MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(An Autonomous Institution – UGC, Govt.of India)

Recognizes under 2(f) and 12(B) of UGC ACT 1956

(Affiliated to JNTUH, Hyderabad, Approved by AICTE –Accredited by NBA & NAAC-"A" Grade-ISO 9001:2015 Certified)

MATHEMATICS-II

B.Tech – I Year – II Semester

DEPARTMENT OF HUMANITIES AND SCIENCES

Preface

Education in mathematics forms the basis of science and engineering from undergraduate to graduate level, because engineering and science are largely based on mathematical modelling. The level and the quality of mathematics education sets the level of the education as a whole.

Our aim is to develop a complete program for mathematics education in science and engineering from basic undergraduate to graduate education. This includes several courses like

- Mathematics I
- Mathematics II
- Mathematics III
- Probability and Random Processes
- Probability and Statistics

These courses are introduced at different levels/semesters of the engineering program some being common and some based on the branch/specialization chosen by the student.

The purpose of these courses is to arm the student with the necessary ideas and methods, so that when mathematical elements appear in other courses and research work, one can tackle them with confidence, possibly with further independent study into specialized areas. Its major role is to summarize, crystallize, enhance and give a forward orientation to the mathematical methods taught in undergraduate curriculum, with projections to future requirements.

Common Features of the Mathematical Courses:

- The courses are based on a synthesis of mathematics, computation and application.
- The courses are designed basing on new interests and needs of the current scenario, giving a new united presentation from the start based on constructive mathematical methods including a computational methodology.
- These courses are designed at different levels of ambition concerning both mathematical analysis and computation, while keeping a common basic core.
- These courses increase the motivation of the student by applying mathematical methods to interesting and important concrete problems right from their introduction.

- While emphasis may be put on problem solving, these courses gives theoretical and computational methods and builds confidence.
- The course contains most of the traditional material from basic courses in analysis, linear algebra, applied mathematics and higher engineering mathematics.
- Emphasis is put on giving the student a solid understanding of basic mathematical concepts.
- The student acquires solid skills of implementing and applying the computational methods learnt.

The idea is that making the student comfortable with both advanced mathematical concepts and modern computational techniques, will open a wealth of possibilities of applying mathematics to problems of real interest.

Mathematics II : Course Description

Mathematics II covers a wide range of mathematical concepts that are useful in almost all engineering core subjects: numerical methods, interpolation, curve fitting, partial differential equations, multiple integrals and vector calculus. Topics includethe roots of algebraic, non-algebraic equations and predicting the value at an intermediate point from a given discrete data, the most appropriate relation of the data variables using curve fitting and this method of data analysis helps engineers to understand the system for better interpretation and decision making, first order linear and non-linear partial differential equations, multiple integrals, the line, surface, volume integrals and converting them from one form to another using vector integral theorems.

SYLLABUS

B. TECH- I- YEAR- II-SEM **L T/P/D C**

3 1/-/- 4

(R20A0022)MATHEMATICS-II

(Common to all branches)

Course Objectives:

1. The aim of numerical methods is to provide systematic methods for solving problems in a numerical form using the given initial data,also used to find the roots of an equation and to solve differential equations.
2. The objective of interpolation is to find an unknown function which approximates the given data points and the objective of curve fitting is to find the relation between the variables x and y from given data and such relationships which exactly pass through the data (or) approximately satisfy the data under the condition of sum of least squares of errors.
3. PDE aims at forming a function with many variables and also their solution methods, Method of separation of variables technique is learnt to solve typical second order PDE.
4. Evaluation of multiple integrals.
5. In many engineering fields the physical quantities involved are vector valued functions. Hence the vector calculus aims at basic properties of vector-valued functions and their applications to line, surface and volume integrals.

UNIT – I: Solutions of algebraic, transcendental equations and Interpolation

Solution of algebraic and transcendental equations: Introduction, Bisection Method, Method of false position, Newton-Raphson method and their graphical interpretations.

Interpolation: Introduction, errors in polynomial interpolation, Finite differences - Forward differences, Backward differences, Central differences. Newton's formulae for interpolation, Gauss's central difference formulae, Interpolation with unevenly spaced points - Lagrange's Interpolation.

UNIT – II: Numerical Methods

Numerical integration : Generalized quadrature - Trapezoidal rule, Simpson's 1/3rd and Simpson's 3/8th rules.

Numerical solution of ordinary differential equations: Solution by Taylor's series method, Euler's method, Euler's modified method, Runge-Kutta fourth order method.

Curve fitting : Fitting a straight line, second degree curve, exponential curve, power curve by method of least squares.

UNIT III: Partial Differential Equations

Introduction, formation of partial differential equation by elimination of arbitrary constants and arbitrary functions, solutions of first order Lagrange's linear equation and non-linear equations, Charpit's method, Method of separation of variables for second order equations and applications of PDE to one dimensional equation (Heat equation).

Unit IV: Double and Triple Integrals

Double and triple integrals (Cartesian and polar), Change of order of integration in double integrals, Change of variables (Cartesian to polar).

Unit V: Vector Calculus

Introduction, Scalar point function and vector point function, Directional derivative, Gradient, Divergence, Curl and their related properties, Laplacian operator, Line integral - work done, Surface integrals, Volume integral. Green's theorem, Stoke's theorem and Gauss's Divergence theorems (Statement & their Verification).

Text Books:

i) Higher Engineering Mathematics by B V Ramana ., Tata McGraw Hill.

ii) Higher Engineering Mathematics by B.S. Grewal, Khanna Publishers.

iii) Mathematical Methods by S.R.K Iyenger, R.K.Jain, Narosa Publishers.

Reference Books:

i) Elementary Numerical Analysis by Atkinson-Han, Wiley Student Edition.

ii) Advanced Engineering Mathematics by Michael Greenberg –Pearson publishers.

iii) Introductory Methods of Numerical Analysis by S.S. Sastry, PHI

Course Outcomes: After learning the concepts of this paper the student will be able to independently

1. Find the roots of algebraic, non algebraic equations and predict the value at an intermediate point from a given discrete data.
2. Find the most appropriate relation of the data variables using curve fitting and this method of data analysis helps engineers to understand the system for better interpretation and decision making.
3. Solve first order linear and non-linear partial differential equations which are very important in engineering field.
4. Evaluate multiple integrals; hence this concept can be used to evaluate Volumes and Areas of an object.
5. Evaluate the line, surface, volume integrals and converting them from one to another using vector integral theorems.

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Mathematics-II:CO-PO Mapping

CO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1	3	2	-	2	1	-	-	1	-	-	-	2
CO2	2	2	-	1	2	-	-	1	-	-	-	1
CO3	3	3	-	2	2	-	-	1	-	-	-	2
CO4	3	3	-	2	2	-	-	1	-	-	-	1
CO5	3	2	-	2	2	-	-	1	-	-	-	1
	2.8	2.4	0	1.8	1.8	0	0	1	0	0	0	1.4

UNIT-I

SOLUTIONS OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS AND INTERPOLATION

Introduction

Using mathematical modeling, most of the problems in Engineering, Physical and Economical sciences can be formulated in terms of systems of linear or non linear equations, ordinary or partial differential equations or Integral equations. In majority of the cases, the solutions to these problems in analytical form are non-existent or difficult or not amenable for direct interpretation. In all such problems, numerical analysis provides approximate solutions practical and amenable for analysis. Numerical analysis does not strive for exaxtness instead it yields approximations with specified degree of accuracy. The early disadvantages of the several numbers of computations involved has been removed through high speed computation using computers, giving results which are accurate, reliable and fast. Numerical is not only a science but also an 'art' because the choice of 'appropriate' procedure which 'best' suits to a given problem yields 'good' solutions.

Approximations curve is the graph of data obtained through measurement of observation. Curve fitting is the process of finding the "best fit" curve since different approximation curves can be obtained for the same data. Least squares method is the best curve fitting by a sum of exponentials, linear weighted and non-linear weighted least squares approximation.

Solution of Algebraic and Transcendental equations

Introduction:

Polynomial function: A function $f(x)$ is said to be a polynomial function, if $f(x)$ is a polynomial in x. ie, $f(x) = a_0x^n + a_1x^{n-1} + \cdots \ldots \ldots \ldots .. + a_{n-1}x + a_n$

where $a_0 \neq 0$, the co-efficients $a_0, a_1 \ldots\ldots\ldots a_n$ are real constants and n is a non-negative integer.

Algebraic function: A function which is a sum (or) difference (or) product of two polynomials is called an algebraic function. Otherwise, the function is called a transcendental (or) non-algebraic function.

Eg: $f(x) = x^3 - 4x^2 + 5x - 2$ is a algebraic equation

Eg: $f(x) = xcosx - e^x = 0$ is a Transcndental equation

Root of an equation: A number α is called a root of an equation $f(x) = 0$ if $f(\alpha) = 0$. We also say that α is a zero of the function.

Graphically the root of an equation are the points where the graph $y = f(x)$ cuts the x-axis.

Methods to find the roots of an equation $f(x) = 0$

1.Direct methods 2.Iterative methods

1. Direct methods : We know the solution of the polynomial equations such as linear equation $ax + b = 0$, and quadratic equation $ax^2 + bx + c = 0$,using direct methods or analytical methods. Analytical methods for the solution of cubic and quadratic equations are also available.But we are unable to find roots of higher order (from fourth order) algebraic equations,and also transcendental euations.So,in such cases we go for Numerical methods i.e Iterative methods.

2. Iterative methods: The following are some iteravative methods to find an approximate root of an equation

(1) **Bisection Method**

(2) **Regula- Falsi Method**

(3) **Newton Raphson method**

Intermediate value theorem: If *f* is a real-valued continuous function on the interval [*a*, *b*], and *u* is a number between *f*(*a*) and *f*(*b*), then there is a *c* ∈ [*a*, *b*] such that *f*(*c*) = *u*.

