(\frac{1}{sinhz})$

$= \frac{(\pi i-\pi i)}{sinh(\pi i)} = \frac{0}{0}$ (indeterminant form)

$= lt_{z\to\pi i}\left(\frac{1}{coshz}\right) = \left(\frac{1}{cosh(\pi i)}\right) = \frac{1}{-1} = -1$

$R_2 \ = -1$

<u>Similarly Residue at $z = -\pi i$</u> is $R_3 = \ -1$

By residue theorem $\int f(z)dz = 2\pi i(sum\ of\ residues\ )$

$$\int \frac{1}{sinhz}dz = 2\pi i(1-1-1\ ) = -2\pi i$$

Evaluation of Real Definite Integrals by Contour Integration:

In this section, we consider the evaluation of certain types of real definite integrals. These integrals often arise in physical problems. To evaluate these integrals, we apply Residue theorem which is simple than the usual methods of integration. The process of evaluating a definite integral by making the parts of integration about a suitable contour (curve) in the complex plane is called contour integration.

Type I: Integrals of the type $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$

Procedure: put $z = e^{i\theta}$

Differentiate on both sides w.r.t $'\theta'$

$$\frac{dz}{d\theta} = ie^{i\theta} \Rightarrow \frac{dz}{ie^{i\theta}} = d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

We know that $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2}$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-\frac{1}{z}}{2i}$

Also since $0 \le \theta \le 2\pi \Rightarrow \theta$ travels on the entire unit circle & $|z| = \left|e^{i\theta}\right| = 1$

$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta = \int_C F\left[\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right]\frac{dz}{iz} = \int_C f(z)\, dz$ (say) ____ (1)

Where 'C' is the unit circle $|z| = 1$

By Residue Theorem : $\int_C f(z)\, dz = 2\pi i \times [\text{sum of the residues}]$ ______ (2)

From (1) & (2)

$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta = 2\pi i \times [\text{sum of the residues}]$

Problems:

1. **Show by the method of residues $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{\pi}{\sqrt{a^2-b^2}}$ (a>b>0)**

   Show that $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$

Sol: we can write $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ ________ (1)

Let C be the unit circle i.e., C: $|z| = 1$

Put $z = e^{i\theta}$

Differentiate on both sides

$$\frac{dz}{d\theta} = ie^{i\theta} = iz \Rightarrow d\theta = \frac{dz}{iz}$$

$$cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

Substitute all above values in equation (1) than

$$\int_0^{\pi} \frac{d\theta}{a + b\,cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a + b\,cos\theta} = \frac{1}{2}\int_C \frac{1}{a + b\left[\frac{z^2+1}{2z}\right]}\frac{dz}{iz}$$

$$= \frac{1}{\cancel{2}i}\int_C \frac{\cancel{2}}{bz^2+2az+b}dz \quad \_\_\_\_\_\_ (2)$$

Let $f(z) = \frac{1}{bz^2+2az+b}$

The poles of $f(z)$ are given by $bz^2 + 2az + b = 0$

∴ The poles of f(z) are $z = \frac{-a\pm\sqrt{a^2-b^2}}{b}$

Which are poles of order '1'.

Let $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

Since a>b>0 $\Rightarrow |\beta| > 1 \Rightarrow 1 > \frac{1}{|\beta|} \Rightarrow \frac{1}{|\beta|} < 1$

But we know that product of the roots $\frac{c}{a} = \frac{b}{b} = 1$

i.e., α.β=1

$$\Rightarrow |\alpha.\beta| = 1$$

$$\Rightarrow |\alpha| = \frac{1}{|\beta|} < 1$$

$$\Rightarrow |\alpha| < 1$$

∴ $'\alpha'$ lies inside the unit circle 'c'

Residue of $f(z)$at $z = \alpha$:

$$R_1 = Res_{z=\alpha}[f(z)] = Lt_{z\to\alpha}(z-\alpha)f(z)$$

$$= Lt_{z\to\alpha}(z-\alpha)\frac{1}{bz^2+2az+b}$$

$$= Lt_{z\to\alpha}\cancel{(z-\alpha)}\frac{1}{b\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{1}{b(\alpha-\beta)}$$

$$=\frac{1}{2\sqrt{a^2-b^2}} \qquad \left(\because\ \alpha-\beta=\frac{2\sqrt{a^2-b^2}}{b}\right)$$

By Residue theorem

$$\int_c f(z)\,dz=\int_c \frac{1}{bz^2+2az+b}=2\pi i\times[sum\ of\ the\ residues\ ]$$

$$=\cancel{2}\pi i\times\frac{1}{\cancel{2}\sqrt{a^2+b^2}} \quad \text{\_\_\_\_\_\_\_\_\_\_ (3)}$$

Sub. (3) in (2)

$$\int_0^{\pi}\frac{d\theta}{a+b\cos\theta}=\frac{1}{i}\int_c \frac{1}{bz^2+2az+b}dz=\frac{1}{\cancel{i}}\times\pi\cancel{i}\times\cancel{2}\pi i\times\frac{1}{\cancel{2}\sqrt{a^2+b^2}}=\frac{\pi}{\sqrt{a^2+b^2}}$$

2. Evaluate $\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)(x^2+4)}dx$ using Residue theorem.

Sol: To evaluate the given integral, we consider

$$\int_c \frac{z^2}{(z^2+1)(z^2+4)}dz=\int_c f(z)dz$$

Where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to R.

Observe that the integrand has simple poles at $z=\pm i, z=\pm 2i$.

But $z=i, z=2i$ are the only two poles lie inside C.

The residue of $f(z)$ at $z=i$ is given by

$$\lim_{z\to i}[(z-i)f(z)]=\lim_{z\to i}\left[(z-i)\frac{z^2}{(z-i)(z+i)(z^2+4)}\right]$$

$$=\lim_{z\to i}\frac{z^2}{(z+i)(z^2+4)}=\frac{-1}{(2i)(3)}=\frac{-1}{6i}$$

The residue of $f(z)$ at $z=2i$ is given by

$$\lim_{z\to 2i}[(z-2i)f(z)]=\lim_{z\to 2i}\left[\frac{z^2}{(z+2i)(z^2+1)}\right]$$

$$=\frac{-4}{(-4+1)(4i)}=\frac{1}{3i}$$

Thus by Residue theorem,

$$\int_c f(z)dz=2\pi i(Sum\ of\ the\ residues\ within\ C)$$

$$= 2\pi i\left(\frac{-1}{6i}+\frac{1}{3i}\right) = 2\pi\left(\frac{1}{3}-\frac{1}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}$$

i.e., $\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = \frac{\pi}{3}$ (since on real axis $z = x$) ____(1)

Hence by making $R \to \infty$, equation (1) becomes

$\int_{-\infty}^{\infty} f(x)dx + \lim_{z\to\infty}\int_{C_R} f(z)dz = \frac{\pi}{3}$ ____(2)

When $R \to \infty, |z| \to \infty$

$\therefore \int_{C_R} f(z)dz = 0$ _____(3)

From (2) and (3), we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{3}$$

i.e., $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}$

Assignment Questions

1. Prove that $\int_0^{2\pi} \frac{sin^2\theta\, d\theta}{a+b\, cos\theta} = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right]$ where a > b > 0

2. Show that $\int_0^{2\pi} \frac{d\theta}{a+b\, cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$, a>b>0 using Residue theorem.

3. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\, cos\theta}\, d\theta$ using Residue theorem.

4. Show that $\int_0^{2\pi} \frac{1+4\cos\theta}{17+8\, cos\theta}\, d\theta = 0$

5. Evaluate $\int_0^{2\pi} \frac{1}{5-3\, cos\theta}\, d\theta$ using Residue Theorem.

Type II: Integrals of the type $\int_{-\infty}^{\infty} f(x)dx$ [Integration around semi circle]

To solve the integrals of the type $\int_{-\infty}^{\infty} f(x)$dx, we consider $\int_{-\infty}^{\infty} f(x)dx = \int_{c} f(z)$dz

Where 'C' is the closed contour.

$C = C_R U$ real axis from –R to R [C_R is the semi circle in upper half plane with radius R]

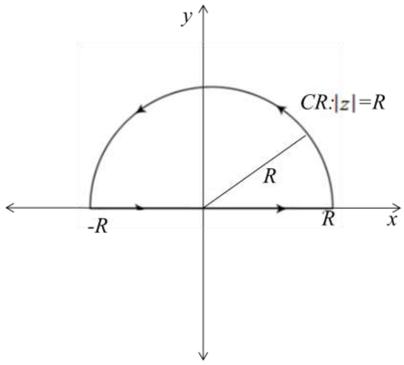
If $f(z)$ has no poles on real axis &on circumference of a circle. But $f(z)$ has some poles Inside curve 'C'. Then by Residue theorem

$$\int_{c} f(z)\text{dz} = 2\pi i \times [sum\ of\ the\ residues\ at\ \text{Interior poles}]$$

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \times [sum\ of\ the\ residues\ at\ \text{Interior poles}]$$

Here we show that $\int_{C_R} |f(z)|dz \to 0$ as $R \to \infty$

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \times [sum\ of\ the\ residues\ at\ \text{Interior poles}]$$



Note: Radius R is taken so large these are the singularities of $f(z)$ lie within semicircle C_R.

1. Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}$

Sol: Here $f(x) = \frac{1}{(x^2+a^2)^2}$

$$f(-x) = \frac{1}{((-x)^2+a^2)^2} = \frac{1}{(x^2+a^2)^2} = f(x)$$

$\therefore f(x)$ is an even function

$$\int_0^{\infty} f(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)dx$$

$$\int_0^{\infty} \frac{1}{(x^2+a^2)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

Now let $\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx = \int_{c} f(z)$dz where $f(z) = \frac{1}{(z^2+a^2)^2}$

& C is the contour consisting of the semi circle C_R of radius R together with the real axis from –R to R.

The poles of $f(z)$ are given by $(\boldsymbol{z^2+a^2})^2$=**0**

$$\Rightarrow z = \pm ai\ ,\pm ai$$

The poles are $z = ai\ , z = -ai$ of order 2

The only pole $z = ai$ lies inside semi circle C_R

<u>Residue of $f(z)$ at $z = ai$</u>

Since $z = ai$ is a pole of order 2

$$R_i = [Res_{z=ai}[f(z)]] = \frac{1}{1!} Lt_{z\to ai}\left[\frac{d}{dz}(z-ai)^2.f(z)\right]$$

$$= Lt_{z\to ai}\left[\frac{d}{dz}(z-ai)^2.\frac{\mathbf{1}}{(\boldsymbol{z^2+a^2})^2}\right]$$

$$= Lt_{z\to ai}\left[\frac{d}{dz}\cancel{(z-ai)^2}.\frac{\mathbf{1}}{(z+ai)^2\cancel{(z-ai)^2}}\right]$$

$$= Lt_{z\to ai}\left[\frac{d}{dz}.\frac{\mathbf{1}}{(z+ai)^2}\right]$$

$$= Lt_{z\to ai}\left[\frac{\mathbf{-2}}{(z+ai)^3}\right]$$

$$R_i = \frac{1}{4a^3 i}$$

Hence by Residue Theorem, $\int_c f(z)\mathrm{dz} = 2\pi i \times [sum\ of\ the\ residues\ at$ Interior poles]

$$= 2\pi i \times \frac{1}{4a^3 i}$$

$$= \frac{\pi}{2a^3}$$

$$\int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = \frac{\pi}{2a^3} \text{ ———— } (2)$$

We know that $\int_{C_R} |f(z)|dz \to 0$ as $R \to \infty$

Hence, $\int_{-\infty}^{\infty} f(x)dx = \frac{\pi}{2a^3}$ ——(3)

Sub. (3) in (1)

$$\int_0^{\infty} \frac{\mathbf{1}}{(\boldsymbol{x^2+a^2})^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathbf{1}}{(\boldsymbol{x^2+a^2})^2}dx = \frac{1}{2}\frac{\pi}{2a^3} = \frac{\pi}{4a^3}$$

Note: Evaluate $\int_0^{\infty} \frac{\mathbf{1}}{(\boldsymbol{x^2+a^2})^2}dx$ using Residue Theorem

Put a=1 in the above problems then we get

$$\int_0^{\infty} \frac{\mathbf{1}}{(\boldsymbol{x^2+a^2})^2}dx = \frac{\pi}{4}$$

Assignment Questions:

1. Using the method of contour integration prove that $\int_0^\infty \frac{1}{x^6+1} dx = \frac{\pi}{3}$ (or) Evaluate $\int_0^\infty \frac{1}{x^6+1} dx$ using the Residue theorem.
2. Evaluate by contour Integration $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$
3. Evaluate by contour Integration $\int_0^\infty \frac{1}{(x^2+1)} dx$
4. Evaluate $\int_0^\infty \frac{\log x}{(x^2+1)} dx$

UNIT-V

CONFORMAL MAPPINGS

Introduction : In this unit we deal the special type of mappings $w = f(z)$, which are called conformal mapping. These mappings are important in engineering mathematics in solving various problems in two dimensional potential theory.

