SYSTEMS & SIGNAL PROCESSING

Lecture Notes

B.TECH
(IIIYEAR –II SEM)
(2020-21)

Prepared by:

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MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution - UGC, Govt. of India)

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(AffiliatedtoJNTUH,Hyderabad,ApprovedbyAICTE-AccreditedbyNBA&NAAC–'A'Grade-ISO9001:2015Certified)

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SYSTEMS & SIGNAL PROCESSING

CONTENTS

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

III B.Tech EEE IISEM

(PROFESSIONAL ELECTIVE – II)

SYSTEMS & SIGNAL PROCESSING

SUBJECT CODE (R18A0463)

OBJECTIVES:

The main objectives of the course are:

- To understand the basic concepts of basic elementary signals and Fourier Series representation.
- To Master the representation of signals in the frequency domain using Fourier transforms and Discrete Fourier transform
- To learn the Mathematical and computational skills needed to understand the principal of Linear System and digital signal processing fundamentals.
- To understand the implementation of the DFT in terms of the FFT.
- To learn the Realization of Digital Filters

UNIT I:

INTRODUCTION TO SIGNALS: Elementary Signals**-** Continuous Time (CT) signals, Discrete Time (DT) signals, Classification of Signals, Basic Operations on signals.

FOURIER SERIES: Exponential Fourier Series, Dirichlet's conditions, Complex Fourier Spectrum.

UNIT II:

FOURIER TRANSFORMS: Fourier transform of arbitrary signal, Fourier transform of standard signals.

Discrete Fourier Transforms: Properties of DFT. Linear Convolution of Sequences using DFT. Computation of DFT: Over-lap Add Method, Over-lap Save Method.

UNIT III:

FAST FOURIER TRANSFORMS: Fast Fourier Transforms (FFT) - Radix-2 Decimation-in-Time and Decimation-in-Frequency FFT Algorithms, Inverse FFT.

UNIT IV:

INTRODUCTION TO LINEAR SYSTEMS: Introduction to Systems, Classification of Systems,

INTRODUCTION TO DIGITAL SIGNAL PROCESSING: Introduction to Digital Signal Processing, Linear Shift Invariant Systems, Stability, and Causality of Discrete time systems

UNIT V:

Z–TRANSFORMS: Concept of Z**-** Transform of a discrete sequence. Region of convergence in Z- Transform

REALIZATION OF DIGITAL FILTERS: Realization of Digital Filters - Direct, Canonic forms.

TEXT BOOKS:

1. Signals, Systems & Communications - B.P. Lathi, BS Publications, 2003.
2. Signals and Systems – A. Anand Kumar, PHI Publications, 3rd edition.

3. Digital Signal Processing, Principles, Algorithms, and Applications: John G. Proakis, Dimitris G. Manolakis, Pearson Education / PHI, 2007.
4. Digital Signal ProcessingA. Anand Kumar, PHI Publications.

REFERENCE BOOKS:

1. Signals & Systems - Simon Haykin and Van Veen,Wiley, 2nd Edition.
2. Fundamentals of Signals and Systems Michel J. Robert, MGH International Edition, 2008.
3. Digital Signal Processing – S.Salivahanan, A.Vallavaraj and C.Gnanapriya, TMH, 2009.
4. Discrete Time Signal Processing – A. V. Oppenheim and R.W. Schaffer, PHI, 2009.

OUTCOMES:

After completion of the course, the student would be able to:

- Understand the basic elementary signals.
- Represent signals in the frequency domain using Fourier Series, Discrete Fourier series, Fourier transform and Discrete Fourier transform techniques.
- Understand the principle of Linear System and digital signal processing fundamentals.
- Implement DFT of any signal using FFT algorithm.
- Realize Digital Filters

UNIT I

INTRODUCTION TO SIGNALS& FOURIER SERIES

- Elementary Signals
  - Continuous Time (CT) signals
  - Discrete Time (DT) signals
- Classification of Signals
- Basic Operations on signals.
- Representation of Fourier Series
  - Exponential Fourier Series
  - Discrete Fourier Series
  - Properties of Discrete Fourier Series

1. INTRODUCTION

Anything that carries information can be called a signal. Signals constitute an important part of our daily life. A signal is defined as a single-valued function of one or more independent variable which contains some information. A signal may also defined be defined as any physical quantity that varies with time, space or any other independent variable. A signal may be represented in time domain or frequency domain. A signal can be function of one or more independent variable. A signal may be a function of time, temperature, pressure, distance etc. If a signal depends on only one independent variable, it is called a one dimensional signal and If a signal depends on two independent variable, it is called a two-dimensional signal. Examples of on1D and 2D signals are shown in figure.

Examples of signals include:

1. A voltage signal: voltage across two points varying as a function of time.
2. A force pattern: force varying as a function of 2-dimensional space.
3. A photograph: color and intensity as a function of 2-dimensional space
4. A video signal: color and intensity as a function of 2-dimensional space and time

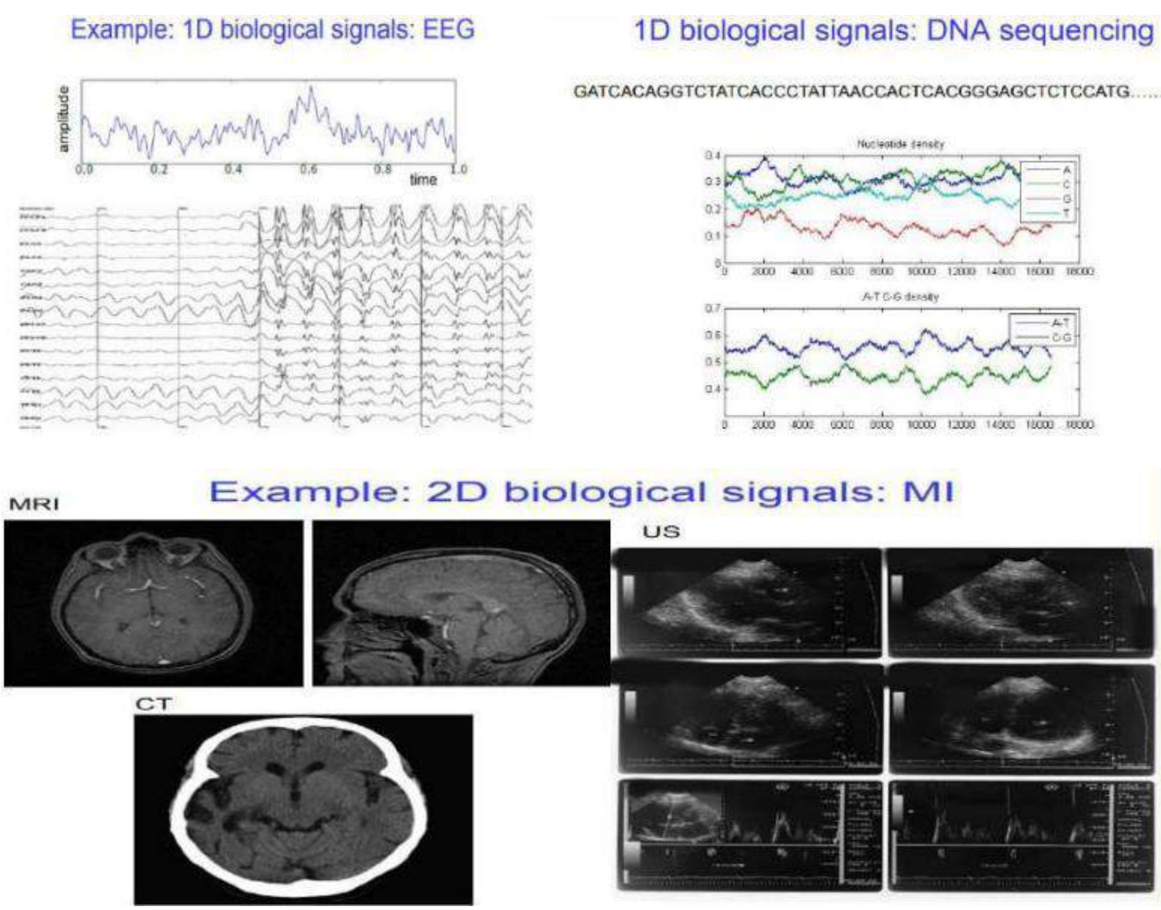


Figure 1.1 a) One Dimensional EEG Signal b) One Dimensional DNA Signal c)Two Dimensional Signal

ELEMENTARY SIGNALS

There are several elementary signals which plays vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals which occur in nature. These elementary signals are also called standard signals.

The standard signals are:

1. Unit step function
2. Unit ramp function
3. Unit parabolic function
4. Unit impulse function
5. Sinusoidal function
6. Real exponential function
7. Complex exponential function, etc

1.2.1 Unit Step Function

The step function is an important signal used for analysis of many systems. The step function is that type of elementary function which exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant forever after application.

If a step function has unity magnitude, then it is called unit step function. The usefulness of the unit-step function lies in the fact that if we want a signal to start at $t = 0$, so that it may have a value of zero for $t < 0$, we only need to multiply the given signal with unit step function $u(t)$. A unit step function is useful as a test signal because the response of the system for a unit step reveals a great deal about how quickly the system responds to a sudden change in the input signal.

The continuous-time unit step function $u(t)$ is defined as:

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

From the above equation for $u(t)$, we can observe that when the argument t in $u(t)$ is less than zero, then the unit step function is zero, and when the argument t in $u(t)$ is greater than or equal to zero, then the unit step function is unity.

The shifted unit step function $u(t - a)$ is defined as:

$$u(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

It is zero if the argument $(t - a) < 0$ and equal to 1 if the argument $(t - a) \geq 0$.
The graphical representations of $u(t)$ and $u(t - a)$ are shown in Figure 1.2[(a) and (b)].

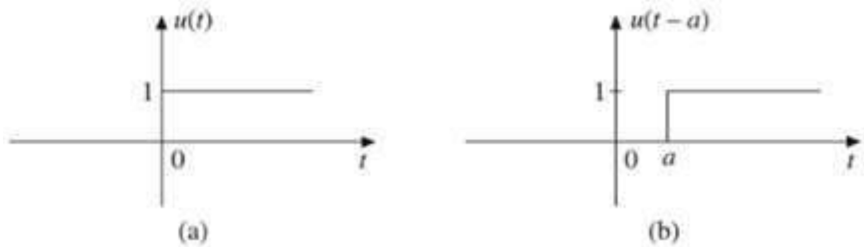


Figure 1.2 (a) Unit step function, (b) Delayed unit step function.

1.2.2 Unit Ramp Function

The continuous-time unit ramp function $r(t)$ is that function which starts at $t = 0$ and increases linearly with time and is defined as:

$$r(t) = \begin{cases} t & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or $$r(t) = t\,u(t)$$

The unit ramp function has unit slope. It is a signal whose amplitude varies linearly. It can be obtained by integrating the unit step function. That means, a unit step signal can be obtained by differentiating the unit ramp signal.

i.e. $$r(t) = \int u(t)\,dt = \int dt = t \quad \text{for } t \geq 0$$

$$u(t) = \frac{d}{dt} r(t)$$

The delayed unit ramp signal $r(t - a)$ is given by

$$r(t-a) = \begin{cases} t - a & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or $$r(t-a) = (t-a)\,u(t-a)$$

The graphical representations of $r(t)$ and $r(t-a)$ are shown in Figure 1.3[(a) and (b)].

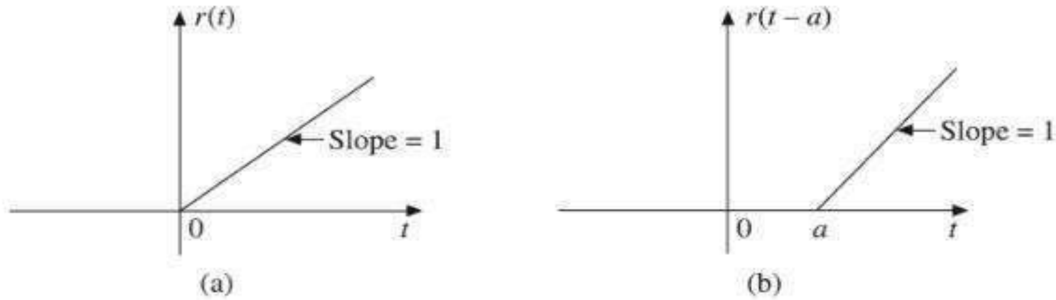


Figure 1. 3 (a) Unit ramp signal. (b) Delayed unit ramp signal.

The discrete-time unit ramp sequence $r(n)$ is defined as

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or

$$r(n) = n\,u(n)$$

The shifted version of the discrete-time unit-ramp sequence $r(n-k)$ is defined as

1.2.3 Unit Parabolic Function

The continuous-time unit parabolic function $p(t)$, also called unit acceleration signal starts at $t = 0$, and is defined as:

$$p(t) = \begin{cases} \dfrac{t^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

or

$$p(t) = \frac{t^2}{2}\,u(t)$$

The shifted version of the unit parabolic sequence $p(t-a)$ is given by

$$p(t-a) = \begin{cases} \dfrac{(t-a)^2}{2} & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

or

$$p(t-a) = \frac{(t-a)^2}{2}\,u(t-a)$$

The graphical representations of $p(t)$ and $p(t-a)$ are shown in Figure 1.4[(a) and (b)].

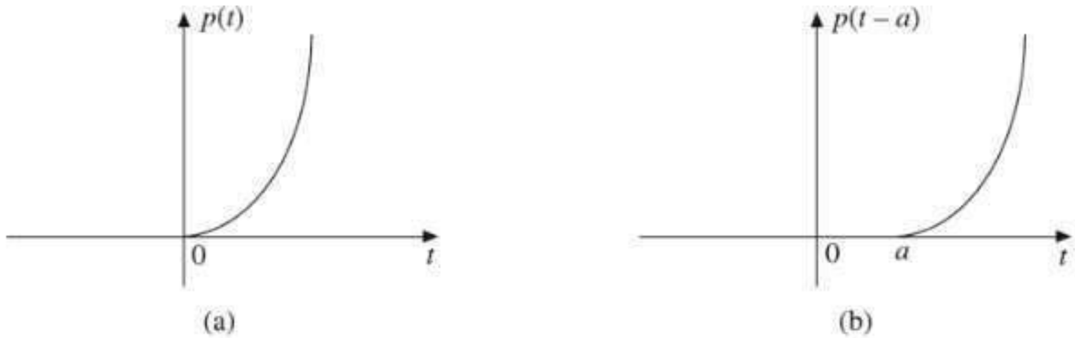


Figure 1. 4 (a) Unit parabolic signal, (b) Delayed parabolic signal.

The unit parabolic function can be obtained by integrating the unit ramp function or double integrating the unit step function.

$$p(t) = \iint u(t)\,dt = \int r(t)\,dt = \int t\,dt = \frac{t^2}{2} \quad \text{for } t \geq 0$$

1.2.4 Unit Impulse Function

The unit impulse function is the most widely used elementary function used in the analysis of signals and systems. The continuous-time unit impulse function $\delta(t)$, also called Dirac delta function, plays an important role in signal analysis. It is defined as:

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

and $$\delta(t) = 0 \quad \text{for } t \neq 0$$

i.e. as $$\delta(t) = \begin{cases} 1 & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$

That is, the impulse function has zero amplitude everywhere except at $t = 0$. At $t = 0$, the amplitude is infinity so that the area under the curve is unity. $\delta(t)$ can be represented as a limiting case of a rectangular pulse function.

As shown in Figure 1.5(a),

$$x(t) = \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

$$\delta(t) = \underset{\Delta \to 0}{\text{Lt}}\ x(t) = \underset{\Delta \to 0}{\text{Lt}}\ \frac{1}{\Delta}[u(t) - u(t - \Delta)]$$

A delayed unit impulse function $\delta(t - a)$ is defined as:

$$\delta(t - a) = \begin{cases} 1 & \text{for } t = a \\ 0 & \text{for } t \neq a \end{cases}$$

The graphical representations of $\delta(t)$ and $\delta(t - a)$ are shown in Figure 1.5[(b) and (c)].

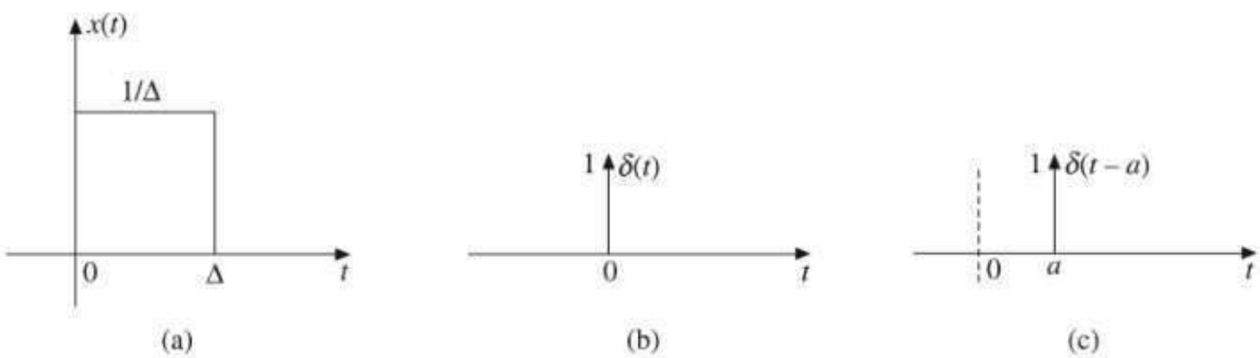


Figure 1. 5 (a) $\delta(t)$ as limiting case of a pulse, (b) Unit impulse, (c) Delayed unit impulse.

If unit impulse function is assumed in the form of a pulse, then the following points may be observed about a unit impulse function.

(i) The width of the pulse is zero. This means the pulse exists only at $t = 0$.

(ii) The height of the pulse goes to infinity.

(iii) The area under the pulse curve is always unity.

(iv) The height of arrow indicates the total area under the impulse.

The integral of unit impulse function is a unit step function and the derivate of unit step function is a unit impulse function.

$$u(t) = \int_{-\infty}^{\infty} \delta(t)\, dt$$

and
$$\delta(t) = \frac{d}{dt} u(t)$$

Properties of continuous-time unit impulse function

1. It is an even function of time t, i.e. $\delta(t) = \delta(-t)$
2. $\int_{-\infty}^{\infty} x(t)\,\delta(t)\,dt = x(0)$; $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$
3. $\delta(at) = \frac{1}{|a|}\delta(t)$
4. $x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0) = x(t_0)$; $x(t)\,\delta(t) = x(0)\,\delta(t) = x(0)$
5. $x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

1.2.5 Sinusoidal Signal

A continuous-time sinusoidal signal in its most general form is given by

$$x(t) = A \sin(\omega t + \phi)$$

where

A = Amplitude
ω = Angular frequency in radians
ϕ = Phase angle in radians

Figure 1.5 shows the waveform of a sinusoidal signal. A sinusoidal signal is an example of a periodic signal. The time period of a continuous-time sinusoidal signal is given by

$$T = \frac{2\pi}{\omega}$$

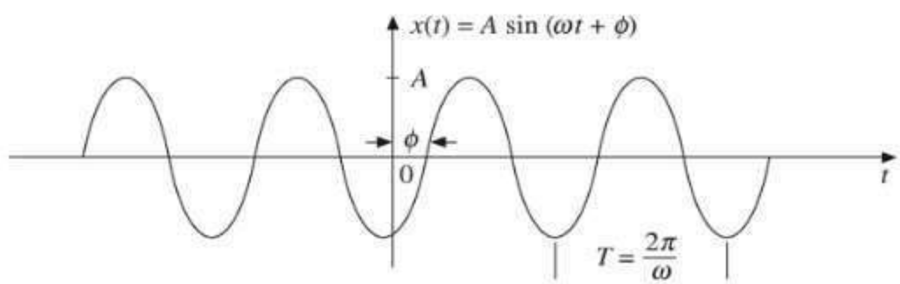


Figure 1.5 Sinusoidal waveform.

1.2.6 Real Exponential Signal

A continuous-time real exponential signal has the general form as:

$$x(t) = Ae^{\alpha t}$$

where both A and α are real.

The parameter A is the amplitude of the exponential measured at $t = 0$. The parameter α can be either positive or negative. Depending on the value of α, we get different exponentials.

1. If $\alpha = 0$, the signal $x(t)$ is of constant amplitude for all times.
2. If α is positive, i.e. $\alpha > 0$, the signal $x(t)$ is a growing exponential signal.
3. If α is negative, i.e. $\alpha < 0$, the signal $x(t)$ is a decaying exponential signal.

These three waveforms are shown in Figure 1. 6[(a), (b) and (c)].

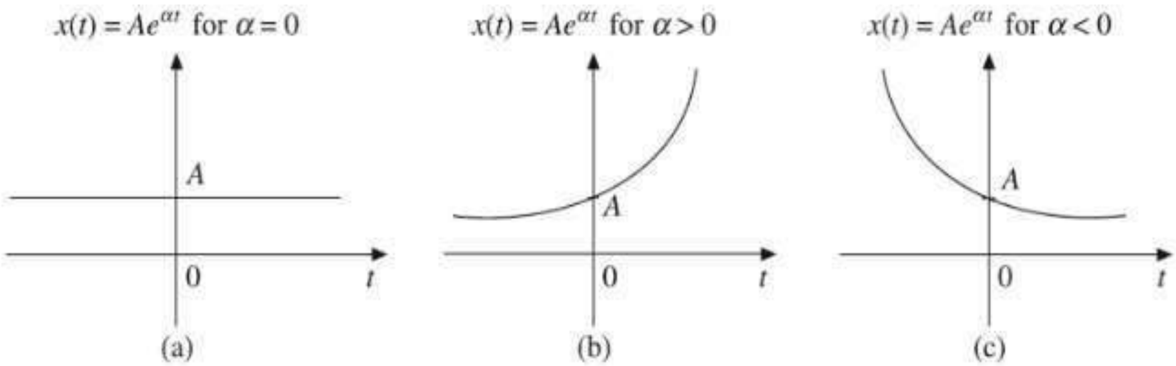


Figure 1. 6 **Continuous-time real exponential signals** $x(t) = Ae^{\alpha t}$ **for (a)** $\alpha = 0$, **(b)** $\alpha > 0$, **(c)** $\alpha < 0$.

1.2.7 Complex Exponential Signal

The complex exponential signal has a general form as

$$x(t) = Ae^{st}$$

where A is the amplitude and s is a complex variable defined as

$$s = \sigma + j\omega$$

Therefore, $$x(t) = Ae^{st} = Ae^{(\sigma + j\omega)t} = Ae^{\sigma t}e^{j\omega t}$$

$$= Ae^{\sigma t}[\cos \omega t + j \sin \omega t]$$

Depending on the values of σ and ω, we get different waveforms as shown in Figure 1.7

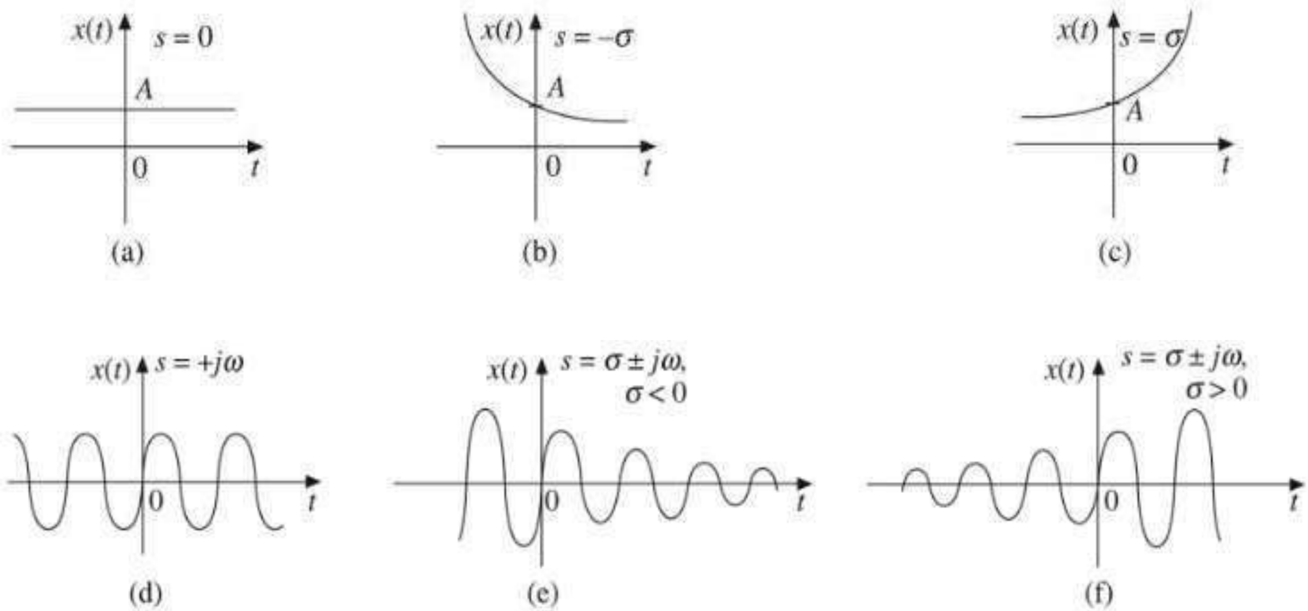


Figure 1. 7 Complex exponential signals.

1.2.8 Rectangular Pulse Function

The unit rectangular pulse function $\Pi(t/\tau)$ shown in Figure 1.16 is defined as

$$\Pi\left(\frac{t}{\tau}\right)=\begin{cases}1 & \text{for } |t|\leq\dfrac{\tau}{2}\\ 0 & \text{otherwise}\end{cases}$$

It is an even function of t.

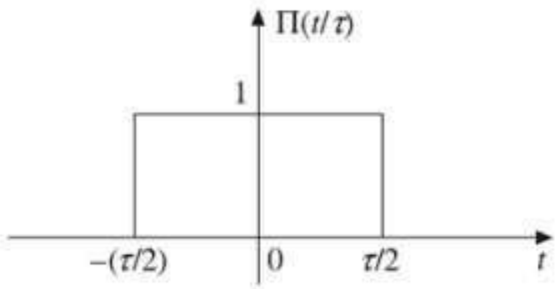


Figure 1.8 Rectangular pulse function.

1.3 ELEMENTARY DISCRETE-TIME SIGNALS

There are several elementary signals which play vital role in the study of signals and systems. These elementary signals serve as basic building blocks for the construction of more complex signals. Infact, these elementary signals may be used to model a large number of physical signals, which occur in nature. These elementary signals are also called standard signals.

The standard discrete-time signals are as follows:

1. Unit step sequence
2. Unit ramp sequence
3. Unit parabolic sequence
4. Unit impulse sequence
5. Sinusoidal sequence
6. Real exponential sequence
7. Complex exponential sequence

1.3.1 Unit Step Sequence

The step sequence is an important signal used for analysis of many discrete-time systems. It exists only for positive time and is zero for negative time. It is equivalent to applying a signal whose amplitude suddenly changes and remains constant at the sampling instants forever after application. In between the discrete instants it is zero. If a step function has unity magnitude, then it is called unit step function.

The usefulness of the unit-step function lies in the fact that if we want a sequence to start at $n = 0$, so that it may have a value of zero for $n < 0$, we only need to multiply the given sequence with unit step function $u(n)$.

The discrete-time unit step sequence $u(n)$ is defined as:

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

The shifted version of the discrete-time unit step sequence $u(n - k)$ is defined as:

$$u(n-k) = \begin{cases} 1 & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

It is zero if the arugment $(n - k) < 0$ and equal to 1 if the arugment $(n - k) \geq 0$.

The graphical representation of $u(n)$ and $u(n - k)$ is shown in Figure 1.3[(a) and (b)].

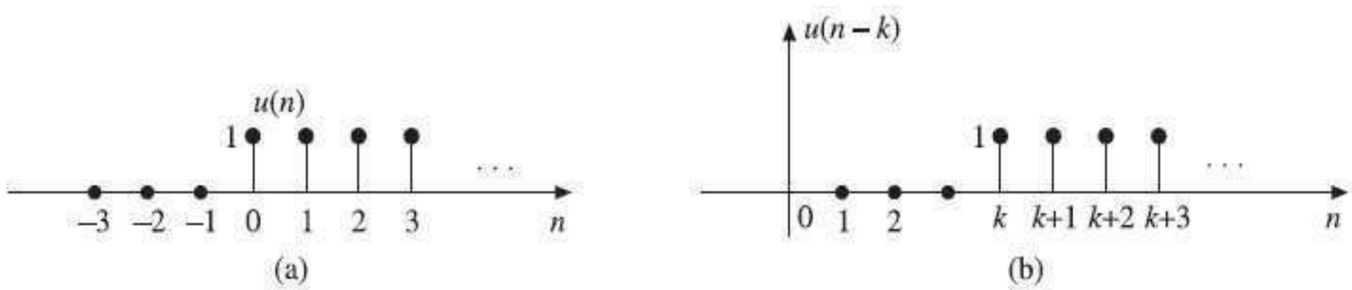


Figure 1.3 Discrete-time (a) Unit step function (b) Shifted unit step function.

1.3.2 Unit Ramp Sequence

The discrete-time unit ramp sequence $r(n)$ is that sequence which starts at $n = 0$ and increases linearly with time and is defined as:

$$r(n) = \begin{cases} n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$r(n) = nu(n)$$

It starts at $n = 0$ and increases linearly with n.

The shifted version of the discrete-time unit ramp sequence $r(n - k)$ is defined as:

$$r(n-k) = \begin{cases} n-k & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases}$$

or $$r(n - k) = (n - k)\, u(n - k)$$

The graphical representation of $r(n)$ and $r(n - 2)$ is shown in Figure 1.4[(a) and (b)].

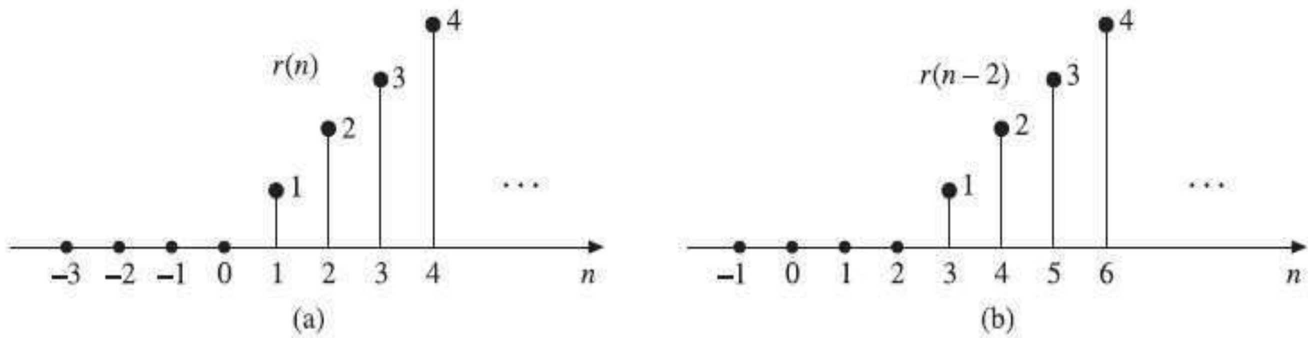


Figure 1.4 Discrete-time (a) Unit ramp sequence (b) Shifted ramp sequence.

1.3.3 Unit Parabolic Sequence

The discrete-time unit parabolic sequence $p(n)$ is defined as:

$$p(n) = \begin{cases} \dfrac{n^2}{2} & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

or $$p(n) = \frac{n^2}{2} u(n)$$

The shifted version of the discrete-time unit parabolic sequence $p(n - k)$ is defined as:

$$p(n-k) = \begin{cases} \frac{(n-k)^2}{2} & \text{for } n \geq k \\ 0 & \text{for } n < k \end{cases} \quad \text{or} \quad p(n-k) = \frac{(n-k)^2}{2} u(n-k)$$

The graphical representation of $p(n)$ and $p(n - 3)$ is shown in Figure 1.5[(a) and (b)].

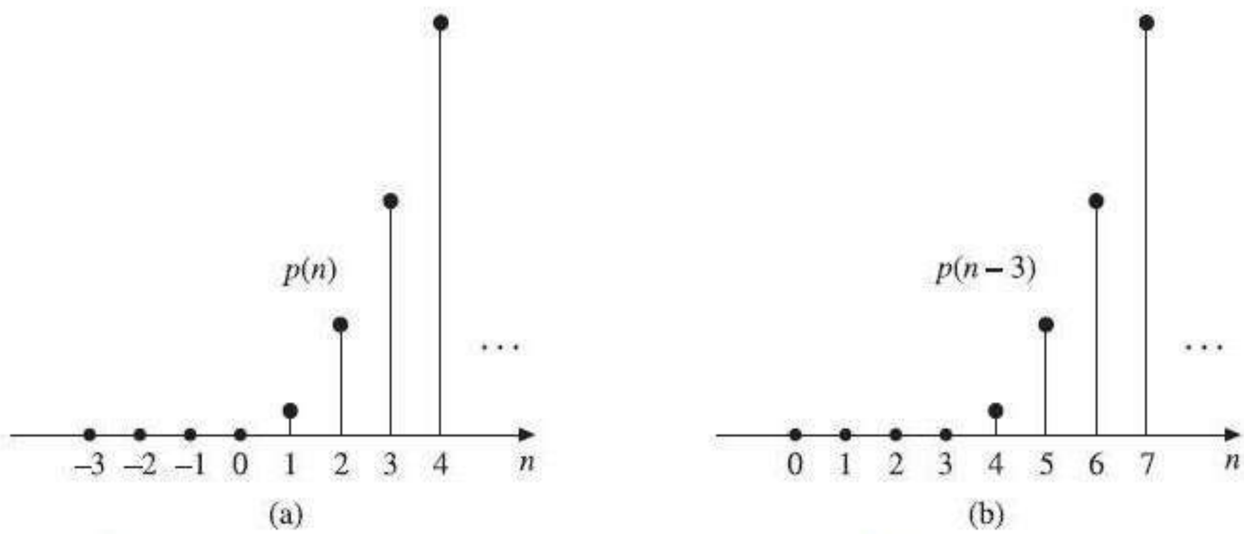


Figure 1.5 Discrete-time (a) Parabolic sequence (b) Shifted parabolic sequence.

1.3.4 Unit Impulse Function or Unit Sample Sequence

The discrete-time unit impulse function $\delta(n)$, also called unit sample sequence, is defined as:

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

This means that the unit sample sequence is a signal that is zero everywhere, except at $n = 0$, where its value is unity. It is the most widely used elementary signal used for the analysis of signals and systems.

The shifted unit impulse function $\delta(n - k)$ is defined as:

$$\delta(n - k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k \end{cases}$$

The graphical representation of $\delta(n)$ and $\delta(n - k)$ is shown in Figure 1.6[(a) and (b)].

1.3.5 Sinusoidal Sequence

The discrete-time sinusoidal sequence is given by

$$x(n) = A \sin(\omega n + \phi)$$

where A is the amplitude, ω is angular frequency, ϕ is phase angle in radians and n is an integer.

The period of the discrete-time sinusoidal sequence is:

$$N = \frac{2\pi}{\omega} m$$

where N and m are integers.

All continuous-time sinusoidal signals are periodic, but discrete-time sinusoidal sequences may or may not be periodic depending on the value of ω.

For a discrete-time signal to be periodic, the angular frequency ω must be a rational multiple of 2π. The graphical representation of a discrete-time sinusoidal signal is shown in Figure 1.7.

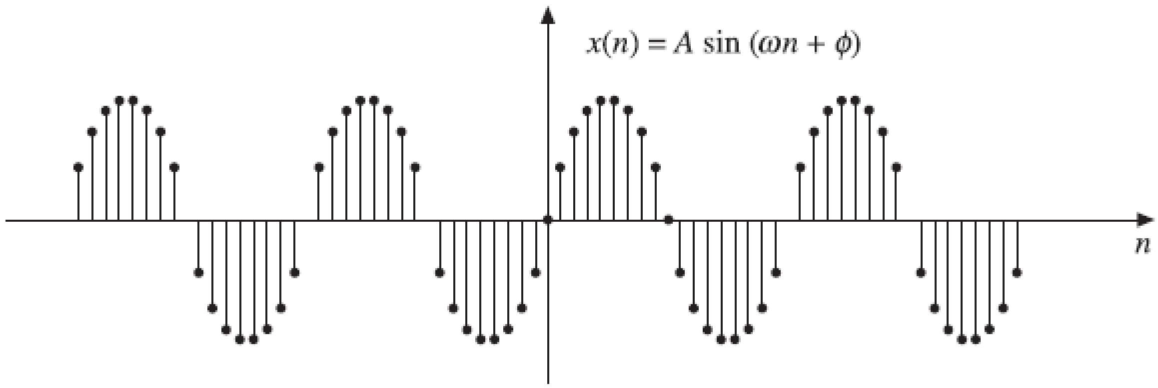


Figure 1.7 Discrete-time sinusoidal signal.

1.3.6 Real Exponential Sequence

The discrete-time real exponential sequence a^n is defined as:

$$x(n) = a^n \quad \text{for all } n$$

Figure 1.8 illustrates different types of discrete-time exponential signals.

When $a > 1$, the sequence grows exponentially as shown in Figure 1.8(a).

When $0 < a < 1$, the sequence decays exponentially as shown in Figure 1.8(b).

When $a < 0$, the sequence takes alternating signs as shown in Figure 1.8[(c) and (d)].

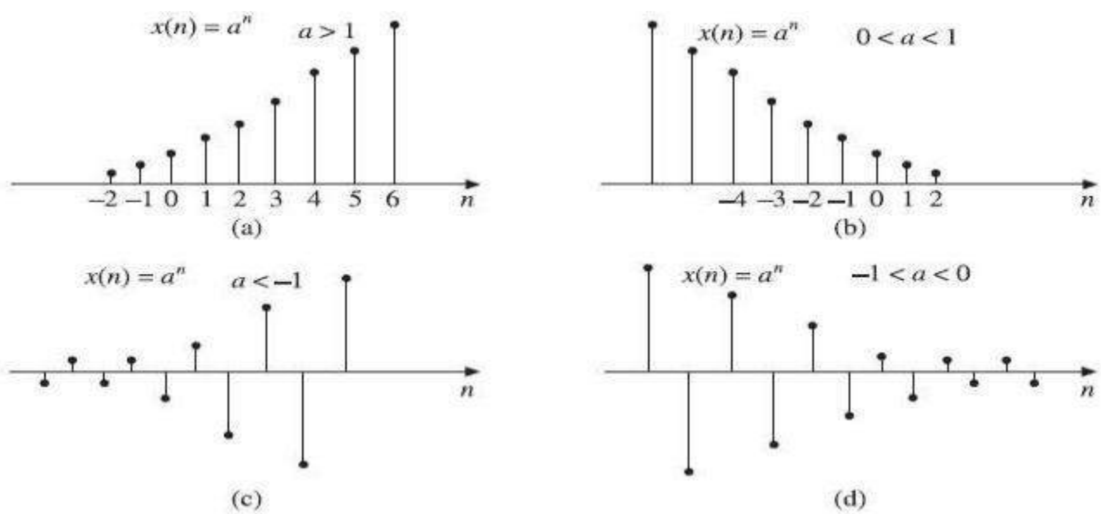


Figure 1.8 Discrete-time exponential signal a^n for (a) $a > 1$ (b) $0 < a < 1$ (c) $a < -1$ (d) $-1 < a < 0$.

1.3.7 Complex Exponential Sequence

The discrete-time complex exponential sequence is defined as:

$$x(n) = a^n e^{j(\omega_0 n+\phi)}$$
$$= a^n \cos(\omega_0 n + \phi) + ja^n \sin(\omega_0 n + \phi)$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequence are sinusoidal.

For $|a| > 1$, the amplitude of the sinusoidal sequence exponentially grows as shown in Figure 1.9(a).

For $|a| < 1$, the amplitude of the sinusoidal sequence exponentially decays as shown in Figure 1.9(b).

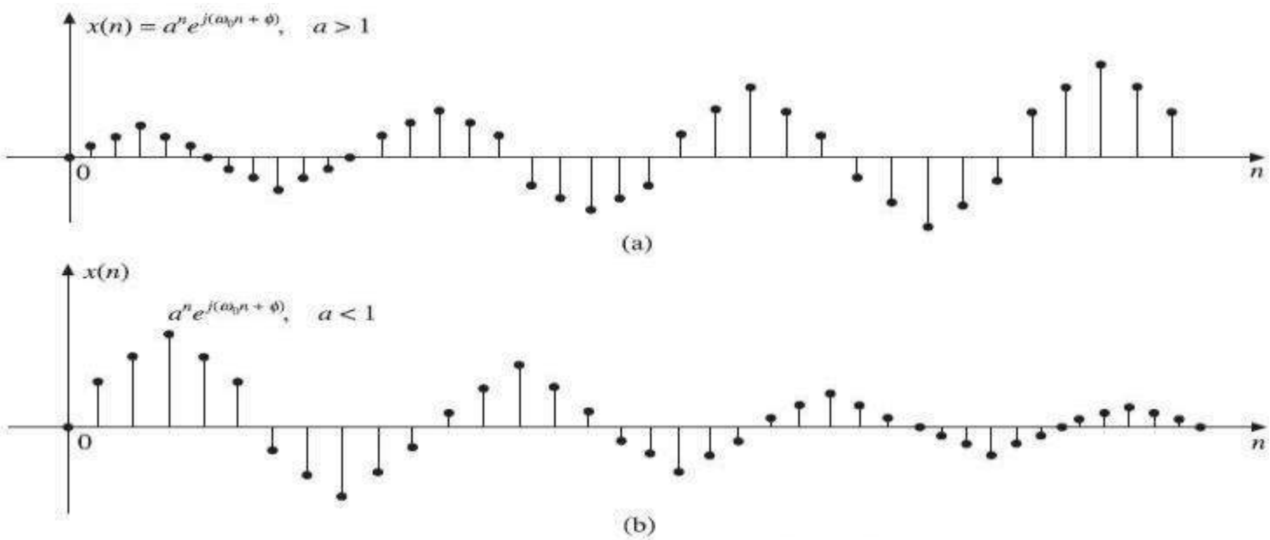


Figure 1.9 Complex exponential sequence $x(n) = a^n e^{j(\omega_0 n + \phi)}$ for (a) $a > 1$ (b) $a < 1$.

Classification of the Signals:

Based upon their nature characteristics in the time domain, the signals may be broadly classified as under

(a) Continuous0time signals

(b) Discrete –time signals

- Continuous-Time (CT) Signals: They may be de ned as continuous in time and continuous in amplitude as shown in Figure 1.5.1. Ex: Speech, audio signals etc..
- Discrete Time (DT) Signals: Discretized in time and Continuous in amplitude. They may also be defined as sampled version of continuous time signals. Ex: Rail track signals.