Bisection method or Half-interval method:

Bisection method is a simple iteration method to find an approximate root of an equation. Suppose that given equation of the form is $f(x) = 0$.

In this method first we choose two points $x_{0,}x_1$ such that $f(x_0)$ and $f(x_1)$ will have opposite signs (i.e $f(x_0).f(x_1) < 0$) then the root lies in interval (x_0, x_1). Now we bisect this interval at x_2 ,if $f(x_2) = 0$ then x_2 is a root of an eqution otherwise the root lies in (x_0, x_2) or(x_2, x_1) accordingly $f(x_0).f(x_2) < 0$ and $f(x_2).f(x_1) < 0$.

Assume that $f(x_0).f(x_2) < 0$ then the root lies in interval (x_0, x_2) , now we bisect this interval at x_3, ,if $f(x_3) = 0$ then x_3 is a root of an eqution otherwise the root lies in (x_0, x_3) or(x_3, x_2) accordingly $f(x_0).f(x_3) < 0$ and $f(x_3).f(x_2) < 0$.

We continue this procedure till the root is found to the desired accuracy.

Solved Problems

1. Using bisection method, find the negative root of $x^3 - 4x + 9 = 0$

Sol:

Given $f(x) = x^3 - 4x + 9$

$$f(-1) = -1 + 4 + 9 = 12 > 0$$
$$f(-2) = -8 + 8 + 9 = 9 > 0$$
$$f(-3) = -27 + 12 + 9 = -6 < 0$$

Since $f(-2) > 0$ and $f(-3) < 0$ therefore root lies in interval (-2,-3)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e x_2=$\frac{-2-3}{2}$= -2.5 , $f(-2.5) > 0$

Since $f(-2) > 0$ $f(-2.5) > 0$ $f(-3) < 0$ therefore root lies in (-2.5,-3)

Bisect this interval to get next approximation x_3

i.e x_3=$\frac{-2.5-3}{2}$=-2.75 , $f(-2.75) < 0$

Since $f(-2.5) > 0$ $f(-2.75) < 0$ $f(-3) < 0$ therefore root lies in (-2.5,-2.75)

Bisect this interval to get next approximation x_4

$i.e$ x_4=$\frac{-2.5-2.75}{2}$=--2.625, $f(-2.625) < 0$

Since $f(-2.5) > 0$ $f(-2.625) > 0$ $f(-2.75) < 0$ therefore root lies in (-2.625,-2.75)

Bisect this interval to get next approximation x_5

i.e x_5=$\frac{-2.625-2.75}{2}$=--2.6875 , $f(-2.6875) < 0$

Since $f(-2.625) > 0$ $f(-2.6875) > 0$ $f(-2.75) < 0$ therefore root lies in (-2.6875,-2.75)

Bisect this interval to get next approximation x_6

i.e x_6=$\frac{-2.6875-2.75}{2}$=--2.7187, $f(-2.7187) < 0$

We continue this procedure till the root is found to the desired accuracy. (stop the procedure when two successive approximations are same up to four decamal places)

2. Find a root of an equation 3x= e^x using bisection method.

Sol :

Let $f(x) = 3x - e^x$

$f(1) = 0.281718 > 0$

$f(2) = -1.389056 < 0$

Since f(1)>0 and f(2)<0 therefore root lies in interval (1,2)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e $x_2 = \frac{x_0 + x_1}{2} = 1.5$ $f(1.5) > 0$

Since f(1)>0 f(1.5)>0 f(2)<0 therefore root lies in (1.5,2)

Bisect this interval to get next approximation x_3

i.e $x_3 = \frac{1.5+2}{2}$=1.75 $f(x_3) = f(1.75) < 0$

Since f(1.5)>0 f(1.75)<0 f(2)<0 therefore root lies in (1.5,1.75)

Bisect this interval to get next approximation x_4

$i.e$ x_4=$\frac{1.5+1.75}{2}$=1.625, $f(1.625) = 1.666 > 0$

Continuing like above up to 12 iterations we get

$$x_{11} = 1.512323$$

and

$$x_{12} = 1.512208$$

Therfore we got two successive iterations same up to three decimal places

∴Approximate root= 1.512

3. Find a root of an equation x $\log_{10} x$= 1.2 using bisection method which lies between 2 and 3

Sol:

Given $f(x) = x \ log_{10} x - 1.2$

$$f(1) = -1.2 < 0$$

$$f(2) = -0.59 < 0$$

$$f(3) = 0.23 > 0$$

Since $f(2) > 0$ and $f(3) < 0$ therefore root lies in interval (2,3)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e $x_2=\frac{2+3}{2}$ =2.5

$Here\ f(2.5) < 0$

Since $f(2) < 0$ $f(2.5) < 0$ $f(3) > 0$ therefore root lies in (2.5,3)

Bisect this interval to get next approximation x_3

i.e$x_3=\frac{2.5+3}{2}$=2.75 $Here\ f(x_3) = f(2.75) > 0$

Continuing like above ,we get $x_9 = 2.7453$ x_{10}=2.7406

$\therefore Approximate\ root = 2.74$

4. Find a root of an equation x = cosx using bisection method.

Sol:

$Given\ fx) = x - cos(x)$

$f(0) = 0 - cos0 = -1 < 0$

$f(1) = 1 - cos1 = 0.4597 > 0$

then one root must be lies between in $(0\ ,1)$

Here $f(1)$value is near to zero so

$f(0.9) = 0.2784 > 0$

$f(0.8) = 0.1033 > 0$

$f(0.7) = -0.0648 < 0$

Since $f(0.7) < 0$ and $f(0.8) > 0$ therefore root lies in interval (0.7,0.8)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e , $x_2 = \frac{x_0+x_1}{2} = \frac{0.7+0.8}{2} = 0.75$ $f(0.75) = 0.0183 > 0$

Since $f(0.7) < 0$ $f(0.75) > 0$ $f(0.8) > 0$ therefore root lies in (0.7,0.75)

Bisect this interval to get next approximation x_3

$$i.e\ x_3 = \frac{x_2 + x_0}{2} = \frac{0.7 + 0.75}{2} = 0.725 \qquad f(0.725) = -0.0235 < 0$$

Since $f(0.7) < 0$ $f(0.725) < 0$ $f(0.75) > 0$ therefore root lies in (0.725,0.75)

Bisect this interval to get next approximationx_4

$$i.e\ x_4 = \frac{x_2 + x_3}{2} = \frac{0.725 + 0.75}{2} = 0.7375 \qquad f(0.7375) = -0.0027 < 0$$

Since $f(0.725) < 0$ $f(0.7375) < 0$ f(0.75)>0 therefore root lies in (0.7375,0.75)

Bisect this interval to get next approximation x_5

$$i.e\ x_5 = \frac{x_2 + x_4}{2} = \frac{0.7375 + 0.75}{2} = 0.7425 \qquad f(0.7425) = 0.0057 > 0$$

We continue this procedure till the root is found to the desired accuracy. (stop the procedure when two successive approximations are same up to four decamal places)

$The\ required\ approximate\ root = 0.7392.$

5. Find a root of an equation $3x = cosx + 1$ using bisection method.

Sol:

Given $f(x) = 3x - \cos x - 1$

$$f(0) = -2 < 0$$
$$f(1) = 1.4597 > 0$$
$$f(0.5) = -0.3776 < 0$$

Since $f(0.5) < 0$ and $f(1) > 0$ therefore root lies in interval (0.5,1)= (x_0, x_1)

Bisect this interval to get next approximation x_2

i.e $x_2 = \frac{x_0 + x_1}{2} = \frac{0.5+1}{2} = 0.75 \qquad f(0.75) = 0.5183 > 0$

Since $f(0.5) < 0$ $f(0.75) > 0$ $f(1) > 0$ therefore root lies in (0.5,0.75)

Bisect this interval to get next approximation x_3

$i.e\ x_3 = \frac{x_2 + x_0}{2} = \frac{0.5+0.75}{2} = 0.625 \qquad f(0.625) = 0.06403 > 0$

Since $f(0.5) < 0$ $f(0.625) > 0$ $f(0.75) > 0$ therefore root lies in (0.5,0.625)

Bisect this interval to get next approximation x_4 $i.e\ x_4 = \frac{x_0 + x_3}{2} = \frac{0.5+0.625}{2} = 0.5625 \qquad f(0.5625) = -0.1584 < 0$

Since $f(0.5) < 0$ $f(0.5625) < 0$ $f(0.625) > 0$ therefore root lies in (0.5625,0.625)

Bisect this interval to get next approximation x_5

$i.e\ x_5 = \frac{x_3 + x_4}{2} = \frac{0.5625+0.625}{2} = 0.59375 \qquad f(0.59375) = -0.0475 < 0$

We continue this procedure till the root is found to the desired accuracy.