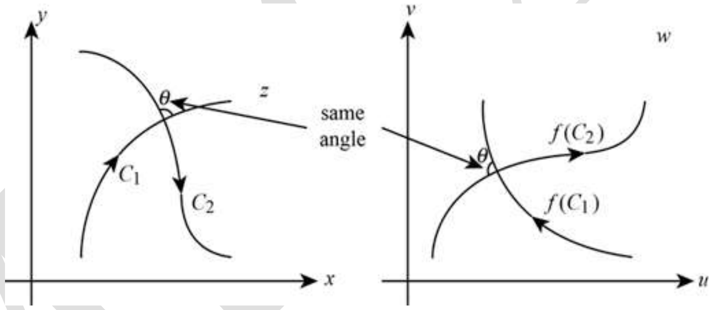
Basic Defintions:

Mapping or transformation from Z-plane to W-plane :

The correspondence defined by the equation $w = f(z)$ between the points in the Z-plane and W-plane is called "Mapping" from Z-plane to the W-plane.

Conformal mapping :

Suppose under the transformation $w = f(z)$, the poinrt $P(x_0, y_0)$ of the Z-plane is mapped in to the point $P'(u_0, v_0)$ of the W-plane. Suppose C_1 and C_2 are any two curves intersecting at the point $P(x_0, y_0)$. Suppose the mapping $w = f(z)$ takes C_1 and C_2 in to the curves c_1' and c_2' which are intersecting at $P'(u_0, v_0)$. If the transformation is such that the angles between C_1 and C_2 at (x_0, y_0) is equal both in magnitude and direction to the angel between c_1' and c_2' at (u_0, v_0) ,then it is said to be conformal transformation at (x_0, y_0) .



Definition : A mapping w=f(z) is said to be conformal in a domain D if it is conformal at every point of D.

Isogonal Transformation :

If the transformation preserves the only magnitude but not necessarily sense (direction) then it is called isogonal mapping.

Sufficient conditions for w=f(z) to represent a conformal mapping :

Theorem : A map w=f(z) is conformal at a point z_0 if f(z) is analytic at z_0 and $f'(z_0) \neq 0$.

Critical point : the points where $f'(z) = 0$ are called critical points.

Ordinary point : the points where $f'(z) \neq 0$ are called ordinary points.

Ex: Find the critical points of $f(z) = z^2$

Sol: $f'(z) = 0$

$\Rightarrow 2z = 0$

$\Rightarrow z = 0$

$\therefore\ z = 0$ is called critical points.

Ex 2: Find the critical points of $f(z) = cosz$

$f'(z) = sinz$

$f'(z) = 0$

$sinz = 0$

$z = n\pi$ where $n = 0, \pm 1, \pm 2 - - - -$

$z = n\pi$ are called critical points of $cosz$

Examples for conformal mappings

1.$w = f(z) = e^z$

We know that $f(z) = e^z$ is analytic everywhere and $f'(z) = e^z \neq 0\ \forall z$

$\therefore f(z)$ is conformal at every point

2.$w = f(z) = z^2 - z + 1$ is conformal mapping because it is a polynomial.

3.$w = f(z) = e^{2z} - 2iz + 3$ is conformal mapping.

Standard Transformations :

1. Translation
2. Expansion or Contraction
3. Inversion

1.**Translation** : the mapping $w = z + c$ where c is any complex constant, is called a translation.

Note : Circles are mapped onto circles under this transformation.

2.**Expansion (or) contraction and rotation(Magnification)** : The mapping $w = cz$ is called contraction and rotation (or) expansion. Under this transformation, any figure in Z-plane is transformed into, geometrically, a similar figure in the W-plane.

Note : if $|c| = 1$ then $w = cz$ is called a pure rotation, since in this case there is no expansion or contraction, but just a rotation through an angle of α.

Example

Prove that circles are invariant under the linear transformation $w = az + c$ (or) prove that circles are mapped to circles under $w = az + c$.

Sol: Given the linear transformation $= az + c$, where a & c are complex constants.

Consider the circle in Z-plane is $A(x^2 + y^2) + Bx + Cy + D = 0$ -------(1)

We have transformation $w = az + c$

$\Rightarrow\ u + iv\ = a(x + iy) + c_1 + ic_2$

Comparing real and imaginary parts

$\Rightarrow u = ax + c_1\ , v = ay + c_2$

$\Rightarrow\ x = \frac{u-c_1}{a}\ , y = \frac{v-c_2}{a}$ -------------------- (2)

Substitute (2) in (1) then we get

$$\Rightarrow A\left[\left(\frac{u-c_1}{a}\right)^2 + \left(\frac{v-c_2}{a}\right)^2\right] + B\left(\frac{u-c_1}{a}\right) + C\left(\frac{v-c_2}{a}\right) + D = 0$$

$$\Rightarrow A'(u^2 + v^2) + B'u + C'v + D' = 0$$

Which is a circle in the W-plane.

Where $A' = \frac{A}{a^2}\ , B' = \frac{B-2Ac_1}{a}\ , C' = \frac{C-2Ac_2}{a}$,

$$D' = D + A\left(\frac{c_1^2 + c_2^2}{a^2}\right) - \frac{Bc_1}{a} - \frac{cc_2}{a}$$

Therefore, circles are mapped on to the circles under the transformation $w = az + c$.

3.Inversion : The mapping $w = \frac{1}{z}$ is called inversion mapping.

Example : the transformation $w = \frac{1}{z}$ maps every straight line or circle onto a circle or straight line.

Proof : let $A(x^2 + y^2) + Bx + Cy + D = 0$ -----------(1) is a circle (or) straight line (if A=0) in Z-plane.

Here A,B,C,D are real numbers.

If A=0, & B & C$\neq$ 0 (at least one) then equation (1) represents straight line.

If $A \neq 0$ then equation (1) represents straight line.