- Digital Signals: Discretized in time and quantized in amplitude. They may also be defined as quantized version of discrete signals.

$$x(n) = x(n + N_0) \qquad (1.2)$$

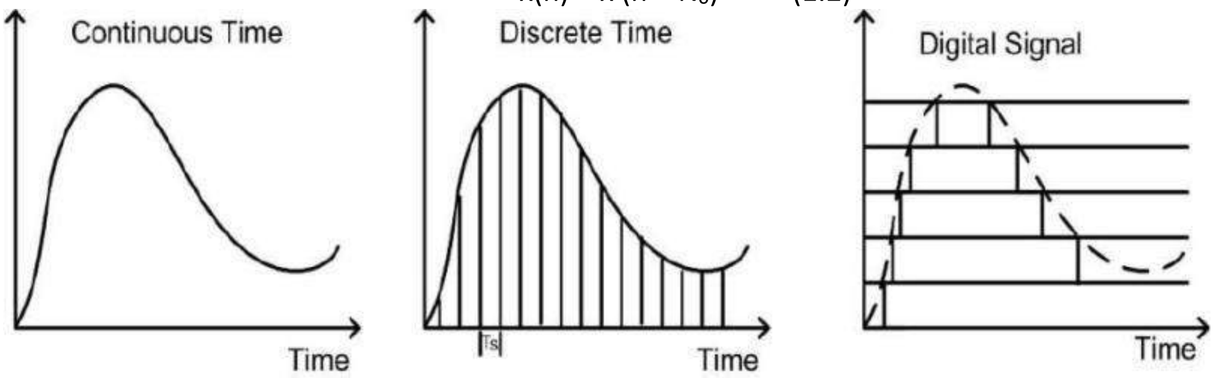


Figure :Description of Continuous, Discrete and Digital Signals

Both Continuous, Discrete and Digital Signals may be further classified into several categories depending upon the criteria and for its classification. Broadly the signals are classified into the following categories

1. Deterministic and Random signals
2. Periodic and Aperiodic Signals
3. Even and Odd Signals
4. Power and Energy Signals
5. Causal and non causal

Continuous-time and Discrete-time Signals:

Deterministic and Random signal

A deterministic signal is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table. Because of this the future values of the signal can be calculated from past values with complete confidence. On the other hand, a random signal has lot of uncertainty about its behavior. The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals.

Periodic Signals

A CT signal x(t) is said to be periodic if it satisfies the following condition

$$x(t) = x(t + T_0) \qquad (1.1)$$

The smallest positive value of T_0 that satisfies the periodicity condition Eq.(1.1), is referred as the fundamental period of x(t). The reciprocal of fundamental period of a signal is fundamental frequency f_0.

Likewise, a DT signal x[n] is said to be periodic if it satisfies

The smallest positive value of N0 that satis es the periodicity condition Eq.(1.2) is referred to as the fundamental period of x [n].

Note: All periodic signals are ever lasting signals i.e. they start at -1 and end at +1 as shown in below Figure.

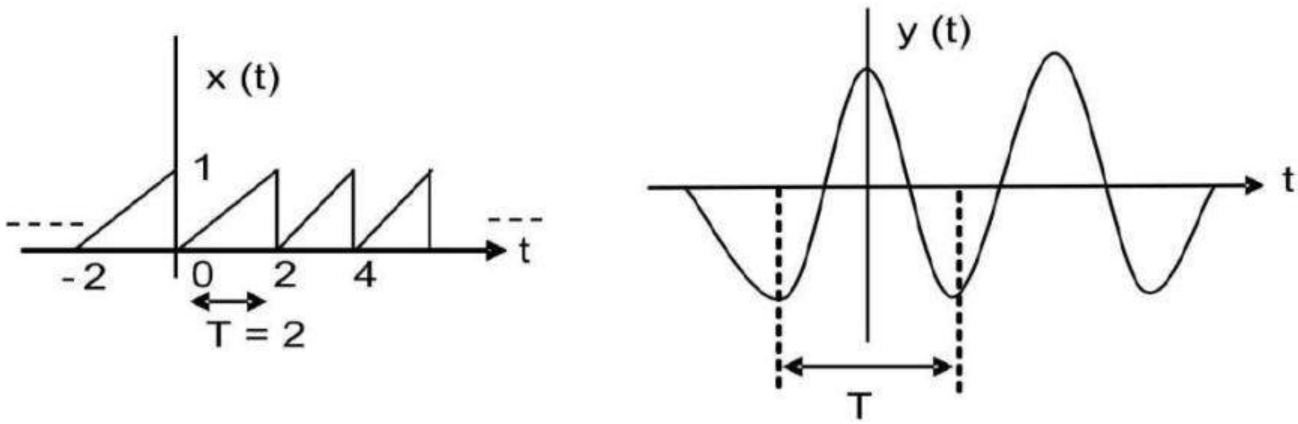
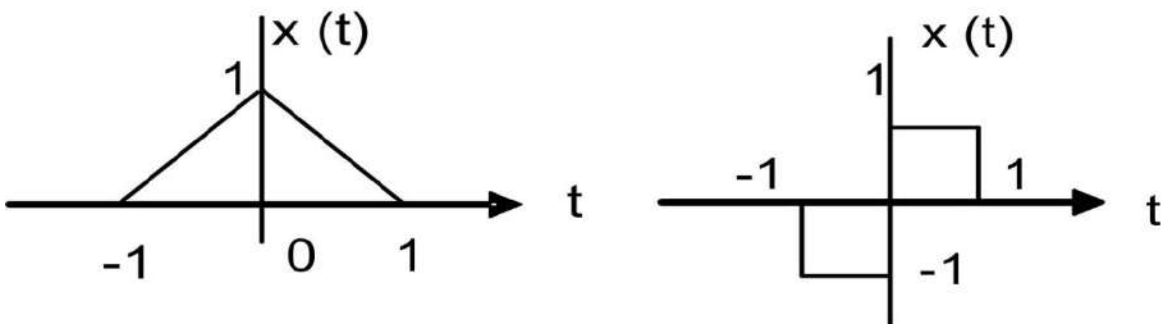


Figure : A typical periodic signal

Even and Odd Signals

Any signal can be called even signal if it satisfies x(t) = x(-t) or x(n) = x(-n). Similarly any signal can be called odd signal if it not satisfies x(t) = x(-t) or x(n) = x(-n). Below Figure shows an example of an even and odd signal whereas Figure 1.3 shows neither even nor odd signal.



Any signal X(t) can be expressed in terms of even component $X_e(t)$ and odd component $X_0(t)$.

$X(t) = X_e(t)+X_o(t)$, $X_e(t)+=(X(t) + X(-t)) / 2$, $X_o(t)+=(X(t) - X(-t)) / 2$

Energy and Power signals

A signal x(t) (or) x(n) is called an energy signal if total energy has a non - zero finite value

i.e. 0 < Ex < 1 and Pavg = 0
A signal is called a power signal if it has non - zero nite power i.e. 0 < Px < 1 and E = 1.
A signal can't be both an energy and power signal simultaneously. The term instantaneous power is reserved for the true rate of change of energy in a system. All periodic signals are power signals and all finite durations signals are energy signals.

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j).$$

$$P_x = \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2$$

A signal is referred to as a power signal if the power P_x satisfies the condition

$$0 < P_x < \infty$$

Causal and non causal

A continuous time signal xt) is said to be causal if x(t)=0 fort<= otherwise the signal is non causal

1.5 CLASSIFICATION OF DISCRETE-TIME SIGNALS

The signals can be classified based on their nature and characteristics in the time domain. They are broadly classified as: (i) continuous-time signals and (ii) discrete-time signals.

The signals that are defined for every instant of time are known as continuous-time signals. The continuous-time signals are also called analog signals. They are denoted by $x(t)$. They are continuous in amplitude as well as in time. Most of the signals available are continuous-time signals.

The signals that are defined only at discrete instants of time are known as discrete-time signals. The discrete-time signals are continuous in amplitude, but discrete in time. For discrete-time signals, the amplitude between two time instants is just not defined. For discrete-time signals, the independent variable is time n. Since they are defined only at discrete instants of time, they are denoted by a sequence $x(nT)$ or simply by $x(n)$ where n is an integer.

Figure 1.18 shows the graphical representation of discrete-time signals. The discrete-time signals may be inherently discrete or may be discrete versions of the continuous-time signals.

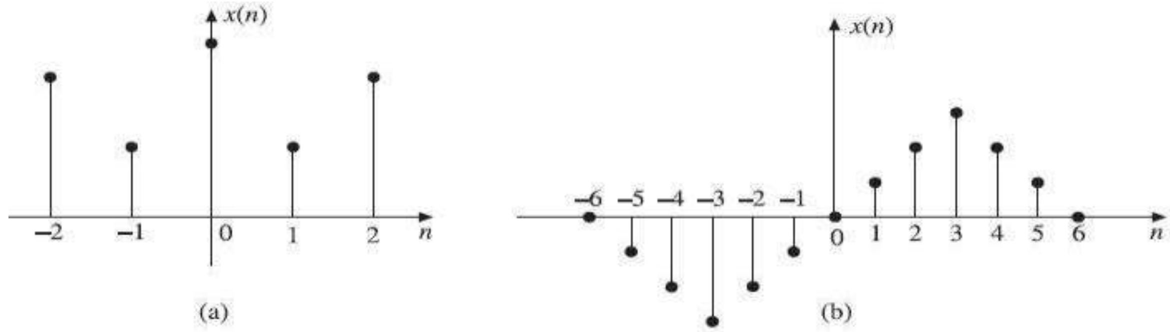


Figure 1.18 Discrete-time signals.

Both continuous-time and discrete-time signals are further classified as follows:

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals
5. Even and odd signals

1.5.1 Deterministic and Random Signals

A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time is called deterministic signal. A deterministic signal can be completely represented by mathematical equation at any time and its nature and amplitude at any time can be predicted.

Examples: Sinusoidal sequence $x(n) = \cos \omega n$, Exponential sequence $x(n) = e^{j\omega n}$, ramp sequence $x(n) = \alpha n$.

A signal characterized by uncertainty about its occurrence is called a non-deterministic or random signal. A random signal cannot be represented by any mathematical equation. The behaviour of such a signal is probabilistic in nature and can be analyzed only stochastically. The pattern of such a signal is quite irregular. Its amplitude and phase at any time instant cannot be predicted in advance. A typical example of a non-deterministic signal is thermal noise.

1.5.2 Periodic and Non-periodic Sequences

A signal which has a definite pattern and repeats itself at regular intervals of time is called a periodic signal, and a signal which does not repeat at regular intervals of time is called a non-periodic or aperiodic signal.

A discrete-time signal $x(n)$ is said to be periodic if it satisfies the condition $x(n) = x(n + N)$ for all integers n.

The smallest value of N which satisfies the above condition is known as fundamental period.

If the above condition is not satisfied even for one value of n, then the discrete-time signal is aperiodic. Sometimes aperiodic signals are said to have a period equal to infinity.

The angular frequency is given by

$$\omega = \frac{2\pi}{N}$$

∴ Fundamental period $$N = \frac{2\pi}{\omega}$$

The sum of two discrete-time periodic sequences is always periodic.

Some examples of discrete-time periodic/non-periodic signals are shown in Figure 1.19.

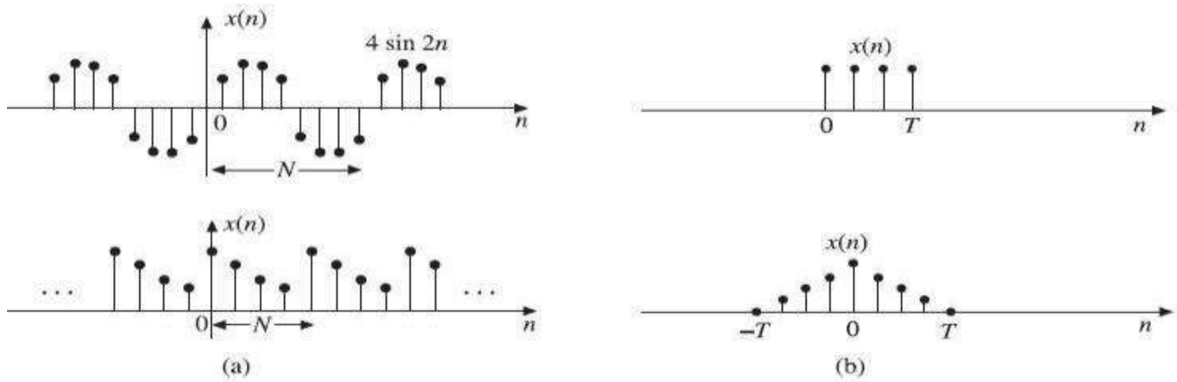


Figure 1.19 Examples of discrete-time: (a) Periodic and (b) Non-periodic signals.

EXAMPLE Show that the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic only if $\omega_0/2\pi$ is a rational number.

Solution: Given $$x(n) = e^{j\omega_0 n}$$

$x(n)$ will be periodic if $$x(n + N) = x(n)$$

i.e. $$e^{j[\omega_0(n+N)]} = e^{j\omega_0 n}$$

i.e. $$e^{j\omega_0 N}\, e^{j\omega_0 n} = e^{j\omega_0 n}$$

This is possible only if $$e^{j\omega_0 N} = 1$$

This is true only if $$\omega_0 N = 2\pi k$$

where k is an integer.

∴ $$\frac{\omega_0}{2\pi} = \frac{k}{N} \text{ Rational number}$$

This shows that the complex exponential sequence $x(n) = e^{j\omega_0 n}$ is periodic if $\omega_0/2\pi$ is a rational number.

1.5.3 Energy and Power Signals

Signals may also be classified as energy signals and power signals. However there are some signals which can neither be classified as energy signals nor power signals.

The total energy E of a discrete-time signal $x(n)$ is defined as:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

and the average power P of a discrete-time signal $x(n)$ is defined as:

$$P = \underset{N \to \infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

or $P = \frac{1}{N} \sum_{n=0}^{n-1} |x(n)|^2$ for a digital signal with $x(n) = 0$ for $n < 0$.

A signal is said to be an energy signal if and only if its total energy E over the interval $(-\infty, \infty)$ is finite (i.e., $0 < E < \infty$). For an energy signal, average power $P = 0$. Non-periodic signals which are defined over a finite time (also called time limited signals) are the examples of energy signals. Since the energy of a periodic signal is always either zero or infinite, any periodic signal cannot be an energy signal.

A signal is said to be a power signal, if its average power P is finite (i.e., $0 < P < \infty$). For a power signal, total energy $E = \infty$. Periodic signals are the examples of power signals. Every bounded and periodic signal is a power signal. But it is true that a power signal is not necessarily a bounded and periodic signal.

Both energy and power signals are mutually exclusive, i.e. no signal can be both energy signal and power signal.

The signals that do not satisfy the above properties are neither energy signals nor power signals. For example, $x(n) = u(n)$, $x(n) = nu(n)$, $x(n) = n^2u(n)$.

These are signals for which neither P nor E are finite. If the signals contain infinite energy and zero power or infinite energy and infinite power, they are neither energy nor power signals.

If the signal amplitude becomes zero as $|n| \to \infty$, it is an energy signal, and if the signal amplitude does not become zero as $|n| \to \infty$, it is a power signal.

1.5.4 Causal and Non-causal Signals

A discrete-time signal $x(n)$ is said to be causal if $x(n) = 0$ for $n < 0$, otherwise the signal is non-causal. A discrete-time signal $x(n)$ is said to be anti-causal if $x(n) = 0$ for $n > 0$.

A causal signal does not exist for negative time and an anti-causal signal does not exist for positive time. A signal which exists in positive as well as negative time is called a non-casual signal.

$u(n)$ is a causal signal and $u(-n)$ an anti-causal signal, whereas $x(n) = 1$ for $-2 \le n \le 3$ is a non-causal signal.

1.5.5 Even and Odd Signals

Any signal $x(n)$ can be expressed as sum of even and odd components. That is

$$x(n) = x_e(n) + x_o(n)$$

where $x_e(n)$ is even components and $x_o(n)$ is odd components of the signal.

Even (symmetric) signal

A discrete-time signal $x(n)$ is said to be an even (symmetric) signal if it satisfies the condition:

$$x(n) = x(-n) \quad \text{for all } n$$

Odd (anti-symmetric) signal

A discrete-time signal $x(n)$ is said to be an odd (anti-symmetric) signal if it satisfies the condition:

$$x(-n) = -x(n) \quad \text{for all } n$$

Basic Operations on Signals

The signals may undergo several manipulation involving the independent variable or the amplitude of the signal. The basic operation on signals are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal Addition
6. Signal multiplication

1.6.1 Time Shifting

Mathematically, the time shifting of a continuous-time signal $x(t)$ can be represented by

$$y(t) = x(t - T)$$

The time shifting of a signal may result in time delay or time advance. In the above equation if T is positive the shift is to the right and then the shifting delays the signal, and if T is negative the shift is to the left and then the shifting advances the signal. An arbitrary

signal $x(t)$, its delayed version and advanced version are shown in Figure 1.21[(a), (b) and (c)]. Shifting a signal in time means that a signal may be either advanced in the time axis or delayed in the time axis.

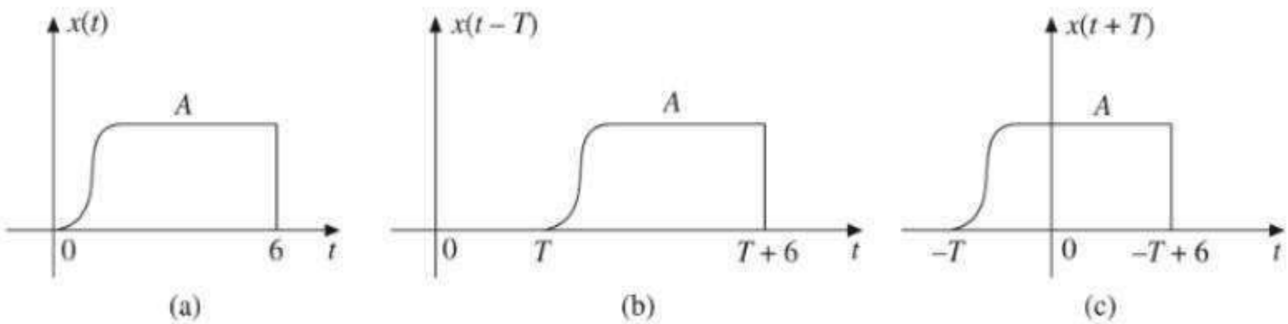


Figure 1.21 (a) Signal, (b) Its delayed version, (c) Its time advanced version.

1.6.2 Time Reversal

The time reversal, also called time folding of a signal $x(t)$ can be obtained by folding the signal about $t = 0$. This operation is very useful in convolution. It is denoted by $x(-t)$. It is obtained by replacing the independent variable t by $(-t)$. Folding is also called as the

reflection of the signal about the time origin $t = 0$. Figure 1.23(a) shows an arbitrary signal $x(t)$, and Figure 1.23(b) shows its reflection $x(-t)$.

The signal $x(-t + 3)$ obtained by shifting the reversed signal $x(-t)$ to the right by 3 units (delay by 3 units) is shown in Figure 1.23(c). The signal $x(-t - 3)$ obtained by shifting the reversed signal $x(-t)$ to the left by 3 units (advance by 3 units) is shown in Figure 1.23(d).

1.6.3 Amplitude Scaling

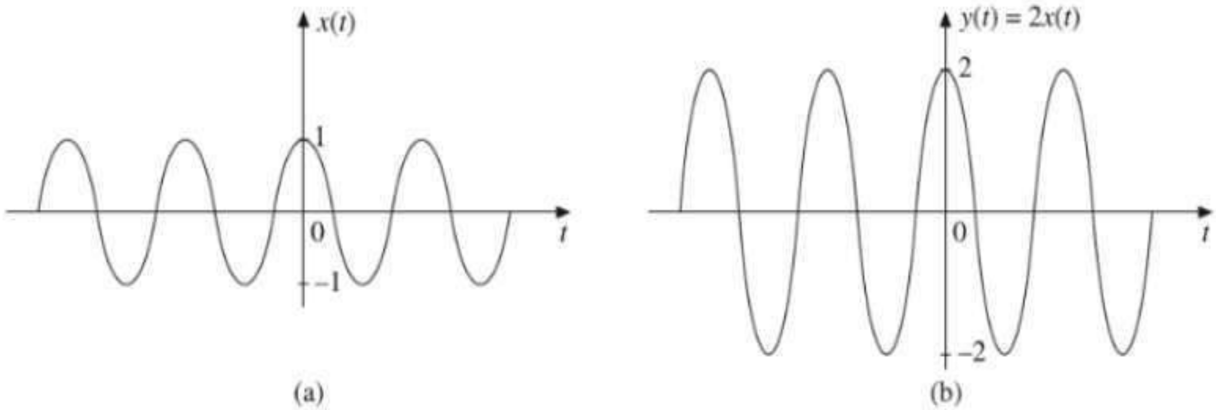
The amplitude scaling of a continuous-time signal $x(t)$ can be represented by

$$y(t) = Ax(t)$$

where A is a constant.

The amplitude of $y(t)$ at any instant is equal to A times the amplitude of $x(t)$ at that instant, but the shape of $y(t)$ is same as the shape of $x(t)$. If $A > 1$, it is amplification and if $A < 1$, it is attenuation.

Here the amplitude is rescaled. Hence the name amplitude scaling. Figure 1.35(a) shows an arbitrary signal $x(t)$ and Figure 1.35(b) shows $y(t) = 2x(t)$.



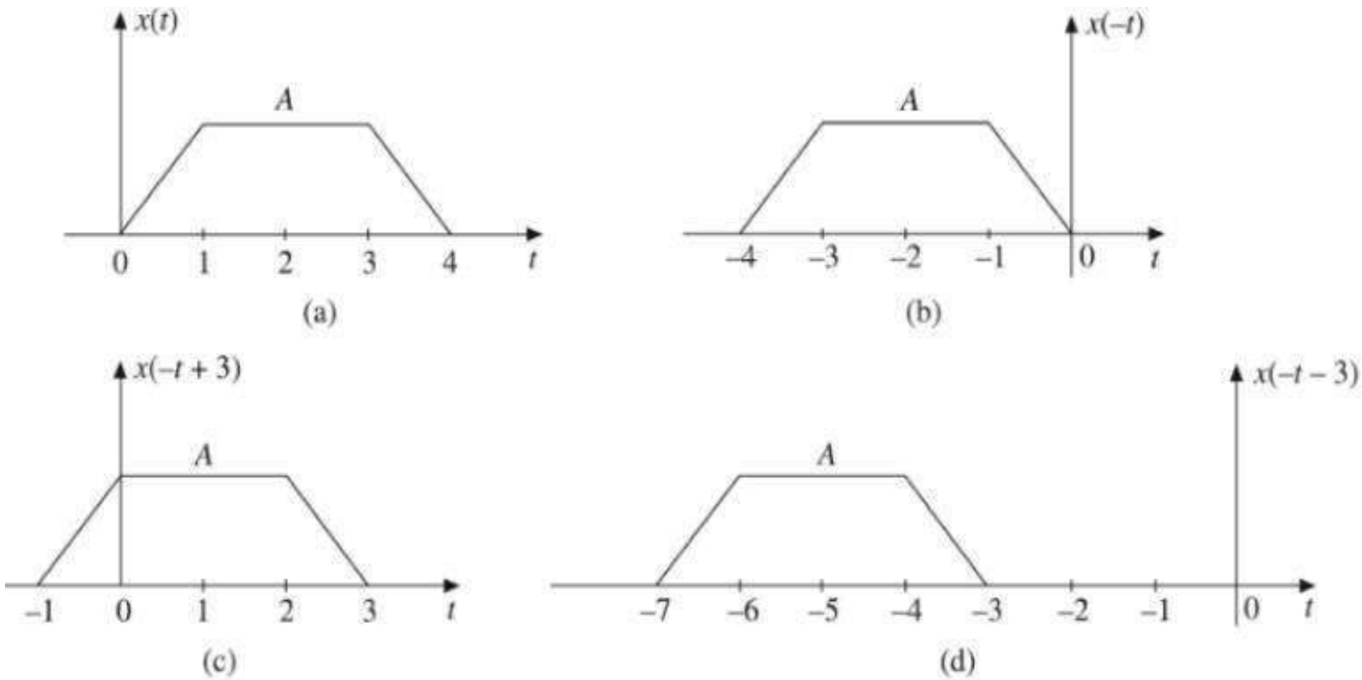


Figure 1.23 (a) An arbitrary signal $x(t)$, (b) Time reversed signal $x(-t)$, (c) Time reversed

1.6.4 Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a signal $x(t)$ can be accomplished by replacing t by at in it. Mathematically, it can be expressed as:

$$y(t) = x(at)$$

If $a > 1$, it results in time compression by a factor a and if $a < 1$, it results in time expansion by a factor a because with that transformation a point at 'at' in signal $x(t)$ becomes a point at 't' in $y(t)$.

Consider a signal shown in Figure 1.37(a). For a transformation $y(t) = x(2t)$, the time compressed signal is as shown in Figure 1.37(b) and for a transformation $y(t) = x(t/2)$ the time expanded signal is as shown in Figure 1.37(c).

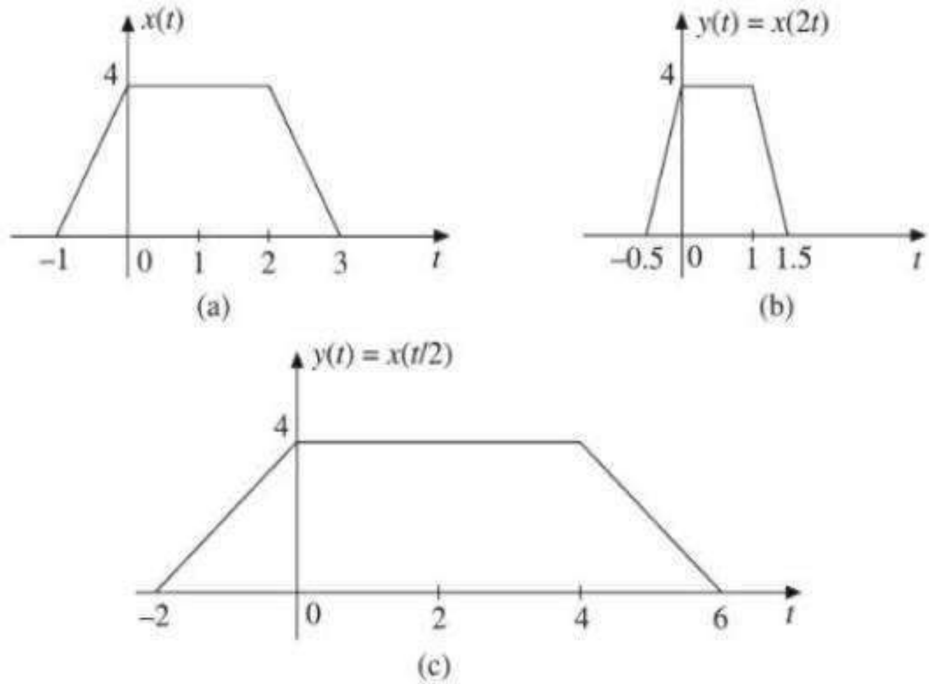


Figure 1.37 (a) Original signal, (b) Compressed signal, (c) Enlarged signal.

1.6.5 Signal Addition

The sum of two continuous-time signals $x_1(t)$ and $x_2(t)$ can be obtained by adding their values at every instant of time. Similarly, the subtraction of one continuous-time signal $x_2(t)$ from another signal $x_1(t)$ can be obtained by subtracting the value of $x_2(t)$ from that of $x_1(t)$ at every instant. Consider two signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.39[(a) and (b)].

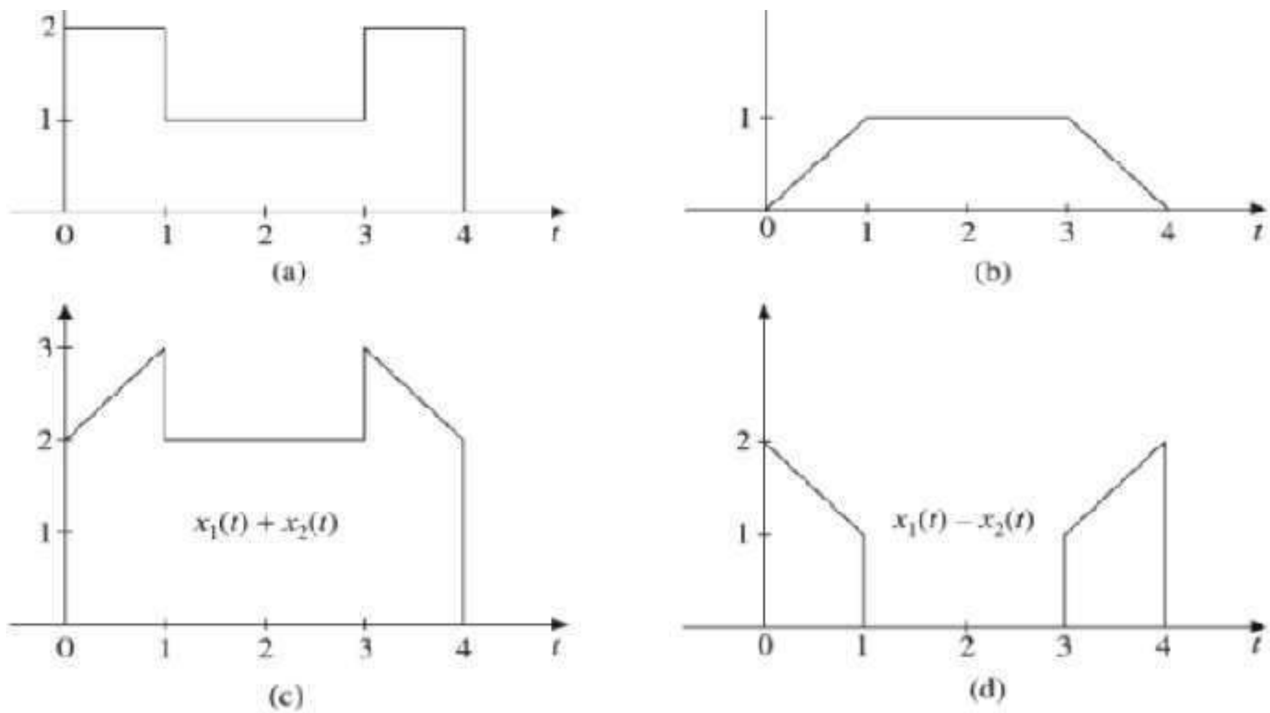


Figure 1.39 Addition and subtraction of continuous-time signals.

1.6.6 Signal Multiplication

The multiplication of two continuous-time signals can be performed by multiplying their values at every instant. Two continuous-time signals $x_1(t)$ and $x_2(t)$ shown in Figure 1.40[(a) and (b)] are multiplied as shown below to obtain $x_1(t)\,x_2(t)$ shown in Figure 1.40(c).

For $0 \le t \le 1$ $\quad x_1(t) = 2$ and $x_2(t) = 1$

Hence $x_1(t)\,x_2(t) = 2 \times 1 = 2$

For $1 \le t \le 2$ $\quad x_1(t) = 1$ and $x_2(t) = 1 + (t - 1)$

Hence $x_1(t)\,x_2(t) = (1)[1 + (t - 1)] = 1 + (t - 1)$

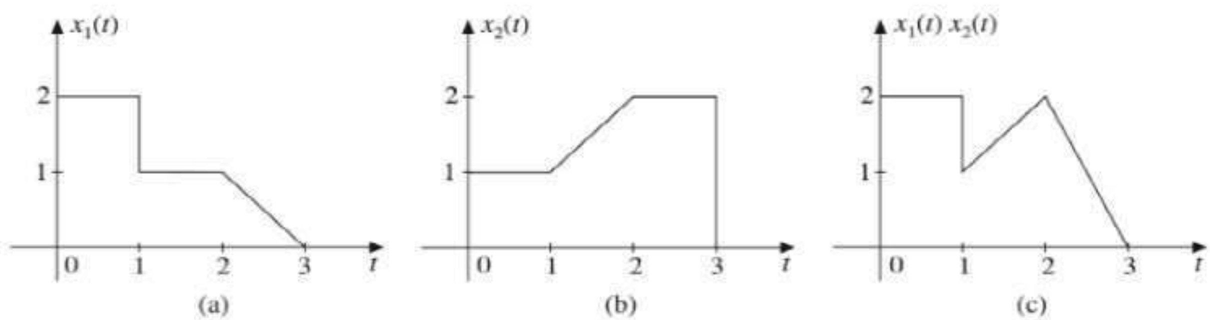


Figure 1.40 Multiplication of continuous-time signals.

1.7 BASIC OPERATIONS ON SEQUENCES

When we process a sequence, this sequence may undergo several manipulations involving the independent variable or the amplitude of the signal.

The basic operations on sequences are as follows:

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

The first three operations correspond to transformation in independent variable n of a signal. The last three operations correspond to transformation on amplitude of a signal.

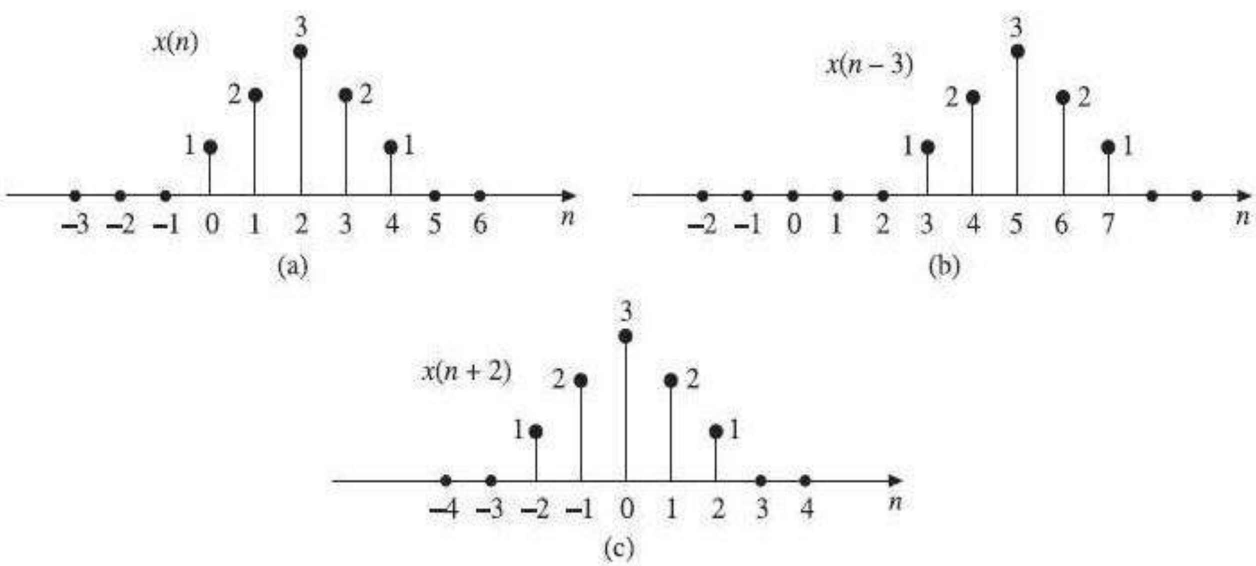
1.7.1 Time Shifting

The time shifting of a signal may result in time delay or time advance. The time shifting operation of a discrete-time signal $x(n)$ can be represented by

$$y(n) = x(n - k)$$

This shows that the signal $y(n)$ can be obtained by time shifting the signal $x(n)$ by k units. If k is positive, it is delay and the shift is to the right, and if k is negative, it is advance and the shift is to the left.

An arbitrary signal $x(n)$ is shown in Figure 1.10(a). $x(n - 3)$ which is obtained by shifting $x(n)$ to the right by 3 units (i.e. delay $x(n)$ by 3 units) is shown in Figure 1.10(b). $x(n + 2)$ which is obtained by shifting $x(n)$ to the left by 2 units (i.e. advancing $x(n)$ by 2 units) is shown in Figure 1.10(c).



1.7.2 Time Reversal

The time reversal also called time folding of a discrete-time signal $x(n)$ can be obtained by folding the sequence about $n = 0$. The time reversed signal is the reflection of the original signal. It is obtained by replacing the independent variable n by $-n$. Figure 1.11(a) shows an arbitrary discrete-time signal $x(n)$, and its time reversed version $x(-n)$ is shown in Figure 1.11(b). Figure 1.11[(c) and (d)] shows the delayed and advanced versions of reversed signal $x(-n)$.

The signal $x(-n + 3)$ is obtained by delaying (shifting to the right) the time reversed signal $x(-n)$ by 3 units of time. The signal $x(-n - 3)$ is obtained by advancing (shifting to the left) the time reversed signal $x(-n)$ by 3 units of time.

EXAMPLE Sketch the following signals: (a) $u(n+2)\,u(-n+3)$

Solution:

(a) Given $x(n) = u(n+2)\,u(-n+3)$

The signal $u(n + 2)\,u(-n + 3)$ can be obtained by first drawing the signal $u(n + 2)$ as shown in Figure 1.13(a), then drawing $u(-n + 3)$ as shown in Figure 1.13(b),

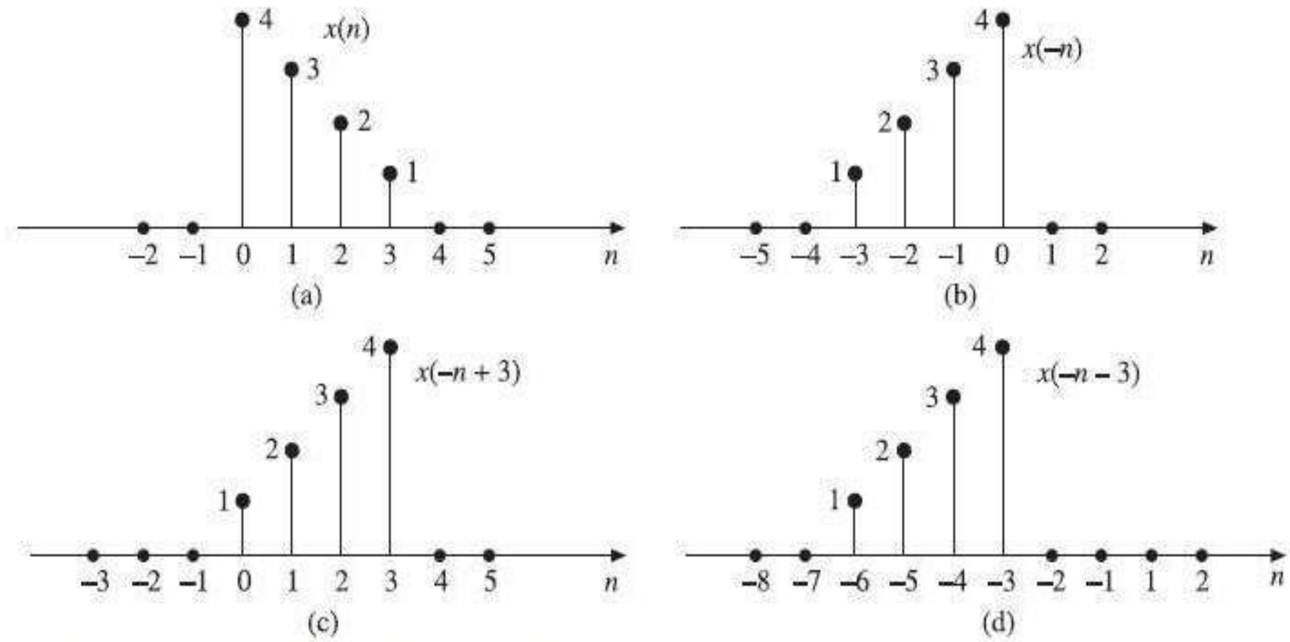


Figure 1.11 (a) Original signal $x(n)$ (b) Time reversed signal $x(-n)$ (c) Time reversed and delayed signal $x(-n+3)$ (d) Time reversed and advanced signal $x(-n-3)$.

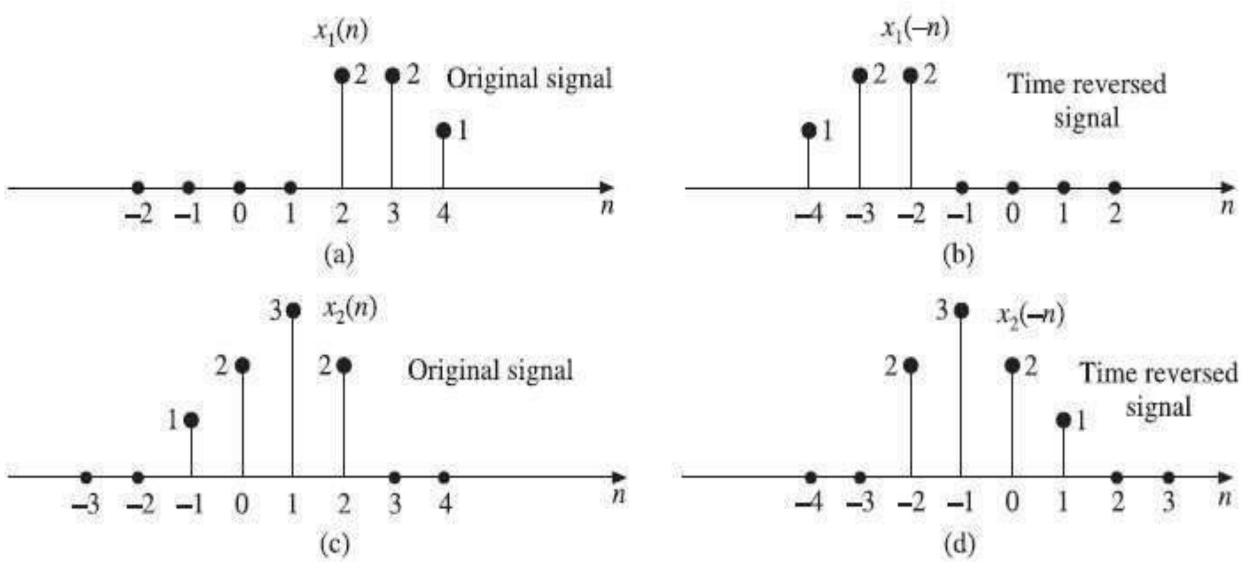


Figure 1.12 Time reversal operations.

1.7.3 Amplitude Scaling

The amplitude scaling of a discrete-time signal can be represented by

$$y(n) = ax(n)$$

where a is a constant.

The amplitude of $y(n)$ at any instant is equal to a times the amplitude of $x(n)$ at that instant. If $a > 1$, it is amplification and if $a < 1$, it is attenuation. Hence the amplitude is rescaled. Hence the name amplitude scaling.