(stop the procedure when two successive approximations are same up to four decamal places)

$So\ the\ required\ approximate\ root = 0.6074.$

6. Find the real root of the equation $x^3 - 5x + 1 = 0$ by bisection method.

Sol: given that $f(x) = \boldsymbol{x^3 - 5x + 1 = 0}$

$f(0) = 1 > 0$,

$f(1) = -3 < 0$

Hence the root lies between 0 and 1

Let the initial approximation be $x_0 = \frac{0+1}{2} = 0.5$

$f(0.5) = -1.375 < 0$

since $f(0) > 0$ and $f(0.5) < 0$

therefore the root lies between 0 and 0.5

The second approximation $x_1 = \frac{0+0.5}{2} = 0.25$

$f(0.25) = -0.234 < 0$

since $f(0) > 0$ $f(0.25) < 0$ $f(0.5) < 0$

therefore the root lies between 0 and 0.25

the third approximation $x_2 = \frac{0+0.25}{2} = 0.125$

Now $f(0.125) = 0.3749 > 0$

$f(0) > 0$ $f(0.125) > 0$ $f(0.25) < 0$

therefore the root lies between 0 and 0.125

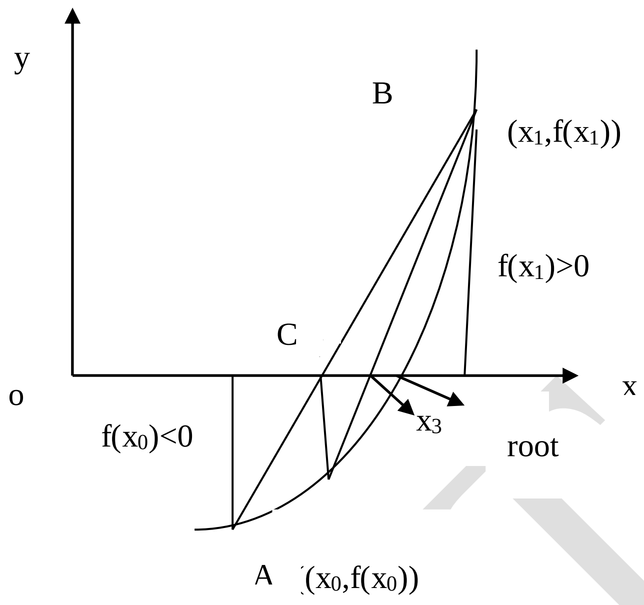
continue this procedure till the desired occurucy is obtained.

False Position Method (Regula – Falsi Method)

Using False position method we find the approximate root of the given equation $f(x)=0$ in in this method first we choose two initial approximate values x_0 *and* x_1 such that $f(x_0)$ and $f(x_1)$ will have opposite signs i.e $f(x_0).f(x_1)$ <0. Therefore the root lies in interaval (x_0, x_1)

Here two cases occur (i) $f(x_0)<0$, $f(x_1)>0$ (ii) $f(x_0)>0$, $f(x_1)<0$

Figure Of Case (I)



Let $A=\left(x_0, f\left(x_0\right)\right)$ *and* $B=\left(x_1, f\left(x_1\right)\right)$ be the points on the curve $y=f\left(x\right)$ Then the equation to the chord AB is $\frac{y-f(x_0)}{x-x_0}=\frac{f(x_1)-f(x_0)}{x_1-x_0}------(1)$

At the point C where the line AB crosses the x – axis, where $f(x)=0\ ie,\ y=0$ substitute $y=0$ in equation (1), then we get

$$x=x_0-\frac{x_1-x_0}{f\left(x_1\right)-f\left(x_0\right)}f\left(x_0\right)\rightarrow(2)$$

x is given by (2) serves as an approximated value of the root, when the interval in which it lies is small. If the new value of x is taken as x_2 then (2) becomes

$$x_2=x_0-\frac{\left(x_1-x_0\right)}{f\left(x_1\right)-f\left(x_0\right)}f\left(x_0\right)$$

$$=\frac{x_0 f\left(x_1\right)-x_1 f\left(x_0\right)}{f\left(x_1\right)-f\left(x_0\right)}\ \text{-------------(3)}$$

Now we decide whether the root lies between

$x_0\ \ and\ \ x_2\ \ \left(or\right)x_2\ \ and\ \ x_1$

In the above graph clearly $f(x_2)<0$

Therefore root lies between x_1 and x_2

We name that interval as $\left(x_1,x_2\right)$

The next approximation is given by $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

This will in general, be nearest to the exact root. We continue this procedure till the root is found to the desired accuracy.

The iteration process based on (3) is known as the method of false position

The successive intervals where the root lies, in the above procedure are named as

$(x_0, x_1), (x_1, x_2), (x_2, x_3)$ etc

Where $x_i < x_{i+1}$ and $f(x_0), f(x_{i+1})$ are of opposite signs.

Also $x_{i+1} = \dfrac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$

Case(ii) $f(x_0)>0,\ f(x_1)<0$

Repeate same procedure as case(i).

Solved Problems

1. By using Regula - Falsi method, find an approximate root of the equation $x^4 - x - 10 = 0$ that lies between 1.8 and 2. Carry out three approximations

Sol :

Let $f(x) = x^4 - x - 10$ and $x_0 = 1.8, x_1 = 2$

Then $f(x_0) = f(1.8) = -1.3 < 0$ and $f(x_1) = f(2) = 4 > 0$

Since $f(x_0)$ and $f(x_1)$ are of opposite signs, therefore the equation $f(x) = 0$ has a root between x_0 *and* x_1

The next approximation to the root is

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.8 - \frac{2 - 1.8}{4 + 1.3} \times (-1.3)$$

$$= 1.849$$

We find that $f(x_2) = -0.161$, since $f(x_2)$ *and* $f(x_1)$ are of opposite signs. Hence the root lies between x_2 *and* x_1 and the next approximation of the root is

$$x_3 = x_2 - \left[\frac{x_1 - x_2}{f(x_1) - f(x_2)}\right].f(x_2)$$

$$= 1.8490 - \left[\frac{2 - 1.849}{0.159}\right] \times (-0.159)$$

$$= 1.8548$$

We find that $f(x_3) = f(1.8548)$

$$= -0.019$$

Since $f(x_3)$ *and* $f(x_2)$ are of the same sign. Hence, the root does not lie between x_2 *and* x_3.But $f(x_3)$ *and* $f(x_1)$ are of opposite signs. So the root lies between x_3 *and* x_1 and the next approximation to the root is $x_4 = x_3 - \left[\frac{x_1 - x_3}{f(x_1) - f(x_3)}\right] f(x_3)$

$= 1.8548 - \frac{2-1.8548}{4+0.019} \times (-0.019) = 1.8557$

$$x_3 \approx x_4$$

The approximate value of x = 1.8557

2. Find an approximate root of the equation $f(x) = logx - cosx$ by using Regula-Falsi method.

Sol : Given equation is $f(x) = logx - cosx$

$f(1) = log1 - cos1 = -0.5403 < 0$

$f(2) = log2 - cos2 = 1.1093 > 0$

Since $f(1) < 0$ and $f(2) > 0$ Therefore the root lies in interval $(1,2) = (x_0, x_1)$

Since $f(x_0) = -0.5403 < 0$ and $f(x_1) = 1.1093 > 0$

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 1.3275$$

$f(x_2) = f(1.3275) = 0.04239 > 0$

Since $f(x_0) = -0.5403 < 0$, $f(x_2) = 0.04239 > 0, f(x_1) = 1.1093 > 0$

Therefore the root lies in interval $(x_0, x_2) = (1,1.3275)$

The next approximation is

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = 1.3035$$

Continue the procedure until the successive approximations are same up to four decimal places

3. Find an approximate root of the equation $f(x) = e^x sinx - 1 = 0$ by using Regula-Falsi method.

Sol: Given equation is $\boldsymbol{f(x) = e^x sinx - 1 = 0}$

$$f(0) = -1 < 0$$

$$f(1) = 1.2873 > 0$$

Since $f(0) < 0$ and $f(1) > 0$

Therefore the root lies in interval $(0,1) = (x_0, x_1)$

$f(x_0) = -1 < 0$ and $f(x_1) = 1.2873 > 0$

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 0.4372$$

$f(x_2) = f(0.4372) = -0.3444 < 0$

$f(x_1) = 1.2873 > 0$ and $f(x_2) = -0.3444 < 0$

Therefore the root lies in interval $(0.4372,1) = (x_1, x_2)$

The next approximation is

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = 0.556$$

Continue the procedure until the successive approximations are same up to four decimal places

4. Find an approximate root of the equation $f(x) = 2x - \log_{10} x - 7 = 0$ by using Regula-Falsi method.

Sol: Given equation is $\boldsymbol{f(x) = 2x - log_{10}^{x} - 7 = 0}$

$$f(1) = -5 < 0$$

$$f(2) = -3.3010 < 0$$

$$f(3) = -1.4771 < 0$$

$$f(4) = 0.3979 > 0$$

Since $f(3) < 0$ and $f(4) > 0$

Therefore the root lies in interval $(3,4) = (x_0, x_1)$

$f(x_0) = -1.4771 < 0$ and $f(x_1) = 0.3979 > 0$

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 3.7878$$

$f(x_2) = -0.0028 < 0$

$f(x_1) = 0.3979 > 0$ and $f(x_2) = -0.0228 < 0$

Therefore the root lies in interval $(3.7878,4) = (x_2, x_1)$

The next approximation is

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)} = 3.7893$$

Continue the procedure until the successive approximations are same up to four decimal places

5. Find a root of an equation 3x= e^x using False position method.

Sol : Let $f(x) = 3x - e^x$

Then $f(0) = -1,\ f(0.1) = -0.8, \ldots$
$f(0.6) = -0.0221192 < 0,\ f(0.7) = 0.086247 > 0$

Since $f(0.6).\ f(0.7) < 0$ and these values are near to zero

Therefore the root lies in the interval (0.6,0.7)= (x_0, x_1)

By False position method

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 3.7878$$

$$= \frac{0.6 f(0.7) - 0.7 f(0.6)}{f(0.7) - f(0.6)}$$

=0.620451

Since $f(x_0)$<0 $f(x_2)$ = f(0.620451)=0.001587>0 $f(x_1)$>0

Therefore the root lies in the interval (0.6, 0.620451)= (x_0, x_2)

The next approximation to the root is given by

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

$$=\frac{0.6f(0.620451)-0.620451f(0.6)}{f(0.620451)-f(0.6)}$$

=0.619083

f (0.619083)=0.000025>0

∴The Approximate root is 0.6190

6.**Find the root of $x\log_{10}x - 1.2 = 0$ using Regula falsi method.**

Sol:

$$f(x) = x\log_{10}x - 1.2$$

Here

$$f(2) = -0.59 < 0,$$

$$f(3) = 0.23 > 0$$

Since $f(2) < 0$ and $f(3) > 0$ the root lies in the interval (2,3) = (x_0, x_1)

The next approximation to the root is given by

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$=\frac{2f(3)-3f(2)}{f(3)-f(2)}$$

=2.7195

Since $f(x_0)$<0 $f(x_2)$ = f(2.7195)=-0.0184<0 $f(x_1)$>0

Therefore the root lies in the interval(2.7195,3)= (x_2, x_1)

The next approximation to the root is given by

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$$= \frac{2.7195\ f(3)-3f(2.7195)}{f(3)-f(2.7195)}$$

=2.7403

$f(2.7403) = -0.000302 < 0$

Clearly $f(2.7403)$ is nearly equal to zero up to 3 decimal places

∴The Approximate root is 2.740

Newton Raphson Method:

The Newton- Raphson method is a powerful and eligant method to find the root of an equation. This method is generally used to improve the results obtained by the previous methods.