We have $z = x + iy$ and $\bar{z} = x - iy$

$z.\bar{z} = x^2 + y^2$

$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$ ---------------------(2)

Substitute (2) in (1) then

$$Az\bar{z} + B\left(\frac{z+\bar{z}}{2}\right) + C\left(\frac{z-\bar{z}}{2i}\right) + D = 0$$

Substitute $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$\Rightarrow A\frac{1}{w\overline{w}} + B\left(\frac{\frac{1}{w}+\frac{1}{\overline{w}}}{2}\right) + C\left(\frac{\frac{1}{w}-\frac{1}{\overline{w}}}{2i}\right) + D = 0$$

Now multiply the above equation by $w\overline{w}$

$$\Rightarrow A + B\left(\frac{w+\overline{w}}{2}\right) + C\left(\frac{w-\overline{w}}{2i}\right) + Dw\overline{w} = 0$$

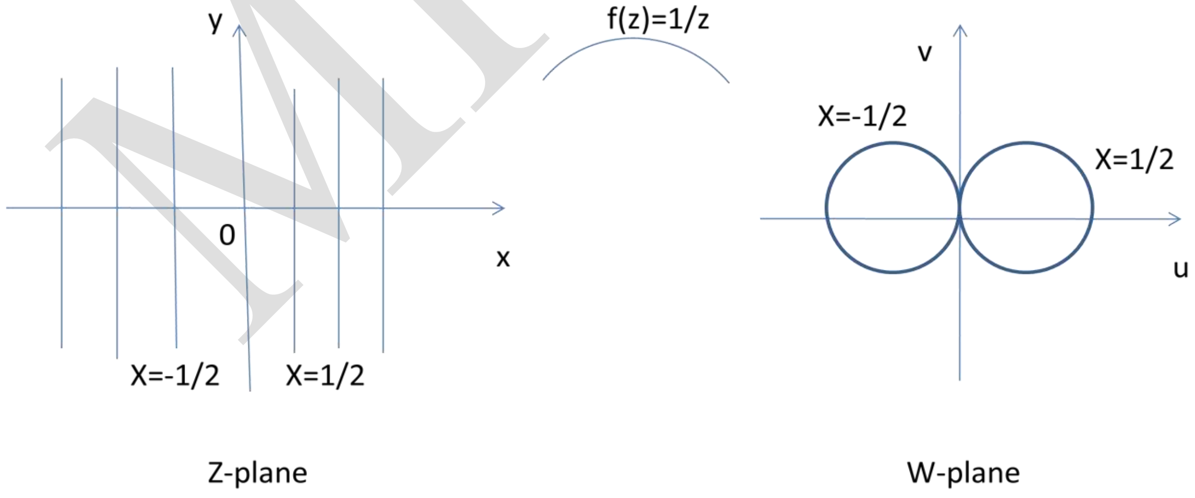
$\Rightarrow A + Bu - Cv + D(u^2+v^2) = 0$--------(3)

Where $= \frac{w+\overline{w}}{2}$, $v = \frac{w-\overline{w}}{2i}$, $u^2 + v^2 = w\overline{w}$

Equation (3) represents a circle in W-plane if $D \neq 0$

Equation (3) represents a straight line in W-plane if $D = 0$ and B & $C \neq 0$ (at least one)

Therefore general equation of circle or straight is transformed to general equation of straight line or circle under the transformation $w = \frac{1}{z}$.



Some special conformal Transformations :

1. $w = z^2$ 2. $w = e^z$ 3. $w = logz$

Problems :

1.Find the points at which $w = coshz$ is not conformal.

Sol : given $w = f(z) = coshz$

$f'(z) = sinhz$

$f'(z) = 0$

$sinhz = 0$

$$\frac{e^z - e^{-z}}{2} = 0$$

$\Rightarrow e^{2z} - 1 = 0$

$\Rightarrow z = \pm n\pi i$ where $n = 0, \pm 1, \pm 2 - - - - -$

Therefore critical points of $f(z)$ are $z = \pm n\pi i,\ n = 0, \pm 1, \pm 2 - - - - -$

Therefore $f(z)$ is not conformal at $z = \pm n\pi i$.

2.Find the image of $|z| = 2$ under the transformation $w = 3z$.

Sol: given $|z| = 2$

$\Rightarrow |x + iy| = 2$

$\Rightarrow \sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4$ which is a circle with center (0,0) & r=2.

It is required to find the image of circle $|z| = 2$ i.e $x^2 + y^2 = 4$ -------(1)

under the mapping $w = 3z$.

Let $w = u + iv$ and $z = x + iy$

Given transformation is $w = 3z$

$u + iv = 3(x + iy)$

Comparing real and imaginary parts then

$u = 3x$ &$v = 3y$

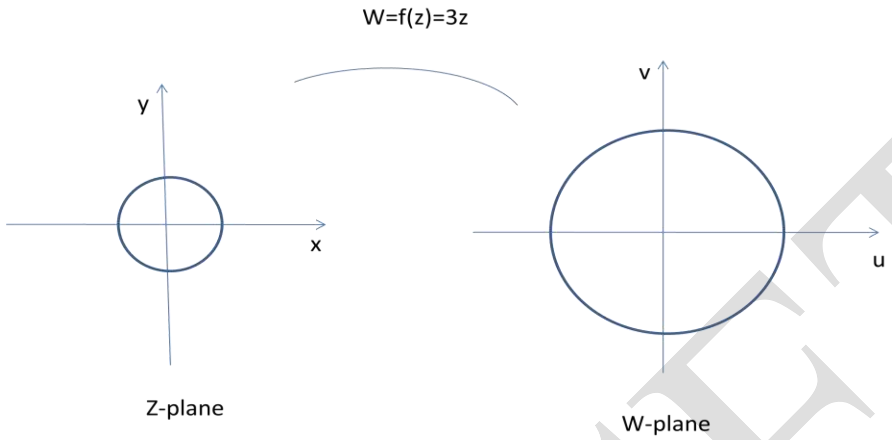
$x = \frac{u}{3}$ and $y = \frac{v}{3}$

Substitute x & y values in (1) then

$$\left(\frac{u}{3}\right)^2 + \left(\frac{v}{3}\right)^2 = 4$$

$$u^2 + v^2 = 36$$

Which is a circle in the W-plane with center at (0,0) & r=6.



3.under the transformation $w = \frac{1}{z}$,find the image of the circle $|z - 2i| = 2$.

Sol : $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$x = \frac{u}{u^2+v^2} \quad y = \frac{-v}{u^2+v^2} \quad \text{-----------(1)}$$

$$|z - 2i| = 2.$$

$$|x + iy - 2i| = 2.$$

$$x^2 + (y-2)^2 = 4 \text{--------------(2)}$$

Which is a circle with center (0,2) and $r = 2$.

Substitute (1) in (2)

$$\Rightarrow 1 + 4v = 0$$

$$\Rightarrow v = \frac{-1}{4}$$

Which is a straight line parallel to X-axis in the W-plane.

4.Find the image of the infinite strip $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$.

Sol: here it is required to find the image of infinite strip $\mathbf{0} < y < \frac{\mathbf{1}}{\mathbf{2}}$ in Z-plane under the map $w = \frac{1}{z}$.