Figure 1.15(a) shows a signal $x(n)$ and Figure 1.15(b) shows a scaled signal $y(n) = 2x(n)$.

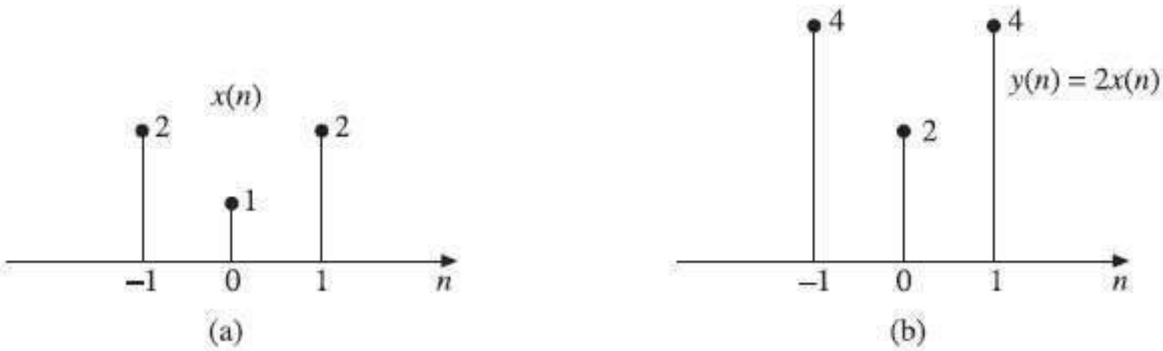


Figure 1.15 Plots of (a) Signal $x(n)$ (b) $y(n) = 2x(n)$.

1.7.4 Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a discrete-time signal $x(n)$ can be accomplished by replacing n by an in it. Mathematically, it can be expressed as:

$$y(n) = x(an)$$

When $a > 1$, it is time compression and when $a < 1$, it is time expansion.

Let $x(n)$ be a sequence as shown in Figure 1.16(a). If $a = 2$, $y(n) = x(2n)$. Then

$$y(0) = x(0) = 1$$
$$y(-1) = x(-2) = 3$$
$$y(-2) = x(-4) = 0$$
$$y(1) = x(2) = 3$$
$$y(2) = x(4) = 0$$

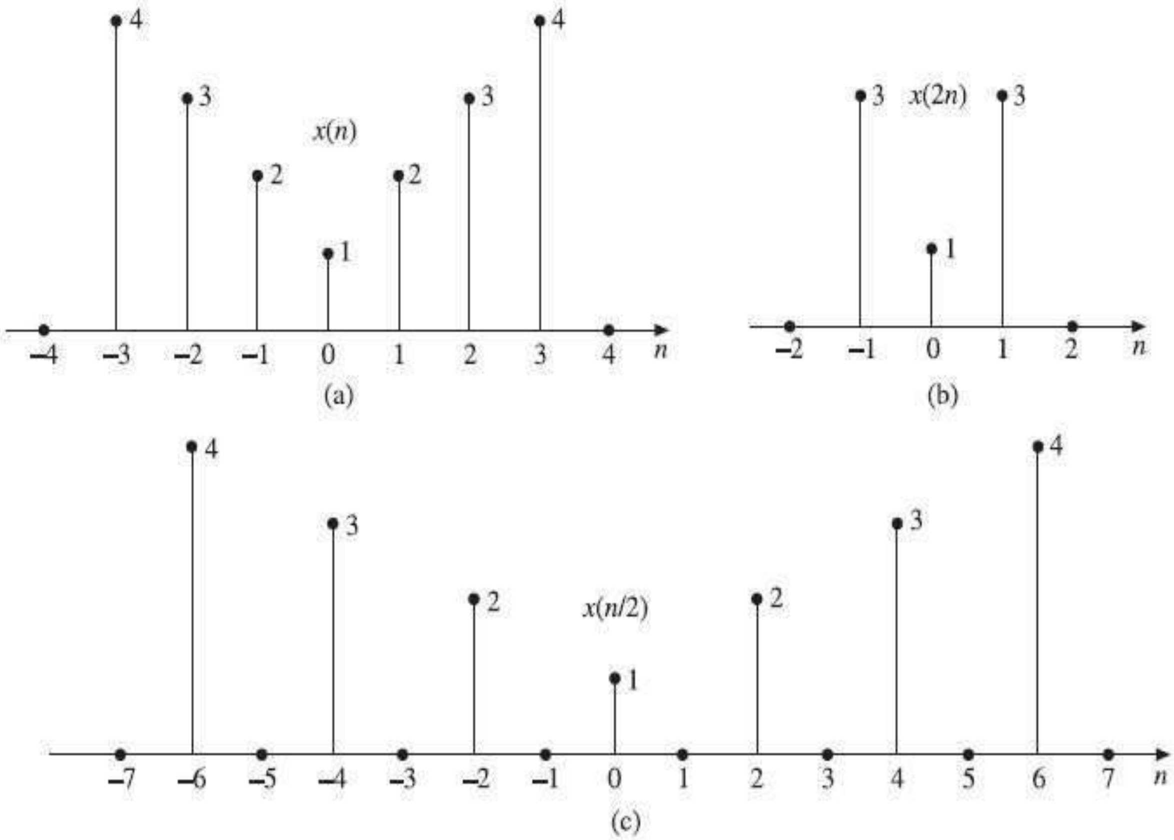


Figure 1.16 Discrete-time scaling (a) Plot of $x(n)$ (b) Plot of $x(2n)$ (c) Plot of $x(n/2)$.

1.7.5 Signal Addition

In discrete-time domain, the sum of two signals $x_1(n)$ and $x_2(n)$ can be obtained by adding the corresponding sample values and the subtraction of $x_2(n)$ from $x_1(n)$ can be obtained by subtracting each sample of $x_2(n)$ from the corresponding sample of $x_1(n)$ as illustrated below.

If $\quad x_1(n) = \{1, 2, 3, 1, 5\}$ and $x_2(n) = \{2, 3, 4, 1, -2\}$

Then $\quad x_1(n) + x_2(n) = \{1 + 2, 2 + 3, 3 + 4, 1 + 1, 5 - 2\} = \{3, 5, 7, 2, 3\}$

and $\quad x_1(n) - x_2(n) = \{1 - 2, 2 - 3, 3 - 4, 1 - 1, 5 + 2\} = \{-1, -1, -1, 0, 7\}$

1.7.6 Signal Multiplication

The multiplication of two discrete-time sequences can be performed by multiplying their values at the sampling instants as shown below.

If $\quad x_1(n) = \{1, -3, 2, 4, 1.5\}$ and $x_2(n) = \{2, -1, 3, 1.5, 2\}$

Then

$$x_1(n)\, x_2(n) = \{1 \times 2, -3 \times -1, 2 \times 3, 4 \times 1.5, 1.5 \times 2\}$$

$$= \{2, 3, 6, 6, 3\}$$

FOURIER SERIES

INTRODUCTION:

The representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions is called Fourier series. The Fourier analysis is also sometimes called the harmonic analysis. Fourier series is applicable only for periodic signals. It cannot be applied to non periodic signals. A periodic signal is one which repeats itself at regular intervals of time, i.e periodically over -∞ to ∞. Three important classes of Fourier series methods are available. They are

1. Trigonometric Form
2. Exponential Form
3. Cosine Form

In the representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions, if the orthogonal functions are exponential functions, then it is called exponential Fourier series. Similarly, in the representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions, if the orthogonal functions are trigonometric functions, then it is called trigonometric Fourier series.

Exponential Fourier series:

The exponential Fourier series is the most widely used form of Fourier series. In this, the function x(t) is expressed as a weighted sum of the complex exponential functions. The complex exponential form is more general and usually more convenient and more compact. So, it

The set of complex exponential functions

$$\{e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2, \ldots\}$$

forms a closed orthogonal set over an interval (t_0, t_0+T) where $T = (2\pi/\omega_0)$ for any value of t_0, and therefore it can be used as a Fourier series. Using Euler's identity, we can write

$$A_n \cos(n\omega_0 t + \theta_n) = A_n\left[\frac{e^{j(n\omega_0 t+\theta n)} + e^{-j(n\omega_0 t+\theta n)}}{2}\right]$$

Substituting this in the definition of the cosine Fourier representation, we obtain

$$x(t) = A_0 + \sum_{n=1}^{\infty} \frac{An}{2}\left[e^{j(n\omega_0 t+\theta n)} + e^{-j(n\omega_0 t+\theta n)}\right]$$

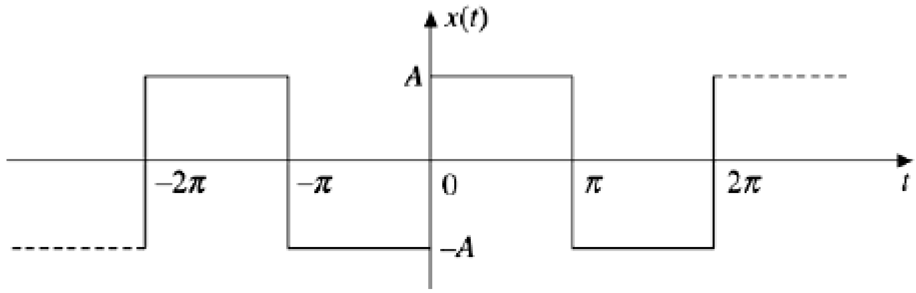
$$= A_0 + \sum_{n=1}^{\infty} \frac{An}{2}\left[e^{jn\omega_0 t}e^{j\theta n} + e^{-jn\omega_0 t}e^{-j\theta n}\right]$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{jn\omega_0 t}e^{j\theta n}\right) + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{-jn\omega_0 t}e^{-j\theta n}\right)$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{j\theta n}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{An}{2}e^{-j\theta n}\right)e^{-jn\omega_0 t}$$

Letting n = -K in the second summation of the above equation, we have is used almost exclusively, and it finds extensive application in communication theory.

1) Obtain the exponential Fourier Series for the wave form shown in below figure



$$x(t) = \begin{cases} A & 0 \le t \le \pi \\ -A & \pi \le t \le 2\pi \end{cases}$$

Let

$$t_0 = 0,\ t_0 + T = 2\pi$$

and Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

Exponential Fourier series

$$C_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} A\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} -A\,dt = 0$$

$$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

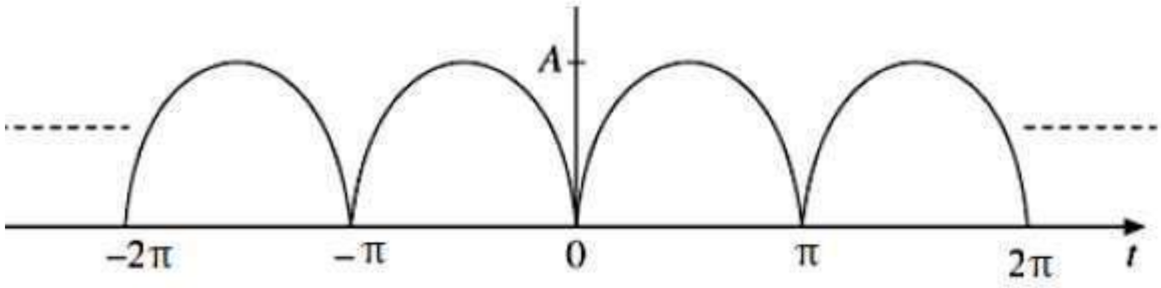
$$= \frac{1}{2\pi}\int_0^{\pi} Ae^{-jnt}\,dt + \frac{1}{2\pi}\int_{\pi}^{2\pi} -Ae^{-jnt}\,dt$$

$$= -\frac{A}{j2n\pi}[(-1)^n - 1] - [1 - (-1)^n] = -j\frac{A}{2n\pi}$$

$$C_n = \begin{cases} \left(-j\frac{2A}{\pi n}\right) & \text{for odd n} \\ 0 & \text{for even n} \end{cases}$$

Solution: The periodic waveform shown in fig with a period T= 2π can be expressed as:

2) Find the exponential Fourier series for the full wave rectified sine wave given in below figure.



Solution: The waveform shown in fig can be expressed over one period(0 to π) as:

$$x(t) = A \sin \omega t \text{ where } \omega = \frac{2\pi}{2\pi} = 1$$

because it is part of a sine wave with period = 2π

$$x(t) = A \sin \omega t \quad 0 \le t \le \pi$$

The full wave rectified sine wave is periodic with period T = π

Let

$$t_0 = 0,\ t_0 + T = 0 + \pi = \pi$$

and Fundamental frequency $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2$

The exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2nt}$$

where $C_n = \frac{1}{T}\int_0^T x(t) e^{-jn\omega_0 t}\, dt$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t\, e^{-j2nt}\, dt = \frac{A}{\pi}\int_0^{\pi} \sin t\, e^{-j2nt}\, dt$$

$$= \frac{A}{j2\pi}\left[\frac{e^{j(1-2n)t} - e^0}{j(1-2n)} - \frac{e^{-j(1-2n)t} - e^0}{-j(1-2n)}\right]$$

$$\therefore C_n = \frac{2A}{\pi(1-4n^2)}$$

$$C_0 = \frac{1}{T}\int_0^T x(t)\, dt$$

$$= \frac{1}{\pi}\int_0^{\pi} A \sin t\, dt = \frac{A}{\pi}[-\cos t]_0^{\pi} = \frac{2A}{\pi}$$

The exponential Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{j2nt} = \frac{2A}{\pi} + \frac{2A}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{e^{j2nt}}{1-4n^2}\right)$$

Complex Fourier Spectrum

The Fourier spectrum of a periodic signal x(t) is a plot of its Fourier coefficients versus frequency ω. It is in two parts: (a) Amplitude spectrum and (b) phase spectrum. The plot of the amplitude of Fourier coefficients verses frequency is known as the amplitude spectra, and the plot of the phase of Fourier coefficients verses frequency is known as phase spectra. The two plots together are known as Fourier frequency spectra of x(t).This type of representation is also called frequency domain representation. The Fourier spectrum exists only at discrete frequencies $n\omega_o$, where n=0,1,2,….. Hence it is known as discrete spectrum or line spectrum. The envelope of the spectrum depends only upon the pulse shape, but not upon the period of repetition.

The below figure (a) represents the spectrum of a trigonometric Fourier series extending from 0 to ∞, producing a one-sided spectrum as no negative frequencies exist here. The figure (b) represents the spectrum of a complex exponential Fourier series extending from $-\infty to \infty$, producing a two-sided spectrum. The amplitude spectrum of the exponential Fourier series is symmetrical Fourier series is symmetrical about the vertical axis. This is true for all periodic functions.

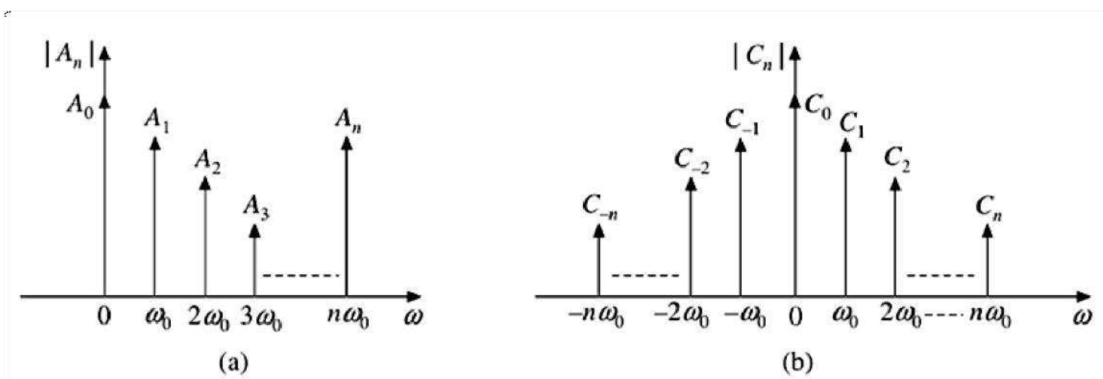


Fig: Complex frequency spectrum for (a) Trigonometric Fourier series and (b) complex exponential Fourier series. If C_n is a general complex number, then

$$C_n = |C_n| \; e^{j\theta n} \;\&\; C_{-n} = |C_n| \; e^{-j\theta n} \;\&\; C_n = |C_{-n}|$$

The magnitude spectrum is symmetrical about the vertical axis passing through the origin, and the phase spectrum is ant symmetrical about the vertical axis passing through the origin. So the magnitude spectrum exhibits even symmetry and phase spectrum exhibits odd symmetry. When x(t) is real , then $C_{-n}=C_n^*$, the complex conjugate of C_n.

DISCRETE FOURIER SERIES

The Fourier series representation of a periodic discrete-time sequence is called discrete Fourier series(DFS).Consider the discrete time signal x(n) that is periodic with period N defined by x(n)=x(n+KN) for any integer value of k. The periodic function x(n) can be synthesized as a linear combination of complex exponentials.

Exponential form of Discrete Fourier Series

A real periodic discrete time signal x(n) of period N can be expressed as a weighted sum of complex exponential sequences. The exponential form of the Fourier series for a periodic discrete time signal is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)nk} \quad \text{for all } n$$

where the coefficients $X(k)$ are expressed as:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk} \quad \text{for all } k$$

These equations for $x(n)$ and $X(k)$ are called DFS synthesis and analysis pair. Hence, $X(k)$ and $x(n)$ are periodic sequences.

The equivalent form for $X(k)$ is:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}$$

EXAMPLE Find both the exponential of the DFS representation of $x(n)$ shown in Figure

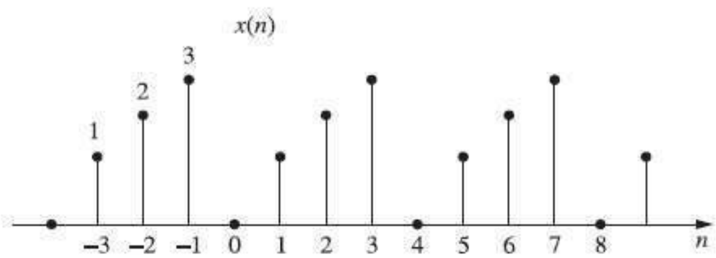


Figure x(n) for Example

Solution: To determine the exponential form of the DFS, we have

$$W_N^k = e^{-j(2\pi/N)k}$$

Given $N = 4$

$$\therefore \qquad W_4^0 = 1,\ W_4^1 = e^{-j(2\pi/4)1} = e^{-j(\pi/2)}$$

$$W_4^1 = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = (W_4^1)(W_4^1) = (-j)(-j) = -1$$

$$W_4^3 = (W_4^2)(W_4^1) = (-1)(-j) = j$$

$$W_4^4 = (W_4^2)(W_4^2) = (-1)(-1) = 1$$

The exponential form of DFS is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,W_N^{-nk} \quad \text{for all } n$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{j(\omega_0)nk}$$

where $$X(k) = \sum_{n=0}^{N-1} x(n)\,W_N^{nk} = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk} \quad \text{for all } k$$

For $k = 0$, $$X(0) = \sum_{n=0}^{3} x(n)W_4^{(0)n} = x(0) + x(1) + x(2) + x(3) = 0 + 1 + 2 + 3 = 6$$

For $k = 1$, $$X(1) = \sum_{n=0}^{3} x(n)W_4^{(1)n} = x(0)W_4^0 + x(1)W_4^1 + x(2)W_4^2 + x(3)W_4^3$$

$$= 0(1) + (1)(-j) + (2)(-1) + (3)(j) = -2 + j2$$

For $k = 2$, $$X(2) = \sum_{n=0}^{3} x(n)W_4^{(2)n} = x(0)W_4^0 + x(1)W_4^2 + x(2)W_4^4 + x(3)W_4^6$$

$$= 0 + 1(-1) + 2(1) + 3(-1) = -2$$

For $k = 3$, $$X(3) = \sum_{n=0}^{3} x(n)W_4^{(3)n} = x(0)W_4^0 + x(1)W_4^3 + x(2)W_4^6 + x(3)W_4^9$$

$$= 0 + 1(j) + 2(-1) + 3(-j) = -2 - j2$$

The complex exponential form of the Fourier series is:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,W_4^{-nk}$$

UNIT-II

FOURIER TRANSFORMS & DISCRETE FOURIER TRANSFORMS

- Fourier transform of arbitrary signal,
- Fourier transform of standard signals
- Properties of Fourier Transform.
- Discrete Fourier Transform
- Properties of DFT
- Linear Convolution of Sequences using DFT
- Computation of DFT: Over-lap Add Method & Over-lap Save Method.

FOURIER TRANSFORM

INTRODUCTION:

Using exponential form of Fourier series, any continuous –time periodic signal x(t) can be represented as a linear combination of complex exponentials and the Fourier coefficients are discrete. Fourier series can deal only with the periodic signals. This is the major drawback of Fourier series. However, all the naturally produced signals which need processing will be in the form of non-periodic or aperiodic signals. Therefore, the applicability of the Fourier series is limited.

Fourier Transform is a transformation technique which transforms signals from the continuous-time domain to the corresponding frequency domain and vice versa and which applies for both periodic as well as aperiodic signals. Fourier transform can be developed by finding the Fourier series of a periodic function and then tending to infinity. The Fourier Transform derived in this chapter is called the continuous-time Fourier transform (CTFT) . The Fourier Transform is an extremely useful mathematical tool and is extensively used in the analysis of linear time –invariant (LTI) systems, cryptography, signal analysis, signal processing, astronomy etc. Several applications ranging from RADAR to spread spectrum communication employ Fourier transform.

The magnitude of X(w) is given by $|X(w)| = \sqrt{X_R(w)^2 + X_I(w)^2}$

The phase of X(w) is given by $\angle X(w) = \tan^{-1}\frac{X_I(w)}{X_R(w)}$

The plot of $|X(w)|$ versus w is known as amplitude spectrum and the plot of $\angle X(w)$ versus w is known as phase spectrum. The amplitude spectrum and phase spectrum together is called frequency spectrum.

EXISTANCE FOURIER TRANSFORM:

The Fourier Transform does not exist for all aperiodic functions. The conditions for function x(t) to have Fourier Transform, called Dirichlet's conditions are:

1. x(t) is absolutely integrable over the interval- ∞ to ∞, that is $\int_{-\infty}^{\infty} |x(t)| dt < \infty$

2.x(t) has a finite number of discontinuities in every finite time interval, Further, each of these discontinuities must be finite.

3.x(t) has a finite number of maxima and minima in every finite time interval.

Dirichlet's condition is a sufficient condition but necessary condition. This means, Fourier transform will definitely exist for functions which satisfy these conditions. On the other hand, in some cases, Fourier transform can be found with the use of impulses even for functions like step functions, sinusoidal function, etc which do not satisfy the convergence condition.

FOURIER TRANSFORM OF STANDARD SIGNALS

1. **Impulse Function $\boldsymbol{\delta(t)}$**

Given $x(t) = \delta(t)$,

$$\delta(t) = \begin{cases} 1 \text{ for } t = 0 \\ 0 \text{ for } t \neq 0 \end{cases}$$

Then

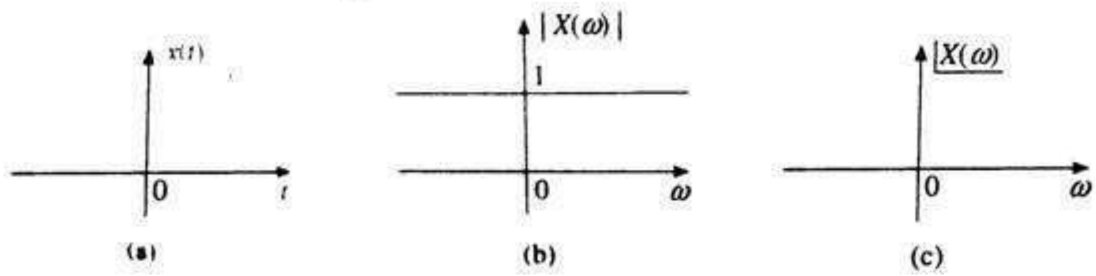
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt = e^{-j\omega t}\big|_{t=0} = 1$$

$$\therefore \qquad F[\delta(t)] = 1 \qquad \text{or } \delta(t) \overset{FT}{\leftrightarrow} 1$$

Hence , the Fourier Transform of a unit impulse function is unity.

$$|X(\omega)| = 1 \quad \text{for all } w$$
$$\lfloor X(w) = 0 \quad \text{for all } w$$

The impulse functions with its magnitude and phase spectra are shown in below figure:



Similarly,

$$F[\delta(t - t_o)] = \int_{-\infty}^{\infty} \delta(t - t_o)e^{-j\omega t}dt = e^{-j\omega t_0} \text{ i.e. } \delta(t - t_o) \overset{FT}{\leftrightarrow} e^{-j\omega t_o}$$

2. **Single Sided Real exponential function $\boldsymbol{e^{-at}u(t)}$**

Given $x(t) = e^{-at}u(t)$, $\qquad u(t) = \begin{cases} 1 \text{ for } t \geq 0 \\ 0 \text{ for } t < 0 \end{cases}$

Then

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = \left[\frac{e^{(-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} = \frac{e^{-\infty} - e^{0}}{-(a+j\omega)}$$

$$= \frac{0-1}{-(a+j\omega)} = \frac{1}{a+j\omega}$$

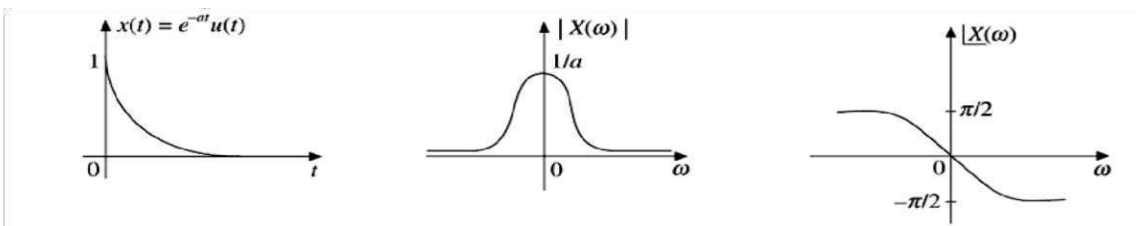
$\therefore F[e^{-at}u(t)] = \frac{1}{a+j\omega}$ or $e^{-at}u(t) \overset{FT}{\leftrightarrow} \frac{1}{a+j\omega}$

Now, $X(\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{(a+j\omega)(a-j\omega)}$

$$= \frac{a-j\omega}{a^2+\omega^2} = \frac{a}{a^2+\omega^2} - j\frac{\omega}{a^2+\omega^2} = \frac{1}{\sqrt{a^2+\omega^2}}\angle -\tan^{-1}\frac{\omega}{a}$$

$$\therefore |X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \angle X(\omega) = -\tan^{-1}\frac{\omega}{a} \text{ for all } \omega$$

Figure shows the single-sided exponential function with its magnitude and phase spectra.



(a) (b) (c)

3. Double sided real exponential function e^{-at}

Given $x(t) = e^{-a|t|}$

$$\therefore x(t) = e^{-a|t|} = \begin{cases} e^{-a(-t)} = e^{at} \text{ for } t \le 0 \\ e^{-at} = e^{-at} \text{ for } t \ge 0 \end{cases}$$

$$= e^{-a(-t)}u(-t) + e^{-at}u(t)$$

$$= e^{at}u(-t) + e^{-at}u(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \int_0^{\infty} e^{-(a-j\omega)t}dt + \int_0^{\infty} e^{-(a+j\omega)t}dt = \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty}$$

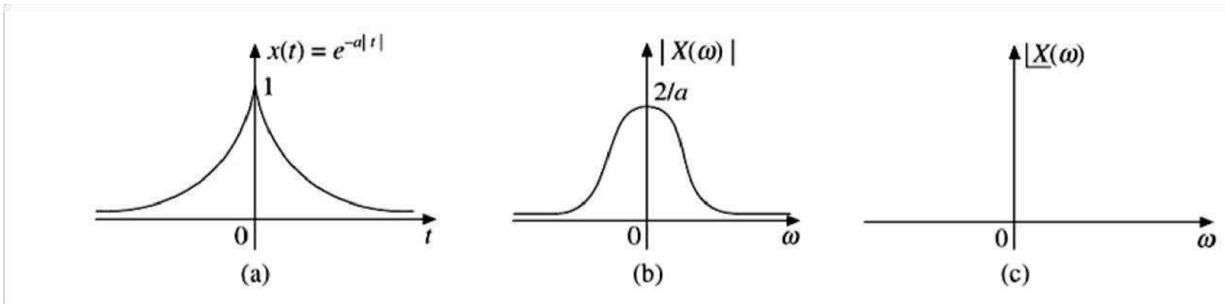
$$= \frac{e^{-\infty} - e^{-0}}{-(a-j\omega)} + \frac{e^{-\infty} - e^{-0}}{-(a+j\omega)} = \frac{1}{a-j\omega} + \frac{1}{a+j\omega} = \frac{2a}{a^2+\omega^2}$$

$$\therefore F\left(e^{-a[t]}\right) = \frac{2a}{a^2+\omega^2} \text{ or } e^{-a[t]} \overset{FT}{\leftrightarrow} \frac{2a}{a^2+\omega^2}$$

$$\therefore [X(\omega)] = \frac{2a}{a^2+\omega^2} \text{ for all } \omega$$

And $\angle X(\omega) = 0 \quad \text{for all}\,\omega$

A Two sided exponential function and its amplitude and phase spectra are shown in figures below:

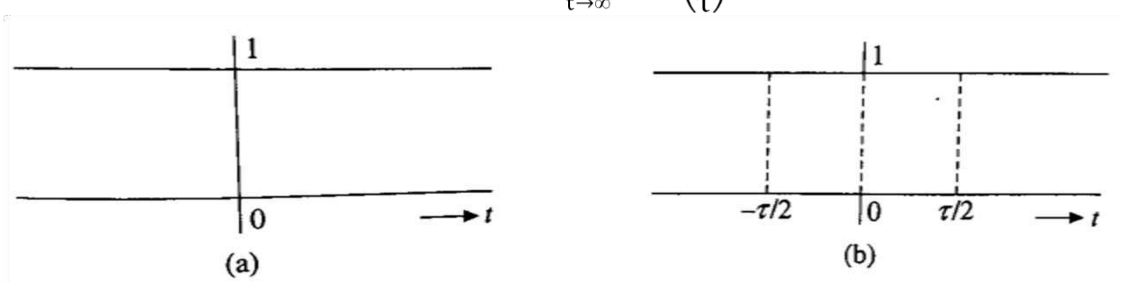


4. Constant Amplitude (1)

Let $\quad x(t) = 1 \qquad -\infty \le t \le \infty$

The waveform of a constant function is shown in below figure .Let us consider a small section of constant function, say, of duration τIf we extend the small duration to infinity, we will get back the original function.Therefore

$$x(t) = \underset{t\to\infty}{\text{Lt}}\left[\text{rect}\left(\frac{t}{\tau}\right)\right]$$



Where $\quad \text{rect}\left(\frac{t}{\tau}\right) = \begin{cases} 1 & \text{for} \frac{-\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & \text{elesewhere} \end{cases}$

By definition, the Fourier transform of x(t) is:

$X(\omega) = F[x(t)] = F\left[\underset{t\to\infty}{\text{Lt}}\,\text{rect}\left(\frac{t}{\tau}\right)\right] = \underset{t\to\infty}{\text{Lt}}\,F\left[\text{rect}\left(\frac{t}{\tau}\right)\right]$

$$= \underset{t\to\infty}{\text{Lt}}\int_{-\tau/2}^{\tau/2}(1)\,e^{-j\omega t}\,dt = \underset{t\to\infty}{\text{Lt}}\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{-\tau/2}^{\tau/2}$$

$$= \underset{t\to\infty}{\text{Lt}}\left[\frac{e^{-j\omega(\tau/2)} - e^{j\omega(\tau/2)}}{-j\omega}\right] = \underset{t\to\infty}{\text{Lt}}\left\{\frac{2\sin[\omega\left(\frac{\tau}{2}\right)]}{\omega}\right\} = \underset{t\to\infty}{\text{Lt}}\left\{\tau\frac{\sin[\omega\left(\frac{\tau}{2}\right)]}{\omega\left(\frac{\tau}{2}\right)}\right\}$$

$$= \underset{t\to\infty}{\text{Lt}}\ \tau\ \text{sa}(\tfrac{\omega\tau}{2}) = 2\pi\left[\underset{t\to\infty}{\text{Lt}}\ \frac{\tau/2}{\pi}\text{sa}(\tfrac{\omega\tau}{2})\right]$$

Using the sampling property of the delta function $\left\{\text{i. e.} \left[\underset{t\to\infty}{\text{Lt}}\ \frac{\tau/2}{\pi}\text{sa}(\frac{\omega\tau}{2})\right] = \delta(\omega)\right\}$, we get

$$X(\omega) = F\left[\underset{t\to\infty}{\text{Lt}}\ \text{rect}\left(\frac{t}{\tau}\right)\right] = 2\pi\delta(\omega)$$

5. Signum function (sgn(t))

The signum function is denoted by sgn(t) and is defined by

$$\text{sgn}(t) = \begin{cases} 1 & \text{for } t > 0 \\ -1 & \text{for } t < 0 \end{cases}$$

This function is not absolutely integrable. So we cannot directly find its Fourier transform. Therefore, let us consider the function $e^{-a|t|}\,\text{sgn}(t)$ and substitute the limit $a \to 0$ to obtain the above sgn(t)

Given $x(t) = \text{sgn}(t) = \underset{a\to 0}{\text{Lt}}\ e^{-a|t|}\,\text{sgn}(t) = \underset{a\to 0}{\text{Lt}}\,[\,e^{-at}u(t) - e^{-at}u(-t)$∴

$X(\omega) = F[\text{sgn}(t)] = \int_{-\infty}^{\infty} \underset{a\to 0}{\text{Lt}}\,[\,e^{-at}u(t) - e^{-at}u(-t)e^{-j\omega t}dt$

$$= \underset{a\to 0}{\text{Lt}}\left[\int_{-\infty}^{\infty} e^{-at}e^{-j\omega t}u(t)dt - \int_{-\infty}^{\infty} e^{at}e^{-j\omega t}u(-t)dt\,\right]$$

$$= \underset{a\to 0}{\text{Lt}}\left[\int_{0}^{\infty} e^{-(a+j\omega)t}dt - \int_{-\infty}^{0} e^{(a-j\omega)t}dt\right] = \underset{a\to 0}{\text{Lt}}\left[\int_{0}^{\infty} e^{-(a+j\omega)t}dt - \int_{0}^{\infty} e^{-(a-j\omega)t}dt\,\right]$$

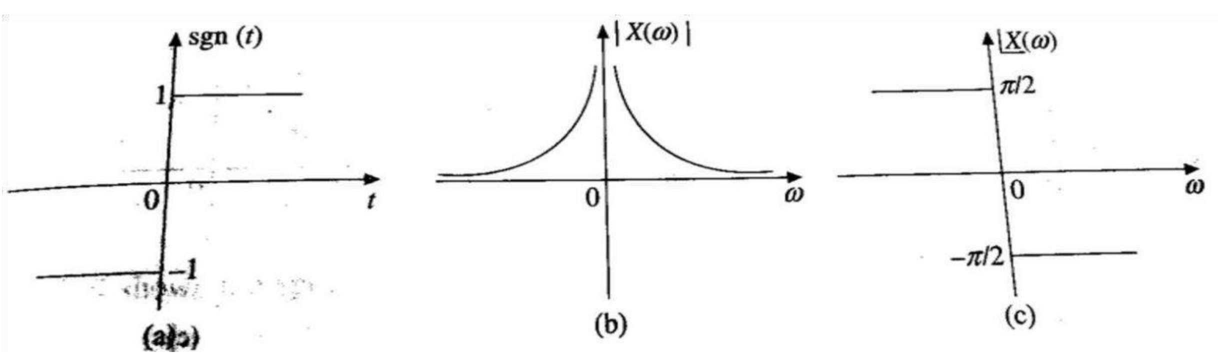
$$= \underset{a\to 0}{\text{Lt}}\left\{\left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_0^{\infty} - \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)}\right]_0^{\infty}\right\} = \underset{a\to 0}{\text{Lt}}\left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega}\right] = \frac{1}{j\omega} - \frac{1}{-j\omega} = \frac{2}{j\omega}$$

$$F[\text{sgn}(t)] = \frac{2}{j\omega}$$

$$\text{sgn}(t) \overset{FT}{\leftrightarrow} \frac{2}{j\omega}$$

∴ $|X(\omega)| = \frac{2}{\omega}$ and $\angle X(\omega) = \frac{\pi}{2}\ for\ \omega < 0\ and\ -\frac{\pi}{2}\ for\ \omega > 0$

Figure below shows the signum function and its magnitude and phase spectra



6. **Unit step function u(t)**

The unit step function is defined by

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

since the unit step function is not absolutely integrable, we cannot directly find its Fourier transform. So express the unit step function in terms of signum function as: $u(t) = \frac{1}{2} + \frac{1}{2}\, sgn(t)$ x(t)= u(t) = $\frac{1}{2}[1 + sgn(t)]$

$X(\omega) = F[u(t)] = F\left\{\frac{1}{2}[1 + sgn(t)]\right\}$

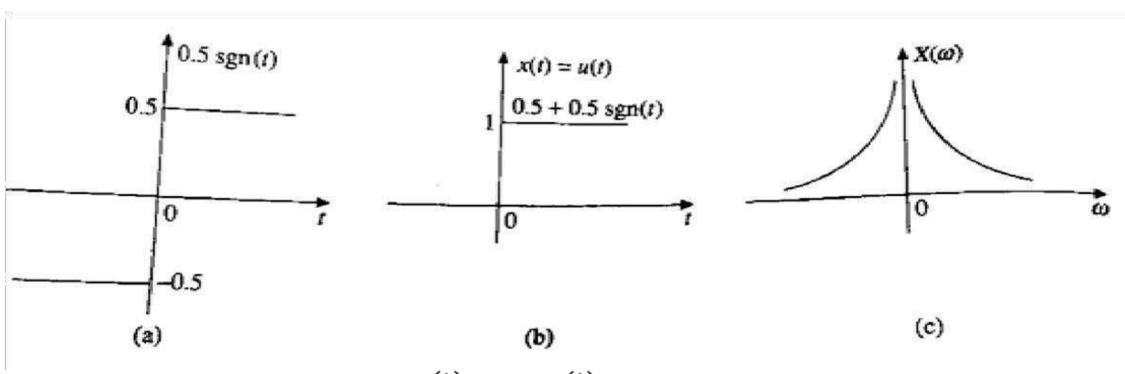
$= \frac{1}{2}\{F[1] + F[sgn(t)]\}$

We know that F[1] = $2\pi\delta(\omega)$ and F[sgn(t)] = $\frac{2}{j\omega}$

$F[u(t)] = \frac{1}{2}\left[2\pi\delta(\omega) + \frac{2}{j\omega}\right] = \pi\delta(\omega) + \frac{1}{j\omega}$

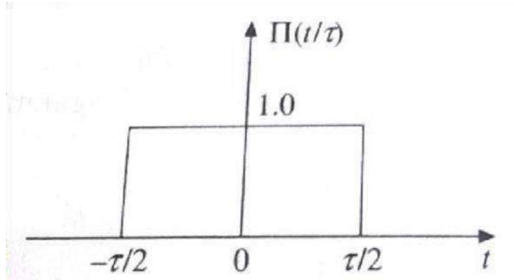
$u(t) \overset{FT}{\leftrightarrow} \pi\delta(\omega) + \frac{1}{j\omega}$

$\therefore$ $|X(\omega)| = \infty$ at $\omega = 0$ and is equal to 0 at $\omega = -\infty$ and $\omega = \infty$



7. **Rectangular pulse (Gate pulse) $\prod\left(\frac{t}{\tau}\right)$ or rect$\left(\frac{t}{\tau}\right)$**

Consider a rectangular pulse as shown in below figure. This is called a unit gate function and is defined as



x(t) = rect$\left(\frac{t}{\tau}\right) = \prod\left(\frac{t}{\tau}\right) = \begin{cases} 1 & for\ |t| < \frac{\tau}{2} \\ 0 & otherwise \end{cases}$

Then X(ω)= F[x(t)] = F $\left[\prod\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} \prod\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$

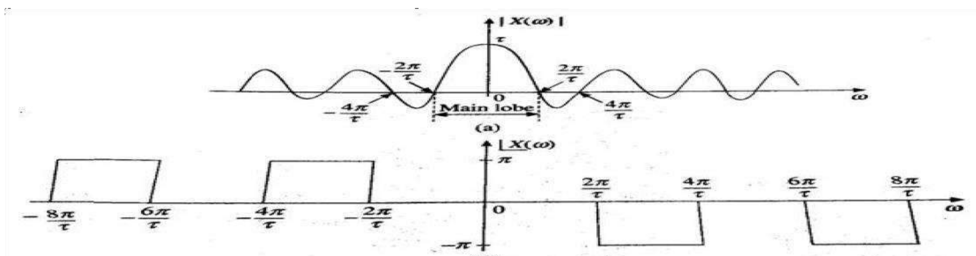
$$= \int_{-\tau/2}^{\tau/2} (1) e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_{\tau/2}^{\tau/2} = \frac{e^{-j\omega\,(\tau/2)} - e^{j\omega\,(\tau/2)}}{-j\omega}$$

$$= \frac{\tau}{\omega(\tau/2)}\left[\frac{e^{j\omega\,(\tau/2)} - e^{-j\omega\,(\tau/2)}}{2j}\right] = \tau\left[\frac{\sin\,\omega(\tau/2)}{\omega(\tau/2)}\right]$$

$$= \tau\ sinc\ \omega(\tau/2)$$

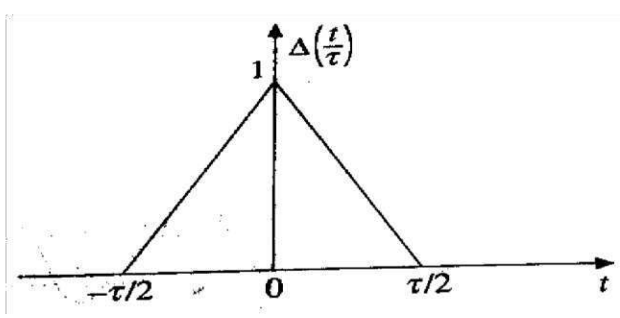
$\therefore$ F$\left[\prod\left(\frac{t}{\tau}\right)\right] = \tau\ sinc\ \omega(\tau/2)$, that is rect$\left(\frac{t}{\tau}\right) = \prod\left(\frac{t}{\tau}\right) \overset{FT}{\leftrightarrow} \tau\ sinc\ \omega(\tau/2)$

Figure shows the spectra of the gate function



8. Triangular Pulse $\Delta\left(\frac{t}{\tau}\right)$

Consider the triangular pulse as shown in below figure. It is defined as:



$$x(t) = \Delta\left(\frac{t}{\tau}\right) = \begin{cases} \frac{1}{\tau/2}\left(t + \frac{\tau}{2}\right) = \left(1 + 2\frac{t}{\tau}\right) & \text{for } -\frac{\tau}{2} < t < 0 \\ \frac{1}{\tau/2}\left(t - \frac{\tau}{2}\right) = \left(1 - 2\frac{t}{\tau}\right) & \text{for } 0 < t < \frac{\tau}{2} \\ 0 & \text{elsewhere} \end{cases}$$

τ

i.e. as $x(t) = \Delta\left(\frac{t}{\tau}\right) = \begin{cases} 1 - \frac{2|t|}{\tau} & \text{for } |t| < \frac{\tau}{2} \\ 0 & \text{otherwise} \end{cases}$

Then $X(\omega) = F[x(t)] = F\left[\Delta\left(\frac{t}{\tau}\right)\right] = \int_{-\infty}^{\infty} \Delta\left(\frac{t}{\tau}\right) e^{-j\omega t}\, dt$

$$= \int_{-\tau/2}^{0} \left(1 + \frac{2t}{\tau}\right) e^{-j\omega t}\, dt + \int_{0}^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$= \int_{0}^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{j\omega t}\, dt + \int_{0}^{\tau/2} \left(1 - \frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$= \int_{0}^{\tau/2} e^{j\omega t}\, dt - \int_{0}^{\tau/2} \left(\frac{2t}{\tau}\right) e^{j\omega t}\, dt + \int_{0}^{\tau/2} e^{-j\omega t}\, dt - \int_{0}^{\tau/2} \left(\frac{2t}{\tau}\right) e^{-j\omega t}\, dt$$

$$= \int_{0}^{\tau/2} [e^{j\omega t} + e^{-j\omega t}]\, dt - \frac{2}{\tau}\int_{0}^{\tau/2} t[e^{j\omega t} + e^{-j\omega t}]\, dt = \int_{0}^{\tau/2} 2\cos\omega t\, dt - \frac{2}{\tau}\int_{0}^{\tau/2} 2t\cos\omega t\, dt$$

$$= 2\left[\frac{\sin\omega t}{\omega}\right]_{0}^{\tau/2} - \frac{4}{\tau}\left[\left[t\frac{\sin\omega t}{\omega}\right]_{0}^{\tau/2} + \left[\frac{\cos\omega t}{\omega^2}\right]_{0}^{\tau/2}\right] = \frac{2}{\omega}\left[\sin\omega\frac{\tau}{2}\right] - \frac{4}{\omega\tau}\left[\frac{\tau}{2}\sin\frac{\omega\tau}{2}\right] - \frac{4}{\omega^2\tau}\left[\cos\frac{\omega\tau}{2} - 1\right]$$

$$= \frac{4}{\omega^2\tau}\left[1 - \cos\frac{\omega\tau}{2}\right] = \frac{4}{\omega^2\tau}\left[2\sin^2\frac{\omega\tau}{4}\right]$$

$$= \frac{8}{\omega^2\tau}\left(\frac{\omega\tau}{4}\right)^2 \frac{\sin^2\left(\frac{\omega\tau}{4}\right)}{\left(\frac{\omega\tau}{4}\right)} = \frac{\tau}{2} sinc^2\left(\frac{\omega\tau}{4}\right)$$

$F\left[\Delta\left(\frac{t}{\tau}\right)\right] = \frac{\tau}{2} sinc^2\left(\frac{\omega\tau}{4}\right)$ Or $\Delta\left(\frac{t}{\tau}\right) \overset{FT}{\leftrightarrow} \frac{\tau}{2} sinc^2\left(\frac{\omega\tau}{4}\right)$

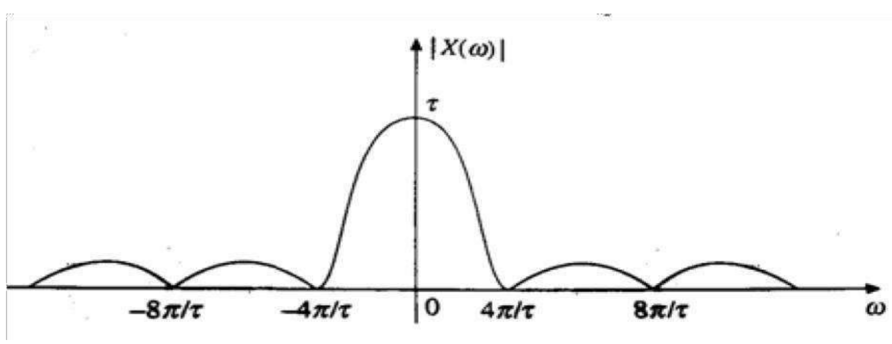


Figure shows the amplitude spectrum of a triangular pulse.