Let x_0 be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root which implies that $f(x_1)=0$. We use Taylor's theorem and expand $f(x_1)= f(x_0+h)=0$

$$\Rightarrow f(x_0)+hf^1(x_0)=0$$

$$\Rightarrow h = -\frac{f(x_0)}{f^1(x_0)}$$

Substituting this in x_1 ,we get

$$x_1 \quad = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$\therefore x_1$ is a better approximation than x_0

Successive approximations are given by

$x_2, x_3 \ldots\ldots\ldots x_{n+1}$ where $x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$

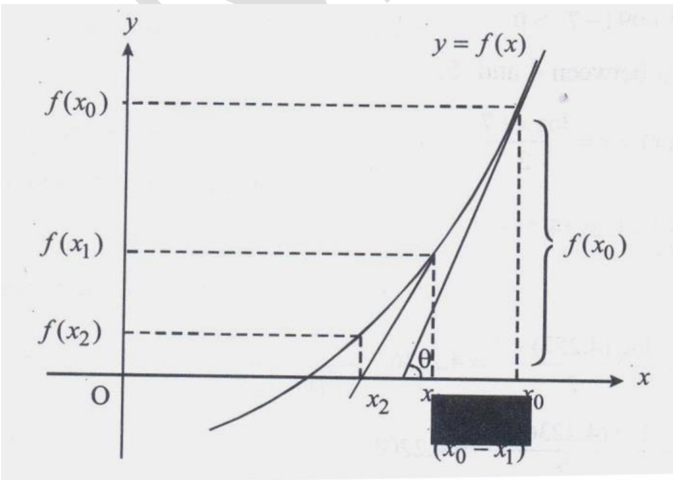
Geometrical Interpretation

From below diagram tanθ=$\frac{opp}{adj}$=$\frac{f(x0)}{xo-x1}$.................(1)

But slope =tanθ $= f^I(x_0)$ ....(2)

From (1) and (2) we have

$$x_1 \quad = x_0 - \frac{f(x_0)}{f^1(x_0)}$$



Solved Problems

1.Find a root of an equation x+$\log_{10} x$ = 2 using Newton raphson method.

Sol:

Given $\boldsymbol{f(x) = x + log_{10} x - 2}$**,** $f'(x) = 1 + \frac{\log_{10} e}{x}$

Here

$$\boldsymbol{f}(1) = -1 < 0$$

$$f(2) = 0.301 > 0$$

Since $\boldsymbol{f}(1) < 0$ and $f(2) > 0$ the root lies in the interval (1,2)

Here $f(2)$is near to zero

Choose $x_0 = 1.8$ and $f(1.8) = 0.0553 > 0$

By Newton Raphson method,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \quad \text{for } i = 0,1,2 \ldots..$$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$=1.8 $-\frac{0.0555}{1.2412}$ =1.7552

Now $f(x_1) = f(1.7552) = -0.00013$ and $f'(x_1) = f'(1.7552) = 1.2473$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$ =1.7555

Now $f(1.7555) = -0.00000012$

Hence approximate root is **1.7555(coorect to 4 decimal places)**

2.Using Newton – Raphson method

a) Find square root of a number

b) Find reciprocal of a number

Sol: a) <u>**Square root:-**</u>

Let $f(x) = x^2 - N = 0$, where N is the number whose square root is to be found. The solution to $f(x)$ is then $x = \sqrt{N}$

Here $f'(x) = 2x$

By Newton-Raphson technique

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} = x_i - \frac{x_i^2 - N}{2x_i}$$

$$\Rightarrow x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

Using the above iteration formula the square root of any number N can be found to any desired accuracy. For example, we will find the square root of $N = 24$.

Let the initial approximation be $x_0 = 4.8$

$$x_1 = \frac{1}{2}\left(4.8 + \frac{24}{4.8}\right) = \frac{1}{2}\left(\frac{23.04+24}{4.8}\right) = \frac{47.04}{9.6} = 4.9$$

$$x_2 = \frac{1}{2}\left(4.9 + \frac{24}{4.9}\right) = \frac{1}{2}\left(\frac{24.01+24}{4.9}\right) = \frac{48.01}{9.8} = 4.898$$

$$x_3 = \frac{1}{2}\left(4.898 + \frac{24}{4.898}\right) = \frac{1}{2}\left(\frac{23.9904+24}{4.898}\right) = \frac{47.9904}{9.796} = 4.898$$

Since, $x_2 \approx x_3$ there fore the solution to $f(x) = x^2 - 24 = 0$ is 4.898. That means, the square root of 24 is 4.898

b) Reciprocal:-

$\therefore$ The reciprocal of Let $f(x) = \frac{1}{x} - N = 0$ where N is the number whose reciprocal is to be found

The solution to $f(x)$ is then $= \frac{1}{N}$. Also, $f^1(x) = \frac{-1}{x^2}$

To find the solution for $f(x) = 0$, apply Newton – Raphson method

$$x_{i+1} = x_i - \frac{\left(\frac{1}{x_i} - N\right)}{-1/x_i^2} = x_i(2 - x_iN)$$

For example, the calculation of reciprocal of 22 is as follows

Assume the initial approximation be $x_0 = 0.045$

$$\therefore x_1 = 0.045(2 - 0.045 \times 22)$$
$$= 0.045(2 - 0.99)$$
$$= 0.0454(1.01) = 0.0454$$
$$x_2 = 0.0454(2 - 0.0454 \times 22)$$
$$= 0.0454(2 - 0.9988)$$
$$= 0.0454(1.0012) = 0.04545$$
$$x_3 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(1.0001) = 0.04545$$

$$x_4 = 0.04545(2 - 0.04545 \times 22)$$
$$= 0.04545(2 - 0.99998)$$

$= 0.04545(1.00002)$

$= 0.0454509$

∴Reciprocal of 22 is 0.04545

3.Find by Newton's method, the real root of the equation $xe^x = cosx$ correct to three decimal places.

Sol: Let $cosx - xe^x = f(x)$

Then $f(0)=1>0$, $f(0.5)=0.053>0$ $f(0.6)=-0.267<0$

So root of $f(x)$ lies between 0.5 and 0.6

Here $f(0.5)$ value is near to zero

So we take $x_0 = 0.5$ and $f^1(x) = -\sin x - (x+1)e^x$

∴ By Newton Raphson method,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \qquad \text{for } i = 0,1,2 \ldots.$$

First approximation is given by

$$x_1 = x_0 - \frac{f(x_0)}{f^1(x_0)}$$

$$= 0.5 - \frac{0.53222}{-2.952507} = 0.68026$$

The second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f^1(x_1)}$$

$$= 0.68026 - \frac{0.56569}{-3.946485}$$

$$= 0.536920$$

∴ Continue like above we have $x_3 =$ 0.51809 $x_4 =$ 0.517757

Approximate Root = 0.517

4. Find a root of an equation e^x sinx = 1 using Newton Raphson method

Sol : $f(x) =$ **e^x sinx -1,** $f'(x) = (cosx + sinx)e^x$

$f'(x) = (cosx + sinx)e^x$

$$f(0) = -1 < 0$$

$$f(0.1) = -0.8 < 0$$

$$f(0.5) = -0.209561 < 0$$

$$f(0.6) = 0.028846 > 0$$

Since $f(0.5) < 0$ and $f(0.6) > 0$ the root lies in the interval(0.5,0.6)

but $f(0.6)$ value is near to zero

So choose $x_0 = 0.6$

By applying **Newton Raphson method**,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \quad \text{for i=0,1,2.....}$$

First approximation $x_1 = x_0 - \dfrac{f(x_0)}{f^1(x_0)}$

$$= 0.6 - \frac{0.028846}{2.532705} = 0.58861$$

The second approximation $x_2 = x_1 - \dfrac{f(x_1)}{f^1(x_1)}$

$$= 0.588611 - \frac{0.000196}{2.498513}$$

$$= 0.588533$$

∴Approximate root is 0.588

5. Using Newton – Raphson method

a) Derive formula for cube root of a number

b) Find cube root of 15.

Sol: Let f(x) = $x^3 = N$ where N is the real number whose root to be found.

Solution to f(x) is then $x^3 = N \quad f'(x) = 3x^2$

Newton Raphson formula to find $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x^3 - N}{3x^2}$

Here $f(2) = -7 < 0$ and $f(2.5) = 0.625 > 0$

so one root lies between (2,2.5)

take initial approx value is $x_0 = 2$

using Newton Raphson formula $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

$$x_1 = 2 - \frac{(2)^3 - 15}{3(2)^2} = 2.58333$$

$$x_2 = 2.58333 - \frac{(2.58333)^3 - 15}{3(2.58333)^2} = 2.47144$$

$$x_3 = 2.47144 - \frac{(2.47144)^3 - 15}{3(2.47144)^2} = 2.46622$$

$$x_4 = 2.46622 - \frac{(2.46622)^3 - 15}{3(2.46622)^2} = 2.46621$$

$\therefore\ x_3 \cong x_4 = 2.466221$(upto 4 decimal places) is the required approximate root.

6.Find a real root of the equation 3 x = cos x + 1 Using Newton Raphson method.

Sol :

$f(x) = 3x - \cos x - 1$, $f^{'}(\mathrm{x}) = 3 + \sin x$

$$f(0) = -2 < 0$$

$$f(1) = 1.4597 > 0$$

$\therefore$ The root lies between 0 and 1.