Given transformation $w = \frac{1}{z}$

$$z = \frac{1}{w}$$

$$x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

Comparing real and imaginary parts

$x = \frac{u}{u^2+v^2}$ $y = \frac{-v}{u^2+v^2}$ ----------------------(1)

Given strip in Z-plane is $\mathbf{0} < y < \frac{\mathbf{1}}{\mathbf{2}}$

If $y = 0$ then $v = 0$ (from (1))

If $y = \frac{1}{2}$ then $u^2 + v^2 + 2v = 0$

$$u^2 + (v + 1)^2 = 1$$

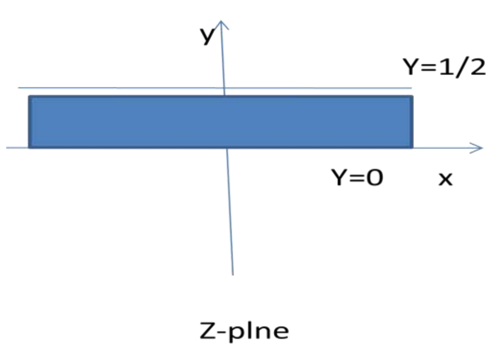
Which is a circle with center (0,-1) & r=1

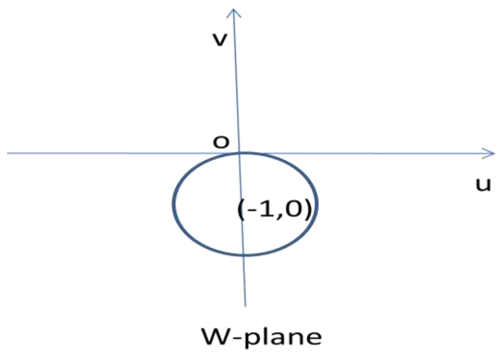
Therefore under the transformation $w = \frac{1}{z}$

The straight $y = 0$ is transformed to line $v = 0$ and

The straight $y = \frac{1}{2}$ is transformed to a circle $u^2 + (v + 1)^2 = 1$

Hence the infinite strip $\mathbf{0} < y < \frac{\mathbf{1}}{\mathbf{2}}$ in Z-plane is mapped in to the region between line V=0 and the circle $u^2 + (v + 1)^2 = 1$ in W-plane under the transformation $w = \frac{1}{z}$.





5.show that the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$ is the lemniscate $\rho^2 = cos2\emptyset$.

Sol: It is required to find the image of hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$

given transformation $w = \frac{1}{z}$

let $z = re^{i\theta}$

$w = Re^{i\emptyset}$

$$Re^{i\emptyset} = \frac{1}{re^{i\theta}}$$

$$Re^{i\emptyset} = \frac{1}{r}e^{-i\theta}$$

$R = \frac{1}{r}, \emptyset = -\theta$

Given hyperbola is $x^2 - y^2 = 1$

$r^2cos^2\theta - r^2sin^2\theta = 1$

$r^2(cos^2\theta - sin^2\theta) = 1$

$r^2cos2\theta = 1$

$\frac{1}{\rho^2}\cos(-2\emptyset) = 1 \quad (\rho = \frac{1}{r}, \emptyset = -\theta)$

$\rho^2 = cos2\emptyset$

Therefore hyperbola $x^2 - y^2 = 1$ in the Z-plane is mapped in to lemniscates $\rho^2 = cos2\emptyset$ in the W-plane.

6.Find and plot the image of the traingularregion with vertices at (0,0) (1,0)(0,1) under the transformation $w = (1 - i)z + 3$.

Sol: Given transformation is $w = (1 - i)z + 3$

$u + iv = (1 - i)(x + iy) + 3$

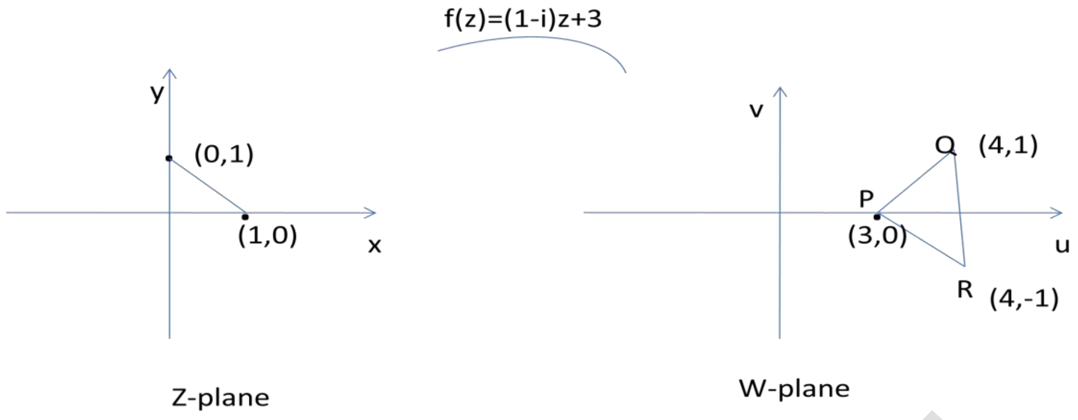
$u + iv = (x + y + 3) + i(y - x)$

$u = x + y + 3$ and $v = y - x$ ---------------------(1)

When $(x, y) = (0,0)$ then $(u, v) = (3,0)$ in W-plane

When $(x, y) = (1,0)$ then $(u, v) = (4, -1)$ in W-plane

When $(x, y) = (0,1)$ then $(u, v) = (4,1)$ in W-plane



7.Find and plot the rectangular region $0 \le x \le 2, 0 \le y \le 2$ under transformation $w = \sqrt{2}e^{\frac{i\pi}{4}}z + (1-2i)$.

Sol: Given transformation is $w = \sqrt{2}e^{\frac{i\pi}{4}}z + (1-2i)$

$$u + iv = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(x+iy) + (1-2i)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(x+iy) + (1-2i)$$

$$= (1+i)(x+iy) + (1-2i)$$

$$= (x-y) + i(x+y) + (1-2i)$$

$$u + iv = (x-y+1) + i(x+y-2)$$

$u = x - y + 1$ and $v = x + y - 2$ ----------(1) which is a given transformation

Under this transformation we have to find the image of rectangular region $0 \le x \le 2, 0 \le y \le 2$ in Z-plane.