Fourier Transform of Periodic Signal

The periodic functions can be analyzed using Fourier series and that non-periodic function can be analyzed using Fourier transform. But we can find the Fourier transform of a periodic function also. This means that the Fourier transform can be used as a universal mathematical tool in the analysis of both non-periodic and periodic waveforms over the entire interval. Fourier transform of periodic functions may be found using the concept of impulse function.

We know that using Fourier series , any periodic signal can be represented as a sum of complex exponentials. Therefore, we can represent a periodic signal using the Fourier integral. Let us consider a periodic signal x(t) with period T. Then, we can express x(t) in terms of exponential Fourier series as:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n \, e^{jn\omega_0 t}$$

The Fourier transform of x(t) is:

$$X(\omega) = F[x(t)] = F\left[\sum_{n=\infty}^{\infty} C_n \, e^{jn\omega_0 t}\right] = \sum_{n=\infty}^{\infty} C_n \, F\left[e^{jn\omega_0 t}\right]$$

Using the frequency shifting theorem, we have

$$F\left[1e^{jn\omega_0 t}\right] = F\,[1]\,\big|_{\omega=\omega-n\omega_0} = s\pi\delta(\omega - n\omega_0)$$

$$X(\omega) = 2\pi \sum_{n=\infty}^{\infty} C_n \, \delta(\omega - n\omega_0)$$

Where $C_n s$ are the Fourier coefficients associated with x(t) and are given by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\omega_0 t}\, dt$$

Thus, the Fourier transform of a periodic function consists of a train of equally spaced impulses. These impulses are located at the harmonic frequencies of the signal and the strength of each impulse is given as $2\pi C_n$.

Solved Problems:

1. Find the Fourier transform of the signals $e^{3t}u(t)$

Solution: Given $x(t) = e^{3t}u(t)$

The given signal is not absolutely integrable.

That is $\int_{-\infty}^{\infty} e^{3t}u(t) = \infty$.

Therefore, Fourier transform of $x(t) = e^{3t}u(t)$ does not exist.

2. Find the Fourier transform of the signals $\cos\omega_o t u(t)$

Solution:

Given $x(t) = \cos\omega_o t\, u(t)$

i.e. $= \frac{e^{j\omega_o t}+e^{-j\omega_o t}}{2} u(t)$

$\therefore$ $X(\omega) = F[\cos\omega_o t\, u(t)] = \int_{-\infty}^{\infty} \frac{e^{j\omega_o t}+e^{-j\omega_o t}}{2} u(t)\, e^{-j\omega t}\, dt$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{-j(\omega-\omega_o)t}\, dt + \int_0^{\infty} e^{-j(\omega+\omega_o)t}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-j(\omega-\omega_o)t}}{-j(\omega-\omega_o)} + \frac{e^{-j(\omega+\omega_o)t}}{-j(\omega+\omega_o)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{-e^{0}}{-j(\omega-\omega_o)} + \frac{-e^{0}}{-j(\omega+\omega_o)}\right]$$

With impulses of strength π at $\omega=\omega_o$ and $\omega=-\omega_o$

$$\therefore X(\omega) = \frac{1}{2}\left[\frac{1}{j(\omega-\omega_o)} + \frac{1}{j(\omega+\omega_o)} + \pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{1}{2}\left[\frac{j2\omega}{(j\omega)^2+\omega_o^2} + \pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{j\omega}{(j\omega)^2+\omega_o^2} + \frac{1}{2}\left[\pi\delta(\omega-\omega_o) + \pi\delta(\omega+\omega_o)\right]$$

3: Find the Fourier transform of the signals $\sin\omega_o t\, u(t)$

Solution:

Given $x(t) = \sin\omega_o t\, u(t)$

i.e. $= \frac{e^{j\omega_o t}- e^{-j\omega_o t}}{2j} u(t)$

$\therefore$ $X(\omega) = F[\sin\omega_o t\, u(t)] = \int_{-\infty}^{\infty} \frac{e^{j\omega_o t}-e^{-j\omega_o t}}{2j} u(t)\, e^{-j\omega t}\, dt$

$$= \frac{1}{2j}\left[\int_0^{\infty} e^{-j(\omega-\omega_o)t}\, dt - \int_0^{\infty} e^{-j(\omega+\omega_o)t}\, dt\right]$$

$$= \frac{1}{2j}\left[\frac{e^{-j(\omega-\omega_o)t}}{-j(\omega-\omega_o)} - \frac{e^{-j(\omega+\omega_o)t}}{-j(\omega+\omega_o)}\right]_0^{\infty}$$

$$= \frac{1}{2j}\left[\frac{-e^{0}}{-j(\omega-\omega_o)} - \frac{-e^{0}}{-j(\omega+\omega_o)}\right]$$

With impulses of strength π at $\omega=\omega_o$ and $\omega=-\omega_o$

$$\therefore X(\omega) = \frac{1}{2j}\left[\frac{1}{j(\omega-\omega_o)} - \frac{1}{j(\omega+\omega_o)} + \pi\delta(\omega-\omega_o) - \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{1}{2j}\left[\frac{j2\omega_o}{(j\omega)^2+\omega_o^2} + \pi\delta(\omega-\omega_o) - \pi\delta(\omega+\omega_o)\right]$$

$$= \frac{\omega_o}{(j\omega)^2+\omega_o^2} - j\frac{\pi}{2}\left[\delta(\omega-\omega_o) + \delta(\omega+\omega_o)\right]$$

4. Find the Fourier transform of the signals $e^{-t}\sin 5t\, u(t)$

Solution:

Given $x(t) = e^{-t}\sin 5t\, u(t)$

$$x(t) = e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right)u(t)$$

$$\therefore \quad X(\omega) = F[e^{-t}\sin 5t\, u(t)] = F\left[e^{-t}\left(\frac{e^{j5t} - e^{-j5t}}{2j}\right)u(t)\right]$$

$$= \frac{1}{2j}\int_{-\infty}^{\infty}[e^{-t}(e^{j5t} - e^{-j5t})u(t)]\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2j}\left[\frac{e^{-[1+j(\omega-5)]t}}{-[1+j(\omega-5)]} - \frac{e^{-[1+j(\omega+5)]t}}{-[1+j(\omega+5)]}\right]_0^{\infty}$$

$$= \frac{1}{2j}\left[\frac{1}{[1+j(\omega-5)]} - \frac{1}{[1+j(\omega+5)]}\right]$$

$$= \frac{5}{[1+j(\omega-5)][1+j(\omega+5)]} = \frac{5}{(1+j\omega)^2+25}\text{[neglecting impulses]}$$

5. Find the Fourier transform of the signals $e^{-2t}\cos 5t\, u(t)$

Solution:

Given $x(t) = e^{-2t}\cos 5t\, u(t)$

$$x(t) = e^{-2t}\left(\frac{e^{j5t} + e^{-j5t}}{2}\right)u(t)$$

$$\therefore \quad X(\omega) = F[e^{-2t}\cos 5t\, u(t)] = F\left[e^{-2t}\left(\frac{e^{j5t} - e^{-j5t}}{2}\right)u(t)\right]$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}[\, e^{-2t}(e^{j5t} - \, e^{-j5t})u(t)]\, e^{-j\omega t}\, dt$$

$$= \frac{1}{2}\left[\int_{0}^{\infty} e^{-[2+j(\omega-5)]t}\, dt + \int_{0}^{\infty} e^{-[2+j(\omega+5)]t}\, dt\right]$$

$$= \frac{1}{2}\left[\frac{e^{-[2+j(\omega-5)]t}}{-[2+j(\omega-5)]} - \frac{e^{-[2+j(\omega+5)]t}}{-[2+j(\omega+5)]}\right]_{0}^{\infty}$$

$$= \frac{1}{2}\left[\frac{1}{[1+j(\omega-5)]} - \frac{1}{[1+j(\omega+5)]}\right]$$

$$= \frac{1}{2}\left[\frac{2(2+j\omega)}{(2+j\omega)^2+25}\right] = \left[\frac{2+j\omega}{(2+j\omega)^2+25}\right]$$ [neglecting impulses]

DISCRETE FOURIER TRANSFORM

INTRODUCTION

Any periodic function can be expressed in a Fourier series representation. The discrete-time Fourier transform (DTFT) $X(\omega)$ of a discrete-time sequence $x(n)$ is a periodic continuous function of ω with a period of 2π. So it cannot be processed by a digital system. For processing by a digital system it should be converted into discrete form. The DFT converts the continuous function of ω to a discrete function of ω. Thus, DFT allows us to perform frequency analysis on a digital computer.

The DFT of a discrete-time signal $x(n)$ is a finite duration discrete frequency sequence. The DFT sequence is denoted by $X(k)$. The DFT is obtained by sampling one period of the Fourier transform $X(\omega)$ of the signal $x(n)$ at a finite number of frequency points. This sampling is conventionally performed at N equally spaced points in the period $0 \leq \omega \leq 2\pi$ or at $\omega_k = 2\pi k/N$; $0 \leq k \leq N-1$. We can say that DFT is used for transforming discrete-time sequence $x(n)$ of finite length into discrete frequency sequence $X(k)$ of finite length.

The DFT is important for two reasons. First it allows us to determine the frequency content of a signal, that is to perform spectral analysis. The second application of the DFT is to perform filtering operation in the frequency domain.

Let $x(n)$ be a discrete-time sequence with Fourier transform $X(\omega)$, then the DFT of $x(n)$ denoted by $X(k)$ is defined as:

$$X(k) = X(\omega)\big|_{\omega=(2\pi k/N)}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

The DFT of $x(n)$ is a sequence consisting of N samples of $X(k)$. The DFT sequence starts at $k = 0$, corresponding to $\omega = 0$, but does not include $k = N$ corresponding to $\omega = 2\pi$ (since the sample at $\omega = 0$ is same as the sample at $\omega = 2\pi$). Generally, the DFT is defined along

with number of samples and is called N-point DFT. The number of samples N for a finite duration sequence $x(n)$ of length L should be such that $N \geq L$.

The DTFT is nothing but the Z-transform evaluated along the unit circle centred at the origin of the z-plane. The DFT is nothing but the Z-transform evaluated at a finite number of equally spaced points on the unit circle centred at the origin of the z-plane.

To calculate the DFT of a sequence, it is not necessary to compute its Fourier transform, since the DFT can be directly computed.

DFT The N-point DFT of a finite duration sequence $x(n)$ of length L, where $N \geq L$ is defined as:

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = \sum_{n=0}^{N-1} x(n)W_N^{nk}; \quad \text{for } k = 0, 1, 2, \ldots, N-1$$

IDFT The Inverse Discrete Fourier transform (IDFT) of the sequence $X(k)$ of length N is defined as:

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi nk/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk}; \quad \text{for } n = 0, 1, 2, \ldots, N-1$$

where $W_N = e^{-j(2\pi/N)}$ is called the twiddle factor.

The N-point DFT pair $x(n)$ and $X(k)$ is denoted as:

$$x(n) \xleftrightarrow[N]{\text{DFT}} X(k)$$

EXAMPLE Compute the DFT of each of the following finite length sequences considered to be of length N:

(a) $x(n) = \delta(n)$

(b) $x(n) = \delta(n - n_0)$, where $0 < n_0 < N$

(c) $x(n) = a^n$, $0 \leq n \leq N - 1$

(d) $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

Solution:

(a) Given $x(n) = \delta(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk}$$

$$= \sum_{n=0}^{N-1} \delta(n)e^{-j(2\pi/N)nk} = 1$$

i.e. $X(k) = 1$ for $0 \leq k \leq N - 1$.

(b) Given $x(n) = \delta(n - n_0)$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j(2\pi/N)nk} = \sum_{n=0}^{N-1} \delta(n - n_0)e^{-j(2\pi/N)nk}$$

$$= e^{-j(2\pi/N)n_0k} \quad \text{for } 0 \leq k \leq N-1$$

(c) Given $x(n) = a^n$

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j(2\pi/N)nk} = \sum_{n=0}^{N-1} [ae^{-j(2\pi/N)k}]^n \text{ for } 0 \le k \le N-1$$

$$= \frac{1 - a^N e^{-j2\pi k}}{1 - ae^{-j(2\pi/N)k}}$$

(d) Given $x(n) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j(2\pi/N)2nk} + \sum_{n=0}^{(N/2)-1} x(2n+1) e^{-j(2\pi/N)(2n+1)k}$$

$$= \sum_{n=0}^{(N/2)-1} x(2n) e^{-j(4\pi/N)nk} = \sum_{n=0}^{(N/2)-1} e^{-j4\pi kn/N}$$

EXAMPLE (a) Find the 4-point DFT of $x(n) = \{1, -1, 2, -2\}$ directly.

(b) Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ directly.

Solution:

(a) Given sequence is $x(n) = \{1, -1, 2, -2\}$. Here the DFT $X(k)$ to be found is $N =$ 4-point and length of the sequence $L = 4$. So no padding of zeros is required.

We know that the DFT $\{x(n)\}$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{N-1} x(n) e^{-j(2\pi/N)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$\therefore \quad X(0) = \sum_{n=0}^{3} x(n) e^0 = x(0) + x(1) + x(2) + x(3) = 1 - 1 + 2 - 2 = 0$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)n} = x(0) + x(1) e^{-(j\pi/2)} + x(2) e^{-j\pi} + x(3) e^{-j(3\pi/2)}$$

$$= 1 + (-1)(0 - j) + 2(-1 - j0) - 2(0 + j)$$

$$= -1 - j$$

$$X(2)=\sum_{n=0}^{3} x(n)e^{-j\pi n}=x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}+x(3)e^{-j3\pi}$$

$$=1-1(-1-j0)+2(1-j0)-2(-1-j0)=6$$

$$X(3)=\sum_{n=0}^{3} x(n)e^{-j(3\pi/2)n}=x(0)+x(1)e^{-j(3\pi/2)}+x(2)e^{-j3\pi}+x(3)e^{-j(9\pi/2)}$$

$$=1-1(0+j)+2(-1-j0)-2(0-j)=-1+j$$

$$\therefore \quad X(k)=\{0,\ -1-j,\ 6,\ -1+j\}$$

(b) Given DFT is $X(k) = \{4, 2, 0, 4\}$. The IDFT of $X(k)$, i.e. $x(n)$ is given by

$$x(n)=\frac{1}{N}\sum_{k=0}^{N-1} X(k)W_N^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)nk}$$

i.e. $$x(n)=\frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(\pi/2)nk}$$

$$\therefore \quad x(0)=\frac{1}{4}\sum_{k=0}^{3} X(k)e^{0}=\frac{1}{4}[X(0)+X(1)+X(2)+X(3)]$$

$$=\frac{1}{4}[4+2+0+4]=2.5$$

$$x(1)=\frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(\pi/2)k}=\frac{1}{4}\left[X(0)+X(1)e^{j(\pi/2)}+X(2)e^{j\pi}+X(3)e^{j(3\pi/2)}\right]$$

$$=\frac{1}{4}[4+2(0+j)+0+4(0-j)]=1-j0.5$$

$$x(2)=\frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k}=\frac{1}{4}\left[X(0)+X(1)e^{j\pi}+X(2)e^{j2\pi}+X(3)e^{j3\pi}\right]$$

$$=\frac{1}{4}[4+2(-1+j0)+0+4(-1+j0)]=-0.5$$

$$x(3)=\frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(3\pi/2)k}=\frac{1}{4}\left[X(0)+X(1)e^{j(3\pi/2)}+X(2)e^{j3\pi}+X(3)e^{j(9\pi/2)}\right]$$

$$=\frac{1}{4}[4+2(0-j)+0+4(0+j)]=1+j0.5$$

$$x_3(n)=\{2.5,\ 1-j0.5,\ -0.5,\ 1+j0.5\}$$

EXAMPLE (a) Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.
(b) Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution:

(a) Given $x(n) = \{1, -2, 3, 2\}$.
Here $N = 4$, $L = 4$. The DFT of $x(n)$ is $X(k)$.

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{3} x(n) e^{-j(2\pi/4)nk} = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)nk}, \quad k = 0, 1, 2, 3$$

$$X(0) = \sum_{n=0}^{3} x(n) e^{0} = x(0) + x(1) + x(2) + x(3) = 1 - 2 + 3 + 2 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j(\pi/2)n} = x(0) + x(1) e^{-j(\pi/2)} + x(2) e^{-j\pi} + x(3) e^{-j(3\pi/2)}$$
$$= 1 - 2(0 - j) + 3(-1 - j0) + 2(0 + j) = -2 + j4$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$
$$= 1 - 2(-1 - j0) + 3(1 - j0) + 2(-1 - j0) = 4$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j(3\pi/2)n} = x(0) + x(1) e^{-j(3\pi/2)} + x(2) e^{-j3\pi} + x(3) e^{-j(9\pi/2)}$$
$$= 1 - 2(0 + j) + 3(-1 - j0) + 2(0 - j) = -2 - j4$$

$$\therefore \quad X(k) = \{4, -2 + j4, 4, -2 - j4\}$$

(b) Given $X(k) = \{1, 0, 1, 0\}$

Let the IDFT of $X(k)$ be $x(n)$.

$$\therefore \quad x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j(2\pi/N)nk}$$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{0} = \frac{1}{4}\big[X(0) + X(1) + X(2) + X(3)\big] = \frac{1}{4}[1+0+1+0] = 0.5$$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(\pi/2)k} = \frac{1}{4}\Big[X(0) + X(1)e^{j(\pi/2)} + X(2)e^{j\pi} + X(3)e^{j(3\pi/2)}\Big]$$

$$= \frac{1}{4}[1+0+e^{j\pi}+0] = \frac{1}{4}[1+0-1+0] = 0$$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\pi k} = \frac{1}{4}\Big[X(0) + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\Big]$$

$$= \frac{1}{4}[1+0+e^{j2\pi}+0] = \frac{1}{4}[1+0+1+0] = 0.5$$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j(3\pi/2)k} = \frac{1}{4}\Big[X(0) + X(1)e^{j(3\pi/2)} + X(2)e^{j3\pi} + X(3)e^{j(9\pi/2)}\Big]$$

$$= \frac{1}{4}[1+0+e^{j3\pi}+0] = \frac{1}{4}[1+0-1+0] = 0$$

$\therefore$ The IDFT of $X(k) = \{1, 0, 1, 0\}$ is $x(n) = \{0.5, 0, 0.5, 0\}$.

EXAMPLE Compute the DFT of the 3-point sequence $x(n) = \{2, 1, 2\}$. Using the same sequence, compute the 6-point DFT and compare the two DFTs.

Solution: The given 3-point sequence is $x(n) = \{2, 1, 2\}$, $N = 3$.

$$\text{DFT}\, x(n) = X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n=0}^{2} x(n) e^{-j(2\pi/3)nk}, \quad k = 0, 1, 2$$

$$= x(0) + x(1) e^{-j(2\pi/3)k} + x(2) e^{-j(4\pi/3)k}$$

$$= 2 + \left(\cos\frac{2\pi}{3}k - j\sin\frac{2\pi}{3}k \right) + 2\left(\cos\frac{4\pi}{3}k - j\sin\frac{4\pi}{3}k \right)$$

When $k = 0$, $\quad X(k) = X(0) = 2 + 1 + 2 = 5$

When $k = 1$, $\quad X(k) = X(1) = 2 + \left(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3} \right) + 2\left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3} \right)$

$$= 2 + (-0.5 - j0.866) + 2(-0.5 + j0.866)$$

$$= 0.5 + j0.866$$

When $k = 2$, $\quad X(k) = X(2) = 2 + \left(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3} \right) + 2\left(\cos\frac{8\pi}{3} - j\sin\frac{8\pi}{3} \right)$

$$= 2 + (-0.5 + j0.866) + 2(-0.5 - j0.866)$$

$$= 0.5 - j0.866$$

$\therefore$ 3-point DFT of $x(n) = X(k) = \{5, 0.5 + j0.866, 0.5 - j0.866\}$

MATRIX FORMULATION OF THE DFT AND IDFT

If we let $W_N = e^{-j(2\pi/N)}$, the defining relations for the DFT and IDFT may be written as:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk}, \quad n = 0, 1, 2, \ldots, N-1$$

The first set of N DFT equations in N unknowns may be expressed in matrix form as:

$$\mathbf{X} = \mathbf{W_N x}$$

Here $\mathbf{X}$ and $\mathbf{x}$ are $N \times 1$ matrices, and $\mathbf{W_N}$ is an $N \times N$ square matrix called the DFT matrix. The full matrix form is described by

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}$$

THE IDFT FROM THE MATRIX FORM

The matrix $\mathbf{x}$ may be expressed in terms of the inverse of $\mathbf{W_N}$ as:

$$\mathbf{x} = \mathbf{W_N^{-1}}\,\mathbf{x}$$

The matrix $\mathbf{W_N^{-1}}$ is called the IDFT matrix. We may also obtain $\mathbf{x}$ directly from the IDFT relation in matrix form, where the change of index from n to k and the change in the sign of the exponent in $e^{j(2\pi/N)nk}$ lead to the conjugate transpose of $\mathbf{W_N}$. We then have

$$\mathbf{x} = \frac{1}{\mathbf{N}}\left[\mathbf{W_N^*}\right]^T \mathbf{x}$$

Comparison of the two forms suggests that $\mathbf{W_N^{-1}} = \frac{1}{N}\left[\mathbf{W_N^*}\right]^T$.

This very important result shows that $\mathbf{W_N^{-1}}$ requires only conjugation and transposition of $\mathbf{W_N}$, an obvious computational advantage.

USING THE DFT TO FIND THE IDFT

Both the DFT and IDFT are matrix operations and there is an inherent symmetry in the DFT and IDFT relations. In fact, we can obtain the IDFT by finding the DFT of the conjugate sequence and then conjugating the results and dividing by N, Mathematically,

$$x(n) = \text{IDFT}[X(k)] = \frac{1}{N}[\text{DFT}\{X^*(k)\}]^*$$

This result involves the conjugate symmetry and duality of the DFT and IDFT, and suggests that the DFT algorithm itself can also be used to find the IDFT. In practice, this is indeed what is done.

EXAMPLE Find the DFT of the sequence

$$x(n) = \{1, 2, 1, 0\}$$

Solution: The DFT $X(k)$ of the given sequence $x(n) = \{1, 2, 1, 0\}$ may be obtained by solving the matrix product as follows. Here $N = 4$.

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & W_N^3 \\ W_N^0 & W_N^2 & W_N^4 & W_N^6 \\ W_N^0 & W_N^3 & W_N^6 & W_N^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

The result is DFT $\{x(n)\} = X(k) = \{4, -j2, 0, j2\}$.

EXAMPLE Find the DFT of $x(n) = \{1, -1, 2, -2\}$.

Solution: The DFT, $X(k)$ of the given sequence $x(n) = \{1, -1, 2, -2\}$ can be determined using matrix as shown below.

$$X(k) = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1-j \\ 6 \\ -1+j \end{bmatrix}$$

$\therefore$ DFT $\{x(n)\} = X(k) = \{0, -1-j, 6, -1+j\}$

EXAMPLE Find the 4-point DFT of $x(n) = \{1, -2, 3, 2\}$.

Solution: Given $x(n) = \{1, -2, 3, 2\}$, the 4-point DFT$\{x(n)\} = X(k)$ is determined using matrix as shown below.

$$\text{DFT}\{x(n)\} = X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2+j4 \\ 4 \\ -2-j4 \end{bmatrix}$$

$\therefore$ $\text{DFT}\{x(n)\} = X(k) = \{4, -2+j4, 4, -2-j4\}$

EXAMPLE Find the IDFT of $X(k)=\{4, -j2, 0, j2\}$ using DFT.

Solution: Given $X(k) = \{4, -j2, 0, j2\}$ $\therefore$ $X^*(k) = \{4, j2, 0, -j2\}$

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$ first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ j2 \\ 0 \\ -j2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}[X(k)] = x(n) = \frac{1}{4}[4, 8, 4, 0]^* = \frac{1}{4}[4, 8, 4, 0] = [1, 2, 1, 0]$$

EXAMPLE Find the IDFT of $X(k) = \{4, 2, 0, 4\}$ using DFT.

Solution: Given $X(k) = \{4, 2, 0, 4\}$

$\therefore$ $X^*(k) = \{4, 2, 0, 4\}$

The IDFT of $X(k)$ is determined using matrix as shown below.

To find IDFT of $X(k)$, first find $X^*(k)$, then find DFT of $X^*(k)$, then take conjugate of DFT $\{X^*(k)\}$ and divide by N.

$$\text{DFT}[X^*(k)] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4+j2 \\ -2 \\ 4-j2 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}[10, 4+j2, -2, 4-j2]^* = \{2.5, 1-j0.5, -0.5, 1+j0.5\}$$

EXAMPLE Find the IDFT of $X(k) = \{1, 0, 1, 0\}$.

Solution: Given $X(k) = \{1, 0, 1, 0\}$, the IDFT of $X(k)$, i.e. $x(n)$ is determined using matrix as shown below.

$$X^*(k) = \{1, 0, 1, 0\}^* = \{1, 0, 1, 0\}$$

$$\text{DFT}\{X^*(k)\} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\therefore \quad \text{IDFT}\{X(k)\} = x(n) = \frac{1}{4}[\text{DFT}\{X^*(k)\}]^* = \frac{1}{4}\{2, 0, 2, 0\} = \{0.5, 0, 0.5, 0\}$$

UNIT III

FAST FOURIER TRANSFORM

- Decimation-in-Time FFT Algorithm
- Decimation-in-Frequency FFT Algorithms
- Decimation-in-Time Inverse FFT.
- Decimation-in-Frequency Inverse FFT

INTRODUCTION

The N-point DFT of a sequence $x(n)$ converts the time domain N-point sequence $x(n)$ to a frequency domain N-point sequence $X(k)$. The direct computation of an N-point DFT requires $N \times N$ complex multiplications and $N(N-1)$ complex additions. Many methods were developed for reducing the number of calculations involved. The most popular of these is the Fast Fourier Transform (FFT), a method developed by Cooley and Turkey. The FFT may be defined as an algorithm (or a method) for computing the DFT efficiently (with reduced number of calculations). The computational efficiency is achieved by adopting a divide and conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs and then combining them to give the total transform. Based on this basic approach, a family of computational algorithms were developed and they are collectively known as FFT algorithms. Basically there are two FFT algorithms; Decimation-in-time (DIT) FFT algorithm and Decimation-in-frequency (DIF) FFT algorithm. In this chapter, we discuss DIT FFT and DIF FFT algorithms and the computation of DFT by these methods.

FAST FOURIER TRANSFORM

The DFT of a sequence $x(n)$ of length N is expressed by a complex-valued sequence $X(k)$ as

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}, \quad k = 0, 1, 2, \dots, N-1$$

Let W_N be the complex valued phase factor, which is an Nth root of unity given by

$$W_N = e^{-j2\pi/N}$$

Thus, $X(k)$ becomes $$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}, \quad k = 0, 1, 2, \dots, N-1$$

Similarly, IDFT is written as $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}, \quad n = 0, 1, 2, \dots, N-1$$

DECIMATION IN TIME (DIT) RADIX-2 FFT

In Decimation in time (DIT) algorithm, the time domain sequence $x(n)$ is decimated and smaller point DFTs are computed and they are combined to get the result of N-point DFT.

In general, we can say that, in DIT algorithm the N-point DFT can be realized from two numbers of $N/2$-point DFTs, the $N/2$-point DFT can be realized from two numbers of $N/4$-point DFTs, and so on.

In DIT radix-2 FFT, the N-point time domain sequence is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-point DFTs, the 4-point DFTs, from the results of 4-point DFTs, the 8-point DFTs and so on are computed until we get N-point DFT.

For performing radix-2 FFT, the value of r should be such that, $N = 2^m$. Here, the decimation can be performed m times, where $m = \log_2 N$. In direct computation of N-point DFT, the total number of complex additions are $N(N-1)$ and the total number of complex multiplications are N^2. In radix-2 FFT, the total number of complex additions are reduced to $N \log_2 N$ and the total number of complex multiplications are reduced to $(N/2) \log_2 N$.

Let $x(n)$ be an N-sample sequence, where N is a power of 2. Decimate or break this sequence into two sequences $f_1(n)$ and $f_2(n)$ of length $N/2$, one composed of the even indexed values of $x(n)$ and the other of odd indexed values of $x(n)$.

Let $x(n)$ be an N-sample sequence, where N is a power of 2. Decimate or break this sequence into two sequences $f_1(n)$ and $f_2(n)$ of length $N/2$, one composed of the even indexed values of $x(n)$ and the other of odd indexed values of $x(n)$.

Given sequence $x(n)$: $x(0), x(1), x(2), \ldots, x\left(\frac{N}{2}-1\right), \ldots, x(N-1)$

Even indexed sequence $f_1(n) = x(2n)$: $x(0), x(2), x(4), \ldots, x(N-2)$

Odd indexed sequence $f_2(n) = x(2n+1)$: $x(1), x(3), x(5), \ldots, x(N-1)$

We know that the transform $X(k)$ of the N-point sequence $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \qquad k = 0, 1, 2, \ldots, N-1$$

Breaking the sum into two parts, one for the even and one for the odd indexed values, gives

$$X(k) = \sum_{n \text{ even}}^{N-2} x(n)\, W_N^{nk} + \sum_{n \text{ odd}}^{N-1} x(n)\, W_N^{nk}, \qquad k = 0, 1, 2, \ldots, N-1$$

When n is replaced by $2n$, the even numbered samples are selected and when n is replaced by $2n + 1$, the odd numbered samples are selected. Hence,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{k(2n+1)}$$

Rearranging each part of $X(k)$ into $(N/2)$-point transforms using

$$W_N^{2nk} = (W_N^2)^{nk} = \left(e^{-j\frac{2\pi}{N}}\right)^{2nk} = \left(e^{-j\frac{2\pi}{N/2}}\right)^{nk} = W_{N/2}^{nk}$$

and $W_N^{(2n+1)k} = W_N^k W_{N/2}^{nk}$, we can write

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n) W_{N/2}^{nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{N/2}^{nk}$$

By definition of DFT, the $N/2$-point DFT of $f_1(n)$ and $f_2(n)$ is given by

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n) W_{N/2}^{kn} \quad \text{and} \quad F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n) W_{N/2}^{kn}$$

$$\therefore \qquad X(k) = F_1(k) + W_N^k F_2(k), \quad k = 0, 1, 2, \ldots, N-1$$

The implementation of this equation for $X(k)$ is shown in Figure 7.1.

This first step in the decomposition breaks the N-point transform into two $(N/2)$-point transforms and the W_N^k provides the N-point combining algebra.

The DFT of a sequence is periodic with period given by the number of points of DFT. Hence, $F_1(k)$ and $F_2(k)$ will be periodic with period $N/2$.

$$\therefore \qquad F_1\left(k + \frac{N}{2}\right) = F_1(k) \quad \text{and} \quad F_2\left(k + \frac{N}{2}\right) = F_2(k)$$

In addition, the phase factor $W_N^{\left(k+\frac{N}{2}\right)} = -W_N^k$.

Therefore, for $k \geq N/2$, $X(k)$ is given by

$$X(k) = F_1\left(k - \frac{N}{2}\right) - W_N^k F_2\left(k - \frac{N}{2}\right)$$

The implementation using the periodicity property is also shown in Figure .1.

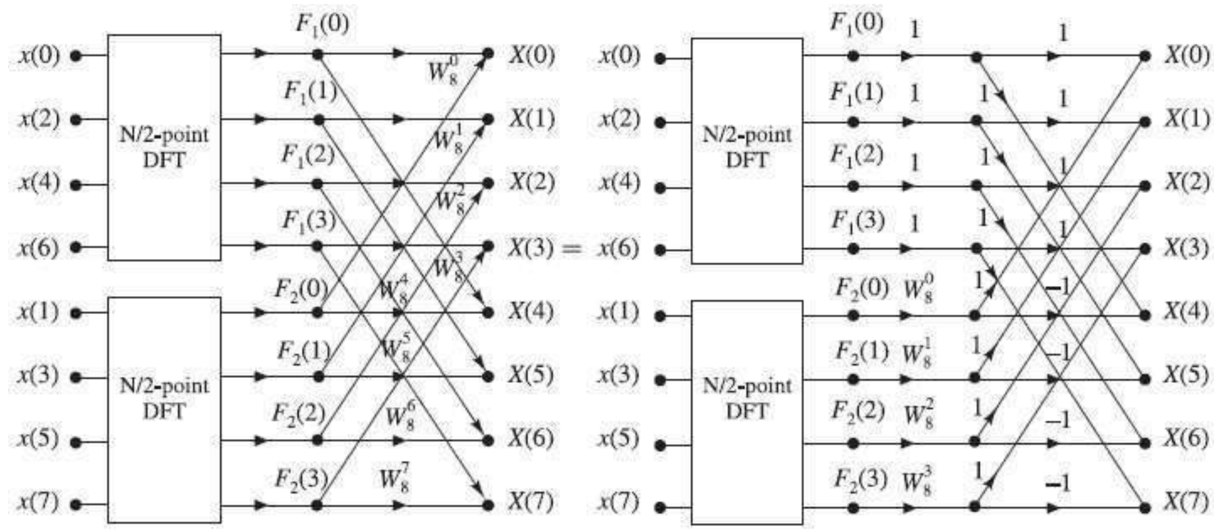


Figure 1 Illustration of flow graph of the first stage DIT FFT algorithm for $N = 8$.

THE 8-POINT DFT USING RADIX-2 DIT FFT

The computation of 8-point DFT using radix-2 FFT involves three stages of computation. Here $N = 8 = 2^3$, therefore, $r = 2$ and $m = 3$. The given 8-point sequence is decimated into four 2-point sequences. For each 2-point sequence, the two point DFT is computed. From the results of four 2-point DFTs, two 4-point DFTs are obtained and from the results of two 4-point DFTs, the 8-point DFT is obtained.

Let the given 8-sample sequence $x(n)$ be $\{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$. The 8-samples should be decimated into sequences of two samples. Before decimation they are arranged in bit reversed order as shown in Table 1.

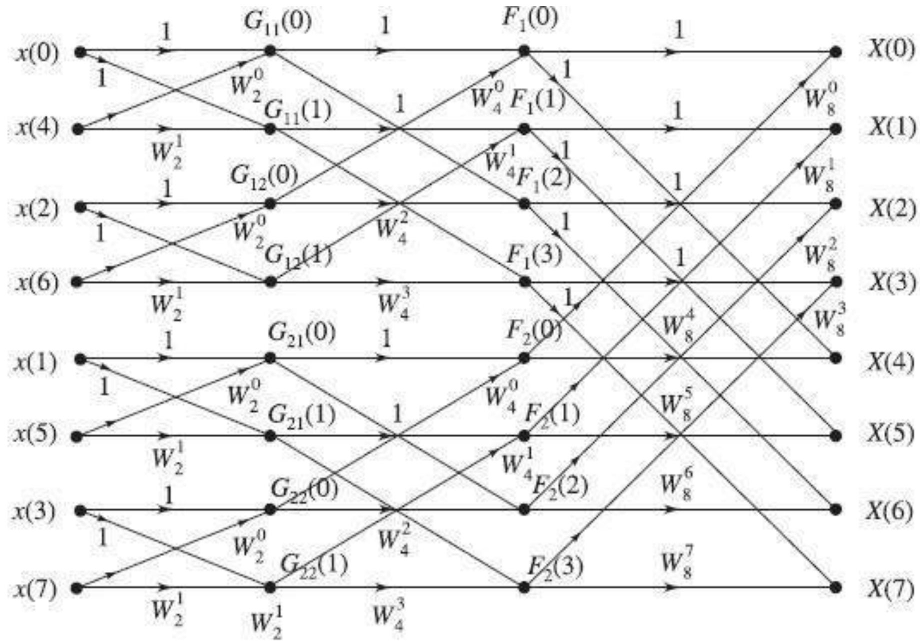


Figure Illustration of complete flow graph obtained by combining all the three stages for $N = 8$.

TABLE 1 Normal and bit reversed order for $N = 8$.

Normal order		*Bit reversed order*	
$x(0)$	$x(000)$	$x(0)$	$x(000)$
$x(1)$	$x(001)$	$x(4)$	$x(100)$
$x(2)$	$x(010)$	$x(2)$	$x(010)$
$x(3)$	$x(011)$	$x(6)$	$x(110)$
$x(4)$	$x(100)$	$x(1)$	$x(001)$
$x(5)$	$x(101)$	$x(5)$	$x(101)$
$x(6)$	$x(110)$	$x(3)$	$x(011)$
$x(7)$	$x(111)$	$x(7)$	$x(111)$

The $x(n)$ in bit reversed order is decimated into 4 numbers of 2-point sequences as shown below.

(i) $x(0)$ and $x(4)$
(ii) $x(2)$ and $x(6)$
(iii) $x(1)$ and $x(5)$
(iv) $x(3)$ and $x(7)$

Using the decimated sequences as input, the 8-point DFT is computed. Figure 7.5 shows the three stages of computation of an 8-point DFT.

The computation of 8-point DFT of an 8-point sequence in detail is given below. The 8-point sequence is decimated into 4-point sequences and 2-point sequences as shown below.

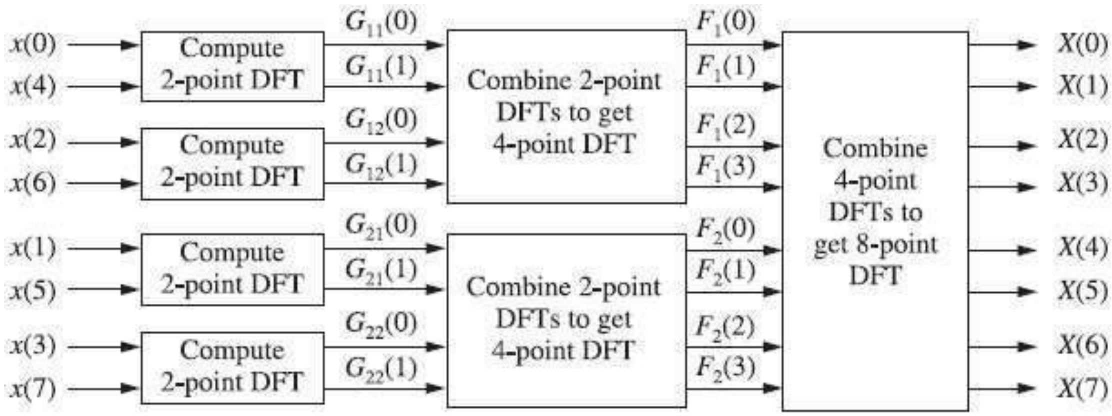


Figure Three stages of computation in 8-point DFT.