Let $x_0 = \frac{0+1}{2} = 0.5$

using Newton Raphson formula,we have

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)} \quad \text{for } i = 0,1,2 \ldots..$$

$$f(x_0) = f(0.5) = -0.3776$$

$$f'(x_0) = (f'(0.5) = 3.47942$$

First approximate root $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ $=0.5 - \frac{-0.3776}{3.47942} = 0.6085$

$$f(0.6085) = 0.0049$$

$$f'(0.6085) = 3.57164$$

Second approximation is $x_2 = x_1 - \frac{f(x_1)}{f^{'}(x_1)}$ $=$ $0.6085 - \frac{0.0049}{3.57164}$=0.60713

Similarly we can find other approcimations $x_3, x_4, ---$ to the root

7. Find the root between 0 and 1 of the equation $x^3 - 6x + 4 = 0$ correct to five decimal places.

Sol: Let $f(x) = x^3 - 6x + 4$

$$f(0) = 4 > 0 \ and\ f(1) = -1 < 0$$

therefore the root lies between 0 and 1.

here the root is nearer to 1.

So, let x_0=1

$f'(x) = 3x^2 - 6$, $f'(1) = -3$

The first approximation to the required root is

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{2}{3}$ =0.66666

Second approximation is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.73015$$

Third approximation is given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.73204$$

Fourth approximation is given by

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.73205$$

The root is 0.73205 correct to five decimal places

Order Of Convergence

The deviation from the approximate root with actual root is called Error.

Error at n th ,(n+1) iterations are

$$e_n = x_n - \alpha \;; \qquad e_{n+1} = x_{n+1} - \alpha$$

If $e_{n+1} \leq k\; e_n{}^p$ then the method is said to be of order 'p'.

Note:

1.The method converges very fast if 'k' is very very small and 'p' is large.

2.Regula falsi and iteration methods converge Linearly.

1.Show Bisection method converges Linearly.

Sol: Choose initial approximations a, b such that f(a).f(b)<0

And let first approximation be x_1

Distance between a and x_1 $=x_1$-a $=\frac{a+b}{2}$-a$=\frac{b-a}{2}$

Distance between b and x_1 $=b - x_1 = b - \frac{a+b}{2} = \frac{b-a}{2}$

Here say Root α lies between a and x_1 or b and x_1

$|x_1-\alpha| \leq \frac{b-a}{2}$

After n iterations, we get

$|x_n-\alpha| \leq \frac{b-a}{2^n}$

$|x_{n+1}-\alpha| \leq \frac{1}{2}\frac{b-a}{2^n}$

$e_{n+1} \leq \frac{1}{2} e_n^1$ $\quad \therefore$ *Bisection method converges linearly*

2.Show Newton Raphson method converges Quadratically

Sol: Let $x_r$be the actual root and x_i, $x_{i+1}$are ith,(i+1)th iterations in NRM.Then

$$x_{i+1} = x_i - \frac{f(x_i)}{f^1(x_i)}$$

$$x_{i+1}\ f^1(x_i) = x_i\ f^1(x_i)\text{-f}(x_i)$$

$f(x_i)$= $f^1(x_i)(x_i - x_{i+1})$……………(1)

Taylor's theorem around $x=x_r$

Is given by $f(x_r)=f(x_i+h$

$$= f(x_i)+ (xr - xi)\ f^1(x_i) + \frac{(xr-xi)^2}{2} f^{II}(x_i) + \cdots \text{....(2)}$$

Neglecting higher order terms and sub (1) in (2) ,we get

$$0=f^1(x_i)(x_r\text{-} x_{i+1})+(xr - xi)\ f^1(x_i)+\frac{(xr-xi)^2}{2} f^{II}(x_i)$$

Solving

e_{i+1}=-1/2 $\left(\frac{f^{II}(x_i)}{f^I(X_i)}\right)e_i^2$ Where p=2 and k==-1/2 $\left(\frac{f^{II}(x_i)}{f^I(X_i)}\right)$

INTERPOLATION

Introduction:

If we consider the statement $y = f(x); x_0 \le x \le x_n$ we understand that we can find the value of y, corresponding to every value of x in the range $x_0 \le x \le x_n$. If the function $f(x)$ is single valued and continuous and is known explicitly then the values of $f(x)$ for certain values of x like $x_0, x_1, ...x_n$ can be calculated. The problem now is if we are given the set of tabular values

x :	x_0	x_1	x_2	....	x_n
y :	y_0	y_1	y_2	....	y_n

Satisfying the relation $y = f(x)$ and the explicit definition of $f(x)$ is not known, it is possible to find a simple function say $\phi(x)$ such that $f(x)$ and $\phi(x)$ agree at the set of tabulated points. This process to finding $\phi(x)$ is called interpolation. If $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called interpolating polynomial. In our study we are concerned with polynomial interpolation

OR

Let $x_0, x_1 ----x_n$ be the values x and $y_0, y_1, y_2, ---, y_n$ be the values of y and $y = f(x)$ be a unknown function .The process to find the value of the unknown function $y = f(x)$ when the given value of x and the value of x lies within the limits $x_0 \, to \, x_n$ is called interpolation

Extrapolation:

Let $x_0, x_1 ----x_n$ be the values x and $y_0, y_1, y_2, ---, y_n$ be the values of y and y=f(x) be a unknown function .The process to find the value of the unknown function y=f(x) when the given value of x and the value of x lies outside the range of $x_0 \, to \, x_n$ is called Extrapolation

Note: If the differences of x values are equal in the given data then it is called equal spaced points otherwise it is called unequal spaced points

Note:

i) Suppose a given value of x is nearer to starting value of x then we use Newton's forward interpolation formula.

ii) Suppose a given value of x is nearer to ending value of x then we use Newton's backward interpolation formula.

iii) Suppose a given value of x is nearer to middle value of x then we use Gauss interpolation formula.

iv) Suppose the given data has unequal spaced points then we use Lagrange's interpolation formula

Finite Differences:

Finite differences play a fundamental role in the study of differential calculus, which is an essential part of numerical applied mathematics, the following are the finite differences.

1. Forward Differences 2. Backward Differences 3. Central Differences

1.Forward Differences: The Forward Difference operator is denoted by Δ , The forward differences are usually arranged in tabular columns as shown in the following table called a Forward difference table

Values of x	Values of y	First differences	Second differences	Third differences	Fourth differences
x_o	y_0				
		$\Delta y_0 = y_1 - y_0$			
x_1	y_1		$\Delta^2 y_0 = \Delta y_1 - y_0$		
		$\Delta y_1 = y_2 - y_1$		$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$	
x_2	y_2		$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$		$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
		$\Delta y_2 = y_3 - y_2$		$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	
x_3	y_3		$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$		
x_4	y_4	$\Delta y_3 = y_4 - y_3$			

2. Backward Differences: The Backward Difference operator is denoted by ∇ and the backward difference table is

x	y	∇y	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

3.Central Difference Table: The central difference operator is denoted by δ and the central Difference table is

x	Y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_4$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Symbolic Relations and Separation of symbols:

We will define more operators and symbols in addition to Δ, ∇ and δ already defined and establish difference formulae by Symbolic methods

Definition:- The averaging operator μ is defined by the equation $\mu y_r = \frac{1}{2}[y_{r+1/2} + y_{r-1/2}]$

Definition:- The shift operator E is defined by the equation $Ey_r = y_{r+1}$. This shows that the effect of E is to shift the functional value y_r to the next higher value y_{r+1}. A second operation with E gives $E^2 y_r = E(Ey_r) = E(y_{r+1}) = y_{r+2}$

Generalizing $E^n y^r = y_{r+n}$

Definition:-

Inverse operator E^{-1} is defined as $E^{-1}y_r = y_{r-1}$

In general $E^{-n}y_n = y_{r-n}$

Definition :-

The operator D is defined as $Dy(x) = \frac{d}{dx}[y(x)]$

Relationship Between operators:

i) Relation between Δ *and* E

Proof: We have $\Delta y_0 = y_1 - y_0$

$$= Ey_0 - y_0 = (E-1)y_0$$

$$\Longrightarrow \Delta \cong E - 1 (or) E = 1 + \Delta$$

ii) $\nabla \equiv 1 - E^{-1}$

Pf: We have $\nabla y_1 = y_1 - y_0$

$$\nabla y_1 = y_1 - E^{-1}y_1$$

$$\nabla y_1 = (1 - E^{-1})y_1$$

$$\nabla \equiv 1 - E^{-1}$$

iii) $\delta \equiv E^{1/2} - E^{-1/2}$

Pf : We have $\delta y_{\frac{1}{2}} = y_1 - y_0$

$$= E^{\frac{1}{2}}y_{\frac{1}{2}} - E^{-\frac{1}{2}}y_{\frac{1}{2}}$$

$$\delta y_{\frac{1}{2}} = (E^{\frac{1}{2}} - E^{-\frac{1}{2}})y_{\frac{1}{2}}$$

$$\delta \equiv E^{1/2} - E^{-1/2}$$

iv) $\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)$

Pf: we have $\mu y_r = \frac{1}{2}(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}})$

$$\mu y_r = \frac{1}{2}(E^{\frac{1}{2}}y_r + E^{-\frac{1}{2}}y_r)$$

$$\mu y_r = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})y_r$$

$$\mu = \frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})$$

v) $\mu^2 \equiv 1 + \frac{1}{4}\delta^2$

Pf: L.H.S = $\mu^2 = \left[\frac{1}{2}(E^{\frac{1}{2}} + E^{-\frac{1}{2}})\right]^2$

$$= \frac{1}{4}(E + E^{-1} + 2)$$

$$= \frac{1}{4}\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + 4 \quad \right]$$

$$= \frac{1}{4}(\delta^2 + 4) = \text{R.H.S}$$

vi). Prove that $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

Pf: Let R.H.S= $\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

$$= \frac{1}{2}\delta[\delta + 2\sqrt{1 + \frac{1}{4}\delta^2}\ ]$$

$$= \frac{1}{2}\delta[\delta + \sqrt{4 + \delta^2}\ ]$$

$$= \frac{1}{2}\delta[(E^{\frac{1}{2}} - E^{-\frac{1}{2}}) + \sqrt{4 + (E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2}$$

$$= \frac{1}{2}\delta[(E^{\frac{1}{2}} - E^{-\frac{1}{2}}) + \sqrt{(E^{\frac{1}{2}} + E^{-\frac{1}{2}})^2}$$

$$= \frac{1}{2}\delta\left[\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) + (E^{\frac{1}{2}} + E^{-\frac{1}{2}})\right.$$

$$= \frac{1}{2}\delta.2.E^{\frac{1}{2}}$$

$$= \delta.E^{\frac{1}{2}}$$

$$= (E^{\frac{1}{2}} - E^{-\frac{1}{2}}).E^{\frac{1}{2}}$$

$= E - 1 = \Delta$ =R.H.S.