Put $x = 0$ in (1) then $u = -y + 1, v = y - 2 \Rightarrow y = 2 + v$

$$u = -(2+v) + 1 \Rightarrow v = -u - 1$$

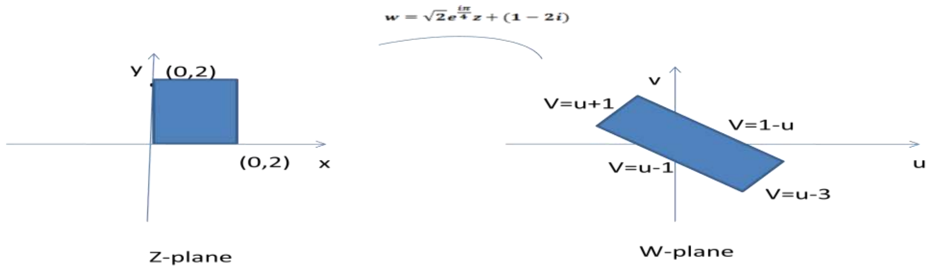
Put $x = 2$ in (1) then $u = 2 - y, v = y - 1 \Rightarrow v = 1 - u$

Put $y = 0$ in (1) then $u = x + 1, v = x - 2 \Rightarrow v = u - 3$

Put $y = 2$ in (1) then $u = x - 1, v = x \Rightarrow v = u + 1$

Thus the region is a rectangle bounded by the lines , $v = -u - 1 \Rightarrow v = 1 - u, v = u - 3$ &

$v = u + 1$



8.Find the image of the region in the Z-plane between the lines $y = 0$ & $y = \pi/2$ under the transformation $w = e^z$.

Sol: Given transformation is $w = e^z$

Let $z = x + iy$ and $w = Re^{i\emptyset}$

$Re^{i\emptyset} = e^{x+iy}$

$Re^{i\emptyset} = e^x . e^{iy}$

$R = e^x$ and $\emptyset = y$ -----(1) which is a given transformation

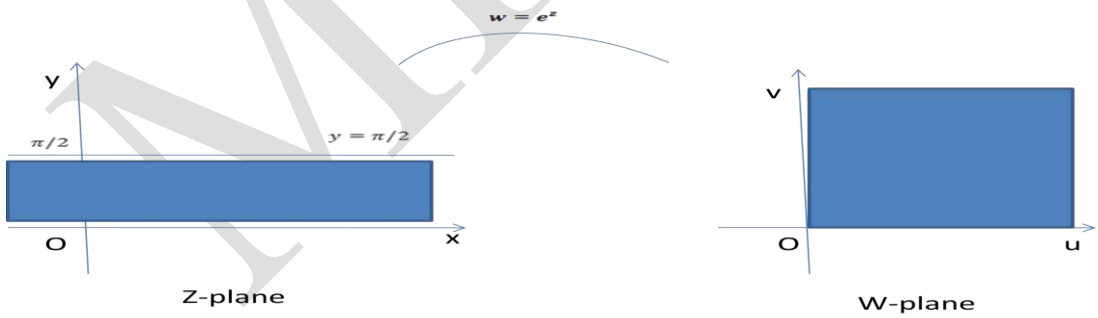
If $y = 0$ then $\emptyset = 0$ (fom (1)) represents radial line making an angle of zero radius with the x-axis .

If $y = \pi/2$ then $\emptyset = \pi/2$ represents radial line making angle of $\pi/2$ radius with the X-axis.

As x increases from $-\infty$ to ∞ then $R = e^x$ (i.e radius) increases from 0 to ∞

$y = \pi/2$ in Z-plane is mapped onto the ray $\emptyset = \pi/2$ excluding origin in W-plane.

Hence the infinite strip bounded by the lines $y = 0$ and $y = \pi/2$ is mapped on to the upper quadrant of W-plane.



Assignment questions :

1.For the mapping $w = \frac{1}{z}$,Find the image of the family of circles $x^2 + y^2 = ax$ where a is real.

2.Show that the transformation $w = \frac{1}{z}$ maps a circle to a circle or to a straight line if the former goes through the origin.

3.Find the image of the domain in the Z-plane to the left of the line $x = -3$ under transformation $w = z^2$.

4.Find and plot the image of the regions

i) $x > 1$ ii) $y > 0$ iii)$0 < y < 1/2$ under transformation $w = 1/z$.

BILINEAR TRANSFORMATION OR MOBIUS TRANSFORMATION

Bilinear transformation : The map $w = T(z) = \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ is called bilinear transformation (or) linear fractional transformation or mobius transformation.

Note :The map $w = \frac{az+b}{cz+d}$ -----(1) where $ad - bc \neq 0$ is bilinear transformation

$\Rightarrow wcz + wd = az + b$

$\Rightarrow wcz - az + dw - b = 0$

$\Rightarrow Azw + Bz + Cw + D = 0$ -----(2)

Where $A = c, B = -a, C = d, D = -b$

Note that $AD - BC = c(-b) - (-a)d = ad - bc \neq 0$

Equation (1) can be written in the form $Azw + Bz + Cw + D = 0$ and $AD - BC \neq 0$

Therefore the form $Azw + Bz + Cw + D = 0$ is also called bilinear transformation

i.e equations (1) and (2) represents bilinear transformation.

- The necessary condition to say that $w = \frac{az+b}{cz+d}$ ---(1) is bilinear transformation is $ad - bc \neq 0$
- The bilinear transformation $w = \frac{az+b}{cz+d}$, $ad - bc \neq 0$ is a bijective from C_∞ to C_∞.
- The inverse of a bilinear is also bilinear.
- The composition of any two bilinear transformation is also bilinear.
- The identity transformation $I(z) = z$ is also bilinear

Properties of Bilinear Transformation

1.A Bilinear transformation is conformal

Proof: Consider the bilinear transformation $w = T(z) = \frac{az+b}{cz+d}$

Differentiate with respect to z

$$\frac{dw}{dz} = T'(z) = \frac{(cz+d)(a) - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

Since $ad - bc \neq 0$

$\Rightarrow \frac{dw}{dz} \neq 0$

$\Rightarrow w = T(z) = \frac{az+b}{cz+d}$ is conformal transformation.

If $ad - bc = 0$ then $\frac{dw}{dz} = 0 \ \forall z$

Then we say that every point of z −plane is critical.

Note : Let the bilinear transformation $w = \frac{az+b}{cz+d}$

For different choices of constants a,b,c,d we get different bilinear transformation as

(i) $w = z + b$ (if $a = 1, c = 0, d = 1$) (translation)
(ii) $w = az + b$(if $c = 0$ &$d = 1$) (Linear translation)
(iii) $w = az$ (if $b = 0, c = 0, d = 1$) (Rotation)
(iv) $w = \frac{1}{z}$ (if $a = 0, b = 1, c = 1, d = 0$) (Inversion)

2.There is a one-one correspondence between all points in two planes.

Proof: Let $w = \frac{az+b}{cz+d}$ -------(1) $ad - bc \neq 0$ be a conformal mapping

From (1) $z = \frac{-dw+b}{cw-a}$ --------(2) is inverse mapping

Since $ad - bc \neq 0$ therefore equation (2) is also represents a bilinear transformation.