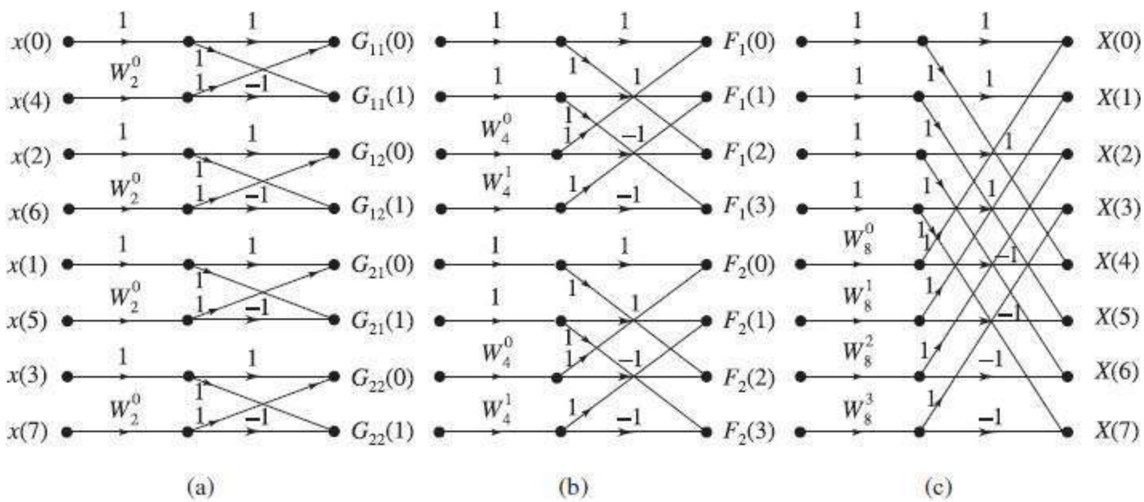


Figure (a)–(c) Flow graphs for implementation of 1st, 2nd and 3rd stages of computation.

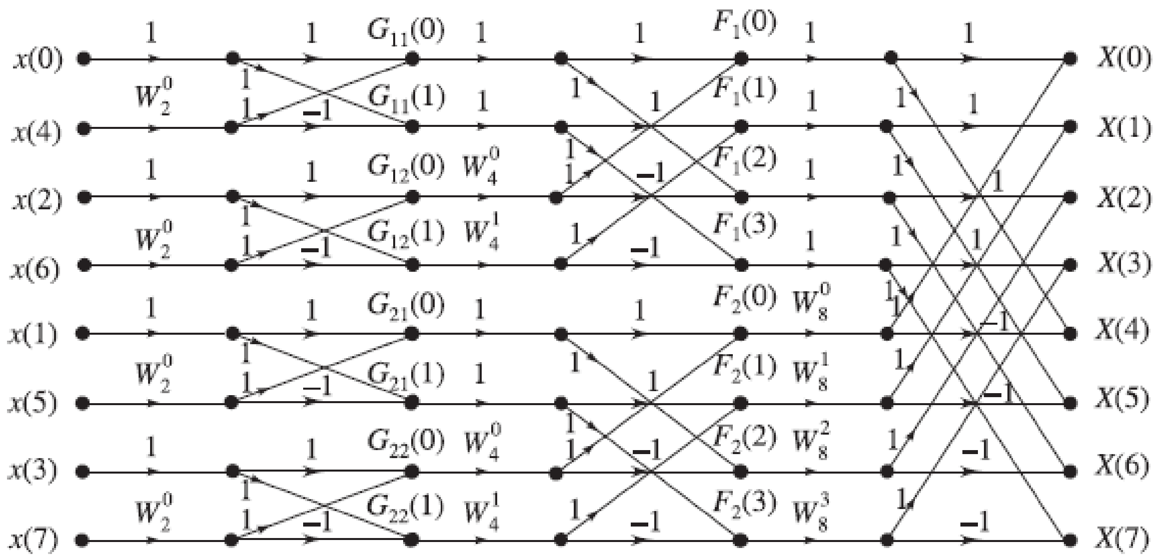


Figure The signal flow graph or butterfly diagram for 8-point radix-2 DIT FFT.

EXAMPLE An 8-point sequence is given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$. Compute the 8-point DFT of $x(n)$ by

(a) Radix-2 DIT FFT algorithm, Also sketch the magnitude and phase spectrum.

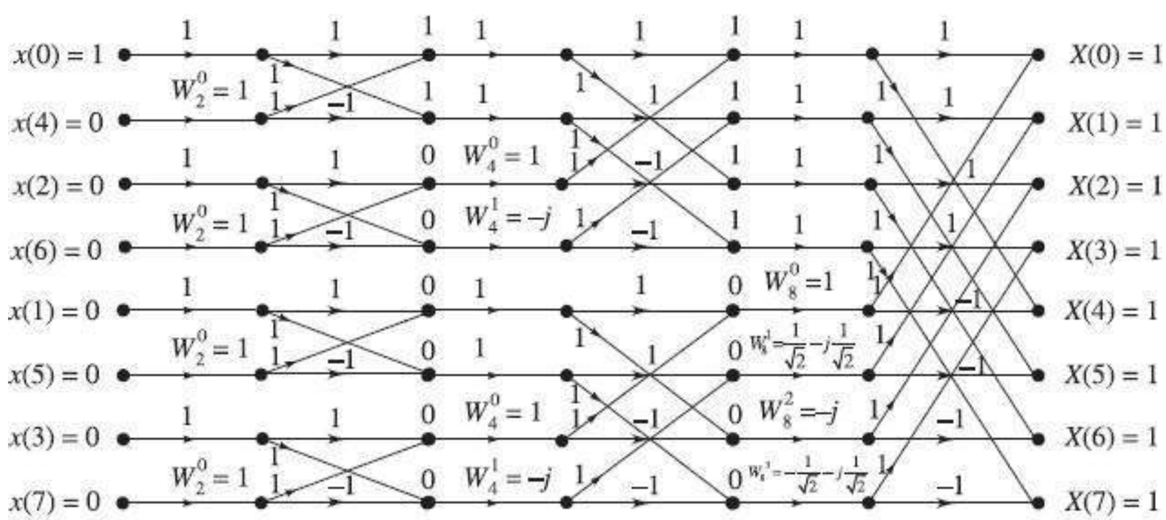


Figure 1 8-point DFT by DIT FFT.

Solution: (a) 8-point DFT by Radix-2 DIT FFT algorithm

The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{2, 2, 2, 2, 1, 1, 1, 1\}$$

The given sequence in bit reversed order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{2, 1, 2, 1, 2, 1, 2, 1\}$$

For DIT FFT, the input is in bit reversed order and the output is in normal order. The computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by Radix-2 DIT FFT algorithm is shown in Figure 2.

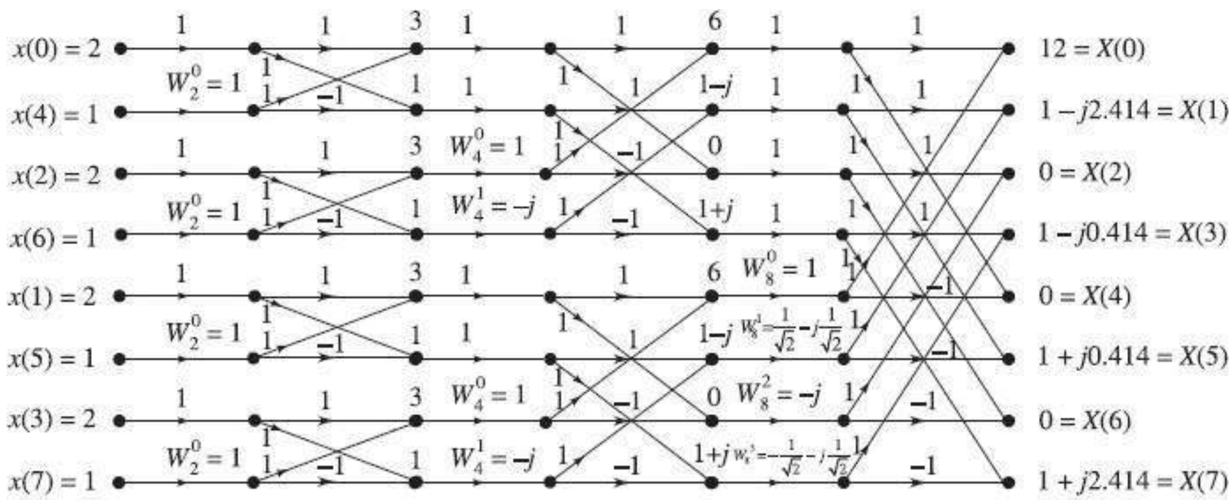


Figure 2 Computation of 8-point DFT of $x(n)$, i.e. $X(k)$ by DIT FFT.

From Figure 2, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{12, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

Magnitude and Phase Spectrum

Each element of the sequence $X(k)$ is a complex number and they are expressed in rectangular coordinates. If they are converted to polar coordinates, then the magnitude and phase of each element can be obtained.

The magnitude spectrum is the plot of the magnitude of each sample of $X(k)$ as a function of k. The phase spectrum is the plot of phase of each sample of $X(k)$ as a function of k. When N-point DFT is performed on a sequence $x(n)$ then the DFT sequence $X(k)$ will have a periodicity of N. Hence, in this example, the magnitude and phase spectrum will have a periodicity of 8 as shown below.

$$X(k) = \{12, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

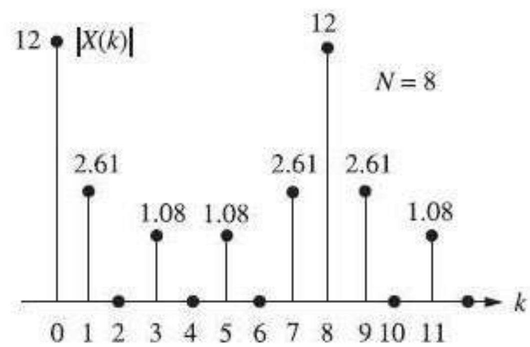
$$= \{12\lfloor 0^\circ, 2.61\lfloor -67^\circ, 0\lfloor 0^\circ, 1.08\lfloor -22^\circ, 0\lfloor 0^\circ, 1.08\lfloor 22^\circ, 0\lfloor 0^\circ, 2.61\lfloor 67^\circ\}$$

$$= \{12\lfloor 0, 2.61\lfloor -0.37\pi, 0\lfloor 0^\circ, 1.08\lfloor -0.12\pi, 0\lfloor 0^\circ, 1.08\lfloor 0.12\pi, 0\lfloor 0^\circ, 2.61\lfloor 0.37\pi\}$$

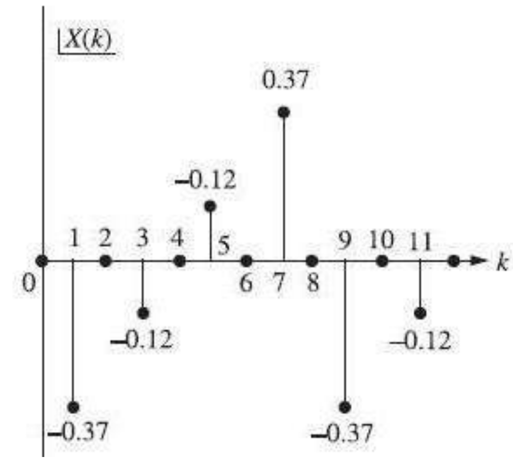
$$\therefore \quad |X(k)| = \{12, 2.61, 0, 1.08, 0, 1.08, 0, 2.61\}$$

$$\lfloor X(k) = \{0, -0.37\pi, 0, -0.12\pi, 0, 0.12\pi, 0, 0.37\pi\}$$

The magnitude and phase spectrum are shown in Figures 7.34(a) and (b).



(a)



(b)

Figure (a) Magnitude spectrum, (b) Phase spectrum.

EXAMPLE Find the 8-point DFT by radix-2 DIT FFT algorithm.

$$x(n) = \{2, 1, 2, 1, 2, 1, 2, 1\}$$

Solution: The given sequence is $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)\}$

$$= \{2, 1, 2, 1, 2, 1, 2, 1\}$$

For DIT FFT computation, the input sequence must be in bit reversed order and the output sequence will be in normal order.

$x(n)$ in bit reverse order is

$$x_r(n) = \{x(0), x(4), x(2), x(6), x(1), x(5), x(3), x(7)\}$$

$$= \{2, 2, 2, 2, 1, 1, 1, 1\}$$

The computation of 8-point DFT of $x(n)$ by radix-2 DIT FFT algorithm is shown in Figure 3

From Figure 7.35, we get the 8-point DFT of $x(n)$ as

$$X(k) = \{12, 0, 0, 0, 4, 0, 0, 0\}$$

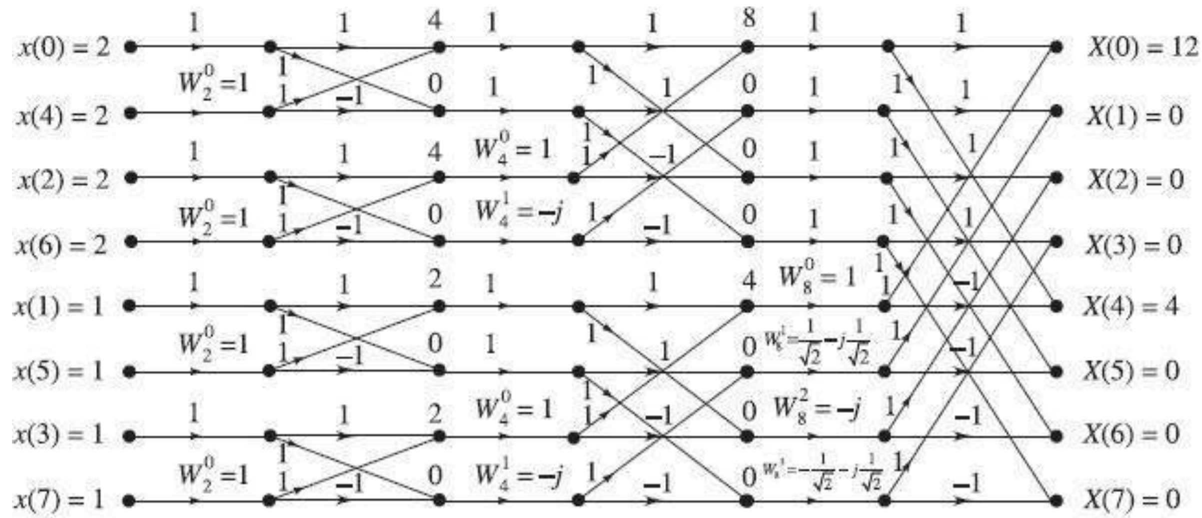


Figure 3 Computation of 8-point DFT of $x(n)$ by radix-2, DIT FFT.

DECIMATION IN FREQUENCY (DIF) RADIX-2 FFT

In decimation in frequency algorithm, the frequency domain sequence $X(k)$ is decimated. In this algorithm, the N-point time domain sequence is converted to two numbers of $N/2$-point sequences. Then each $N/2$-point sequence is converted to two numbers of $N/4$-point sequences. This process is continued until we get $N/2$ numbers of 2-point sequences. Finally, the 2-point DFT of each 2-point sequence is computed. The 2-point DFTs of $N/2$ numbers of 2-point sequences will give N-samples, which is the N-point DFT of the time domain sequence. Here the equations for $N/2$-point sequences, $N/4$-point sequences, etc., are obtained by decimation of frequency domain sequences. Hence this method is called DIF.

To derive the decimation-in-frequency form of the FFT algorithm for N, a power of 2, we can first divide the given input sequence $x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6), x(7)$ into the first half and last half of the points so that its DFT $X(k)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\, W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)}$$

It is important to observe that while the above equation for $X(k)$ contains two summations over $N/2$-points, each of these summations is not an $N/2$-point DFT, since W_N^{nk} rather than $W_{N/2}^{nk}$ appears in each of the sums.

or
$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + W_N^{(N/2)k} \sum_{n=0}^{\frac{N}{2}-1} x\left(n + \frac{N}{2}\right) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) W_N^{nk} + (-1)^k \, x\left(n + \frac{N}{2}\right) W_N^{nk} \right]$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^k x\left(n + \frac{N}{2}\right)\right] W_N^{nk}$$

Let us split $X(k)$ into even and odd numbered samples.

For even values of k, the $X(k)$ can be written as

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k} x\left(n + \frac{N}{2}\right)\right] W_N^{2nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + x\left(n + \frac{N}{2}\right)\right] W_{N/2}^{kn}; \quad \text{for } k = 0, 1, 2, \ldots, \left(\frac{N}{2} - 1\right)$$

For odd values of k, the $X(k)$ can be written as

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) + (-1)^{2k+1} x\left(n + \frac{N}{2}\right)\right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n W_{N/2}^{kn}; \quad \text{for } k = 0, 1, 2, \ldots, \left(\frac{N}{2} - 1\right)$$

The above equations for $X(2k)$ and $X(2k+1)$ can be recognized as $N/2$-point DFTs. $X(2k)$ is the DFT of the sum of first half and last half of the input sequence, i.e. of $\{x(n) + x(n + N/2)\}$ and $X(2k+1)$ is the DFT of the product W_N^n with the difference of first half and last half of the input, i.e. of $\{x(n) - x(n + N/2)\} W_N^n$.

If we define new time domain sequences, $u_1(n)$ and $u_2(n)$ consisting of $N/2$-samples, such that

$$u_1(n) = x(n) + x\left(n + \frac{N}{2}\right); \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{2} - 1$$

and
$$u_2(n) = \left[x(n) - x\left(n + \frac{N}{2}\right)\right] W_N^n; \quad \text{for } n = 0, 1, 2, \ldots, \frac{N}{2} - 1$$

then the DFTs $U_1(k) = X(2k)$ and $U_2(k) = X(2k + 1)$ can be computed by first forming the sequences $u_1(n)$ and $u_2(n)$, then computing the $N/2$-point DFTs of these two sequences to obtain the even numbered output points and odd numbered output points respectively. The procedure suggested above is illustrated in Figure for the case of an 8-point sequence.

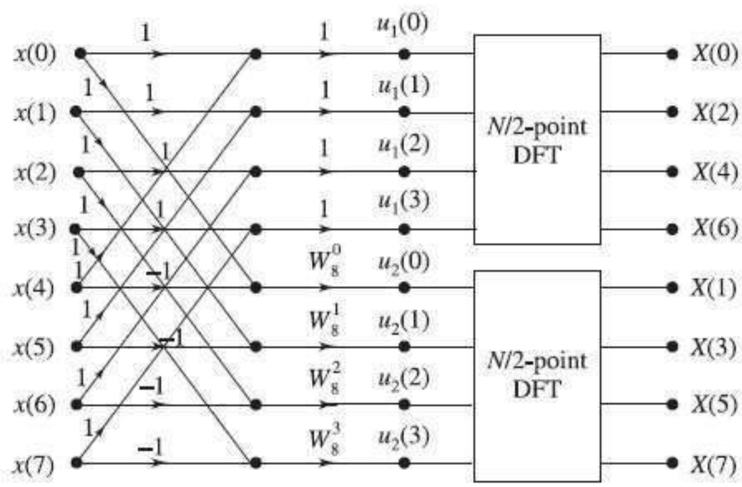


Figure Flow graph of the DIF decomposition of an N-point DFT computation into two $N/2$-point DFT computations $N = 8$.

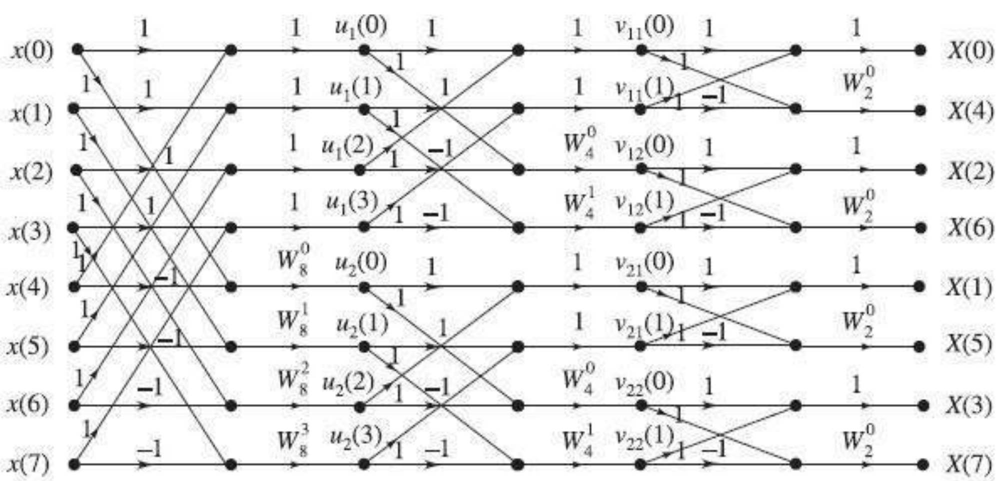


Figure Signal flow graph or butterfly diagram for the 8-point radix-2 DIF FFT algorithm.

The computation of 2-point DFTs is done by the butterfly operation shown in Figure

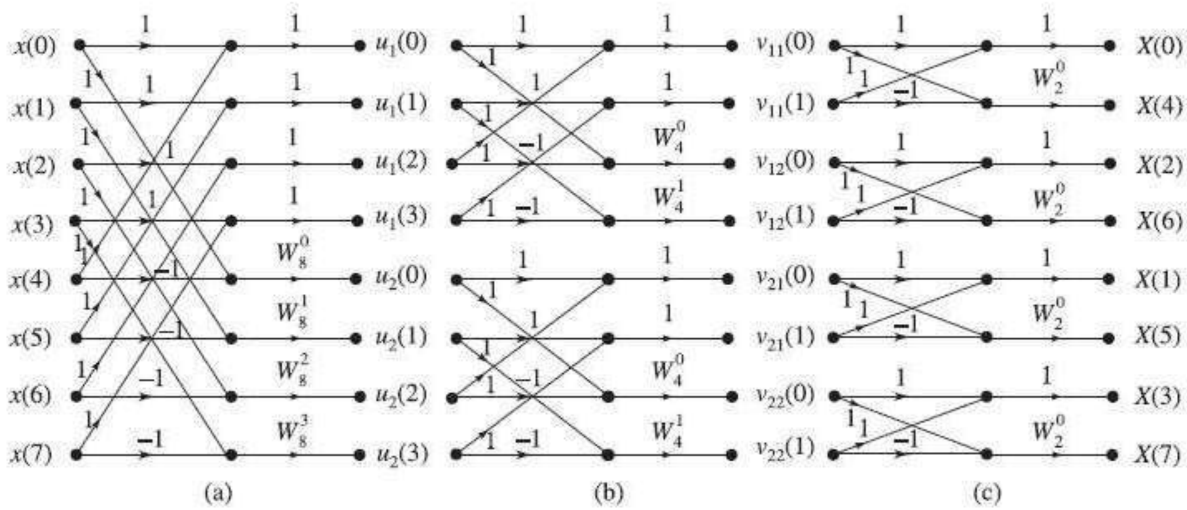


Figure (a)–(c) The first, second and third stages of computation of 8-point DFT by Radix-2 DIF FFT.

Comparison of DIT (Decimation-in-time) and DIF (Decimation-in-frequency) algorithms

Difference between DIT and DIF

1. In DIT, the input is bit reversed while the output is in normal order. For DIF, the reverse is true, i.e. the input is in normal order, while the output is bit reversed. However, both DIT and DIF can go from normal to shuffled data or vice versa.
2. Considering the butterfly diagram, in DIT, the complex multiplication takes place before the add subtract operation, while in DIF, the complex multiplication takes place after the add subtract operation.

Similarities

1. Both algorithms require the same number of operations to compute DFT.
2. Both algorithms require bit reversal at some place during computation.

EXAMPLE An 8-point sequence is given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$. Compute the 8-point DFT of $x(n)$ by

(a) Radix-2 DIF FFT algorithm, Also sketch the magnitude and phase spectrum.

8-point DFT by radix-2 DIF FFT algorithm

For DIF FFT, the input is in normal order and the output is in bit reversed order. The computation of DFT by radix-2 DIF FFT algorithm is shown in Figure

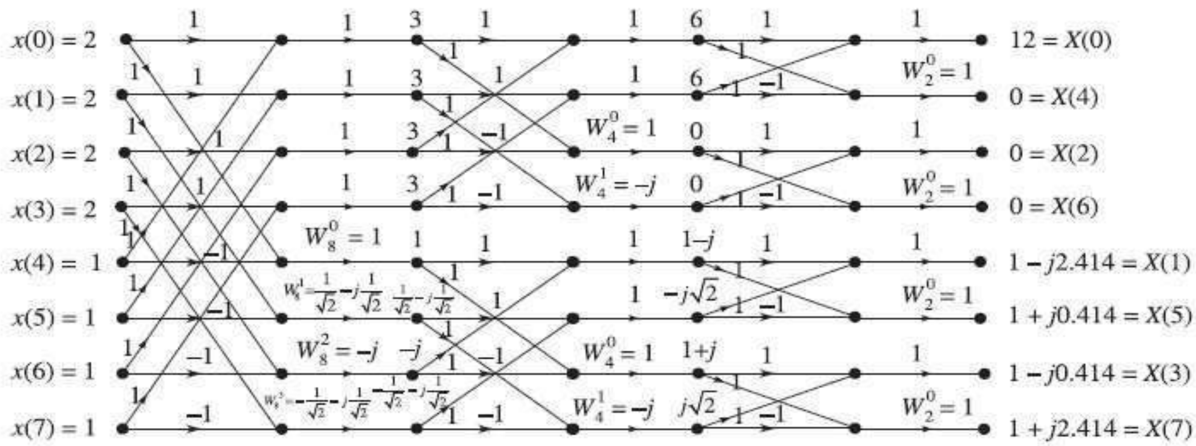


Figure Computation of 8-point DFT of x(*n*) by radix-2 DIF FFT algorithm.

From Figure we observe that the 8-point DFT in bit reversed order is

$$X_r(k) = \{X(0), X(4), X(2), X(6), X(1), X(5), X(3), X(7)\}$$
$$= \{12, 0, 0, 0, 1 - j2.414, 1 + j0.414, 1 - j0.414, 1 + j2.414\}$$

∴ The 8-point DFT in normal order is

$$X(k) = \{X(0), X(1), X(2), X(3), X(4), X(5), X(6), X(7)\}$$
$$= \{12, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

Computation of IDFT through FFT

The IDFT of an N-point sequence $\{X(k)\}$; $k = 0, 1, ..., N - 1$ is defined as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk}$$

Taking the conjugate of the above equation for $x(n)$, we get

$$x^*(n) = \left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk}\right]^* = \frac{1}{N}\sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}$$

Taking the conjugate of the above equation for $x^*(n)$, we get

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)\, W_N^{nk}\right]^*$$

The term inside the square brackets in the above equation for $x(n)$ is same as the DFT computation of a sequence $X^*(k)$ and may be computed using any FFT algorithm. So we can say that the IDFT of $X(k)$ can be obtained by finding the DFT of $X^*(k)$, taking the conjugate of that DFT and dividing by N. Hence, to compute the IDFT of $X(k)$ the following procedure can be followed

1. Take conjugate of $X(k)$, i.e. determine $X^*(k)$.
2. Compute the N-point DFT of $X^*(k)$ using radix-2 FFT.
3. Take conjugate of the output sequence of FFT.
4. Divide the sequence obtained in step-3 by N.

The resultant sequence is $x(n)$.

Thus, a single FFT algorithm serves the evaluation of both direct and inverse DFTs.

EXAMPLE Find the IDFT of the sequence

$$X(k) = \{4, 1 - j2.414, 0, 1 - j0.414, 0, 1 + j0.414, 0, 1 + j2.414\}$$

using DIF algorithm.

Solution: The IDFT $x(n)$ of the given 8-point sequence $X(k)$ can be obtained by finding $X^*(k)$, the conjugate of $X(k)$, finding the 8-point DFT of $X^*(k)$, using DIF algorithm to get

$8x^*(n)$, taking the conjugate of that to get $8x(n)$ and then dividing the result by 8 to get $x(n)$. For DIF algorithm, input $X^*(k)$ must be in normal order. The output will be in bit reversed order for the given $X(k)$.

$$X^*(k) = \{4, 1 + j2.414, 0, 1 + j0.414, 0, 1 - j0.414, 0, 1 - j2.414\}$$

The DFT of $X^*(k)$ using radix-2, DIF FFT algorithm is computed as shown in Figure

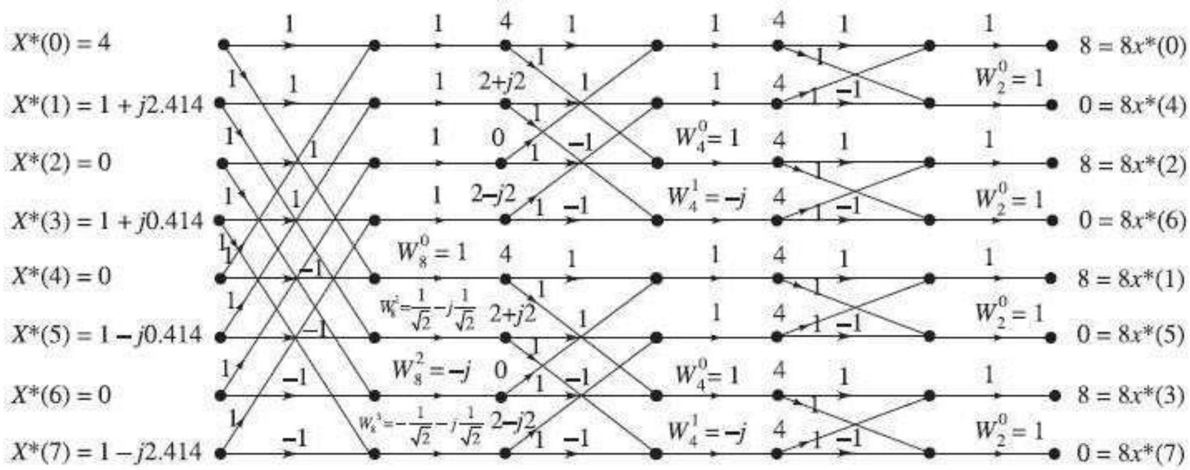


Figure Computation of 8-point DFT of $X^*(k)$ by radix-2 DIF FFT.

From the DIF FFT algorithm of Figure 7.42, we get

$$8x_r^*(n) = \{8, 0, 8, 0, 8, 0, 8, 0\}$$

$$\therefore \qquad 8x_r(n) = \{8, 0, 8, 0, 8, 0, 8, 0\}^* = \{8, 0, 8, 0, 8, 0, 8, 0\}$$

$$\therefore \qquad x(n) = \frac{1}{8}\{8, 8, 8, 8, 0, 0, 0, 0\} = \{1, 1, 1, 1, 0, 0, 0, 0\}$$

EXAMPLE Compute the IDFT of the sequence

$$X(k) = \{7, -0.707 - j0.707, -j, 0.707 - j0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$$

using DIT algorithm.

Solution: The IDFT $x(n)$ of the given sequence $X(k)$ can be obtained by finding $X^*(k)$, the conjugate of $X(k)$, finding the 8-point DFT of $X^*(k)$ using radix-2 DIT FFT algorithm to get $8x^*(n)$, taking the conjugate of that to get $8x(n)$ and then dividing by 8 to get $x(n)$. For DIT FFT, the input $X^*(k)$ must be in bit reverse order. The output $8x^*(n)$ will be in normal order. For the given $X(k)$.

$$X^*(k) = \{7, -0.707 + j0.707, j, 0.707 + j0.707, 1, 0.707 - j0.707, -j, -0.707 - j0.707\}$$

$X^*(k)$ in bit reverse order is

$$X_r^*(k) = \{7, 1, j, -j, -0.707 + j0.707, 0.707 - j0.707, 0.707 + j0.707, -0.707 - j0.707\}$$

The 8-point DFT of $X^*(k)$ using radix-2, DIT FFT algorithm is computed as shown in Figure

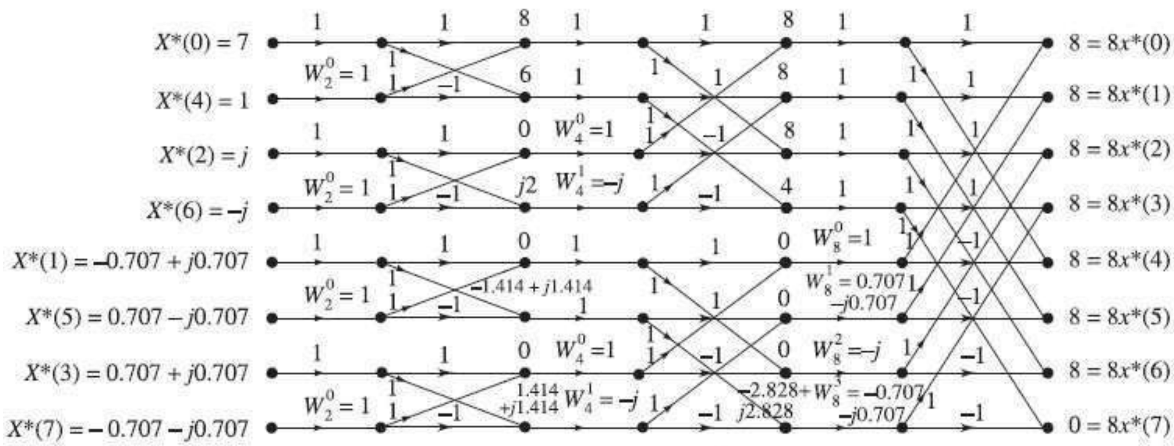


Figure Computation of 8-point DFT of $X^*(k)$ by radix-2, DIT FFT.

From the DIT FFT algorithm of Figure 7.43, we have

$$8x^*(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$$

$$\therefore \quad 8x(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$$

$$\therefore \quad x(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$$

UNIT IV

INTRODUCTION TO LINEAR SYSTEMS & DIGITAL SIGNAL PROCESSING

- Introduction to Systems
- Classification of Systems
- Impulse response
- Transfer function of a LTI system.
- Introduction to Digital Signal Processing
- Linear Shift Invariant Systems,
- Stability and Causality of Discrete time systems

INTRODUCTION TO LINEAR SYSTEMS

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system may also be defined as a set of elements or functional blocks which are connected together and produces an output in response to an input signal. The response or output of the system depends on the transfer function of the system. It is a cause and effect relation between two or more signals.

As signals, systems are also broadly classified into continuous-time and discrete-time systems. A continuous-time system is one which transforms continuous-time input signals into continuous-time output signals, whereas a discrete-time system is one which transforms discrete-time input signals into discrete-time output signals.

Classification

- There are two types of systems : (i) continuous time and (ii) discrete time systems.
- Continuous time (CT) systems handle continuous time signals. Analog filters, amplifiers, attenuators, analog transmitters and receivers etc are examples of continuous time systems.
- Discrete time (DT) systems handle discrete time signals. Fig. 1.6.1 (b) shows such system. Computers, printers, microprocessors, memories, shift registers etc are examples of discrete time systems. They operate only on discrete time signals.

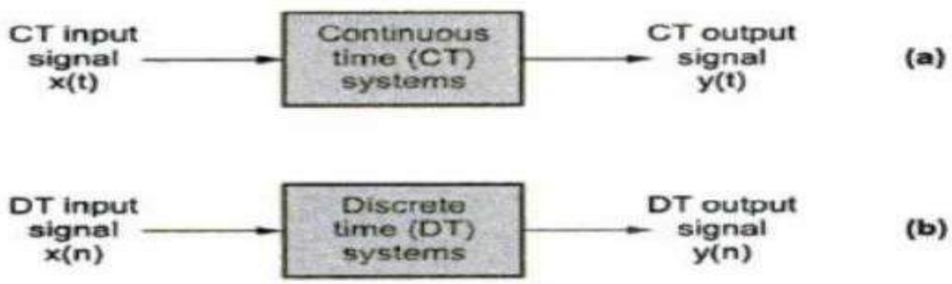


Fig. Two types of systems based on signals they handle

Continuous as well as discrete time systems can be further classified based on their properties. These properties are as follows :

i) Dynamicity property : Static and dynamic systems.

ii) Shift invariance : Time invariant and time variant systems.

iii) Linearity property : Linear and non-linear systems.

iv) Causality property : Causal and non-causal systems.

v) Stability property : Stable and unstable systems.

vi) Invertibility property : Inversible and non-inversible systems.

1 Static and Dynamic Systems (Systems with Memory or without Memory)

> ***Definition :*** *The continuous time system is said to be static or (memoryless, instantaneous) if its output depends upon the present input only.*

The discrete time systems can also be static or dynamic. If output of the discrete time system depends upon the present input sample only, then it is called static or memoryless or instantaneous system. For example,

$$y(n) = 10 \cdot x(n)$$

or

$$y(n) = 15 \cdot x^2(n) + 10x(n)$$

are the static systems. Here the $y(n)$ depends only upon n^{th} input sample. Hence such systems do not need memory for its operation. A system is said to be dynamic if the output depends upon the past values of input also. For example,

$$y(n) = x(n) + x(n-1)$$

2 Time Invariant and Time Variant Systems

Definition : *A continuous time system is time invariant if the time shift in the input signal results in corresponding time shift in the output.*

Let $y(t) = f[x(t)]$ i.e. $y(t)$ is response for $x(t)$. Then if $x(t)$ is delayed by time t_1, then output $y(t)$ will also be delayed by the same time. i.e.,

$$f[x(t-t_1)] = y(t-t_1) \qquad \text{... (1.6.1)}$$

The time variant system do not satisfy above relation. The time invariant systems are also called fixed systems.

Similarly if the input/output characteristics of the discrete time system do not change with shift of time origin, such systems are called shift invariant or time invariant systems. Let the system has input $x(n)$ and corresponding output $y(n)$, i.e. $y(n) = f[x(n)]$. Then the system is shift invariant or time invariant if and only if,

$$f[x(n-k)] = y(n-k)$$

3 Linear and Non-linear Systems

Definition : *A system is said to be linear if it satisfies the superposition principle.*

Consider the two systems defined as follows :

$y_1(t) = f[x_1(t)]$ i.e. $x_1(t)$ is input and $y_1(t)$ is output.

and $y_2(t) = f[x_2(t)]$ i.e. $x_2(t)$ is input and $y_2(t)$ is output.

Then the continuous time system is linear if,

$$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t)$$

Here a_1 and a_2 are arbitrary constants. This condition states that combined response due to $x_1(t)$ and $x_2(t)$ together is same as the sum of individual responses for a linear system.

Similarly, the discrete time system is said to be linear if it satisfies superposition principle. Consider the two systems defined as follows :

$y_1(n) = f[x_1(n)]$ i.e. $x_1(n)$ is input and $y_1(n)$ is output.

$y_2(n) = f[x_2(n)]$ i.e. $x_2(n)$ is input and $y_2(n)$ is output.

Then the discrete time system is linear if,

$$f[a_1 x_1(n) + a_2 x_2(n)] = a_1 y_1(n) + a_2 y_2(n)$$

4 Causal and Non-causal Systems

> **Definition** : *The system is said to be causal if its output at any time depends upon present and past inputs only.*

i.e.,

$$y(t_0) = f\left[x(t);\ t \le t_0\right]$$

Thus the output at time t_0, depends on inputs before t_0. The causal system is not anticipatory. Similarly, a discrete time system is said to be causal if its output at any instant depends upon present and past input samples only. i.e.,

$$y(n) = f\left[x(k);\ k \le n\right]$$

Thus the output is the function of $x(n)$, $x(n-1), x(n-2), x(n-3)$ ... etc. For causal system. The system is non-causal if its output depends upon future inputs also, i.e. $x(n+1)$, $x(n+2)$, $x(n+3)$ ... etc.

Normally all causal systems are physically realizable. There is no system which can generate the output for inputs which will be available in future. Such systems are non-causal, and they are not physically realizable.

.5 Stable and Unstable Systems

> **Definition** : *When every bounded input produces bounded output, then the system is called Bounded Input Bounded Output (BIBO) stable.*

This criteria is applicable for both the continuous time and discrete time systems. The input is said to be bounded if there exists some finite number M_x such that,

$$\left.\begin{aligned} &\text{CT input : } |x(t)| \le M_x < \infty \\ &\text{DT input : } |x(n)| \le M_x < \infty \end{aligned}\right\}$$

Similarly the output is said to be bounded if there exists some finite number M_y such that,

$$\left.\begin{aligned} &\text{CT output : } |y(t)| \le M_y < \infty \\ &\text{DT output : } |y(n)| \le M_y < \infty \end{aligned}\right\}$$

If the system produces unbounded output for bounded input, then it is unstable.

6 Invertability and Inverse Systems

Definition : ***A system is said to be invertible if there is unique output for every unique input.***

Fig. 1.6.3 shows this concept.

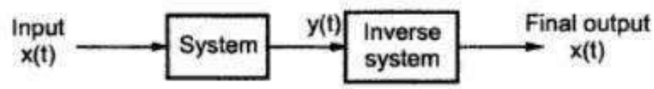


Fig. 1.6.3 Invertible system

If the system is invertible, there exists an inverse system. If these two systems are cascaded as shown in figure, then final output is same as input.

If the system is denoted by H, then its inverse system is denoted by H^{-1}. Then cascading the two systems gives,

$$H\,H^{-1} = 1$$

Frequency Response of LTI Systems

The LTI systems form an important class in communication. The amplitude and phase response, realizability, bandwidth, distortion during transmission of signal are all very important concepts related to design and implementation of systems.

Frequency Response

The frequency response of the system gives the variation of magnitude and phase of the system output with respect to frequency on application of input. We know that the output y (t) of the system is given as,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

This equation gives time response of the LTI system.

The RHS of the above equation represents convolution of input signal x (t) and impulse response h (t). By applying fourier transform to above equation,

$$F[y(t)] = F\left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right]$$

We know that convolution of two functions is transformed into multiplication of their fourier transforms. By applying this to above equation,

$$y(t) = K\,x(t-t_0)$$

Here, K = constant represents change in amplitude.

& t_0 = time delay in transmission of signal through a system.