vii) Relation between the Operators D and E

Using Taylor's series we have, $y(x+h) = y(x) + hy^1(x) + \frac{h^2}{2!}y^{11}(x) + \frac{h^3}{3!}y^{111}(x) + ----$

This can be written in symbolic form

$$Ey_x = \left[1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + ----\right] y_x = e^{hD}.y_x$$

$$E = e^{hd}$$

❖ If $f(x)$ is a polynomial of degree n and the values of x are equally spaced then $\Delta^n f(x)$ is a constant

Note:

1. As $\Delta^n f(x)$ is a constant, it follows that $\Delta^{n+1} f(x) = 0, \Delta^{n+2} f(x) = 0, ...$
2. The converse of above result is also true. That is, if $\Delta^n f(x)$ is tabulated at equal spaced intervals and is a constant, then the function $f(x)$ is a polynomial of degree n
3. $\Delta^2 \mathrm{f(x)} = \Delta(\Delta(\mathrm{f(x)})$

Solved Problems :

1.Evaluate

$(i)\Delta \cos x$

$(ii)\Delta^2 \sin(px+q)$

$(iii)\Delta^n e^{ax+b}$

(iv). If the interval of difference is unity then prove that

$$\Delta[x(x+1)(x+2)(x+3) = 4(x+1(x+2)(x+3))$$

Sol: Let h be the interval of differencing

$(i)\Delta \cos x = \cos(x+h) - \cos x$

$$= -2\sin\left(x + \frac{h}{2}\right)\sin\frac{\mathrm{h}}{2}$$

$(ii)\Delta \sin(px+q) = \sin[p(x+h)+q] - \sin(px+q)$

$$= 2\cos\left(px + q + \frac{ph}{2}\right)\sin\frac{ph}{2}$$

$$= 2\sin\frac{ph}{2}\sin\left(\frac{\pi}{2} + px + q + \frac{ph}{2}\right)$$

$$\Delta^2 \sin(\mathrm{px+q}) = 2\sin\tfrac{\mathrm{ph}}{2}\Delta\left[\sin\left[\mathrm{px+q}+\tfrac{1}{2}(\pi+\mathrm{ph})\right]\right]$$

$$= \left[2\sin\frac{ph}{2}\right]^2 \sin\left[px + q + \frac{1}{2}(\pi + ph)\right]$$

$(iii)\ \Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}$

$$= e^{(ax+b)}(e^{ah-1})$$

$$\Delta^2 e^{ax+b} = \Delta[\Delta(e^{ax+b})] = \Delta[(e^{ah}-1)(e^{ax+b})]$$
$$= (e^{ah}-1)^2\Delta(e^{ax+b})$$
$$= (e^{ah}-1)^2 e^{ax+b}$$

Proceeding on, we get $\Delta^n\left(e^{ax+b}\right) = \left(e^{ah}-1\right)^n e^{ax+b}$

iv) Let $f(x) = x(x+1)(x+2)(x+3)$

given $h = 1$

we know that $\Delta f(x) = f(x+h) - f(x)$

$$\Delta[x(x+1)(x+2)(x+3)] = (x+1)(x+2)(x+3)(x+4))$$
$$-x(x+1)(x+2)(x+3)$$
$$= (x+1)(x+2)(x+3)[x+4-x]$$
$$= \mathbf{4}(x+1)(x+2)(x+3)$$

2.Find the missing term in the following data

x	0	1	2	3	4
y	1	3	9	-	81

Why this value is not equal to 3^3 . Explain

Sol: Consider $\Delta^4 y_0 = 0$

$\Rightarrow y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$

Substitute given values, we get

$81 - 4y_3 + 54 - 12 + 1 = 0 \Rightarrow y_3 = 31$

From the given data we can conclude that the given function is $y = 3^x$. To find y_3, we have to assume that y is a polynomial function, which is not so. Thus we are not getting $y = 3^3 = 27$

Equally Spaced : If the differences of x values are equal in the given data then it is called equal spaced points otherwise it is called unequal spaced points

Newton's Forward Interpolation Formula: Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), ---x_n, y_n)$ of x and y. It is required to find a polynomial of nth degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0 + ih$ (i = 0, 1, 2…..n) then the **Newton's forward interpolation formula** is given by

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + --$$
$$+ \frac{p(p-1)(p-2)----(p-(n-1))}{n!}\Delta^n y_0$$

where $p = \frac{x-x_0}{h}$

Note : this formula is used when value of x is located near beginning of tabular values

Solved Problems :

1.Find the melting point of the alloy containing 54% of lead, using appropriate interpolation formula

Percentage of lead(p)	50	60	70	80
Temperature $(Q°c)$	205	225	248	274

Sol: The difference table is

x	y	Δ	Δ^2	Δ^3
50	205			
		20		
60	225		3	
		23		0
70	248		3	
		26		
80	274			

Let temperature = $f(x)$

We have $x = 54, x_0 = 50, h = 10 \quad p = \frac{x-x_0}{h} = 0.4$

By Newton's forward interpolation formula

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ----$$

$$f(54) = 205 + 0.4(20) + \frac{0.4(0.4-1)}{2!}(3) + \frac{(0.4)(0.4-1)(0.4-2)}{3!}(0)$$

=205+8-0.36 =212.64. Melting point = 212.64

2. The population of a town in the decimal census was given below. Estimate the population for the 1895

Year x	1891	1901	1911	1921	1931
Populationin thousands	46	66	81	93	101

Sol: The forward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

$$46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{6} - (-5) + \frac{(0.4-1)0.4(0.4-2)}{6}(2)$$

given $h = 10, x_0 = 1891, x = 1985$ then $p = 2/5 = 0.4$

By Newton's forward interpolation formula

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p+1)}{2!}\Delta^2 y_0 + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_0 + --$$

$$f(1895) = +\frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

=54.45 thousands

3. Find y (1.6) using Newton's Forward difference formula from the table

x	**1**	**1.4**	**1.8**	**2.2**
y	**3.49**	**4.82**	**5.96**	**6.5**

Sol: The difference table is

X	y	Δy	Δ^2y	Δ^3y
1	3.49			
1.4	4.82	1.33	-0.81	
1.8	5.96	1.14	-0.60	-1.41
2.2	6.5	0.54		

Let $= 1.6$, x_0=1, h=1.4 -1=0.4, $p = \frac{x-x_0}{h} = \frac{3}{2}$

Using Newton's forward difference formula, we have

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p+1)}{2!}\Delta^2 y_0 + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_0 + ----$$

$$f\,(1.6) = 3.49 + 3/2(1.33) + \frac{\frac{3}{2}\frac{5}{2}}{2}(-0.81) + \frac{\frac{3}{2}\frac{57}{22}}{6}(-1.41)$$

$$= 4.9656$$

4.Find the cubic polynomial which takes the following values

X	**0**	**1**	**2**	**3**
Y=f(x)	**1**	**2**	**1**	**10**

Hence evaluate f(4).

Sol: The forward difference table is given by

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	**1**			
		1		
1	**2**		-2	
		-1		12
2	**1**		10	
		9		
3	**10**			

$P = \dfrac{x-0}{h} = x \;\; ; h = 1$

Using newton's forward interpolation formula, we get

$$Y = y_0 + \frac{x}{1}\Delta y_0 + \frac{x(x-1)}{1.2}\Delta^2 y_0 + \frac{x(x-1)(x-2)}{1.2.3}\Delta^3 y_0$$

$$= 1 + x(1) + \frac{x(x-1)}{2}(-2) + \frac{x(x-1)(x-2)}{6}(12)$$

$$= 2x^3 - 7x^2 + 6x + 1$$

Which is the required polynomial.

To compute f(4), we take $x_n=3$, x=4

So that $p = \dfrac{x-x_n}{h} = 1$

Using Newton's backward interpolation formula , we get

$$Y_4 = y_3 + p\nabla y_3 + \frac{p(p+1)}{1.2}\nabla^2 y_3 + \frac{p(p+1)(p+2)}{1.2.3}\nabla^3 y_3$$

$$= 10+9+10+12$$

= 41

Which is the same value as that obtained by substituting x=4 in the cubic polynomial $2x^3-7x^2+6x+1$.