From (1) , it is clear that to each point in the Z-plane except $z = \frac{-d}{c}$ there corresponds a unique point in the W-plane.

Invariant or Fixed point : A point z_0 is said to be a fixed point of a bilinear transformation $w = T(z)$ if $T(z_0) = z_0$.

Ex 1: For the map $W = T(z) = z$

Every point is a fixed point

Ex2: For the map $W = \frac{1}{z}$

the fixed point are obtained by $T(z) = z$

$\Rightarrow \frac{1}{z} = z$

$\Rightarrow z^2 - 1 = 0$

$\Rightarrow z = \pm 1$,therefore $z = \pm 1$ are fixed points

- **Finding the Bilnear Transformation whose fixed point are α and β are given by $w = \frac{\gamma z - \alpha\beta}{z-(\alpha+\beta)+\gamma}$**

Prop 3. Every bilinear transformation maps the totality of circles and straight lines in Z-plane onto the totality of circles and straight lines the W-plane.

OR

Every bilinear transformation maps circles and straight lines into circles and straight lines

Proof: Let the bilinear transformation $w = T(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$

(i). If $c = 0$ then $T(z) = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right) = Az + B$ where $= \frac{a}{d}$,$B = \frac{b}{d}$

Clearly T is linear.

We know that image of any region in the Z-plane under the linear transformation has the same.

i.e the transformation $w = T(z)$ transforms circles & straight lines in to circles and straight lines.

(ii). If $c \neq 0$ then

$T(z) = \left(\frac{a}{c}\right) + \left(\frac{bc-ad}{c^2}\right) . \frac{1}{z+\frac{d}{c}}$

Let $T_1(z) = z + \frac{d}{c}$, $T_2(z) = \frac{1}{z}$, $T_3(z) = \frac{bc-ad}{c^2} . z$, $T_4(z) = \frac{a}{c} + z$

Therefore $T(z) = T_4 o T_3 o T_2 o T_1$

We know that (i) the inversion transformation maps circles and straight lines in to circles and straight lines.

(i)The translation and rotation are linear transforms.

Therefore the transformation transforms circles and straight lines into circles and straight lines.

Since every bilinear transformation is a composition of translation,rotation and inversion.

Hence bilinear transformation $T(z)$ is a of translation,rotation and inversion.

Therefore bilinear transformation $T(z)$, circles and straight lines into circles and straight lines.

Cross Ratio :

For three distinct points z_1, z_2, z_3 in C_∞ then the cross ratio of z, z_1, z_2, z_3 is denoted by

(z, z_1, z_2, z_3) and defined by $(z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

Prop4: The cross ratio is invariant under a bilinear transformation

(or)

A bilinear transformation preserves cross ratio property of four points.

Proof : Let the bilinear transformation $w = T(z) = \frac{az+b}{cz+d}$ where $ad - bc \neq 0$ where $a, b, c, d \in \mathbb{C}$

Let $T(z_k) = w_k$ for $k = 1,2,3$

It is required to prove that $(z, z_1, z_{2,} z_3) = (T(z), T(z_1), T(z_2), T(z_3))$

i.e $(w, w_1, w_{2,} w_3) = (z, z_1, z_{2,} z_3)$

now $w - w_k = T(z) - T(z_k)$ where $k = 1,2,3$

$$= \frac{az+b}{cz+d} - \frac{az_k+b}{cz_k+d}$$

$$= \frac{(az+b)(cz_k+d)-(az_k+b)(cz+d)}{(cz+d)(cz_k+d)}$$

$$w - w_k = \frac{(ad-bc)(z-z_k)}{(cz+d)(cz_k+d)}$$

$$w_i - w_j = \frac{(ad-bc)(z_i-z_j)}{(cz_i+d)(cz_j+d)}$$

Let the cross ratio of $w, w_1, w_{2,} w_3$

$$(w, w_1, w_{2,} w_3) = \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}$$

$$= \frac{\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)} \cdot \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}}{\frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)} \cdot \frac{(ad-bc)(z_3-z_1)}{(cz_3+d)(cz+d)}}$$

$$= \frac{(z-z_1)(z_2-z_3)}{(z_1-z_1)(z_3-z)}$$

$$= (z, z_1, z_{2,} z_3)$$

Therefore $(z, z_1, z_{2,} z_3) = (T(z), T(z_1), T(z_2), T(z_3))$

Note1: To find the bilinear transformation $= T(z)$, we can use the condition

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Note 2: To find the bilinear transformation we can also use the formula $w = \frac{az+b}{cz+d}$

Note 3: $\frac{\infty-i}{\infty-w} = \log_{n\to\infty}\frac{n-i}{n-w} = 1$,similarly $\frac{\infty-w}{\infty-i} = 1$

Note 4: $\frac{i-\infty}{\infty-w} = -1$, $\frac{\infty-i}{w-\infty} = -1$

Problems:

1: Find the Bilinear transformation which maps the point (-1,0,1) in to the points (0,i,3i)

Soln: let $z_1 = -1, z_2 = 0, z_3 = 1$

$$w_1 = -1, w_2 = 0, w_3 = 1$$

we know that $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\frac{(\ w-0\ )(i-3i\ )}{(0-\ i\ )(3i-w)} = \frac{(z+1)(\ 0-1\ )}{(-1-0)(1-\ Z\ )}$$

$$\frac{(\ w\ )(i-3i\ )}{(-\ i\ )(3i-w)} = \frac{(z+1)}{(1-\ Z\ )}$$

$$\frac{(\ 2\,w\ )}{(3i-w)} = \frac{(z+1)}{(1-\ Z\ )}$$

$$(\ 2\,\mathrm{w}\ )(1-\ \mathrm{Z}) = (\mathrm{z}+1)(3\mathrm{i}-\mathrm{w})$$

$$2\,\mathrm{w} - 2\,\mathrm{w}\,\mathrm{Z} = 3\mathrm{i}\,\mathrm{z} - \mathrm{w}\,\mathrm{z} + 3\mathrm{i} - \mathrm{w}$$

$$\mathrm{w}[2 - 2\mathrm{z} + \mathrm{z} + 1] = 3\mathrm{i}[\mathrm{z}+1]$$

$$\mathrm{w}(-\mathrm{z}+3) = 3\mathrm{i}[\mathrm{z}+1]$$

$$\mathrm{w} = \frac{3\mathrm{i}[\mathrm{z}+1]}{3-\mathrm{z}}$$

$$\mathrm{w} = \mathrm{T}(\mathrm{z}) = \frac{3\mathrm{i}[\mathrm{z}+1]}{3-\mathrm{z}}$$