By taking fourier transform of both sides of above equation

$$Y(f) = F[y(t)] = F\left\{K\,x(t-t_0)\right\}$$

∴ From the time shifting property of FT,

$$Y(f) = K\,X(f)\,e^{-j2\pi f t_0}$$

Transfer function H (f) is given from equation 2.13.2

$$H(f) = \frac{Y(f)}{X(f)}$$

Putting for RHS from equation 2.13.4 in above equation,

$$H(f) = K\,e^{-j2\pi f t_0}$$

This equation gives the transfer function for a distortionless system. It is clear from above equation that, the magnitude of the transfer function is 'K', which is independent of frequency. That is the transfer function has constant amplitude at all frequencies. The phase shift of above equation is,

$$\theta(f) = -2\pi f t_0$$
$$= (-2\pi t_0) f$$

That is the phase shift is linearly proportional to frequency. Here the phase shift is linear at all frequencies. This can be expressed with the example.

Let there be a signal in time domain as

$$x(t) = \cos(2\pi f t)$$

Now let the output signal be same in amplitude but shifted in time by t_0 seconds. i.e.

$$y(t) = \cos[2\pi f(t-t_0)]$$

This equation can also be written as,

$$y(t) = \cos(2\pi f t - 2\pi f t_0) = \cos(2\pi f t - \theta(f))]$$

Thus phase shift of y (t) is,

$$\theta(f) = -2\pi f t_0$$

which is proportional to frequency 'f'.

Response of a Linear System

Impulse Response

Convolution relates input and output of LTI sytem.

It is given as,

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

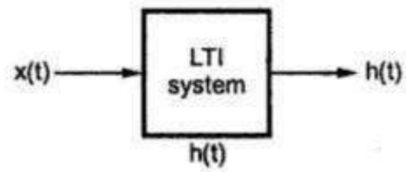


Fig. 1 Input and output of LTI system

Here h (t) is called impulse response of the system.
It is characteristic of a particular system. Impulse response h (t) of the system is obtained at the output by applying unit impulse δ(t) at the input. i.e.,

when $x(t) = \delta(t),\ y(t) = h(t)$

Frequency Response

Frequency response analysis and differential equations etc. can be analyzed with the help of Fourier representations. For example, the fourier transform $X(\omega)$ gives frequency spectrum of the signal. We know that output of the system is,

$$y(t) = x(t) * h(t)$$

By convolution theorem above equation becomes

$$Y(\omega) = X(\omega) \cdot H(\omega) \quad \text{or} \quad Y(f) = X(f) \cdot H(f)$$

and $y(t) = IFT\{X(\omega) \cdot H(\omega)\}$

Thus output $y(t)$ can be obtained by taking the inverse fourier transform of the product $X(\omega)$. Let us now study these aspects.

The convolution is given as,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Let the input be $e^{j\omega t}$, i.e. sinusoid. Then above equation becomes,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, e^{j\omega(t-\tau)}\, d\tau$$

$$= e^{j\omega t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

In the above equation the integral represents fourier transform of $h(\tau)$. i.e., In the above equation the integral represents fourier transform of $h(\tau)$. i.e.,

$$y(t) = e^{j\omega t} H(\omega)$$

Here $H(\omega)$ is the Fourier transform of $h(t)$. The above equation shows that output $y(t)$ contains the same signal as input $e^{j\omega t}$ multiplied by $H(\omega)$. This $H(\omega)$ is called frequency response of the system.

Again consider the convolution,

$$y(t) = x(t) * h(t)$$

By convolution property of Fourier transform we can write above equation as,

$$Y(\omega) = X(\omega)\, H(\omega) \quad \text{or} \quad Y(f) = X(f) \cdot H(f)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \quad \text{or} \quad H(f) = \frac{Y(f)}{X(f)}$$

Here $H(\omega)$ represent the frequency response of the LTI-CT system. These functions are also called as system transfer functions.

Example .1 : *The impulse response of the continuous time system is given as,*

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

Determine the frequency response and plot the magnitude phase plots.

Solution : Take Fourier transform of the given impulse response. i.e.,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \frac{1}{RC} e^{-t/RC}\, u(t)\, e^{-j\omega t}\, dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t/RC}\, e^{-j\omega t}\, dt$$

$$= \frac{1}{RC} \int_{0}^{\infty} e^{-t\left(j\omega+\frac{1}{RC}\right)}\, dt$$

$$= \frac{1}{RC} \left(-\frac{1}{j\omega+\frac{1}{RC}}\right) \left[e^{-t\left(j\omega+\frac{1}{RC}\right)}\right]_0^{\infty}$$

$$= \frac{1/RC}{j\omega+1/RC} = \frac{1}{1+j\omega RC}$$

Now let us determine the magnitude and phase of $H(\omega)$. Let us rearrange above equation as,

$$H(\omega) = \frac{1}{1+j\omega RC} \times \frac{1-j\omega RC}{1-j\omega RC} = \frac{1-j\omega RC}{1+(\omega RC)^2}$$

$$= \frac{1}{1+(\omega RC)^2} + j\frac{-\omega RC}{1+(\omega RC)^2}$$

Thus $H(\omega)$ is expressed into its real and imaginary parts. Now magnitude can be obtained as,

$$|H(\omega)| = \left\{\frac{1}{[1+(\omega RC)^2]^2} + \frac{(\omega RC)^2}{[1+(\omega RC)^2]^2}\right\}^{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

This is the magnitude response of the given system. And the phase response will be,

$$\angle H(\omega) = tan^{-1}\left\{\frac{(-\omega RC)\,/\,[1+(\omega RC)^2]}{1\,/\,[1+(\omega RC)^2]}\right\}$$

$$= -\,tan^{-1}\,(\omega RC)$$

Let $RC = 1$, then magnitude and phase response will be,

$$|H(\omega)| = \frac{1}{\sqrt{1+\omega^2}}$$

Fig. .2 shows the magnitude and phase response as given by above equations.

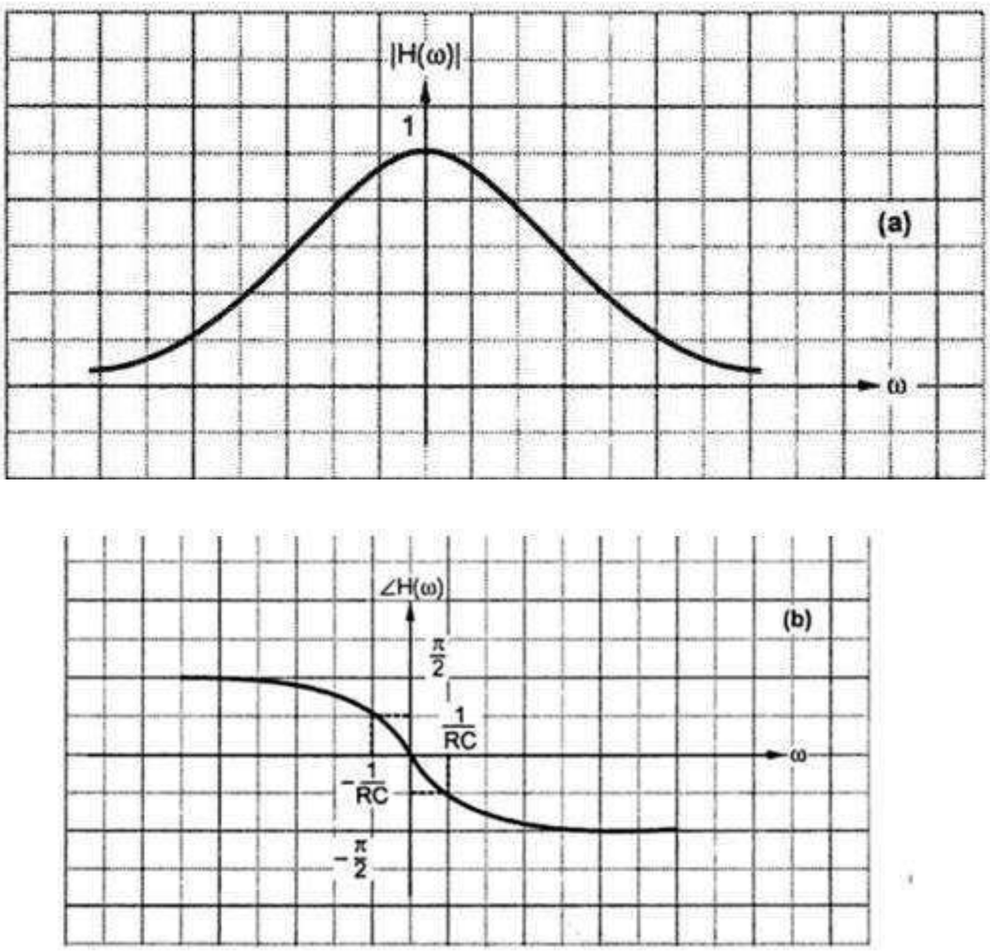


Fig. .2 (a) Magnitude response (b) Phase response

In this figure observe that the magnitude response is symmetric but phase response is antisymmetric. Magnitude response is monotonically decreasing. Hence this is a lowpass filter.

Example .2 : *The system produces the output of $y(t) = e^{-t}u(t)$ for an input of $x(t) = e^{-2t}\,u(t)$. Determine the impulse response and frequency response of the system.*

Solution : Here $\quad y(t) = e^{-t}\,u(t)$

and $\quad x(t) = e^{-2t}\,u(t)$

Consider the standard Fourier transform pair,

$$e^{-at}\,u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

Hence fourier transforms of $y(t)$ and $x(t)$ will be,

$$Y(\omega) = \frac{1}{1+j\omega}$$

and

$$X(\omega) = \frac{1}{2+j\omega}$$

From equation 4.5.3 we can obtain the transfer function as,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Putting the values of $X(\omega)$ and $Y(\omega)$,

Putting the values of $X(\omega)$ and $Y(\omega)$,

$$H(\omega) = \frac{1/(1+j\omega)}{1/(2+j\,\omega)} = \frac{2+j\omega}{1+j\omega}$$

Let us multiply the numerator and denominator by $1-j\omega$ i.e.,

$$H(\omega) = \frac{2+j\omega}{1+j\omega} \times \frac{1-j\omega}{1-j\omega}$$

$$= \frac{2+(\omega)^2}{1+(\omega)^2} + j\,\frac{-\omega}{1+(\omega)^2}$$

Hence magnitude of $H(\omega)$ will be,

$$|H(\omega)| = \left\{\left[\frac{2+(\omega)^2}{1+(\omega)^2}\right]^2 + \left[\frac{(\omega)}{1+(\omega)^2}\right]^2\right\}^{\frac{1}{2}}$$

Simplifying the above equation we get,

$$|H(\omega)| = \sqrt{\frac{4+(\omega)^2}{1+(\omega)^2}}$$

This is the magnitude response of the system. And the phase response will be,

$$\angle H(\omega) = tan^{-1} \frac{(-\omega) / [1+(\omega)^2]}{[2+(\omega)^2] / [1+(\omega)^2]} = -tan^{-1} \left\{ \frac{\omega}{2+(\omega)^2} \right\}$$

Now consider the transfer function of equation 4.4.8. i.e.,

$$H(\omega) = \frac{2+j\omega}{1+j\omega}$$

Let us rearrange the above equation as,

$$H(\omega) = \frac{1+j\omega+1}{1+j\omega} = 1+ \frac{1}{1+j\omega}$$

Inverse Fourier transform of above equation becomes,

$$h(t) = IFT\{H(\omega)\} = \delta(t)+ e^{-t}\ u(t)$$

This is the impulse response of the given system.

INTRODUCTION TO DIGITAL SIGNAL PROCESSING

Introduction

Signals constitute an important part of our daily life. Anything that carries some information is called a signal. A signal is defined as a single-valued function of one or more independent variables which contain some information. A signal is also defined as a physical quantity that varies with time, space or any other independent variable. A signal may be represented in time domain or frequency domain. Human speech is a familiar example of a signal. Electric current and voltage are also examples of signals. A signal can be a function of one or more independent variables. A signal may be a function of time, temperature, position, pressure, distance etc. If a signal depends on only one independent variable, it is called a one-dimensional signal, and if a signal depends on two independent variables, it is called a two-dimensional signal.

A system is defined as an entity that acts on an input signal and transforms it into an output signal. A system is also defined as a set of elements or fundamental blocks which are connected together and produces an output in response to an input signal. It is a cause-and-effect relation between two or more signals. The actual physical structure of the system determines the exact relation between the input $x(n)$ and the output $y(n)$, and specifies the output for every input. Systems may be single-input and single-output systems or multi-input and multi-output systems.

Signal processing is a method of extracting information from the signal which in turn depends on the type of signal and the nature of information it carries. Thus signal processing is concerned with representing signals in the mathematical terms and extracting information by carrying out algorithmic operations on the signal. Digital signal processing has many advantages over analog signal processing. Some of these are as follows:

Digital circuits do not depend on precise values of digital signals for their operation. Digital circuits are less sensitive to changes in component values. They are also less sensitive to variations in temperature, ageing and other external parameters.

In a digital processor, the signals and system coefficients are represented as binary words. This enables one to choose any accuracy by increasing or decreasing the number of bits in the binary word.

Digital processing of a signal facilitates the sharing of a single processor among a number of signals by time sharing. This reduces the processing cost per signal.

Digital implementation of a system allows easy adjustment of the processor characteristics during processing.

Linear phase characteristics can be achieved only with digital filters. Also multirate processing is possible only in the digital domain. Digital circuits can be connected in cascade without any loading problems, whereas this cannot be easily done with analog circuits.

Storage of digital data is very easy. Signals can be stored on various storage media such as magnetic tapes, disks and optical disks without any loss. On the other hand, stored analog signals deteriorate rapidly as time progresses and cannot be recovered in their original form.

Digital processing is more suited for processing very low frequency signals such as seismic signals.

Though the advantages are many, there are some drawbacks associated with processing a signal in digital domain. Digital processing needs 'pre' and 'post' processing devices like analog-to-digital and digital-to-analog converters and associated reconstruction filters. This increases the complexity of the digital system. Also, digital techniques suffer from frequency limitations. Digital systems are constructed using active devices which consume power whereas analog processing algorithms can be implemented using passive devices which do not consume power. Moreover, active devices are less reliable than passive components. But the advantages of digital processing techniques outweigh the disadvantages in many applications. Also the cost of DSP hardware is decreasing continuously. Consequently, the applications of digital signal processing are increasing rapidly.

The digital signal processor may be a large programmable digital computer or a small microprocessor programmed to perform the desired operations on the input signal. It may also be a hardwired digital processor configured to perform a specified set of operations on the input signal.

DSP has many applications. Some of them are: Speech processing, Communication, Biomedical, Consumer electronics, Seismology and Image processing.

The block diagram of a DSP system is shown in Figure 1.1.

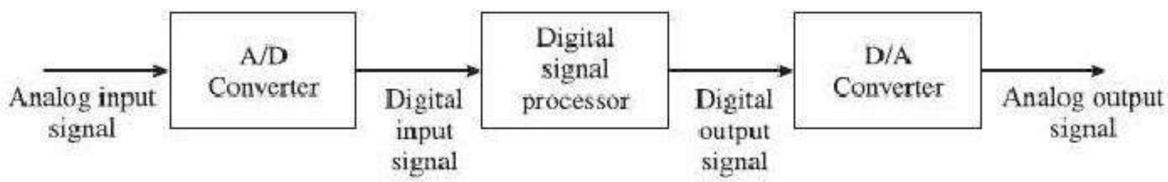


Figure 1.1 Block diagram of a digital signal processing system.

In this book we discuss only about discrete one-dimensional signals and consider only single-input and single-output discrete-time systems. In this chapter, we discuss about various basic discrete-time signals available, various operations on discrete-time signals and classification of discrete-time signals and discrete-time systems.

CLASSIFICATION OF DISCRETE-TIME SYSTEMS

A discrete-time system is represented by a block diagram as shown in Figure 2. An arrow entering the box is the input signal (also called excitation, source or driving function) and an arrow leaving the box is an output signal (also called response). Generally, the input is denoted by $x(n)$ and the output is denoted by $y(n)$.

The relation between the input $x(n)$ and the output $y(n)$ of a system has the form:

$$y(n) = \text{Operation on } x(n)$$

Mathematically,

$$y(n) = T[x(n)]$$

which represents that $x(n)$ is transformed to $y(n)$. In other words, $y(n)$ is the transformed version of $x(n)$.

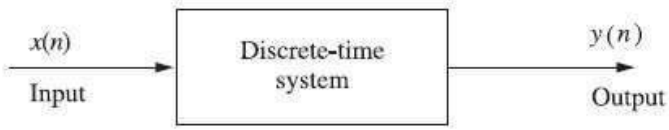


Figure 2 Block diagram of discrete-time system.

Both continuous-time and discrete-time systems are further classified as follows:

1. Static (memoryless) and dynamic (memory) systems
2. Causal and non-causal systems
3. Linear and non-linear systems
4. Time-invariant and time varying systems
5. Stable and unstable systems.
6. Invertible and non-invertible systems
7. FIR and IIR systems

1 Static and Dynamic Systems

A system is said to be static or memoryless if the response is due to present input alone, i.e., for a static or memoryless system, the output at any instant n depends only on the input applied at that instant n but not on the past or future values of input or past values of output.

For example, the systems defined below are static or memoryless systems.

$$y(n) = x(n)$$

$$y(n) = 2x^2(n)$$

In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs or past outputs. A summer or accumulator, a delay element is a discrete-time system with memory.

For example, the systems defined below are dynamic or memory systems.

$$y(n) = x(2n)$$

$$y(n) = x(n) + x(n - 2)$$

$$y(n) + 4y(n - 1) + 4y(n - 2) = x(n)$$

Any discrete-time system described by a difference equation is a dynamic system.

A purely resistive electrical circuit is a static system, whereas an electric circuit having inductors and/or capacitors is a dynamic system.

A discrete-time LTI system is memoryless (static) if its impulse response $h(n)$ is zero for $n \neq 0$. If the impulse response is not identically zero for $n \neq 0$, then the system is called dynamic system or system with memory.

EXAMPLE 1. Find whether the following systems are dynamic or not:

(a) $y(n) = x(n + 2)$ (b) $y(n) = x^2(n)$
(c) $y(n) = x(n - 2) + x(n)$

Solution:

(a) Given $y(n) = x(n + 2)$

The output depends on the future value of input. Therefore, the system is dynamic.

(b) Given $y(n) = x^2(n)$

The output depends on the present value of input alone. Therefore, the system is static.

(c) Given $y(n) = x(n - 2) + x(n)$

The system is described by a difference equation. Therefore, the system is dynamic.

2 Causal and Non-causal Systems

A system is said to be causal (or non-anticipative) if the output of the system at any instant n depends only on the present and past values of the input but not on future inputs, i.e., for a causal system, the impulse response or output does not begin before the input function is applied, i.e., a causal system is non anticipatory.

Causal systems are real time systems. They are physically realizable.

The impulse response of a causal system is zero for $n < 0$, since $\delta(n)$ exists only at $n = 0$,

i.e. $$h(n) = 0 \quad \text{for} \quad n < 0$$

The examples for causal systems are:

$$y(n) = nx(n)$$
$$y(n) = x(n - 2) + x(n - 1) + x(n)$$

A system is said to be non-causal (anticipative) if the output of the system at any instant n depends on future inputs. They are anticipatory systems. They produce an output even before the input is given. They do not exist in real time. They are not physically realizable.

A delay element is a causal system, whereas an image processing system is a non-causal system.

The examples for non-causal systems are:

$$y(n) = x(n) + x(2n)$$
$$y(n) = x^2(n) + 2x(n + 2)$$

EXAMPLE Check whether the following systems are causal or not:

(a) $y(n) = x(n) + x(n-2)$ (b) $y(n) = x(2n)$

(c) $y(n) = \sin[x(n)]$ (d) $y(n) = x(-n)$

Solution:

(a) Given $y(n) = x(n) + x(n-2)$

For $n = -2$ $y(-2) = x(-2) + x(-4)$

For $n = 0$ $y(0) = x(0) + x(-2)$

For $n = 2$ $y(2) = x(2) + x(0)$

For all values of n, the output depends only on the present and past inputs. Therefore, the system is causal.

(b) Given $y(n) = x(2n)$

For $n = -2$ $y(-2) = x(-4)$

For $n = 0$ $y(0) = x(0)$

For $n = 2$ $y(2) = x(4)$

For positive values of n, the output depends on the future values of input. Therefore, the system is non-causal.

(c) Given $y(n) = \sin[x(n)]$

For $n = -2$ $y(-2) = \sin[x(-2)]$

For $n = 0$ $y(0) = \sin[x(0)]$

For $n = 2$ $y(2) = \sin[x(2)]$

For all values of n, the output depends only on the present value of input. Therefore, the system is causal.

(d) Given $y(n) = x(-n)$

For $n = -2$ $y(-2) = x(2)$

For $n = 0$ $y(0) = x(0)$

For $n = 2$ $y(2) = x(-2)$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

3 Linear and Non-linear Systems

A system which obeys the principle of superposition and principle of homogeneity is called a linear system and a system which does not obey the principle of superposition and homogeneity is called a non-linear system.

Homogeneity property means a system which produces an output $y(n)$ for an input $x(n)$ must produce an output $ay(n)$ for an input $ax(n)$.

Superposition property means a system which produces an output $y_1(n)$ for an input $x_1(n)$ and an output $y_2(n)$ for an input $x_2(n)$ must produce an output $y_1(n) + y_2(n)$ for an input $x_1(n) + x_2(n)$.

Combining them we can say that a system is linear if an arbitrary input $x_1(n)$ produces an output $y_1(n)$ and an arbitrary input $x_2(n)$ produces an output $y_2(n)$, then the weighted sum of inputs $ax_1(n) + bx_2(n)$ where a and b are constants produces an output $ay_1(n) + by_2(n)$ which is the sum of weighted outputs.

$$T(ax_1(n) + bx_2(n)] = aT[x_1(n)] + bT[x_2(n)]$$

Simply we can say that a system is linear if the output due to weighted sum of inputs is equal to the weighted sum of outputs.

In general, if the describing equation contains square or higher order terms of input and/or output and/or product of input/output and its difference or a constant, the system will definitely be non-linear.

EXAMPLE Check whether the following systems are linear or not:

(a) $y(n) = n^2 x(n)$

(b) $y(n) = x(n) + \dfrac{1}{2x(n-2)}$

(c) $y(n) = 2x(n) + 4$

(d) $y(n) = x(n) \cos \omega n$

(e) $y(n) = |x(n)|$

(f) $y(n) = \dfrac{1}{N} \sum_{k=0}^{N-1} x(n-k)$

Solution:

(a) Given $y(n) = n^2 x(n)$

$$y(n) = T[x(n)] = n^2 x(n)$$

Let an input $x_1(n)$ produce an output $y_1(n)$.

$$\therefore \quad y_1(n) = T[x_1(n)] = n^2 x_1(n)$$

Let an input $x_2(n)$ produce an output $y_2(n)$.

$$\therefore \quad y_2(n) = T[x_2(n)] = n^2 x_2(n)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[n^2 x_1(n)] + b[n^2 x_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = n^2[ax_1(n) + bx_2(n)]$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

(b) Given
$$y(n) = x(n) + \frac{1}{2x(n-2)}$$
$$y(n) = T[x(n)] = x(n) + \frac{1}{2x(n-2)}$$

For an input $x_1(n)$,
$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{2x_1(n-2)}$$
For an input $x_2(n)$,
$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{2x_2(n-2)}$$

The weighted sum of outputs is:
$$ay_1(n) + by_2(n) = a\left[x_1(n) + \frac{1}{2x_1(n-2)}\right] + b\left[x_2(n) + \frac{1}{2x_2(n-2)}\right]$$
$$= [ax_1(n) + bx_2(n)] + \frac{a}{2x_1(n-2)} + \frac{b}{2x_2(n-2)}$$

The output due to weighted sum of inputs is:
$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] + \frac{1}{2[ax_1(n-2) + bx_2(n-2)]}$$
$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(c) Given
$$y(n) = 2x(n) + 4$$
$$y(n) = T[x(n)] = 2x(n) + 4$$

For an input $x_1(n)$,
$$y_1(n) = T[x_1(n)] = 2x_1(n) + 4$$
For an input $x_2(n)$,
$$y_2(n) = T[x_2(n)] = 2x_2(n) + 4$$

The weighted sum of outputs is:
$$ay_1(n) + by_2(n) = a[2x_1(n) + 4] + b[2x_2(n) + 4] = 2[ax_1(n) + bx_2(n)] + 4(a+b)$$

The output due to weighted sum of inputs is:
$$y_3(n) = T[ax_1(n) + bx_2(n)] = 2[ax_1(n) + bx_2(n)] + 4$$
$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

(d) Given $y(n) = x(n) \cos \omega n$

$$y(n) = T[x(n)] = x(n) \cos \omega n$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = x_1(n) \cos \omega n$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = x_2(n) \cos \omega n$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = ax_1(n) \cos \omega n + bx_2(n) \cos \omega n = [ax_1(n) + bx_2(n)] \cos \omega n$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] \cos \omega n$$

$$y_3(n) = ay_1(n) + by_2(n)$$

The weighted sum of outputs is equal to the output due to weighted sum of inputs. The superposition principle is satisfied. Therefore, the given system is linear.

(e) Given $y(n) = |x(n)|$

$$y(n) = T[x(n)] = |x(n)|$$

For an input $x_1(n)$,

$$y_1(n) = T[x_1(n)] = |x_1(n)|$$

For an input $x_2(n)$,

$$y_2(n) = T[x_2(n)] = |x_2(n)|$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a|x_1(n)| + b|x_2(n)|$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = |ax_1(n) + bx_2(n)|$$

$$y_3(n) \neq ay_1(n) + by_2(n)$$

The weighted sum of outputs is not equal to the output due to weighted sum of inputs. The superposition principle is not satisfied. Therefore, the given system is non-linear.

4 Shift-invariant and Shift-varying Systems

Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied. For discrete-time systems, the time invariance property is called shift invariance.

A system is said to be shift-invariant if its input/output characteristics do not change with time, i.e., if a time shift in the input results in a corresponding time shift in the output as shown in Figure 1.23, i.e.

If $$T[x(n)] = y(n)$$

Then $$T[x(n - k)] = y(n - k)$$

A system not satisfying the above requirements is called a time-varying system (or shift-varying system). A time-invariant system is also called a fixed system.

The time-invariance property of the given discrete-time system can be tested as follows:

Let $x(n)$ be the input and let $x(n - k)$ be the input delayed by k units.

$y(n) = T[x(n)]$ be the output for the input $x(n)$.

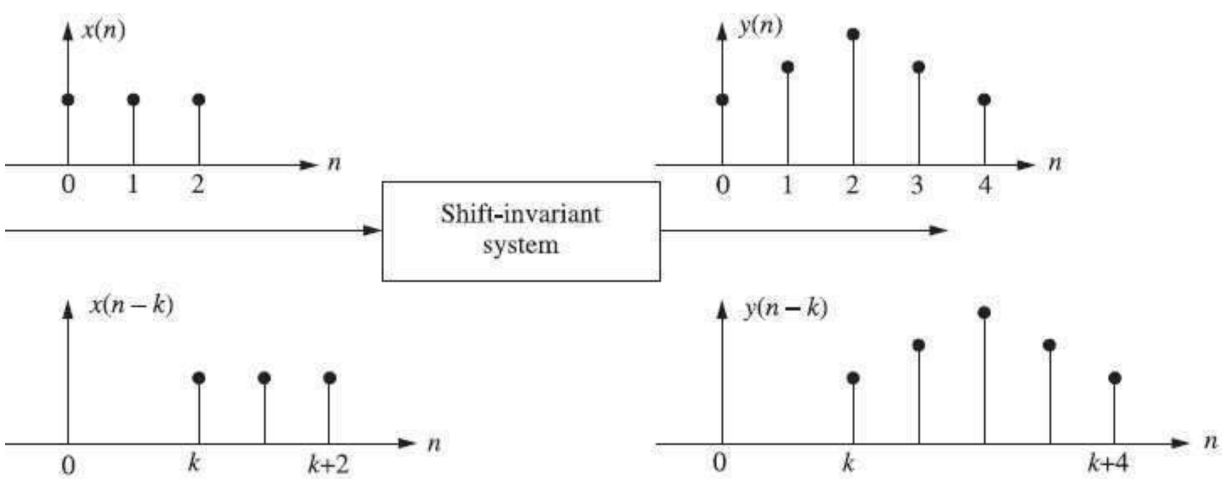


Figure 1.23 Time-invariant system.

$y(n, k) = T[x(n-k)] = y(n)\Big|_{x(n)=x(n-k)}$ be the output for the delayed input $x(n - k)$.

$y(n-k) = y(n)\Big|_{n=n-k}$ be the output delayed by k units.

If

$$y(n, k) = y(n - k)$$

i.e. if delayed output is equal to the output due to delayed input for all possible values of k, then the system is time-invariant.

On the other hand, if

$$y(n, k) \neq y(n - k)$$

i.e. if the delayed output is not equal to the output due to delayed input, then the system is time-variant.

If the discrete-time system is described by difference equation, the time invariance can be found by observing the coefficients of the difference equation.

If all the coefficients of the difference equation are constants, then the system is time-invariant. If even one of the coefficient is function of time, then the system is time-variant.

The system described by

$$y(n) + 3y(n - 1) + 5y(n - 2) = 2x(n)$$

is time-invariant system because all the coefficients are constants.

The system described by

$$y(n) - 2ny(n - 1) + 3n^2y(n - 2) = x(n) + x(n - 1)$$

is time-varying system because all the coefficients are not constant (Two are functions of time).

The systems satisfying both linearity and time-invariant conditions are called **linear, time-invariant** systems, or simply **LTI** systems.

EXAMPLE Determine whether the following systems are time-invariant or not:

(a) $y(n) = x(n/2)$ (b) $y(n) = x(n)$

(c) $y(n) = x^2(n-2)$ (d) $y(n) = x(n) + nx(n-2)$

Solution:

(a) Given $$y(n) = x\left(\frac{n}{2}\right)$$

$$y(n) = T[x(n)] = x\left(\frac{n}{2}\right)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\big|_{x(n)=x(n-k)} = x\left(\frac{n}{2} - k\right)$$

The output delayed by k units is:

$$y(n-k) = y(n)\big|_{n=n-k} = x\left(\frac{n-k}{2}\right)$$

$$y(n, k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

(b) Given $$y(n) = x(n)$$

$$y(n) = T[x(n)] = x(n)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\Big|_{x(n)=x(n-k)} = x(n-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\Big|_{n=n-k} = x(n-k)$$

$$y(n, k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(c) Given $$y(n) = x^2(n-2)$$

$$y(n) = T[x(n)] = x^2(n-2)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\Big|_{x(n)=x(n-k)} = x^2(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\Big|_{n=n-k} = x^2(n-2-k)$$

$$y(n, k) = y(n-k)$$

i.e. the delayed output is equal to the output due to delayed input. Therefore, the system is time-invariant.

(d) Given $$y(n) = x(n) + nx(n-2)$$

$$y(n) = T[x(n)] = x(n) + nx(n-2)$$

The output due to input delayed by k units is:

$$y(n, k) = T[x(n-k)] = y(n)\Big|_{x(n)=x(n-k)} = x(n-k) + nx(n-2-k)$$

The output delayed by k units is:

$$y(n-k) = y(n)\Big|_{n=n-k} = x(n-k) + (n-k)\,x(n-k-2)$$

$$y(n, k) \neq y(n-k)$$

i.e. the delayed output is not equal to the output due to delayed input. Therefore, the system is time-variant.

EXAMPLE Show that the following systems are linear shift-invariant systems:

(a) $y(n) = x\left(\frac{n}{2}\right)$ (b) $y(n) = \begin{cases} x(n) + x(n-2) & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

Solution: To show that a given system is a linear time-invariant system we have to show separately that it is linear and time-invariant.

(a) Given $$y(n) = x\left(\frac{n}{2}\right)$$

For inputs $x_1(n)$ and $x_2(n)$,

$$y_1(n) = x_1\left(\frac{n}{2}\right)$$

$$y_2(n) = x_2\left(\frac{n}{2}\right)$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = ax_1\left(\frac{n}{2}\right) + bx_2\left(\frac{n}{2}\right)$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = ax_1\left(\frac{n}{2}\right) + bx_2\left(\frac{n}{2}\right)$$

$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

$$y(n, k) = y(n)\big|_{x(n)=x(n-k)} = x\left(\frac{n}{2} - k\right)$$

$$y(n-k) = y(n)\big|_{n=n-k} = x\left(\frac{n-k}{2}\right)$$

$$y(n, k) \neq y(n-k)$$

So the system is shift-varying.

Hence the given system is linear but shift-varying. It is not a linear shift-invariant system.

(b) Given $$y(n) = \begin{cases} x(n) + x(n-2) & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

For inputs $x_1(n)$ and $x_2(n)$,

$$y_1(n) = x_1(n) + x_1(n-2) \quad \text{for } n \geq 0$$

$$y_2(n) = x_2(n) + x_2(n-2) \quad \text{for } n \geq 0$$

The weighted sum of outputs is:

$$ay_1(n) + by_2(n) = a[x_1(n) + x_1(n-2)] + b[x_2(n) + x_2(n-2)]$$

The output due to weighted sum of inputs is:

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)] + ax_1(n-2) + bx_2(n-2)$$

$$y_3(n) = ay_1(n) + by_2(n)$$

So the system is linear.

$$y(n, k) = y(n)\big|_{x(n)=x(n-k)} = x(n-k) + x(n-2-k)$$

$$y(n-k) = y(n)\big|_{n=n-k} = x(n-k) + x(n-k-2)$$

$$y(n, k) = y(n-k)$$

So the system is time-invariant. Hence the given system is linear time-invariant.

5 Stable and Unstable Systems

A bounded signal is a signal whose magnitude is always a finite value, i.e. $|x(n)| \leq M$, where M is a positive real finite number. For example a sinewave is a bounded signal. A system is said to be bounded-input, bounded-output (BIBO) stable, if and only if every bounded input produces a bounded output. The output of such a system does not diverge or does not grow unreasonably large.

Let the input signal $x(n)$ be bounded (finite), i.e.,

$$|x(n)| \leq M_x < \infty \quad \text{for all } n$$

where M_x is a positive real number. If

$$|y(n)| \leq M_y < \infty$$

i.e. if the output $y(n)$ is also bounded, then the system is BIBO stable. Otherwise, the system is unstable. That is, we say that a system is unstable even if one bounded input produces an unbounded output.

It is very important to know about the stability of the system. Stability indicates the usefulness of the system. The stability can be found from the impulse response of the system which is nothing but the output of the system for a unit impulse input. If the impulse response is absolutely summable for a discrete-time system, then the system is stable.

BIBO stability criterion

The necessary and sufficient condition for a discrete-time system to be BIBO stable is given by the expression:

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

where $h(n)$ is the impulse response of the system. This is called BIBO stability criterion.

Proof: Consider a linear time-invariant system with $x(n)$ as input and $y(n)$ as output. The input and output of the system are related by the convolution integral.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Taking absolute values on both sides, we have

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \right|$$

Using the fact that the absolute value of the sum of the product of two terms is always less than or equal to the sum of the product of their absolute values, we have

$$\left| \sum_{k=-\infty}^{\infty} x(k)h(n-k) \right| \leq \sum_{k=-\infty}^{\infty} |x(k)||h(n-k)|$$

If the input $x(k)$ is bounded, i.e. there exists a finite number M_x such that,

$$|x(k)| \leq M_x < \infty$$

$$|y(n)| \leq M_x \sum_{k=-\infty}^{\infty} |h(n-k)|$$

Changing the variables by $m = n - k$, the output is bounded if

$$\sum_{n=-\infty}^{\infty} |h(m)| < \infty$$

Replacing m by n, we have

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

which is the necessary and sufficient condition for a system to be BIBO stable.

The conditions for a BIBO stable system are given as follows:

1. If the system transfer function is a rational function, the degree of the numerator should not be larger than the degree of the denominator.
2. The poles of the system must lie inside the unit circle in the z-plane.
3. If a pole lies on the unit circle it must be a single order pole, i.e. no repeated pole lies on the unit circle.

EXAMPLE Check the stability of the system defined by

(a) $y(n) = ax(n-7)$

(b) $y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$

(c) $h(n) = a^n$ for $0 < n < 11$

(d) $h(n) = 2^n u(n)$

(e) $h(n) = u(n)$

Solution:

(a) Given $y(n) = ax(n-7)$

Let $x(n) = \delta(n)$

Then $y(n) = h(n)$

$\therefore$ $h(n) = a\delta(n-7)$

$\therefore$ $h(n) = a$ for $n = 7$

$= 0$ for $n \neq 7$

A system is stable if its impulse response $h(n)$ is absolutely summable.

i.e.

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} a\delta(n-7) = a$$

Hence the given system is stable if the value of a is finite.

(b) Given $$y(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{4}x(n-2)$$

Let $$x(n) = \delta(n)$$

Then $$y(n) = h(n)$$

$$\therefore \quad h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2)$$

A discrete-time system is stable if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

The given $h(n)$ has a value only at $n = 0$, $n = 1$ and $n = 2$. For all other values of n from $-\infty$ to ∞, $h(n) = 0$.

At $n = 0$, $h(0) = \delta(0) + \frac{1}{2}\delta(0-1) + \frac{1}{4}\delta(0-2) = \delta(0) + \frac{1}{2}\delta(-1) + \frac{1}{4}\delta(-2) = 1$

At $n = 1$, $h(1) = \delta(1) + \frac{1}{2}\delta(1-1) + \frac{1}{4}\delta(1-2) = \delta(1) + \frac{1}{2}\delta(0) + \frac{1}{4}\delta(-2) = \frac{1}{2}$

At $n = 2$, $h(2) = \delta(2) + \frac{1}{2}\delta(2-1) + \frac{1}{4}\delta(2-2) = \delta(2) + \frac{1}{2}\delta(1) + \frac{1}{4}\delta(0) = \frac{1}{4}$

$$\therefore \quad \sum_{n=-\infty}^{\infty} |h(n)| = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} < \infty \quad \text{a finite value.}$$

Hence the system is stable.

(c) Given $$h(n) = a^n \quad \text{for } 0 < n < 11$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |a^n| = \sum_{n=0}^{11} a^n = \frac{1-a^{12}}{1-a}$$

This value is finite for finite value of a. Hence the system is stable if a is finite.

(d) Given $$h(n) = 2^n u(n)$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |2^n u(n)| = \sum_{n=0}^{\infty} 2^n = \infty$$

The impulse response is not absolutely summable. Hence this system is unstable.

(e) Given $$h(n) = u(n)$$

For stability,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

In this case,

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \cdots = \infty$$

So the output is not bounded and the system is unstable.

UNIT V

Z–TRANSFORMS & REALIZATION OF DIGITAL FILTERS

- Concept of Z- Transform of a discrete sequence.
- Region of convergence in Z-Transform
- Inverse Z- Transform.
- Solution of Difference Equations Using Z-Transform
- Realization of Digital Filters - Direct and Canonic forms.

INTRODUCTION

A linear time-invariant discrete-time system is represented by difference equations. The direct solution of higher order difference equations is quite tedious and time consuming. So usually they are solved by indirect methods. The Z-transform plays the same role for discrete-time systems as that played by Laplace transform for continuous-time systems. The Z-transform is the discrete-time counterpart of the Laplace transform. It is the Laplace transform of the discretized version of the continuous-time signal $x(t)$. To solve the difference equations which are in time domain, they are converted first into algebraic equations in z-domain using Z-transform, the algebraic equations are manipulated in z-domain and the result obtained is converted back into time domain using inverse Z-transform. The Z-transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and the initial conditions can be introduced in the beginning of the process itself. The Z-transform plays an important role in the analysis and representation of discrete-time Linear Shift Invariant (LSI) systems. It is the generalization of the Discrete-Time Fourier Transform (DTFT). The Z-transform may be one-sided (unilateral) or two-sided (bilateral). It is the one-sided or unilateral Z-transform that is more useful, because we mostly deal with causal sequences. Further, it is eminently suited for solving difference equations with initial conditions.

The *bilateral* or *two-sided* Z-transform of a discrete-time signal or a sequence $x(n)$ is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad \text{where } z \text{ is a complex variable.}$$

The *one-sided* or *unilateral* Z-transform is defined as:

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

If $x(n) = 0$, for $n < 0$, the one-sided and two-sided Z-transforms are equivalent.

In the z-domain, the convolution of two time domain signals is equivalent to multiplication of their corresponding Z-transforms. This property simplifies the analysis of the response of an LTI system to various signals.

Region of convergence (ROC)

For any given sequence, the Z-transform may or may not converge.

The set of values of z or equivalently the set of points in z-plane, for which $X(z)$ converges is called the region of convergence (ROC) of $X(z)$. In general ROC can be $R_{x^-} < |z| < R_{x^+}$ where R_{x^-} can be as small as zero and R_{x^+} can be as large as infinity.

If there is no value of z (i.e. no point in the z-plane) for which $X(z)$ converges, then the sequence $x(n)$ is said to be having no Z-transform.

RELATION BETWEEN DISCRETE-TIME FOURIER TRANSFORM (DTFT) AND Z-TRANSFORM

The Discrete-Time Fourier Transform (DTFT) of a sequence $x(n)$ is given by

$$X(e^{j\omega}) \quad \text{or} \quad X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

For the existence of DTFT, the above summation should converge, i.e. $x(n)$ must be absolutely summable. The Z-transform of the sequence $x(n)$ is given by

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable and is given by $z = re^{j\omega}$ where r is the radius of a circle.

Advantages of Z-transform

1. The Z-transform converts the difference equations of a discrete-time system into linear algebraic equations so that the analysis becomes easy and simple.
2. Convolution in time domain is converted into multiplication in z-domain.
3. Z-transform exists for most of the signals for which Discrete-Time Fourier Transform (DTFT) does not exist.
4. Also since the Fourier transform is nothing but the Z-transform evaluated along the unit circle in the z-plane, the frequency response can be determined.

$$\therefore \qquad X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\omega n}$$

For the existence of Z-transform, the above summation should converge, i.e. $x(n)\, r^{-n}$ must be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty$$

The above equation represents the Discrete-Time Fourier Transform of a signal $x(n)\, r^{-n}$. Hence, we can say that the Z-transform of $x(n)$ is same as the Discrete-Time Fourier Transform of $x(n)\, r^{-n}$.

For the DTFT to exist, the discrete sequence $x(n)$ must be absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

So for many sequences, the DTFT may not exist but the Z-transform may exist. When $r = 1$, the DTFT is same as the Z-transform, i.e. the DTFT is nothing but the Z-transform evaluated along the unit circle centred at the origin of the z-plane.