Newton's Backward Interpolation Formula: Given the set of $(n+1)$ values $(x_0, y_0), (x_1, y_1), ---x_n, y_n)$ of x and y. It is required to find a polynomial of n^{th} degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0+ih$ (i = 0, 1, 2…..n) then the **Newton's backward interpolation formula** is given by

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + --- + \frac{p(p+1)----[p+(n-1)]}{n!}\nabla^n y_0$$

Where $p = \dfrac{x - x_n}{h}$

Note : This formula is used when value of x is located near end of tabular values

Solved Problems :

1. The population of a town in the decimal census was given below. Estimate the population for the 1925

Year x	**1891**	**1901**	**1911**	**1921**	**1931**
Population in thousands	**46**	**66**	**81**	**93**	**101**

Sol: The backward difference table is

x	y	∇	∇^2	∇^3	∇^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

given $h = 10, x_n = 1931, x = 1925$ then $p = \frac{x-x_n}{h} = \frac{1925-1931}{10} = -0.6$

By Newton's backward interpolation formula

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + - + \frac{p(p+1)\ldots.[p+(n-1)]}{n!}\nabla^n y_0$$

$$\therefore f(1925) = 101 + (-0.6)(8) + \frac{(-0.6)(0.4)}{2}(-4)$$

$$+\frac{(-0.6)(0.4)(1.4)}{6}(-1) + \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-3)$$

= 96.21

2.Find $y(42)$ from the following data. Using Newton's interpolation formula

x	**20**	**25**	**30**	**35**	**40**	**45**
y	**354**	**332**	**291**	**260**	**231**	**204**

Sol: since x=42 is located near end of the tabular values therefore we use NBIF

the backward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
20	354					
		-22				
25	332		-19			
				29		
		-41				
30	291		10		-37	
				-8		
		-31				45
35	260		2		8	
				0		
		-29				
40	231		2			
		-27				
45	204					

Given $x = 42$ and $x_n = 45$, $h = 5$, then $p = \frac{x-x_0}{h} = -0.6$

We know that NBIF

$$y_n(x) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n$$

$$y(42) = 204 + (-0.6)(-27) + \frac{(-0.6)(-0.6+1)}{2}(2)+0+\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(8) + \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{120}(45)$$

=234.44

Central Difference Interpolation: The middle part of the forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	y_{-4}					
		Δy_{-4}	$\Delta^2 y_{-4}$			
x_{-3}	y_{-3}					
		Δy_{-3}	$\Delta^2 y_{-3}$	$\Delta^3 y_{-4}$	$\Delta^4 y_{-4}$	$\Delta^5 y_{-4}$
x_{-2}	y_{-2}					
		Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$	$\Delta^4 y_{-3}$	$\Delta^5 y_{-3}$
x_{-1}	y_{-1}					
		Δy_{-1}	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-2}$
x_0	y_0					
		Δy_0	$\Delta^2 y_0$	$\Delta^3 y_{-1}$	$\Delta^4 y_{-1}$	$\Delta^5 y_{-1}$
x_1	y_1					
		Δy_1	$\Delta^2 y_1$	$\Delta^3 y_0$	$\Delta^4 y_0$	
x_2	y_2					
		Δy_2	$\Delta^2 y_2$	$\Delta^3 y_1$		
x_3	y_3					
		Δy_3				
x_4	y_4					

1.Gauss's forward Interpolation Formula: Given the set of $(n+1)$ values $(x_0,y_0),(x_1,y_1),---x_n,y_n)$ of x and y. It is required to find a polynomial of n^th^ degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0+ih$ (i = 0, 1, 2…..n) then the **Gauus Forward interpolation formula** is given by

$$\boldsymbol{y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2} + ---}$$

Where $p = \frac{x-x_0}{h}$

Note:- We observe from the difference table that

$\Delta y_0 = \delta y_{1/2}, \Delta^2 y_{-1} = \delta^2 y_0, \Delta^3 y_{-1} = \delta^3 y_{1/2}, \Delta^4 y_{-2} = \delta^4 y_0$ and so on. Accordingly the formula (4) can be rewritten in the notation of central diff

$$y_p = [y_0 + p\delta y_{1/2} + \frac{p(p-1)}{2!}\delta^2 y_0 + \frac{(p+1)p(p-1)}{3!}\delta^3 y_{1/2} + \frac{(p+1)(p-1)p(p-2)}{4!}\delta^4 y_0 + ----]$$

2.Gauss's Backward Interpolation formula: Given the set of $(n+1)$ values $(x_0,y_0),(x_1,y_1),---x_n,y_n)$ of x and y. It is required to find a polynomial of n^th^ degree $y_n(x)$ such that y and $y_n(x)$ agree at the tabular points with x's equidistant (i.e.) $x_i = x_0+ih$ (i = 0, 1, 2…..n) then the **Gauus Backward interpolation formula** is given by

$$y = y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p(p+1)(p-1)(p+2)}{4!}\Delta^4 y_{-2} + \ldots$$

Note: Gauss forward and Backward formulae used when x is located middle of the tabular values

Solved Problems :

1.**Use Gauss Forward interpolation formula to find $f(3.3)$ from the following table**

x	1	2	3	4	5
$y = f(x)$	15.30	15.10	15.00	14.50	14.00

Sol: the difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1 x_{-2}	15.3 y_{-2}				
		-0.2			
2 x_{-1}	15.1 y_{-1}		0.1		
		-0.1		-0.5	
3 x_0	15.0 y_0		-0.4 $\Delta^2 y_{-1}$		0.9$\Delta^4 y_{-2}$
		-0.5 Δy_0		0.4$\Delta^3 y_{-1}$	
4 x_1	14.5 y_1		0.0		
		-0.5			
5 x_2	14.0 y_2				

Given x=3.3 , x_0=3 ,h=1 hence $p = \frac{x-x_0}{h} = 0.3$

We know that Gauss forward interpolation formula is

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \rightarrow (4)$$

$$=15+(0.3)(0.5)+\frac{(0.3)(0.3-1)}{2}(-0.4) +$$

$$\frac{(0.3)(0.09-1)}{6}(0.4)+\frac{(0.3)(0.09-1)(0.3-2)}{24}(0.9)=14.9$$

2. **Find f (2.5) using following Table**

x	**1**	**2**	**3**	**4**
y	**1**	**8**	**27**	**64**

Sol: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	1			
		7		
2	8		12	
		19		6
3	27		18	
		37		
4	64			

h = 1

$$P = \frac{X - X_0}{h} = \frac{2.5-2}{1} = 0.5$$

Using Gauss Forward interpolation formula,

$$= 8 + (0.5)19 + \frac{(0.5)(-0.5)}{2}(12) + \frac{(0.5-1)(0.5)(1.5+1)}{6}(6)$$

= 15.625

3. Use Gauss forward interpolation formulae to find f(3.3) from the following

x	**1**	**2**	**3**	**4**	**5**
y	**15.30**	**15.10**	**15.00**	**14.50**	**14.00**

Sol:

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
1	15.30				
		-0.20			
2	15.10		0.10		
		-0.10		-0.50	
3	15.00		-0.40		0.90
		-0.50		0.40	
4	14.50		0.00		
		-0.50			
5	14.00				

$$P = \frac{3.3-3}{1} = 0.3$$

$$= 15 + (0.3)(-0.5) + \frac{(0.3)(-0.4)(-0.7)}{2} + (0.3)(0.4)\frac{(-0.7)(1.3)}{6}$$

$$+ \frac{(0.3)(-0.7)(1.3)(-1.3)}{24}(-0.9) = 14.8604925 = 14.9$$

4. Find f(2.36) from the following table

x:	**1.6**	**1.8**	**2.0**	**2.2**	**2.4**	**2.6**
y:	**4.95**	**6.05**	**7.39**	**9.03**	**11.02**	**13.46**

Sol:

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.6	4.95					
		1.1				
1.8	6.05		0.24			
		1.34		0.06		
2.0	7.39		0.3		-0.01	
		1.64		0.05		0.06
2.2 x_0	9.03 y_0		0.35		0.05	
		1.99		0.1		
2.4	11.02		0.45			
		2.44				
2.6	13.46					

here we have $x = 2.36$, $x_0 = 2.2$, $h = 0.2$, $p = \frac{x-x_0}{h} = 0.8$

$$y_p = [y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(p+1)(p-1)p(p-2)}{4!}(\Delta^4 y_{-2}) + ----] \to (4)$$

Substituting all above values in the formula then

$f(2.36) = 9.03 + (0.8)(1.99) +$

$$\frac{(0.8)(0.8-1)}{2}(0.35) + \frac{(0.8+1)(0.8)(0.8-1)}{6}(0.1) + \frac{(0.8+1)(0.8)(0.8-1)(0.8-2)}{24}(0.05)$$

=10.02

5. Find f(22) from the following table using Gauss forward formula

x	20	25	30	35	40	45
y	354	332	291	260	231	204

Sol : the middle part of the difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5

$20x_0$	$354y_0$					
		-22				
25	332		-19			
		-41		29		
30	291		10		-37	
				-8		
		-31				45
35	260		2		8	
		-29				
				0		
40	231		2			
		-27				
45	204					

Given $x = 22$ and $x_0 = 20$, $h = 5$, then $p = \frac{x-x_0}{h} = 0.4$

The Gauss forward formula is

$y = y_0 + p\Delta y_0$

$= 354 + (0.4)(-22)$

= 345.2

6. Find by Gauss's Backward interpolating formula the value of y at x=1936, using the following table.

x	1901	1911	1921	1931	1941	1951
y	12	15	20	27	39	52

Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1901 x_{-3}	12 y_{-3}					
		3				
1911 x_{-2}	15 y_{-2}		2			
		5		0		
1921 x_{-1}	20 y_{-1}		2		3	
		7 Δy_{-1}		$3\Delta^3 y_{-2}$		$-10\Delta^5 y_{-3}$
1931 x_0	27 y_0		5 $\Delta^2 y_{-1}$		$7\Delta^4 y_{-2}$	
		12		-4		
1941 x_1	39 y_1		1			
		13				
1951 x_2	52 y_2					