Which is the required bilinear transformation

2.Find the fixed points (Invariant points) of the transformation

(i) $\boldsymbol{w = \frac{2i-6z}{iz-3}}$

(ii) $\boldsymbol{w = \frac{z-1}{z+1}}$

Soln : The fixed point of transformations are obtained by $w = z$

$$i.e \quad f(z) = z$$

(i) $w = f(z) = \frac{2i-6z}{iz-3}$

$f(z) = z$

$\frac{2i-6z}{iz-3} = z$

$2i - 6z = iz^2 - 3z$

$iz^2 + 3z - 2i = 0$

$z^2 - 3iz - 2 = 0$

It is a quadratic equation

$Z = \frac{3i \pm \sqrt{9i^2-4.1.(-z)}}{2}$

$Z = i, 2i$

Fixed points are $i, 2i$

3. find the bilinear transformations which maps $Z = 0, -i, 2i$ in to $w = 5i, \infty, -i/3$.

soln: let the bilinear transformation be $w = \frac{az+b}{cz+d}$ -----(1)

Given $Z = 0, -i, 2i$ & $w = 5i, \infty, -i/3$

sub above values in (1)

$5i = \frac{b}{d}$; b = 5id ------ (2)

$\infty = \frac{-ai+b}{-ci+d}$; $\frac{1}{0} = \frac{-ai+b}{-ci+d}$; $-ci + d = 0$ ------(3)

$\frac{-i}{3} = \frac{2ai+b}{2ci+d}$; $2c - id = 6ia + 3b$ ------- (4)

Solving (2) (3) & (4) for a, b, c, d

From (2) $b = 5id$

From (3) $c = -id$

Sub b, c values in (4)

$2(-id) - id = 6ia + 15\, id$

$a = -3d$

Sub a, b, c in (1)

$$w = \frac{-3dz + 5id}{-idz + d}$$

$$w = \frac{-3z+5i}{-iz+1}$$

Multiply & divide by i

$$w = \frac{-(3iz+5)}{z+1}$$

Prob4. Find the bilinear transformation that maps the points $(\infty, i, 0)$ into the points $(0, i, \infty)$.

Sol: we know that $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

$$\frac{(w-0)(i-\infty)}{(0-i)(\infty-w)} = \frac{(z-\infty)(i-0)}{(\infty-i)(0-z)}$$

$$w = -\frac{1}{z}$$

Prob 5. Show that transformation $w = \frac{z-i}{z+i}$ maps the real axis in the Z-plane in to the unit circle $|w| = 1$ in the W-plane.

Sol: Given transformation is $\boldsymbol{w} = \frac{z-i}{z+i}$

Unit circle in w-plane is $|w| = 1$

$$\left|\frac{z-i}{z+i}\right| = 1$$

$$|z - i| = |z + i|$$

$$|x + i(y - 1)| = |x + i(y + 1)|$$

$$x^2 + (y-1)^2 = x^2 + (y+1)^2$$

$$x^2 + y^2 - 2y + 1 = x^2 + y^2 + 2y + 1$$

$$4y = 0$$

$y = 0$ which is a real axis in Z-plane.

Prob6. Show that the transformation $w = \frac{z-i}{z+i}$ transforms $|w| \leq 1$ into upper half plane (i.e img(z)>0)

Sol: consider the transformation $w = \frac{z-i}{z+i}$

$$\bar{w} = \frac{\bar{z}+i}{\bar{z}-i}$$

$$w\bar{w}-1=\frac{z-i}{z+i}.\frac{\bar{z}+i}{\bar{z}-i}-1$$

$$=\frac{(\bar{z}+i)(z-i)-(z+i)(\bar{z}-i)}{(\bar{z}-i)(z+i)}$$

$$=\frac{2i(z-\bar{z})}{|z+i|^2}$$

$$w\bar{w}-1=\frac{-4y}{|z+i|^2}$$

$$|w|^2-1=\frac{-4y}{|z+i|^2} \text{ --------------(1)}$$

Given $|\boldsymbol{w}| \leq \mathbf{1}$

if $|w|=1$ then $|w|^2=1 \Rightarrow y=0$ (form (1)) which is a real axis in Z-plane.

therefore circle $|w|=1$ in W-plane transformed straight line $y=0$ in Z-plane.

If $|w|<1$ then $y>0$ (form (1))

i.e img(z)>0

i.e Upper half of Z-plane.

Hence $|w| \leq 1$ is transformed into upper half plane (i.e img(z)>0) unde transformation $w=\frac{z-i}{z+i}$.

Prob7. Show that the relation $w=\frac{5-4z}{4z-2}$ transforms the circle $|z|=1$ into a circle of radius unity in the W-plane.

Sol: Given transformation is $w=\frac{5-4z}{4z-2}$ -------(1)

solving (1) for z

$$\boldsymbol{z}=\frac{5+2w}{4(w+1)}$$

$$|\boldsymbol{z}|=\mathbf{1}$$

$$\left|\frac{5+2w}{4(w+1)}\right|=1$$

$$|\mathbf{5}+\mathbf{2}\boldsymbol{w}|=|\mathbf{4}(\boldsymbol{w}+\mathbf{1})|$$

$$\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{i}\boldsymbol{v}$$

$$|\mathbf{5}+\mathbf{2}\boldsymbol{u}+\mathbf{2}\boldsymbol{i}\boldsymbol{v}|=|\mathbf{4}\boldsymbol{u}+\mathbf{4}\boldsymbol{i}\boldsymbol{v}+\mathbf{1}|$$

$$|(\mathbf{5}+\mathbf{2}\boldsymbol{u})+\mathbf{2}\boldsymbol{i}\boldsymbol{v}|=|(\mathbf{4}\boldsymbol{u}+\mathbf{1})+\mathbf{4}\boldsymbol{i}\boldsymbol{v}|$$

$$\sqrt{(\mathbf{5}+\mathbf{2}\boldsymbol{u})^2+\mathbf{4}\boldsymbol{v}^2}=\sqrt{(\mathbf{4}\boldsymbol{u}+\mathbf{1})^2+\mathbf{16}\boldsymbol{v}^2}$$

$$\boldsymbol{u^2 + v^2 + u - \frac{3}{4} = 0}$$

it is the circle with center $C = (-1/2,0)$ and $r = 1$ in W-plane.

The Image of a circle $|z| = 1$ in Z-plane is a circle $u^2 + v^2 + u - \frac{3}{4} = 0$ in W-plane under the transformation $w = \frac{5-4z}{4z-2}$.