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jwn},$$

which is the discrete-time Fourier transform of $x[n]$. Therefore, DTFT is a special case of the z-transform! Pictorially, we can view DTFT as the z-transform evaluated on the unit circle:

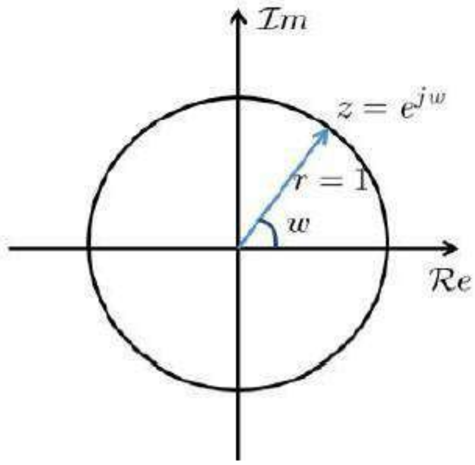


Figure Complex z-plane. The z-transform reduces to DTFT for values of z on the unit circle.

Z-TRANSFORM AND ROC OF FINITE DURATION SEQUENCES

Finite duration sequences are sequences having a finite number of samples. Finite duration sequences may be right-sided sequences or left-sided sequences or two-sided sequences.

1 Right-sided Sequence

A right-sided sequence is one for which $x(n) = 0$ for $n < n_0$, where n_0 is positive or negative but finite. The Z-transform of such a sequence is $X(z) = \sum_{n=n_0}^{\infty} x(n)\, z^{-n}$. The ROC of the above series is the exterior of a cirle. If $n_0 \geq 0$, the resulting sequence is a causal or a positive time sequence. For a causal or a positive finite time sequence, the ROC is entire z-plane except at $z = 0$.

EXAMPLE 1 Find the ROC and Z-transform of the causal sequence

$$x(n) = \{1, 0, -2, 3, 5, 4\}$$
$$\uparrow$$

Solution: The given sequence values are:

$$x(0) = 1,\ x(1) = 0,\ x(2) = -2,\ x(3) = 3,\ x(4) = 5 \text{ and } x(5) = 4.$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(0) + x(1)z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + x(5)z^{-5}$$

$$\therefore \qquad Z[x(n)] = X(z) = 1 - 2z^{-2} + 3z^{-3} + 5z^{-4} + 4z^{-5}$$

The $X(z)$ converges for all values of z except at $z = 0$.

EXAMPLE 2 A finite sequence $x(n)$ is defined as $x(n) = \{5, 3, -2, 0, 4, -3\}$. Find $X(z)$ and its ROC.

Solution: Given $\qquad x(n) = \{5, 3, -2, 0, 4, -3\}$

$$\therefore \qquad x(n) = 5\delta(n) + 3\delta(n-1) - 2\delta(n-2) + 4\delta(n-4) - 3\delta(n-5)$$

The given sequence is a right-sided sequence. So the ROC is entire z-plane except at $z = 0$. Taking Z-transform on both sides of the above equation, we have

$$X(z) = 5 + 3z^{-1} - 2z^{-2} + 4z^{-4} - 3z^{-5}$$

ROC: Entire z-plane except at $z = 0$.

2 Left-sided Sequence

A left-sided sequence is one for which $x(n) = 0$ for $n \geq n_0$ where n_0 is positive or negative, but finite. The Z-transform of such a sequence is $X(z) = \sum_{n=-\infty}^{n_0} x(n)\, z^{-n}$. The ROC of the above series is the interior of a cirle. If $n_0 \leq 0$, the resulting sequence is anticausal sequence. For an anticausal finite duration sequence, the ROC is entire z-plane except at $z = \infty$.

EXAMPLE 3. Find the Z-transform and ROC of the anticausal sequence.

$$x(n) = \{4, 2, 3, -1, -2, \underset{\uparrow}{1}\}$$

Solution: The given sequence values are:

$$x(-5) = 4,\ x(-4) = 2,\ x(-3) = 3,\ x(-2) = -1,\ x(-1) = -2,\ x(0) = 1$$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values, $X(z)$ is:

$$X(z) = x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2) z^2 + x(-1)\, z + x(0)$$

$$\therefore \qquad Z[x(n)] = X(z) = 4z^5 + 2z^4 + 3z^3 - z^2 - 2z + 1$$

The $X(z)$ converges for all values of z except at $z = \infty$.

3 Two-sided Sequence

A sequence that has finite duration on both the left and right sides is known as a two-sided sequence. A two-sided sequence is one that extends from $n = -\infty$ to $n = +\infty$. In general, we can write $X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{-1} x(n)\, z^{-n} + \sum_{n=0}^{\infty} x(n)\, z^{-n}$. The first series converges for

$|z| < R_{x-}$ and the second series converges for $|z| > R_{x+}$. So the ROC of such a sequence $R_{x-} < |z| < R_{x+}$ is a ring. For a two-sided finite duration sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.

EXAMPLE 4 Find the Z-transform and ROC of the sequence

$$x(n) = \{2, 1, -3, 0, \underset{\uparrow}{4}, 3, 2, 1, 5\}$$

Solution: The given sequence values are:

$x(-4) = 2,\ x(-3) = 1,\ x(-2) = -3,\ x(-1) = 0,\ x(0) = 4,\ x(1) = 3,\ x(2) = 2,\ x(3) = 1,\ x(4) = 5$

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For the given sample values,

$$X(z) = x(-4)z^4 + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$
$$= 2z^4 + z^3 - 3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

Example 5 . Consider the signal $x[n] = a^n u[n]$, with $0 < a < 1$. The z-transform of $x[n]$ is

$$X(z) = \sum_{-\infty}^{\infty} a^n u[n] z^{-n} \qquad = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n.$$

Therefore, $X(z)$ converges if $\sum_{n=0}^{\infty} \left(az^{-1}\right)^n < \infty$. From geometric series, we know that

$$\sum_{n=0}^{\infty} \left(rz^{-1}\right)^n = \frac{1}{1 - az^{-1}},$$

with ROC being the set of z such that $|z| > |a|$.

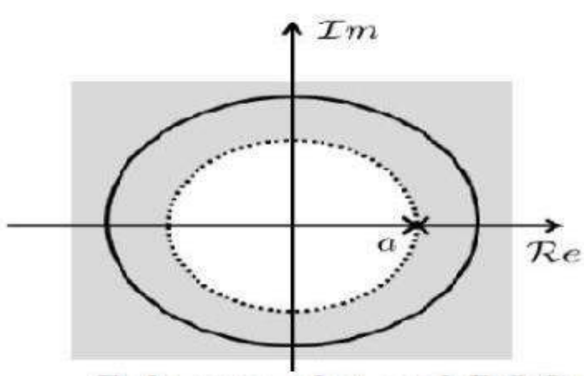


Figure . Pole-zero plot and ROC of Example 5

Example 6 Consider the signal $x[n] = -a^n u[-n-1]$ with $0 < a < 1$. The z-transform of $x[n]$ is

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} \quad = -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n \quad = 1 - \sum_{n=0}^{\infty} \left(a^{-1}z\right)^n.$$

Therefore, $X(z)$ converges when $|a^{-1}z| < 1$, or equivalently $|z| < |a|$. In this case,

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}},$$

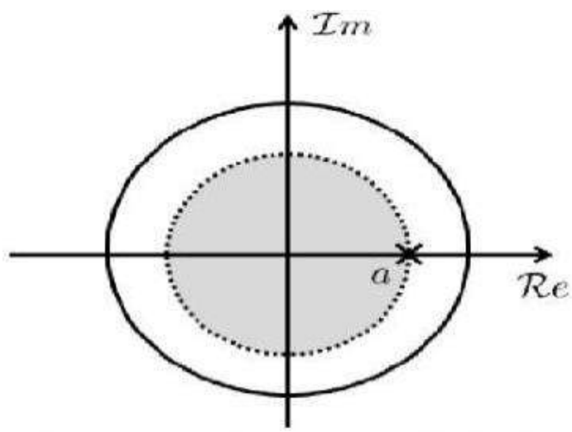


Figure : Pole-zero plot and ROC of Example 6

Example 7. Consider the signal

$$x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n].$$

The z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} \left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right] u[n] z^{-n}$$

$$= 7\sum_{n=-\infty}^{\infty} \left(\frac{1}{3}\right)^n u[n] z^{-n} - 6\sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u[n] z^{-n}$$

$$= 7\left(\frac{1}{1 - \frac{1}{3}z^{-1}}\right) - 6\left(\frac{1}{1 - \frac{1}{2}z^{-1}}\right)$$

For $X(z)$ to converge, both sums in $X(z)$ must converge. So we need both $|z| > |\frac{1}{3}|$ and $|z| > |\frac{1}{2}|$. Thus, the ROC is the set of z such that $|z| > |\frac{1}{2}|$.

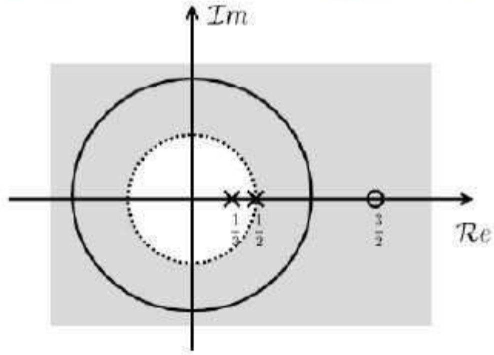


Figure . : Pole-zero plot and ROC of Example 7

EXAMPLE . 8 Find the Z-transform of the following sequences:

(a) $u(n)-u(n-4)$ (b) $u(-n)-u(-n-3)$ (c) $u(2-n)-u(-2-n)$

Solution:

(a) The given sequence is:

$$x(n)=u(n)-u(n-4)$$

From Figure we notice that the sequence values are:

$$x(n)=1, \quad \text{for } 0 \le n \le 3$$
$$=0, \quad \text{otherwise}$$

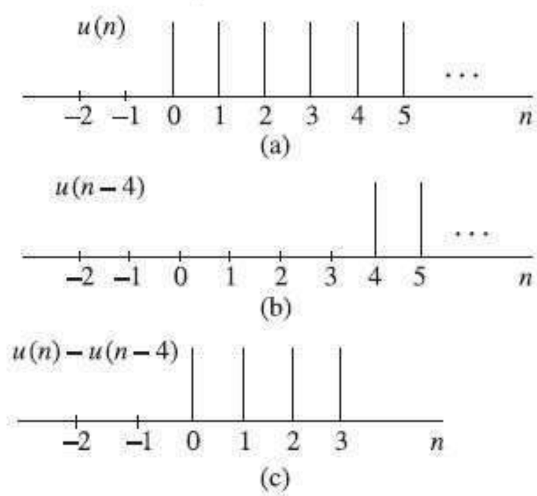


Figure Sequences (a) $u(n)$, (b) $u(n-4)$ and (c) $u(n)-u(n-4)$.

We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z^{-1} + z^{-2} + z^{-3}$$

The ROC is entire z-plane except at $z = 0$.

(b) The given sequence is:

$$x(n) = u(-n) - u(-n-3)$$

From Figure , we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } -2 \le n \le 0$$
$$= 0, \quad \text{otherwise}$$

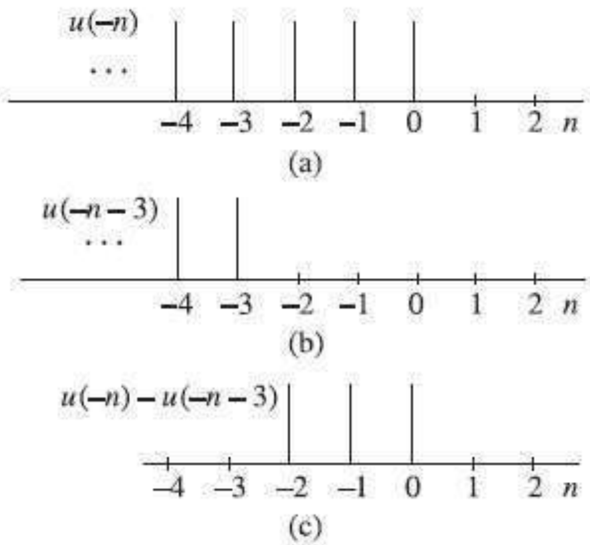


Figure Sequences (a) $u(-n)$, (b) $u(-n-3)$ and (c) $u(-n) - u(-n-3)$.

We know that $$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Substituting the sequence values, we get

$$X(z) = 1 + z + z^2$$

The ROC is entire z-plane except at $z = \infty$.

(c) The given sequence is:

$$x(n) = u(2 - n) - u(-2 - n)$$

From Figure we notice that the sequence values are:

$$x(n) = 1, \quad \text{for } -1 \le n \le 2$$
$$= 0, \quad \text{otherwise}$$

Substituting the sequence values, we get

$$X(z) = z + 1 + z^{-1} + z^{-2}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$.

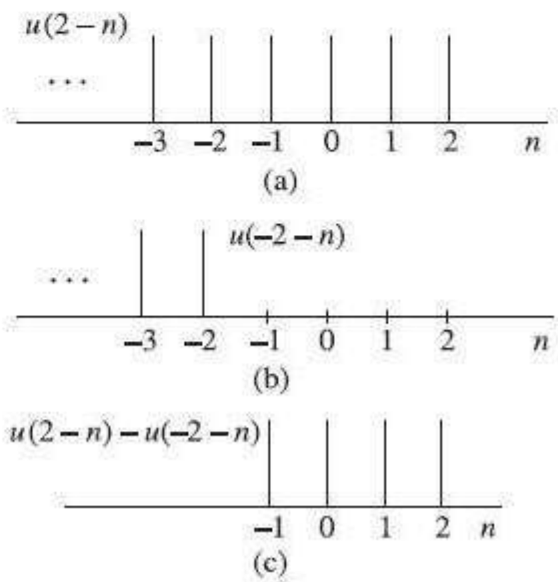
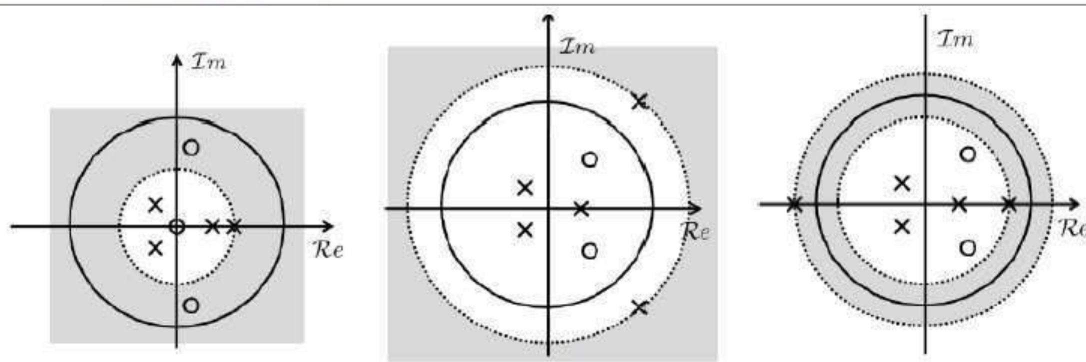


Figure Sequences (a) $u(2 - n)$, (b) $u(-2 - n)$ and (c) $u(2 - n) - u(-2 - n)$.

PROPERTIES OF ROC

1. The ROC is a ring or disk in the z-plane centred at the origin.
2. The ROC cannot contain any poles.
3. If $x(n)$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, i.e. it is the exterior of a circle of radius α.
   If $x(n)$ is a finite duration causal sequence (right-sided sequence), the ROC is entire z-plane except at $z = 0$.
4. If $x(n)$ is an infinite duration anticausal sequence, the ROC is $|z| < \beta$, i.e. it is the interior of a circle of radius β.
   If $x(n)$ is a finite duration anticausal sequence (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.
5. If $x(n)$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.
6. If $x(n)$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.
7. The ROC of an LTI stable system contains the unit circle.
8. The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends up to infinity.
9. $x(n) = \delta(n)$ is the only signal whose ROC is entire z-plane.

Property 10. *A causal discrete-time LTI system is stable if and only if all of its poles are inside the unit circle.*



Causal, Stable　　Causal, Unstable　　Not causal, Stable

INVERSE Z-TRANSFORM

The process of finding the time domain signal $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform which is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\omega n}$$

This is the DTFT of the signal $x(n)\, r^{-n}$. Hence the Inverse Discrete-Time Fourier Transform (IDTFT) of $X(re^{j\omega})$ must be $x(n)\, r^{-n}$. Therefore, we can write

$$x(n) r^{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})\, e^{j\omega n}\, d\omega$$

i.e.
$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^n\, d\omega$$

We have

$$z = re^{j\omega}$$

$\therefore$
$$\frac{dz}{d\omega} = jre^{j\omega}, \text{ i.e. } d\omega = \frac{dz}{jre^{j\omega}}$$

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

Basically, there are four methods that are often used to find the inverse Z-transform. They are:

(a) Power series method or long division method
(b) Partial fraction expansion method
(c) Complex inversion integral method (also known as the residue method)
(d) Convolution integral method

The long division method is simple, but does not give a closed form expression for the time signal. Further, it can be used only if the ROC of the given $X(z)$ is either of the form $|z| > \alpha$ or of the form $|z| < \alpha$, i.e. it is useful only if the sequence $x(n)$ is either purely right-sided or purely left-sided. The partial fraction expansion method enables us to determine the time signal $x(n)$ making use of our knowledge of some basic Z-transform pairs and Z-transform theorems. The inversion integral method requires a knowledge of the theory of complex variables, but is quite powerful and useful. The convolution integral method uses convolution property of Z-transforms and can be used when given $X(z)$ can be written as the product of two functions.

1 Long Division Method

The Z-transform of a two-sided sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The $X(z)$ has both positive powers of z as well as negative powers of z. We cannot obtain a two-sided sequence by long division. If the sequence $x(n)$ is causal, then

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

has only negative powers of z, with ROC; $|z| > \alpha$.

EXAMPLE Find the inverse Z-transform of

$$X(z) = z^3 + 2z^2 + z + 1 - 2z^{-1} - 3z^{-2} + 4z^{-3}$$

Solution: We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \ldots x(-3)\, z^3 + x(-2)\, z^2 + x(-1)\, z^1 + x(0) + x(1)\, z^{-1}$$
$$+ x(2)\, z^{-2} + x(3)\, z^{-3} + \cdots$$

Comparing this $X(z)$ with the given $X(z)$, we have

$$x(n) = \{1, 2, 1, \underset{\uparrow}{1}, -2, -3, 4\}$$

Alternatively, taking inverse Z-transform of $X(z)$, we have

$$x(n) = \delta(n+3) + 2\delta(n+2) + \delta(n+1) + \delta(n) - 2\delta(n-1) - 3\delta(n-2) + 4\delta(n-3)$$

EXAMPLE Determine the inverse Z-transform of

(a) $X(z)=\frac{1}{z-a}$; ROC; $|z|>a$ (b) $X(z)=\frac{1}{1-az^{-1}}$; ROC; $|z|>a$

(c) $X(z)=\frac{1}{1-z^{-4}}$; ROC; $|z|>1$

Solution:

(a) Given $X(z)=\frac{1}{z-a}$; ROC; $|z|>a$

$$=\frac{1}{z(1-az^{-1})}=z^{-1}(1-az^{-1})^{-1}=z^{-1}(1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots)$$

$$=z^{-1}+az^{-2}+a^2z^{-3}+\cdots=\sum_{n=1}^{\infty}a^{n-1}z^{-n}=\sum_{n=0}^{\infty}a^{n-1}u(n-1)z^{-n}$$

$$\therefore \quad x(n)=a^{n-1}u(n-1)$$

(b) Given $X(z)=\frac{1}{1-az^{-1}}$; ROC; $|z|>a$

By Taylor's series expansion, we have

$$X(z)=\frac{1}{1-az^{-1}}=1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots=\sum_{n=0}^{\infty}a^nz^{-n}=\sum_{n=0}^{\infty}a^n u(n)z^{-n}$$

Therefore, $x(n) = a^n u(n)$

(c) From infinite sum formula, we have

$$\sum_{k=0}^{\infty}\alpha^k=\frac{1}{1-\alpha};\ |\alpha|<1$$

Given $X(z)=\frac{1}{1-z^{-4}}=\sum_{k=0}^{\infty}(z^{-4})^k=\sum_{k=0}^{\infty}z^{-4k}\quad\left[|z^{-4}|<1,\ \text{i.e.}\ |z|>1\right]$

Taking inverse Z-transform on both sides, we get

$$x(n)=\sum_{k=0}^{\infty}\delta(n-4k)$$

$\therefore \quad x(n)=1,$ when $n=4k$, i.e. when n is an integer multiple of 4

$=0,$ otherwise

EXAMPLE Using long division, determine the inverse Z-transform of

$$X(z) = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1}; \text{ ROC; } |z| > 1$$

Solution: Since ROC is $|z| > 1$, $x(n)$ must be a causal sequence. For getting a causal sequence, the $N(z)$ and $D(z)$ of $X(z)$ must be put either in descending powers of z or in ascending powers of z^{-1} before performing long division.

In the given $X(z)$ both $N(z)$ and $D(z)$ are already in descending powers of z.

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
z^3 - 3z^2 + 4z + 1 & z^2 + 2z \\
 & \underline{z^2 - 3z + 4 + z^{-1}} \\
 & \quad 5z - 4 - z^{-1} \\
 & \quad \underline{5z - 15 + 20z^{-1} + 5z^{-2}} \\
 & \qquad 11 - 21z^{-1} - 5z^{-2} \\
 & \qquad \underline{11 - 33z^{-1} + 44z^{-2} + 11z^{-3}} \\
 & \qquad\quad 12z^{-1} - 49z^{-2} - 11z^{-3} \\
 & \qquad\quad \underline{12z^{-1} - 36z^{-2} + 48z^{-3} + 12z^{-4}} \\
 & \qquad\qquad -13z^{-2} - 59z^{-3} - 12z^{-4}
\end{array}$$

$$X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \cdots$$

$$\therefore \quad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Writing $N(z)$ and $D(z)$ of $X(z)$ in ascending powers of z^{-1}, we have

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1} = \frac{z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 4z^{-2} + z^{-3}}$$

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
1 - 3z^{-1} + 4z^{-2} + z^{-3} & z^{-1} + 2z^{-2} \\
 & \underline{z^{-1} - 3z^{-2} + 4z^{-3} + z^{-4}} \\
 & \quad 5z^{-2} - 4z^{-3} - z^{-4} \\
 & \quad \underline{5z^{-2} - 15z^{-3} + 20z^{-4} + 5z^{-5}} \\
 & \qquad 11z^{-3} - 21z^{-4} - 5z^{-5} \\
 & \qquad \underline{11z^{-3} - 33z^{-4} + 44z^{-5} + 11z^{-6}} \\
 & \qquad\quad 12z^{-4} - 49z^{-5} - 11z^{-6} \\
 & \qquad\quad \underline{12z^{-4} - 36z^{-5} + 48z^{-6} + 12z^{-7}} \\
 & \qquad\qquad -13z^{-5} - 59z^{-6} - 12z^{-7}
\end{array}$$

$$\therefore \quad X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \cdots$$

$$\therefore \quad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Observe that both the methods give the same sequence $x(n)$.

EXAMPLE Using long division, determine the inverse *Z*-transform of

$$X(z) = \frac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4}; \text{ ROC; } |z| < 1$$

Solution: Since ROC is $|z| < 1$, $x(n)$ must be a non-causal sequence. For getting a non-causal sequence, the $N(z)$ and $D(z)$ must be put either in ascending powers of z or in descending powers of z^{-1} before performing long division.

$$X(z) = \frac{z^2 + z + 2}{z^3 - 2z^2 + 3z + 4} = \frac{2 + z + z^2}{4 + 3z - 2z^2 + z^3}$$

$$\begin{array}{r|l}
 & \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 \\
\hline
4 + 3z - 2z^2 + z^3 & 2 + z + z^2 \\
 & 2 + \frac{3}{2}z - z^2 + \frac{1}{2}z^3 \\
\hline
 & -\frac{1}{2}z + 2z^2 - \frac{1}{2}z^3 \\
 & -\frac{1}{2}z - \frac{3}{8}z^2 + \frac{1}{4}z^3 - \frac{1}{8}z^4 \\
\hline
 & \frac{19}{8}z^2 - \frac{3}{4}z^3 + \frac{1}{8}z^4 \\
 & \frac{19}{8}z^2 + \frac{57}{32}z^3 - \frac{19}{16}z^4 + \frac{19}{32}z^5 \\
\hline
 & -\frac{81}{32}z^3 + \frac{21}{16}z^4 - \frac{19}{32}z^5 \\
 & -\frac{81}{32}z^3 - \frac{243}{128}z^4 + \frac{81}{64}z^5 - \frac{81}{128}z^6 \\
\hline
 & \frac{411}{128}z^4 - 129z^5 + \frac{81}{128}z^6
\end{array}$$

$$X(z) = \frac{1}{2} - \frac{1}{8}z + \frac{19}{32}z^2 - \frac{81}{128}z^3 + \frac{411}{512}z^4 \dots$$

$$x(n) = \left\{ \dots, \frac{411}{512}, -\frac{81}{128}, \frac{19}{32}, -\frac{1}{8}, \underset{\uparrow}{\frac{1}{2}} \right\}$$

2 Partial Fraction Expansion Method

To find the inverse Z-transform of $X(z)$ using partial fraction expansion method, its denominator must be in factored form. It is similar to the partial fraction expansion method used earlier for the inversion of Laplace transforms. However, in this case, we try to obtain the partial fraction expansion of $X(z)/z$ instead of $X(z)$. This is because, the Z-transform of time domain signals have z in their numerators. This method can be applied only if $X(z)/z$ is a proper rational function (i.e. the order of its denominator is greater than the order of its numerator). If $X(z)/z$ is not proper, then it should be written as the sum of a polynomial and a proper function before applying this method. The disadvantage of this method is that, the denominator must be factored. Using known Z-transform pairs and the properties of Z-transform, the inverse Z-transform of each partial fraction can be found.

Consider a rational function $X(z)/z$ given by

$$\frac{X(z)}{z} = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

When $M < N$, it is a proper function.

When $M \geq N$, it is not a proper function, so write it as:

$$\frac{X(z)}{z} = \underbrace{c_0 z^{N-M} + c_1 z^{N-M-1} + \cdots + c_{N-M}}_{\text{polynomial}} + \underbrace{\frac{N_1(z)}{D(z)}}_{\text{Proper rational function}}$$

There are two cases for the proper rational function $X(z)/z$.

CASE 1 $X(z)/z$ has all distinct poles.
When all the poles of $X(z)/z$ are distinct, then $X(z)/z$ can be expanded in the form

$$\frac{X(z)}{z} = \frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots + \frac{C_N}{z - P_N}$$

The coefficients $C_1, C_2, \ldots, C_N$ can be determined using the formula

$$C_k = (z - P_k)\left.\frac{X(z)}{z}\right|_{z = P_k}, \quad k = 1, 2, \ldots, N$$

CASE 2 $X(z)/z$ has l-repeated poles and the remaining N-l poles are simple. Let us say the kth pole is repeated l times. Then, $X(z)/z$ can be written as:

$$\frac{X(z)}{z} = \underbrace{\frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots}_{(N\text{-}l)\text{ terms}} + \frac{C_{k1}}{z - P_k} + \frac{C_{k2}}{(z - P_k)^2} + \cdots + \frac{C_{kl}}{(z - P_k)^l}$$

where
$$C_{kl} = (z - P_k)^l \left.\frac{X(z)}{z}\right|_{z = P_k}$$

\

In general, $$C_{ki} = \frac{1}{(l-i)!} \frac{d^{l-i}}{dz^{l-i}} \left[(z-P_k)^l \frac{X(z)}{z} \right]\Bigg|_{z=P_k}$$

If $X(z)$ has a complex pole, then the partial fraction can be expressed as:

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_1^*}{z-P_1^*}$$

where C_1^* is complex conjugate of C_1 and P_1^* is complex conjugate of P_1.

In other words, complex conjugate poles result in complex conjugate coefficients in the partial fraction expansion.

EXAMPLE 1 Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}}; \text{ ROC; } |z| > 1$$

Solution: Given $$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} = \frac{z}{3z^2-4z+1}$$

$$= \frac{z}{3[z^2-(4z/3)+(1/3)]} = \frac{1}{3} \frac{z}{(z-1)[z-(1/3)]}$$

$\therefore$ $$\frac{X(z)}{z} = \frac{1}{3} \frac{1}{(z-1)[z-(1/3)]} = \frac{A}{z-1} + \frac{B}{z-(1/3)}$$

where A and B can be evaluated as follows:

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = (z-1)\frac{1}{3}\frac{1}{(z-1)[z-(1/3)]}\bigg|_{z=1} = \frac{1}{3}\frac{1}{1-(1/3)} = \frac{1}{2}$$

$$B = \left(z-\frac{1}{3}\right)\frac{X(z)}{z}\bigg|_{z=1/3} = \left(z-\frac{1}{3}\right)\frac{1}{3}\frac{1}{(z-1)[z-(1/3)]}\bigg|_{z=1/3} = \frac{1}{3}\frac{1}{(1/3)-1} = -\frac{1}{2}$$

$\therefore$ $$\frac{X(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-(1/3)}$$

or $$X(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(1/3)}\right]; \text{ ROC; } |z| > 1$$

Since ROC is $|z| > 1$, both the sequences must be causal. Therefore, taking inverse Z-transform, we have

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right]; \text{ ROC; } |z| > 1$$

EXAMPLE 2 Find the inverse Z-transform of

$$X(z) = \frac{z(z-1)}{(z+1)^3 (z+2)}; \text{ ROC; } |z| > 2$$

Solution: Given $$X(z) = \frac{z(z-1)}{(z+1)^3 (z+2)}; \text{ ROC; } |z| > 2$$

$$\therefore \quad \frac{X(z)}{z} = \frac{z-1}{(z+1)^3 (z+2)} = \frac{C_1}{z+1} + \frac{C_2}{(z+1)^2} + \frac{C_3}{(z+1)^3} + \frac{C_4}{z+2}$$

where the constants C_1, C_2, C_3 and C_4 can be obtained as follows:

$$C_4 = (z+2)\frac{X(z)}{z}\bigg|_{z=-2} = \frac{z-1}{(z+1)^3}\bigg|_{z=-2} = \frac{-2-1}{(-2+1)^3} = 3$$

$$C_3 = (z+1)^3 \frac{X(z)}{z}\bigg|_{z=-1} = \frac{z-1}{(z+2)}\bigg|_{z=-1} = \frac{-1-1}{-1+2} = -2$$

$$C_2 = \frac{1}{1!}\frac{d}{dz}\left[(z+1)^3 \frac{X(z)}{z}\right]\bigg|_{z=-1} = \frac{d}{dz}\left(\frac{z-1}{z+2}\right)\bigg|_{z=-1} = \frac{(z+2)(1)-(z-1)(1)}{(z+2)^2}\bigg|_{z=-1} = 3$$

$$C_1 = \frac{1}{2!}\frac{d^2}{dz^2}\left[(z+1)^3 \frac{X(z)}{z}\right]\bigg|_{z=-1} = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{z-1}{z+2}\right)\bigg|_{z=-1}$$

$$= \frac{1}{2!}\frac{d}{dz}\left[\frac{3}{(z+2)^2}\right]\bigg|_{z=-1} = \frac{1}{2}\frac{-3\times 2(z+2)}{(z+2)^4}\bigg|_{z=-1} = \frac{-3(-1+2)}{(-1+2)^3} = -3$$

$$\therefore \quad \frac{X(z)}{z} = \frac{-3}{z+1} + \frac{3}{(z+1)^2} - \frac{2}{(z+1)^3} + \frac{3}{z+2}$$

$$\therefore \quad X(z) = \frac{-3z}{z+1} + \frac{3z}{(z+1)^2} - \frac{2z}{(z+1)^3} + \frac{3z}{z+2}; \text{ ROC; } |z| > 2$$

Since ROC is $|z| > 2$, all the above sequences must be causal. Taking inverse Z-transform on both sides, we have

$$x(n) = -3(-1)^n u(n) + 3n(-1)^n u(n) - 2(n)(n-1)(-1)^n u(n) + 3(-2)^n u(n)$$

$$= [-3 + 3n - 2n(n-1)](-1)^n u(n) + 3(-2)^n u(n)$$

EXAMPLE 3 Determine all possible signals $x(n)$ associated with Z-transform.

$$X(z) = \frac{(1/4)z^{-1}}{[1-(1/2)z^{-1}][1-(1/4)z^{-1}]}$$

Solution: Given $$X(z) = \frac{(1/4)z^{-1}}{[1-(1/2)z^{-1}][1-(1/4)z^{-1}]}$$

Multiplying the numerator and denominator with z^2, we obtain

$$X(z) = \frac{(1/4)z}{[z-(1/2)][z-(1/4)]}$$

Now, $X(z)$ has two poles, one at $z = (1/2)$ and the other at $z = 1/4$ as shown in Figure 3. The possible ROCs are:

(a) ROC; $|z| > \frac{1}{2}$ (b) ROC; $|z| < \frac{1}{4}$ (c) ROC; $\frac{1}{4} < |z| < \frac{1}{2}$

Hence there are three possible signals $x(n)$ corresponding to these ROCs.

Now, $$\frac{X(z)}{z} = \frac{1/4}{[z-(1/2)][z-(1/4)]} = \frac{C_1}{z-(1/2)} + \frac{C_2}{z-(1/4)} = \frac{1}{z-(1/2)} - \frac{1}{z-(1/4)}$$

or $$X(z) = \frac{z}{z-(1/2)} - \frac{z}{z-(1/4)}$$

(a) ROC; $|z| > \frac{1}{2}$

Here both the poles, i.e. $z = (1/2)$ and $z = (1/4)$ correspond to causal terms.

$$\therefore \qquad x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

(b) ROC; $|z| < \frac{1}{4}$

Here both the poles must correspond to anticausal terms.

$$\therefore \qquad x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) + \left(\frac{1}{4}\right)^n u(-n-1)$$

(c) ROC; $\frac{1}{4} < |z| < \frac{1}{2}$

Here the pole at $z = (1/4)$ must correspond to causal term and the pole at $z = (1/2)$ must correspond to anticausal term.

$$\therefore \qquad x(n) = -\left(\frac{1}{2}\right)^n u(-n-1) - \left(\frac{1}{4}\right)^n u(n)$$

The ROCs are shown in Figure 3.

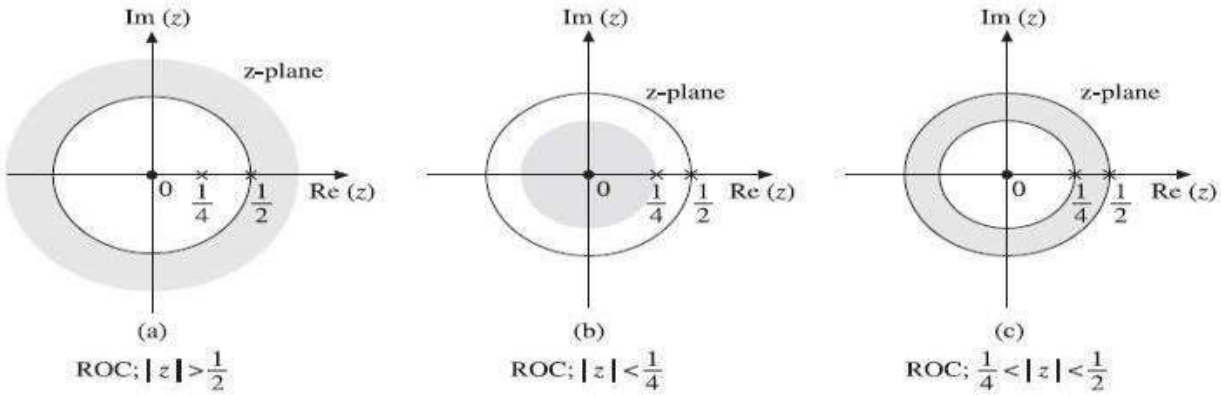


Figure 3. ROCs for Example 3.

EXAMPLE Determine the causal signal $x(n)$ having Z-transform

$$X(z) = \frac{z^2 + z}{[z - (1/2)]^2\,[z - (1/4)]}$$

Solution: Given $X(z) = \dfrac{z^2 + z}{[z - (1/2)]^2\,[z - (1/4)]} = \dfrac{z(z+1)}{[z - (1/2)]^2\,[z - (1/4)]}$

Taking partial fractions of $\dfrac{X(z)}{z}$, we have

$$\frac{X(z)}{z} = \frac{(z+1)}{[z - (1/2)]^2\,[z - (1/4)]} = \frac{A}{[z - (1/2)]^2} + \frac{B}{[z - (1/2)]} + \frac{C}{[z - (1/4)]}$$

$$= \frac{6}{[z - (1/2)]^2} - \frac{20}{[z - (1/2)]} + \frac{20}{[z - (1/4)]}$$

$$\therefore \quad X(z) = 6\frac{z}{[z - (1/2)]^2} - 20\frac{z}{[z - (1/2)]} + 20\frac{z}{[z - (1/4)]}$$

Taking inverse Z-transform on both sides, we have the causal signal

$$x(n) = 6n\left(\frac{1}{2}\right)^{n-1} u(n) - 20\left(\frac{1}{2}\right)^{n} u(n) + 20\left(\frac{1}{4}\right)^{n} u(n)$$

3 Residue Method

The inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where c is a circle in the z-plane in the ROC of $X(z)$. The above equation can be evaluated by finding the sum of all residues of the poles that are inside the circle c. Therefore,

$$x(n) = \sum \text{Residues of } X(z)z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z)\, z^{n-1} \Big|\, z = z_i$$

If $X(z)\, z^{n-1}$ has no poles inside the contour c for one or more values of n, then $x(n) = 0$ for these values.

EXAMPLE Using residue method, find the inverse Z-transform of

$$X(z) = \frac{1 + 2z^{-1}}{1 + 4z^{-1} + 3z^{-2}}; \text{ ROC; } |z| > 3$$

Solution: Given $X(z) = \dfrac{1 + 2z^{-1}}{1 + 4z^{-1} + 3z^{-2}} = \dfrac{z(z+2)}{z^2 + 4z + 3} = \dfrac{z(z+2)}{(z+1)(z+3)}$

$$\therefore \quad x(n) = \sum \text{Residues of } X(z)\, z^{n-1} \text{ at the poles of } X(z)\, z^{n-1} \text{ within } c$$

$$= \sum \text{Residues of } \frac{z(z+2)\, z^{n-1}}{(z+1)\,(z+3)} = \frac{z^n (z+2)}{(z+1)(z+3)} \text{ at the poles of same within } c$$

$$\therefore \quad x(n) = \sum \text{Residues of } \frac{z^n (z+2)}{(z+1)\,(z+3)} \text{ at poles } z = -1 \text{ and } z = -3$$

$$= (z+1)\frac{z^n (z+2)}{(z+1)\,(z+3)}\bigg|_{z=-1} + \frac{(z+3)\, z^n (z+2)}{(z+1)\,(z+3)}\bigg|_{z=-3}$$

$$= \frac{1}{2}(-1)^n\, u(n) + \frac{1}{2}(-3)^n\, u(n)$$

4 Convolution Method

The inverse Z-transform can also be determined using convolution method. In this method, the given $X(z)$ is splitted into $X_1(z)$ and $X_2(z)$ such that $X(z) = X_1(z)\,X_2(z)$. Then, $x_1(n)$ and $x_2(n)$ are obtained by taking the inverse Z-transform of $X_1(z)$ and $X_2(z)$ respectively. Then, $x(n)$ is obtained by performing convolution of $x_1(n)$ and $x_2(n)$ in time domain.

$$\mathrm{Z}[x_1(n) * x_2(n)] = X_1(z)\,X_2(z) = X(z)$$

$$\therefore \quad x(n) = \mathrm{Z}^{-1}[X(z)] = \mathrm{Z}^{-1}[\mathrm{Z}\{x_1(n) * x_2(n)\}] = x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\,x_2(n-k)$$

EXAMPLE Find the inverse Z-transform of $X(z) = \dfrac{z^2}{(z-2)(z-3)}$ using convolution property of Z-transforms.

Solution: Given

$$X(z) = \frac{z^2}{(z-2)(z-3)}$$

Let

$$X(z) = X_1(z)\,X_2(z) = \frac{z}{z-2}\,\frac{z}{z-3}$$

$$\therefore \quad x_1(n) = \mathrm{Z}^{-1}[X_1(z)] = \mathrm{Z}^{-1}\left(\frac{z}{z-2}\right) = 2^n u(n)$$

$$x_2(n) = \mathrm{Z}^{-1}[X_2(z)] = \mathrm{Z}^{-1}\left(\frac{z}{z-3}\right) = 3^n u(n)$$

$$\therefore \quad x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\,x_2(n-k)$$

$$= \sum_{k=0}^{n} 2^k u(k)\,3^{n-k} u(n-k)$$

$$= 3^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = 3^n \left[\frac{1-(2/3)^{n+1}}{1-(2/3)}\right]$$

$$= 3^{n+1}\left[1-\left(\frac{2}{3}\right)^{n+1}\right] = 3^{n+1} u(n) - 2^{n+1} u(n)$$

\

EXAMPLE Find the inverse Z-transform of $X(z)=\frac{z}{(z-1)[z-(1/2)]}$ using convolution property of Z-transforms.

Solution: Given

$$X(z)=\frac{z}{(z-1)[z-(1/2)]}$$

Let

$$X(z)=X_1(z)\,X_2(z)=\frac{z}{(z-1)}\,\frac{1}{[z-(1/2)]}$$

$\therefore$

$$x_1(n)=\mathrm{Z}^{-1}[X_1(z)]=\mathrm{Z}^{-1}\left(\frac{z}{z-1}\right)=u(n)$$

and

$$x_2(n)=\mathrm{Z}^{-1}[X_2(z)]=\mathrm{Z}^{-1}\left[\frac{1}{z-(1/2)}\right]=\left(\frac{1}{2}\right)^{n-1}u(n-1)$$

$\therefore$

$$x_1(n)*x_2(n)=\sum_{k=0}^{n}x_1(k)x_2(n-k)$$

$$=\sum_{k=0}^{n-1}u(k)\left(\frac{1}{2}\right)^{n-1-k}u(n-1-k)$$

$$=\left(\frac{1}{2}\right)^{n-1}\left[\sum_{k=0}^{n-1}\left(\frac{1}{2}\right)^{-k}\right]=\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{n-1}\left[\left(\frac{1}{2}\right)^{-1}\right]^{k}$$

$$=\left(\frac{1}{2}\right)^{n-1}\left\{\frac{1-[(1/2)^{-1}]^{n}}{1-(1/2)^{-1}}\right\}=\left(\frac{1}{2}\right)^{n-1}\left[\frac{1-(1/2)^{-n}}{-1}\right]$$

$$=\left(\frac{1}{2}\right)^{-1}-\left(\frac{1}{2}\right)^{n}\left(\frac{1}{2}\right)^{-1}=2u(n)-2\left(\frac{1}{2}\right)^{n}u(n)$$

TRANSFORM ANALYSIS OF LTI SYSTEMS

The Z-transform plays an important role in the analysis and design of discrete-time LTI systems.

1 System Function and Impulse Response

Consider a discrete-time LTI system having an impulse response $h(n)$ as shown in Figure

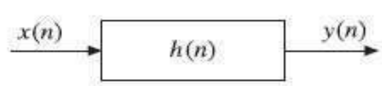


Figure Discrete-time LTI system.

Let us say it gives an output $y(n)$ for an input $x(n)$. Then, we have

$$y(n)=x(n)*h(n)$$

Taking Z-transform on both sides, we get

$$Y(z)=X(z)\,H(z)$$

where

$Y(z)$ = Z-transform of the output $y(n)$
$X(z)$ = Z-transform of the input $x(n)$
$H(z)$ = Z-transform of the impulse response $h(n)$

$$\therefore \qquad H(z)=\frac{Y(z)}{X(z)}$$

$H(z)$ is called the *system function* or the *transfer function* of the LTI discrete system and is defined as:

The ratio of the Z-transform of the output sequence $y(n)$ to the Z-transform of the input sequence $x(n)$ when the initial conditions are neglected.

If the input $x(n)$ is an impulse sequence, then $X(z) = 1$. So $Y(z) = H(z)$. So the transfer function is also defined as the Z-transform of the impulse response of the system.

The poles and zeros of the system function offer an insight into the system characteristics. The poles of the system are defined as the values of z for which the system function $H(z)$ is infinity and the zeros of the system are the values of z for which the system function $H(z)$ is zero.

2 Relationship between Transfer Function and Difference Equation

In terms of a difference equation, an nth order discrete-time LTI system is specified as:

$$\sum_{k=0}^{N} a_k y(n-k)=\sum_{k=0}^{M} b_k x(n-k)$$

Expanding it, we have

$$a_0 y(n)+a_1 y(n-1)+a_2 y(n-2)+\cdots+a_N y(n-N)=b_0 x(n)+b_1 x(n-1)+b_2 x(n-2)+\cdots+b_M x(n-M)$$

Taking Z-transform on both sides and neglecting the initial conditions, we obtain

$$a_0 Y(z)+a_1 z^{-1}Y(z)+a_2 z^{-2}Y(z)+\cdots+a_N z^{-N}Y(z)=b_0 X(z)+b_1 z^{-1}X(z)+b_2 z^{-2}X(z)+\cdots+b_M z^{-M}X(z)$$

i.e. $\quad Y(z)\,[a_0+a_1 z^{-1}+a_2 z^{-2}+\cdots+a_N z^{-N}]=X(z)[b_0+b_1 z^{-1}+b_2 z^{-2}+\cdots+b_M z^{-M}]$

$$\therefore \qquad \frac{Y(z)}{X(z)}=\frac{b_0+b_1 z^{-1}+b_2 z^{-2}+\cdots+b_M z^{-M}}{a_0+a_1 z^{-1}+a_2 z^{-2}+\cdots+a_N z^{-N}}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Now, $Y(z)/X(z) = H(z)$ is called the transfer function of the system or the system function. The frequency response of a system is obtained by substituting $z = e^{j\omega}$ in $H(z)$.

3. STABILITY AND CAUSALITY

We know that the necessary and sufficient condition for a causal linear time-invariant discrete-time system to be BIBO stable is:

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

i.e. an LTI discrete-time system is BIBO stable if its impulse response is absolutely summable.

We also know that for a system to be causal, its impulse response must be equal to zero for $n < 0$ [i.e. $h(n) = 0$ for $n < 0$]. Alternately, if the system is causal, then the ROC for $H(z)$ will be outside the outermost pole.

For a causal LTI system to be stable, all the poles of $H(z)$ must lie inside the unit circle in the z-plane, i.e. for a causal LTI system to be stable, the ROC of the system function must include the unit circle.

EXAMPLE Consider an LTI system with a system function $H(z)=\dfrac{1}{1-(1/2)z^{-1}}$. Find the difference equation. Determine the stability.

Solution: Given

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-(1/2)z^{-1}}=\frac{z}{z-(1/2)}$$

That is

$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)$$

Taking inverse Z-transform on both sides (applying the time shifting property), we get the difference equation

$$y(n)-\frac{1}{2}y(n-1)=x(n)$$

The only pole of $H(z)$ is at $z = 1/2$, i.e., inside the unit cirle. So the system is stable.

EXAMPLE A causal system is represented by $H(z)=\dfrac{z+2}{2z^2-3z+4}$

Find the difference equation and the frequency response of the system.

Solution: Given

$$H(z)=\frac{z+2}{2z^2-3z+4}$$

As the system is causal, $H(z)$ is expressed in negative powers of z.

$$\therefore \quad H(z)=\frac{Y(z)}{X(z)}=\frac{z+2}{2z^2-3z+4}=\frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$$

i.e. $$2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$$

Taking inverse Z-transform on both sides, we have

$$2y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2x(n-2)$$

which is the required difference equation.

Putting $z = e^{j\omega}$ in $H(z)$, we get the frequency response $H(\omega)$ of the system.

$$H(\omega) = \left.\frac{z+2}{2z^2 - 3z + 4}\right|_{z=e^{j\omega}} = \frac{e^{j\omega} + 2}{2e^{j2\omega} - 3e^{j\omega} + 4}$$

$$= \frac{2 + \cos\omega + j\sin\omega}{4 + (2\cos 2\omega - 3\cos\omega) + j(2\sin 2\omega - 3\sin\omega)}$$

EXAMPLE Determine the system function of a discrete-time system described by the difference equation

$$y(n) - \frac{1}{3}y(n-1) + \frac{1}{5}y(n-2) = x(n) - 2x(n-1)$$

Solution: Taking Z-transform on both sides of the given difference equation, we get

$$Y(z) - \frac{1}{3}z^{-1}Y(z) + \frac{1}{5}z^{-2}Y(z) = X(z) - 2z^{-1}X(z)$$

Hence the system function or transfer function of the given system is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - (1/3)z^{-1} + (1/5)z^{-2}} = \frac{z(z-2)}{z^2 - (1/3)z + (1/5)}$$

EXAMPLE A causal system has input $x(n)$ and output $y(n)$. Find the system function, frequency response and impulse response of the system if

$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

and
$$y(n) = \delta(n) - \frac{2}{3}\delta(n-1)$$

Also assess the stability.

Solution: Given
$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

and
$$y(n) = \delta(n) - \frac{2}{3}\delta(n-1)$$

Taking Z-transform of the above equations, we get

$$X(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

and
$$Y(z) = 1 - \frac{2}{3}z^{-1}$$

The system function or the transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1-(2/3)z^{-1}}{1+(1/6)z^{-1}-(1/6)z^{-2}} = \frac{z[z-(2/3)]}{[z-(1/3)][z+(1/2)]}$$

The frequency response of the system is:

$$H(\omega) = \left.\frac{z[z-(2/3)]}{[z-(1/3)][z+(1/2)]}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}[e^{j\omega}-(2/3)]}{[e^{j\omega}-(1/3)][e^{j\omega}+(1/2)]}$$

By partial fraction expansion, we have

$$\frac{H(z)}{z} = \frac{[z-(2/3)]}{[z-(1/3)][z+(1/2)]} = \frac{A}{z-(1/3)} + \frac{B}{z+(1/2)} = \frac{-2/5}{z-(1/3)} + \frac{7/5}{z+(1/2)}$$

$\therefore$
$$H(z) = -\frac{2}{5}\left[\frac{z}{z-(1/3)}\right] + \frac{7}{5}\left[\frac{z}{z+(1/2)}\right]$$

Taking inverse Z-transform on both sides, we get the impulse response as:

$$h(n) = -\frac{2}{5}\left(\frac{1}{3}\right)^n u(n) + \frac{7}{5}\left(-\frac{1}{2}\right)^n u(n)$$

Both the poles of $H(z)$ are inside the unit circle. So the system is stable.

REALIZATION OF DIGITAL FILTERS

INTRODUCTION

Systems may be continuous-time systems or discrete-time systems. Discrete-time systems may be FIR (Finite Impulse Response) systems or IIR (Infinite Impulse Response) systems. FIR systems are the systems whose impulse response has finite number of samples and IIR systems are systems whose impulse response has infinite number of samples. Realization of a discrete-time system means obtaining a network corresponding to the difference equation or transfer function of the system. In this chapter, various methods of realization of discrete-time systems are discussed.

REALIZATION OF DISCRETE-TIME SYSTEMS

To realize a discrete-time system, the given difference equation in time domain is to be converted into an algebraic equation in z-domain, and each term of that equation is to be represented by a suitable element (a constant multiplier or a delay element). Then using adders, all the elements representing various terms of the equation are to be connected to obtain the output. The symbols of the basic elements used for constructing the block diagram of a discrete-time system (adder, constant multiplier and unit delay element) are shown in Figure .

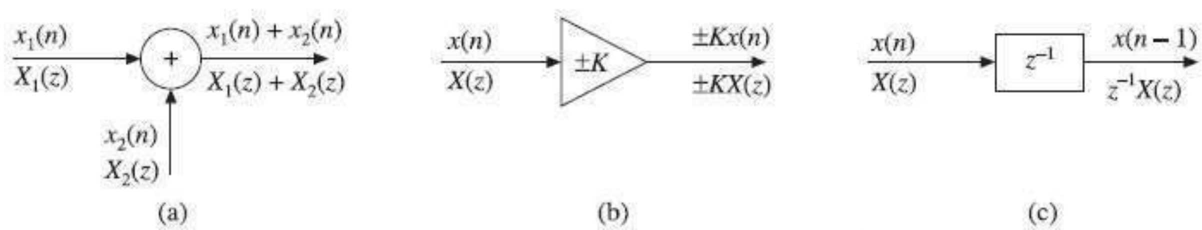


Figure (a) Adder (b) Constant multiplier and (c) Unit delay element.

Adder: An adder is used to add two or more signals. The output of adder is equal to the sum of all incoming signals.

Constant multiplier: A constant multiplier is used to multiply the signals by a constant. The output of the multiplier is equal to the product of the input signal and the constant of the multiplier.

Unit delay element: A unit delay element is used to delay the signal passing through it by one sampling time.

SOLUTION OF DIFFERENCE EQUATIONS USING Z-TRANSFORMS

To solve the difference equation, first it is converted into algebraic equation by taking its Z-transform. The solution is obtained in z-domain and the time domain solution is obtained by taking its inverse Z-transform.

The system response has two components. The source free response and the forced response. The response of the system due to input alone when the initial conditions are neglected is called the forced response of the system. It is also called the steady state response of the system. It represents the component of the response due to the driving force. The response of the system due to initial conditions alone when the input is neglected is called the free or natural response of the system. It is also called the transient response of the system. It represents the component of the response when the driving function is made zero. The response due to input and initial conditions considered simultaneously is called the total response of the system.

For a stable system, the source free component always decays with time. In fact a stable system is one whose source free component decays with time. For this reason the source free component is also designated as the transient component and the component due to source is called the steady state component.

When input is a unit impulse input, the response is called the impulse response of the system and when the input is a unit step input, the response is called the step response of the system.

EXAMPLE 1 A linear shift invariant system is described by the difference equation

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=x(n)+x(n-1)$$

with $y(-1) = 0$ and $y(-2) = -1$.

Find (a) the natural response of the system (b) the forced response of the system for a step input and (c) the frequency response of the system.

Solution:

(a) The natural response is the response due to initial conditions only. So make $x(n) = 0$. Then the difference equation becomes

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=0$$

Taking Z-transform on both sides, we have

$$Y(z)-\frac{3}{4}\,[z^{-1}Y(z)+y(-1)]+\frac{1}{8}\,[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)]=0$$

i.e.
$$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)-\frac{1}{8}=0$$

i.e. $$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)-\frac{1}{8}=0$$

$$\therefore\ Y(z)=\frac{1/8}{1-(3/4)z^{-1}+(1/8)z^{-2}}=\frac{1/8z^2}{z^2-(3/4)z+(1/8)}=\frac{1/8z^2}{[z-(1/2)][z-(1/4)]}$$

The partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z}=\frac{(1/8)z}{[z-(1/2)][z-(1/4)]}=\frac{A}{z-(1/2)}+\frac{B}{z-(1/4)}=\frac{1/4}{z-(1/2)}-\frac{1/8}{z-(1/4)}$$

$$\therefore\quad Y(z)=\frac{1}{4}\frac{z}{z-(1/2)}-\frac{1}{8}\frac{z}{z-(1/4)}$$

Taking inverse Z-transform on both sides, we get the natural response as:

$$y(n)=\frac{1}{4}\left(\frac{1}{2}\right)^n u(n)-\frac{1}{8}\left(\frac{1}{4}\right)^n u(n)$$

(b) To find the forced response due to a step input, put $x(n) = u(n)$. So we have

$$y(n)-\frac{3}{4}y(n-1)+\frac{1}{8}y(n-2)=u(n)+u(n-1)$$

We know that the forced response is due to input alone. So for forced response, the initial conditions are neglected. Taking Z-transform on both sides of the above equation and neglecting the initial conditions, we have

$$Y(z)-\frac{3}{4}z^{-1}Y(z)+\frac{1}{8}z^{-2}Y(z)=U(z)+z^{-1}U(z)=\frac{z}{z-1}+\frac{1}{z-1}$$

i.e. $$Y(z)\left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)=\frac{z+1}{z-1}$$

$$\therefore\ Y(z)=\frac{z+1}{(z-1)[1-(3/4)z^{-1}+(1/8)z^{-2})}=\frac{z^2(z+1)}{(z-1)[z^2-(3/4)z+(1/8)]}=\frac{z^2(z+1)}{(z-1)[z-(1/2)][z-(1/4)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\therefore \quad \frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)[z-(1/2)][z-(1/4)]} = \frac{A}{z-1} + \frac{B}{z-(1/2)} + \frac{C}{z-(1/4)}$$

$$= \frac{16/3}{z-1} - \frac{6}{z-(1/2)} + \frac{5/3}{z-(1/4)}$$

or $$Y(z) = \frac{16}{3}\left(\frac{z}{z-1}\right) - 6\left[\frac{z}{z-(1/2)}\right] + \frac{5}{3}\left[\frac{z}{z-(1/4)}\right]$$

Taking the inverse Z-transform on both sides, we have the forced response for a step input.

$$y(n) = \frac{16}{3}u(n) - 6\left(\frac{1}{2}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

(c) The frequency response of the system $H(\omega)$ is obtained by putting $z = e^{j\omega}$ in $H(z)$.

Here $$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z+1)}{z^2 - (3/4)z + (1/8)}$$

Therefore, $$H(\omega) = \frac{e^{j\omega}(e^{j\omega}+1)}{(e^{j\omega})^2 - (3/4)e^{j\omega} + (1/8)}$$

EXAMPLE 2 (a) Determine the free response of the system described by the difference equation

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=x(n) \text{ with } y(-1)=1 \text{ and } y(-2)=0$$

(b) Determine the forced response for an input

$$x(n)=\left(\frac{1}{4}\right)^n u(n)$$

Solution:

(a) The free response, also called the natural response or transient response is the response due to initial conditions only [i.e. make $x(n) = 0$].

So, the difference equation is:

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=0$$

Taking Z-transform on both sides, we get

$$Y(z)-\frac{5}{6}[z^{-1}Y(z)+y(-1)]+\frac{1}{6}[z^{-2}Y(z)+z^{-1}y(-1)+y(-2)]=0$$

$$Y(z)\left(1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}\right)-\frac{5}{6}+\frac{1}{6}z^{-1}=0$$

$$\therefore \quad Y(z)=\frac{(5/6)-(1/6)z^{-1}}{1-(5/6)z^{-1}+(1/6)z^{-2}}=\frac{5/6[z-(1/5)]z}{z^2-(5/6)z+(1/6)}=\frac{(5/6)z[z-(1/5)]}{[z-(1/2)][z-(1/3)]}$$

Taking partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z}=\frac{5/6[z-(1/5)]}{[z-(1/2)][z-(1/3)]}=\frac{A}{z-(1/2)}+\frac{B}{z-(1/3)}=\frac{3/2}{z-(1/2)}-\frac{2/3}{z-(1/3)}$$

$$\therefore \quad Y(z)=\frac{3}{2}\frac{z}{z-(1/2)}-\frac{2}{3}\frac{z}{z-(1/3)}$$

Taking inverse Z-transform on both sides, we get the free response of the system as:

$$y(n)=\frac{3}{2}\left(\frac{1}{2}\right)^n u(n)-\frac{2}{3}\left(\frac{1}{3}\right)^n u(n)$$

(b) To determine the forced response, i.e. the steady state response, the initial conditions are to be neglected.

The given difference equation is:

$$y(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) = x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) - \frac{5}{6}z^{-1}Y(z) + \frac{1}{6}z^{-2}Y(z) = \frac{z}{z-(1/4)}$$

i.e.,
$$Y(z)\left(1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}\right) = \frac{z}{z-(1/4)}$$

$$\therefore \quad Y(z) = \frac{z}{z-(1/4)}\,\frac{1}{1-(5/6)z^{-1}+(1/6)z^{-2}} = \frac{z^3}{[z-(1/4)][z-(1/2)][z-(1/3)]}$$

Partial fraction expansion of $Y(z)/z$ gives

$$\frac{Y(z)}{z} = \frac{z^2}{[z-(1/4)][z-(1/3)][z-(1/2)]} = \frac{A}{z-(1/4)} + \frac{B}{z-(1/3)} + \frac{C}{z-(1/2)}$$

$$= \frac{3}{z-(1/4)} - \frac{8}{z-(1/3)} + \frac{6}{z-(1/2)}$$

Multiplying both sides by z, we get

$$Y(z) = 3\frac{z}{z-(1/4)} - 8\frac{z}{z-(1/3)} + 6\frac{z}{z-(1/2)}$$

Taking inverse Z-transform on both sides, the forced response of the system is:

$$y(n) = 3\left(\frac{1}{4}\right)^n u(n) - 8\left(\frac{1}{3}\right)^n u(n) + 6\left(\frac{1}{2}\right)^n u(n)$$

EXAMPLE 1 Construct the block diagram for the discrete-time systems whose input-output relations are described by the following difference equations:

(a) $y(n) = 0.7x(n) + 0.3x(n-1)$

(b) $y(n) = 0.5y(n-1) + 0.8x(n) + 0.4x(n-1)$

Solution:

(a) Given $$y(n) = 0.7x(n) + 0.3x(n-1)$$

The system may be realized by using the difference equation directly or by using the Z-transformed version of that. The individual terms of the given difference equation are $0.7x(n)$ and $0.3x(n-1)$. They are represented by the basic elements as shown in Figure

Alternatively

Taking Z-transform on both sides of the given difference equation, we have

$$Y(z) = 0.7X(z) + 0.3z^{-1}X(z)$$

The individual terms of the above equation are: $0.7X(z)$ and $0.3z^{-1}X(z)$.
They are represented by the basic elements as shown in Figure

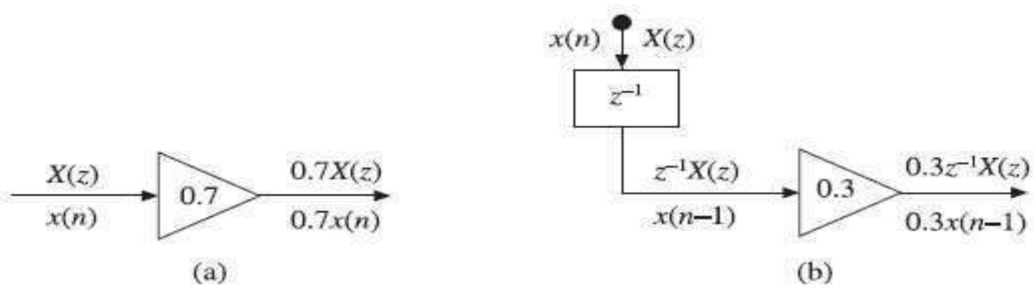


Figure Block diagram representation of (a) $0.7X(z)$ and (b) $0.3z^{-1}X(z)$.

The input to the system is $X(z)$ [or $x(n)$] and the output of the system is $Y(z)$ [or $y(n)$]. The above elements are connected as shown in Figure to get the output $Y(z)$ [or $y(n)$].

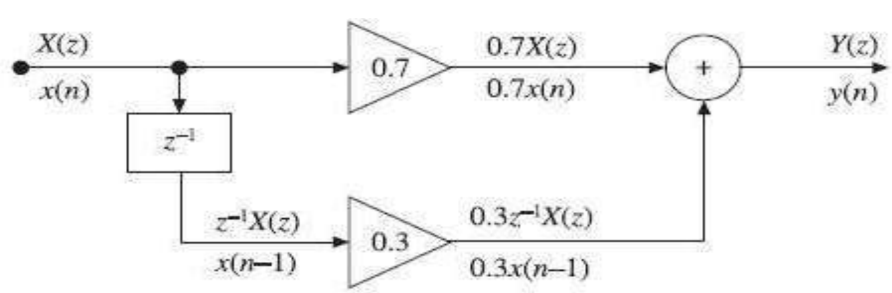


Figure Realization of system described by $y(n) = 0.7x(n) + 0.3x(n-1)$.

STRUCTURES FOR REALIZATION OF IIR SYSTEMS

IIR systems are systems whose impulse response has infinite number of samples. They are designed by using all the samples of the infinite duration impulse response. The convolution formula for IIR systems is given by

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

Since this weighted sum involves the present and all the past input samples, we can say that the IIR system has an infinite memory.

A system whose output $y(n)$ at time n depends on the present input and any number of past values of input and output is called a recursive system. The past outputs are

$$y(n-1),\ y(n-2),\ y(n-3),\ \ldots$$

Hence, for recursive system, the output $y(n)$ is given by

$$y(n) = F[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)]$$

In recursive system, in order to compute $y(n_0)$, we need to compute all the previous values $y(0)$, $y(1)$, $y(2)$, ..., $y(n_0 - 1)$ before calculating $y(n_0)$. Hence, output of recursive system has to be computed in order $[y(0), y(1), y(2), \ldots]$.

Transfer function of IIR system

In general, an IIR system is described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e. in general, IIR systems are those in which the output at any instant of time depends not only on the present and past inputs but also on the past outputs. Hence, in general, an IIR system is of recursive type.

On taking Z-transform of the above equation for $y(n)$, we get

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

i.e.

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

The system function or the transfer function of the IIR system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

The above equations for $Y(z)$ and $H(z)$ can be viewed as a computational procedure (or algorithm) to determine the output sequence $y(n)$ from the input sequence $x(n)$. The computations in the above equation can be arranged into various equivalent sets of difference equations with each set of equations defining a computational procedure or algorithm for implementing the system.

For each set of equations, we can construct a block diagram consisting of delays, adders and multipliers. Such block diagrams are referred to as realization of the system or equivalently as structure for realizing the system.

The main advantage of re-arranging the sets of difference equations (i.e. the main criteria for selecting a particular structure) is to reduce the computational complexity, memory requirements and finite word length effects in computations.

So the factors that influence the choice of structure for realization of LTI system are: computational complexity, memory requirements and finite word length effects in computations.

Computational complexity refers to the number of arithmetic operations required to compute the output value $y(n)$ for the system.

Memory requirements refer to the number of memory locations required to store the system parameters, past inputs and outputs and any intermediate computed values.

Finite-word-length effects or finite precision effects refer to the quantization effects that are inherent in any digital implementation of the system either in hardware or in software.

Although the above three factors play a major role in influencing our choice of the realization of the system, other factors such as whether the structure lends itself to parallel processing or whether the computations can be pipelined may play a role in selecting a specific structure.

The different types of structures for realizing IIR systems are:

1. Direct form-I structure
2. Direct form-II structure
3. Transposed form structure
4. Cascade form structure
5. Parallel form structure
6. Lattice structure
7. Ladder structure

1 Direct Form-I Structure

Direct form-I realization of an IIR system is nothing, but the direct implementation of the difference equation or transfer function. It is the simplest and most straight forward realization structure available.

The difference equation governing the behaviour of an IIR system is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e. $y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M)$

On taking the Z-transform of the above equation for $y(n)$, we get

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

The equation for $Y(z)$ [or $y(n)$] can be directly represented by a block diagram as shown in Figure 4.6 and this structure is called Direct form-I structure. This structure uses separate delays (z^{-1}) for input and output samples. Hence, for realizing this structure more memory is required. The direct form structure provides a direct relation between time domain and z-domain equations.

The structure shown in Figure 4.6 is called a ***non-canonical structure*** because the number of delay elements used is more than the order of the difference equation.

If the IIR system is more complex, that is of higher order, then introduce an intermediate variable $W(z)$ so that

$$W(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

or

$$w(n) = \sum_{k=0}^{M} b_k x(n-k) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-m)$$

$\therefore$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots + W(z)$$

or

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \cdots + w(n)$$

So, the direct form-I structure is in two parts. The first part contains only zeros [that is, the input components either $x(n)$ or $X(z)$] and the second part contains only poles [that is, the output components either $y(n)$ or $Y(z)$]. In direct form-I, the zeros are realized first and poles are realized second.

Limitations of direct form-I

- Since the number of delay elements used in direct form-I is more than (double) the order of the difference equation, it is not effective.
- It lacks hardware flexibility.
- There are chances of instability due to the quantization noise.

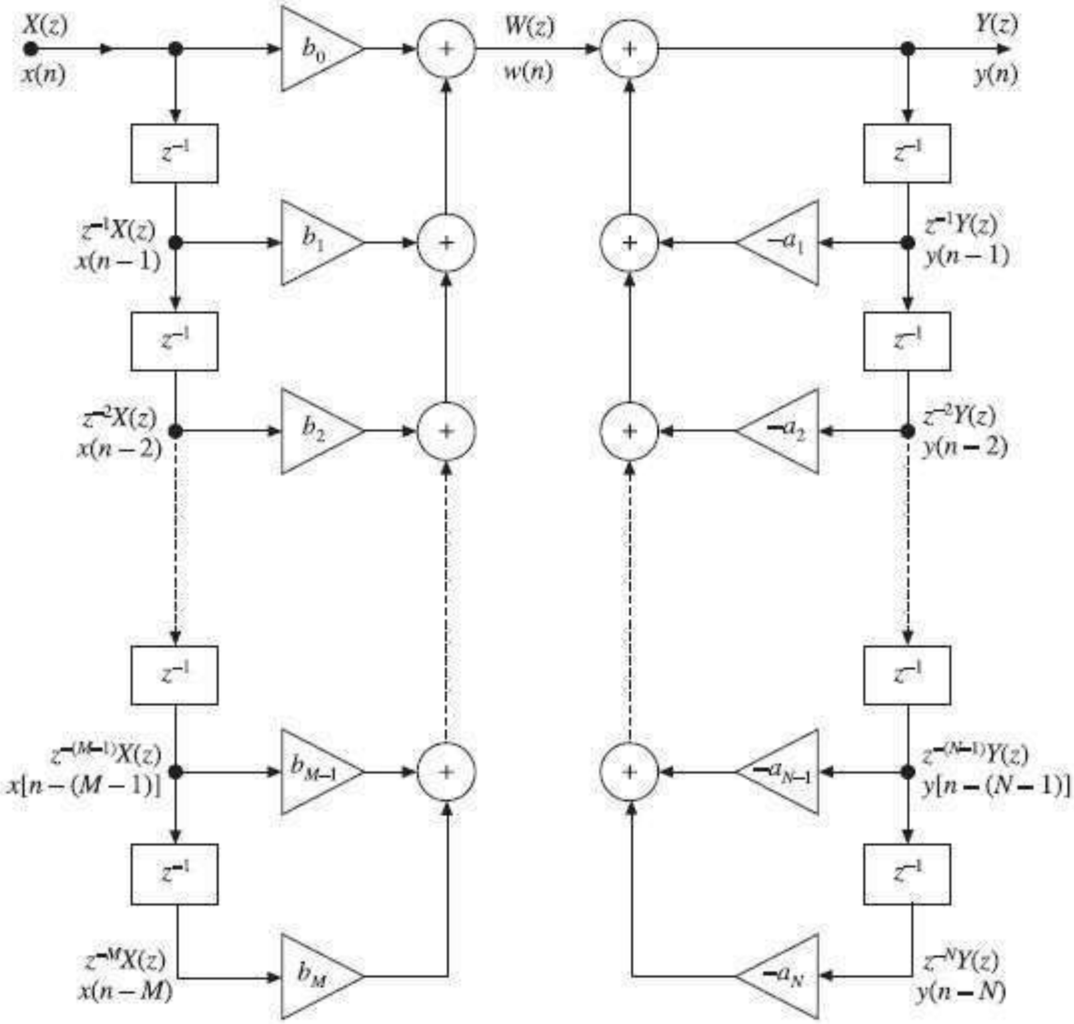


Figure Direct form-I structure.

2 Direct Form-II Structure

The Direct form-II structure is an alternative to direct form-I structure. It is more advantageous to use direct form-II technique than direct form-I, because it uses less number of delay elements than the direct form-I structure.

In direct form-II, an intermediate variable is introduced and the given transfer function is split into two, one containing only poles and the other containing only zeros. The poles [that is, the output components $y(n)$ or $Y(z)$ which is the denominator part of the transfer function] are realized first and the zeros [that is, the input components either $x(n)$ or $X(z)$, which is the numerator part of the transfer function] second.

If the coefficient of the present output sample $y(n)$ or the non-delay constant at denominator is non unity, then transform it to unity. The systematic procedure is given as follows:

Consider the general difference equation governing an IIR system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

i.e.
$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - \cdots - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \cdots + b_M x(n-M)$$

On taking Z-transform of the above equation and neglecting initial conditions, we get

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \cdots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

i.e. $Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$

i.e. $Y(z)[1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}] = X(z)[b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}]$

i.e.
$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Let
$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \frac{W(z)}{X(z)}$$

where
$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

and
$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$

On cross multiplying the above equations, we get

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + \cdots + a_N z^{-N} W(z) = X(z)$$

$\therefore$
$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - \cdots - a_N z^{-N} W(z)$$

and
$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z)$$

The realization of an IIR system represented by these equations in direct form-II is shown in Figure

Advantage of the direct form-II over the direct form-I

The number of delay elements used in direct form-II is less than that of direct form-I.

Limitations of direct form-II

- It also lacks hardware flexibility
- There are chances of instability due to the quantization noise

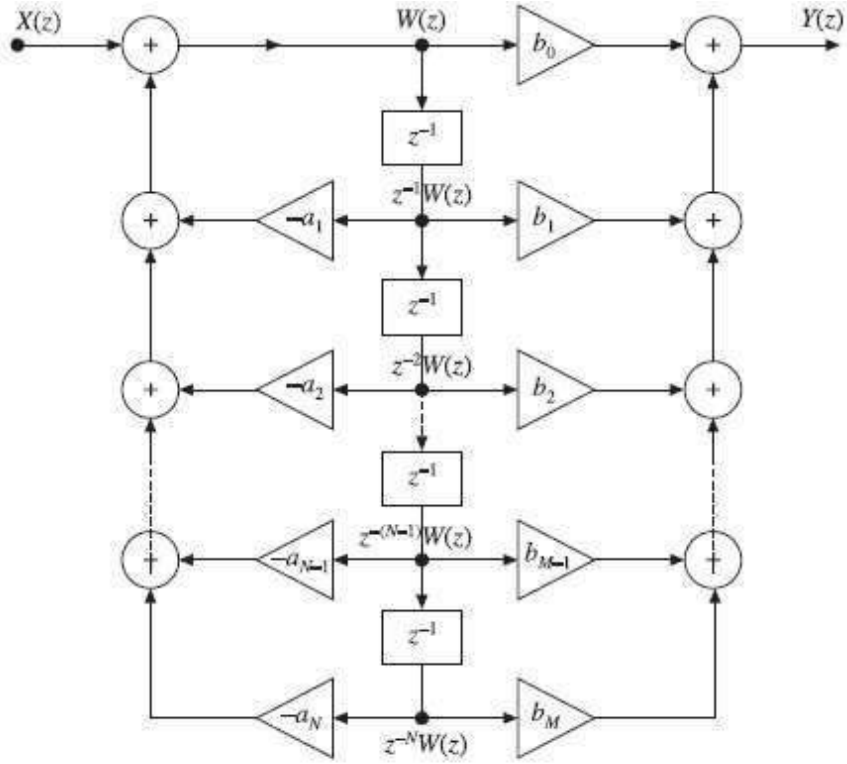


Figure Direct form-II structure of IIR system for $M = N$.

Since the number of delay elements used in direct form-II is the same as that of the order of the difference equation, direct form-II is called a ***canonical structure.***

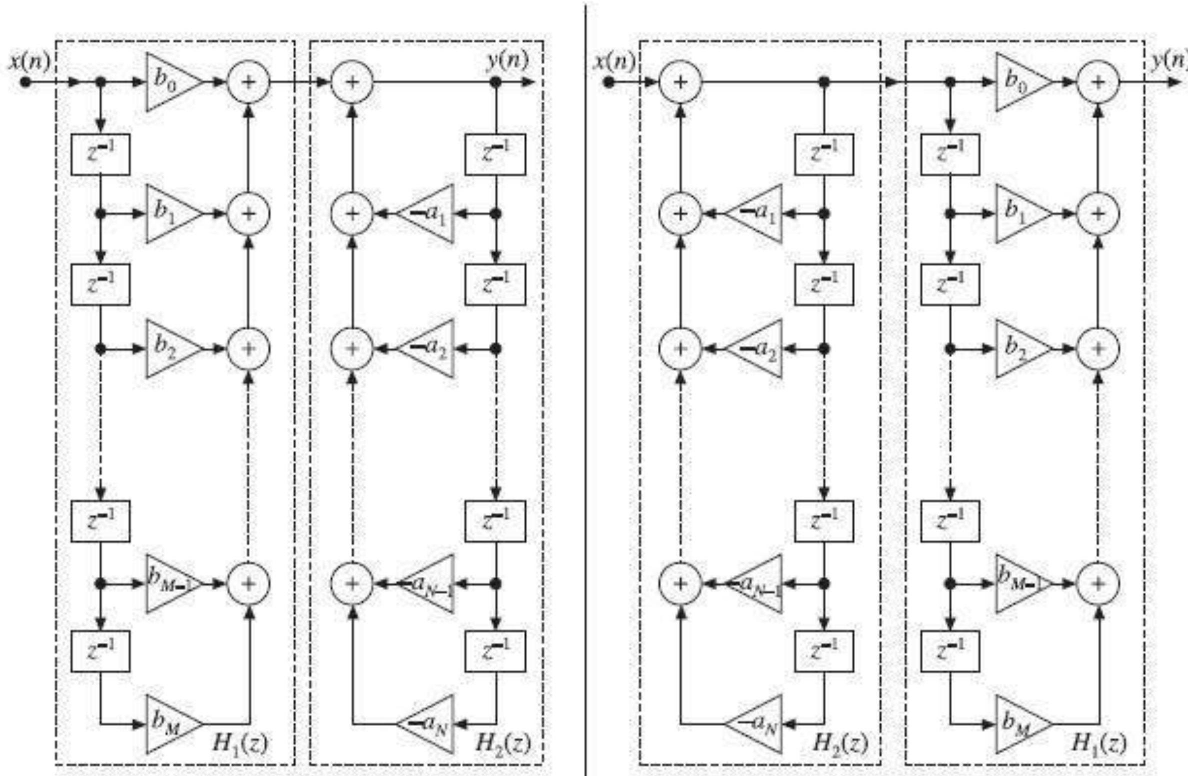
The comparison of direct form-I and direct form-II structures is given in Table .1

TABLE 1 Comparison of direct form-I and direct form-II structures

Direct form-I structure	*Direct form-II structure*
This realization uses separate delays (memory) for both the input and output signal samples.	This realization uses a single delay (memory) for both the input and output signal samples.
For the $(M-1)$th or $(N-1)$th order IIR system, direct form-I requires $M+N-1$ multipliers, $M+N-2$ adders and $M+N-2$ delays.	For the $(M-1)$th or $(N-1)$th order IIR system, direct form-II requires $M+N-1$ multipliers, $M+N-2$ adders and max $[(M-1), (N-1)]$ delays.
It is also called non-canonical, because it requires more number of delays.	It is called canonical, because it requires a minimum number of delays.
It is not efficient in terms of memory requirements compared to direct form-II.	It is more efficient in terms of memory requirements.
Direct form-I can be viewed as two linear time-invariant systems in cascade. The first one is non-recursive and the second one recursive.	Direct form-II can also be viewed as two linear time-invariant systems in cascade. The first one is recursive and the second one non-recursive.

Conversion of direct form-I structure to direct form-II structure

The direct form-I structure can be converted to direct form-II structure by considering the direct form-I structure as cascade of two systems $H_1(z)$ and $H_2(z)$ as shown in Figure (a). By linearity property, the order of cascading can be interchanged as shown in Figure (b).



(a) (b)

EXAMPLE Find the digital network in direct form-I for the system described by the difference equation

$$y(n) = 2x(n) + 0.3\,x(n-1) + 0.5x(n-2) - 0.7\,y(n-1) - 0.9\,y(n-2)$$

Solution: Given difference equation is:

$$y(n) = 2x(n) + 0.3\,x(n-1) + 0.5x(n-2) - 0.7\,y(n-1) - 0.9\,y(n-2)$$

Taking Z-transform on both sides, we have

$$Y(z) = 2X(z) + 0.3\,z^{-1}X(z) + 0.5\,z^{-2}X(z) - 0.7\,z^{-1}Y(z) - 0.9\,z^{-2}Y(z)$$

$$Y(z) + 0.7z^{-1}Y(z) + 0.9z^{-2}Y(z) = 2X(z) + 0.3z^{-1}X(z) + 0.5z^{-2}X(z)$$

i.e. $$Y(z)[1 + 0.7z^{-1} + 0.9z^{-2}] = X(z)[2 + 0.3z^{-1} + 0.5z^{-2}]$$

$\therefore$ $$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + 0.3z^{-1} + 0.5z^{-2}}{1 + 0.7z^{-1} + 0.9z^{-2}}$$

Let $$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} = \frac{2 + 0.3z^{-1} + 0.5z^{-2}}{1 + 0.7z^{-1} + 0.9z^{-2}}$$

where $\frac{W(z)}{X(z)} = \frac{1}{1 + 0.7z^{-1} + 0.9z^{-2}}$ and $\frac{Y(z)}{W(z)} = 2 + 0.3z^{-1} + 0.5z^{-2}$

Cross multiplying the above equations, we get

$$W(z) + 0.7z^{-1}W(z) + 0.9z^{-2}W(z) = X(z)$$

i.e. $$W(z) = X(z) - 0.7z^{-1}W(z) - 0.9z^{-2}W(z)$$

and $$Y(z) = 2W(z) + 0.3z^{-1}W(z) + 0.5z^{-2}W(z)$$

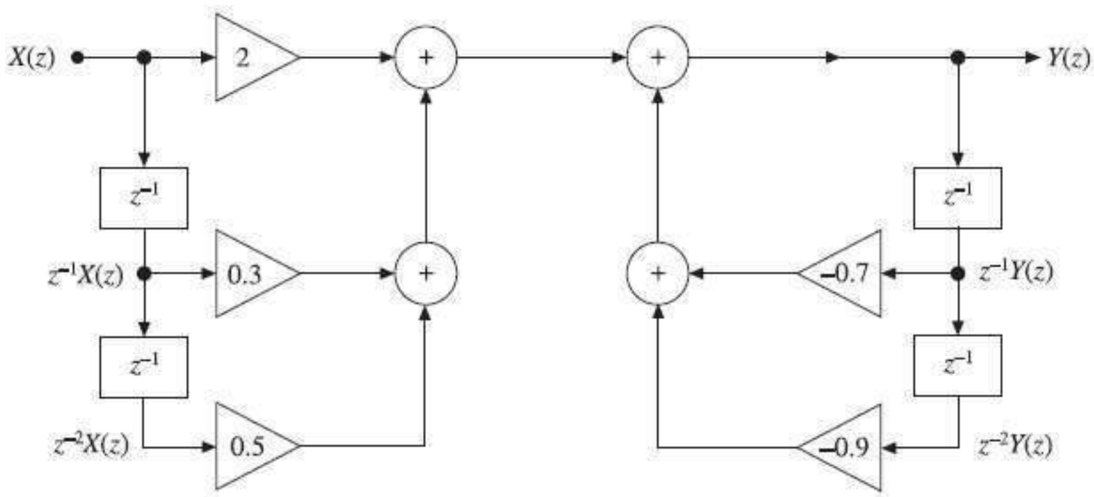


Figure Direct form structure

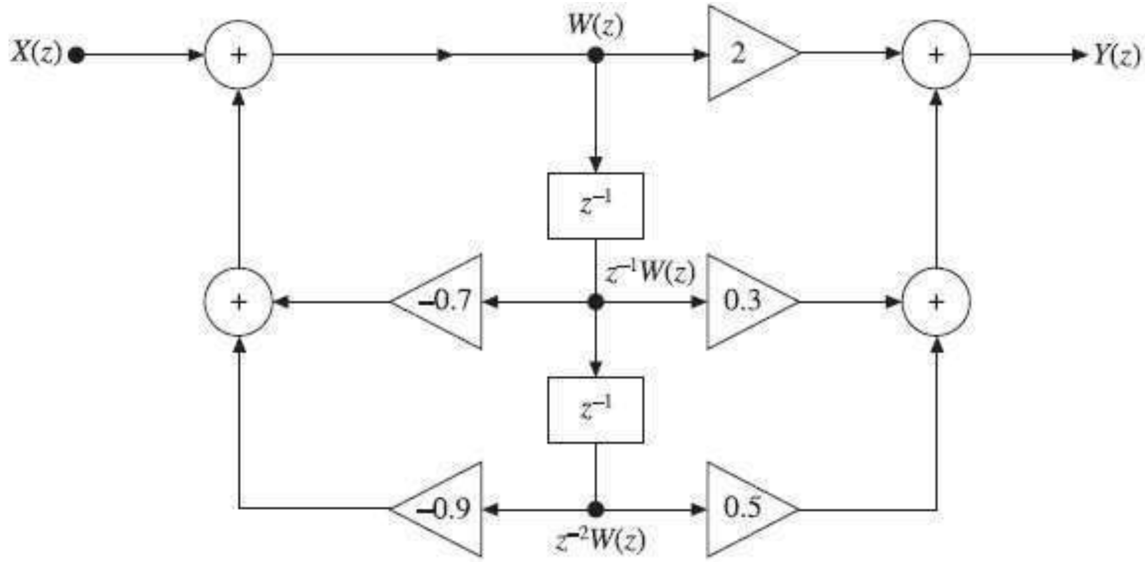


Figure Direct form-II structure (b)