Given x=1936 and let x_0 =1931 and h=10 then $p = \frac{x-x_0}{h} = 0.5$

By Gauss backward interpolation formula we have

$$y = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + ---$$

$$=27+(0.5)(7)+\frac{0.5)(0.5+1)}{2}(5) + \frac{(0.5)(1.5)(-0.5)}{6}(3) + \frac{(0.5)(1.5)(-0.5)(-1.5)}{24}(-7) +$$

$$\frac{(0.5)(1.5)(-0.5)(-1.5)(2.5)}{120}(-10)$$

=32.345

7. Using Gauss back ward difference formula, find y(8) from the following table

x	0	5	10	15	20	25
y	7	11	14	18	24	32

Solution: Solution: The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0 x_{-2}	7 y_{-2}					
		4				
5 x_{-1}	11 y_{-1}		-1			
		3		2		
10 x_0	14 y_0		1		-1	
		4		1		0
15 x_1	18 y_1		2		-1	
		6		0		
20 x_2	24 y_2		2			
		8				
25 x_3	32 y_3					

Given x=8 and let x_0 =10 and h=5 then $p = \frac{x-x_0}{h} = -0.4$

By Gauss backward interpolation formula we have

$$y = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + -$$

$$14+(0.4)(3)+\frac{(-0.4)(-0.4+1)}{2}(1) + \frac{(-0.4)(-0.4+1)(-0.4-1)}{6}(2) + \frac{(-0.4)(-0.4+1)(-0.4-1)(-0.4-2)}{24}(-1)=12.704$$

Lagrange's Interpolation Formula:

Let $f(x)$ be continuous and differentiable $(n+1)$ times in the interval (a,b). Given the $(n+1)$ points as$(x_0, y_0), (x_1, y_1), (x_2, y_2)\ldots.. (x_n, y_n)$ where values of x not necessarily be equally spaced then the interpolating polynomial of degree 'n' say $f(x)$ is given by

$$f(x)=\frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2).........(x_0-x_n)}f(x_0)+\frac{(x-x_0)(x-x_2).....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)........(x-x_n)}{(x_2-x_0)(x_2-x_1).........(x_2-x_n)}f(x_2)+\ \ldots\ldots.+\frac{(x-x_0)(x-x_1)........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1).........(x_n-x_{n-1})}f(x_n)$$

Note : This formula is used when values of x are unequally spaced and equally spaced

Solved Problems:

1. **Using Lagrange formula, calculate $f(3)$ from the following table**

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	19

Sol: Given $x_0=0, x_1=1, x_2=2, x_3=4, x_4=5, x_5=6$

$$f(x_0)=1, f(x_1)=14, f(x_2)=15, f(x_3)=5, f(x_4)=6, f(x_5)=19$$

From Lagrange's interpolation formula

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}f(x_2)$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}f(x_5)$$

Here $x=3$ then

$$f(3)=\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}\times1+$$

$$\frac{(3-0)(3-2)(3-4)(3-5)(3-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}\times14+$$

$$\frac{(3-0)(3-1)(3-4)(3-5)(3-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}\times15+$$

$$\frac{(3-0)(3-1)(3-2)(3-5)(3-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}\times 5+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)}\times 6+$$

$$\frac{(3-0)(3-1)(3-2)(3-4)(3-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}\times 19$$

$$=\frac{12}{240}-\frac{18}{60}\times 14+\frac{36}{48}\times 15+\frac{36}{48}\times 5-\frac{18}{60}\times 6+\frac{12}{40}\times 19$$

$$=0.05-4.2+11.25+3.75-1.8+0.95$$

$$=10$$

$$f(x_3)=10$$

2. Find $f(3.5)$ using Lagrange method of 2^{nd} and 3^{rd} order degree polynomials.

x	1	2	3	4
$f(x)$	1	2	9	28

Sol: By Lagrange's interpolation formula For $n=4$,we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)+$$

$$\therefore f(3.5)=\frac{(3.5-2)(3.5-3)(3.5-4)}{(1-2)(1-3)(1-4)}(1)+\frac{(3.5-1)(3.5-3)(3.5-4)}{(2-1)(2-3)(2-4)}(2)+$$

$$\frac{(3.5-1)(3.5-2)(3.5-4)}{(3-1)(3-2)(3-4)}(9)+\frac{(3.5-1)(3.5-2)(3.5-3)}{(4-1)(4-2)(4-3)}(28)$$

=0.0625+(-0.625)+8.4375+8.75

=16.625

$$Now\ \ f(x)=\frac{(x-2)(x-3)(x-4)}{-6}(1)+\frac{(x-1)(x-3)(x-4)}{2}(2)$$

$$++\frac{(x-1)(x-2)(x-4)}{(-2)}(9)+\frac{(x-1)(x-2)(x-3)}{6}(28)$$

$$= \frac{(x^2-5x+6)(x-4)}{-6} + (x^2-4x+3)(x-4) + \frac{(x^2-3x+2)(x-4)}{-2}(9) + \frac{(x^2-3x+2)(x-3)}{6}(28)$$

$$= \frac{x^3-9x^2+26x-24}{-6} + x^3-8x^2+19x-12 + \frac{x^3-7x^2+14x-8}{-2}(9) + \frac{x^3-6x^2+11x-6}{6}(28)$$

x	0	2	3	6
Y=f(x)	-4	2	14	158

$$= \frac{\left[-x^3+9x^2-26x+24+6x^3-48x^2+114x-72-27x^3+189x^2-378x+216+308x+28x^3-168x^2-168\right]}{6}$$

$$= \frac{6x^3-18x^2+18x}{6} \Rightarrow f(x) = x^3-3x^2+3x$$

$$\therefore f(3.5) = (3.5)^3 - 3(3.5)^2 + 3(3.5) = 16.625$$

3. Find f (4) use Lagrange's interpolation formulae.

Sol : $f(x) = \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} y_1 + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)}$

$Y_2 + \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} y_3 + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} Y_4$

Where $x=4, x_1=0, x_2=2, x_3=3, x_4=6$

$$= \frac{(4-2)(4-3)(4-6)}{(-2)(-3)(-6)} \times (-4) + \frac{(4)(1)(-2)}{2(-1)(-4)} \times (2) + \frac{4\times2\times(-2)}{3\times1\times3(-3)} \times 14$$

$$= \frac{4(2)(1)}{6(4)(3)} \times 158$$

$$= \frac{-4}{9}(-2) + \frac{224}{9} + \frac{158}{9} = \frac{-4-18+224+158}{9}$$

$$= 40$$

4.The following are the measurements T made on curverecorded by the oscilograph representing a change of current I due to a change in condn s of anelectric current

T	1.2	2	2.5	3
I	**1.36**	**0.58**	**0.34**	**0.2**

Sol :

Since data is unequispaced,we use Lagrange's interpolation

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = \frac{(1.6-1.2)(1.6-2)(1.6-3)}{(1.6-1.2)(1.6-2)(1.6-3)}(1.36) + \frac{(1.6-1.2)(1.6-2.5)(1.6-3)}{(2-1.2)(2-2.5)(1.6-3)}(0.58)$$
$$+\frac{(1.6-1.2)(1.6-2)(1.6-3)}{(1.6-1.2)(1.6-2)(1.6-3)}(0.34) + \frac{(1.6-1.2)(1.6-2)(1.6-2.5)}{(1.6-1.2)(1.6-2)(1.6-2.5)}(0.2)$$

$=0.8947 \quad \therefore I = 0.8947$

5.Find the parabola passing through points(0,1),(1,3) and(3,55) using Lagrange's Interpolation Formula.

x	0	1	3
y	1	3	55

Sol : Given Lagrange's interpolation formula is

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1$$
$$+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$y = \frac{(x-1)(x-3)}{(0-1)(0-3)} + \frac{(x-0)(x-3)}{(1-0)(1-3)}(3)$$
$$+\frac{(x-0)(x-1)}{(3-0)(3-1)}(55)$$

$=\frac{1}{6}[48x^2 - 36x + 6]$

$=8x^2 - 6x + 1$

6.A Curve passes through the points (0,18),(1,10),(3,-18) and (6,90). Find the slope of the curve at x=2.

x	0	1	3	6
y	18	10	-18	90

Sol : Given data is

Since data is unequispaced,we use Lagrange's interpolation

$$y=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y=\frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}18+\frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}10$$
$$+\frac{(x)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18)+\frac{(x)(x-1)(x-3)}{(6)(6-1)(6-3)}90$$

$$y=\frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}18+\frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}10$$
$$+\frac{(x)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18)+\frac{(x)(x-1)(x-3)}{(6)(6-1)(6-3)}90$$

$=2x^3-10x^2+18$

$$\therefore \frac{dy}{dx}=6x^2-20x$$

$\therefore Slope\ of\ curve\ at\ x=2\ is\ 6(2)^2-20(2)=-16$

UNIT-II

NUMERICAL METHODS

Numerical Integration

Introduction :

The process of evaluating a definite integral from a set of tabulated values of the integrand $f(x)$, which is not known explicitly is called Numerical Integration.

Newton –Cote's Quadrature Formula:

We want to find Definite integral form $\int_a^b f(x)dx$, where $f(x\ )$ is unknown explicitly,then

We replace $f(x)$ with interpolating polynomial.

Here we replace with Newton Forward Interpolation formula

Divide the interval (a, b) into n sub intervals of width h so that

$a\ = x_0 < x_1 = x_0 + h \ldots\ldots\ldots\ldots < x_n = x_n + h = b$ Then

$$y_n(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots\ldots + \frac{p(p-1)(p-2)\ldots\ldots(p-(n-1))}{n!}\Delta^n y_0$$

Where $p = \frac{x-x_0}{h}$ $\quad hdp = dx \quad a\,t\; x = x_0 \Rightarrow \quad p = 0\ and\ x = x_n \Rightarrow\ p = n$

$$\therefore \int_a^b f(x)dx = \int_{x_0}^{x_n} y_n(x)\,\mathrm{dx} = h\int_{x_0}^{x_n}\left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots\right)dp$$

$$= h\int_0^n\left(y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \ldots\ldots\ldots\right)dp$$

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n}{12}(2\mathrm{n}-3)\Delta^2 y_0 + \frac{n}{24}(\mathrm{n}-2)^2\Delta^3 y_0 + \ldots\right]$$

This is Newton Cotes Quadrature Formula.

Derive Trapezoidal Rule for numerical integration of $\int_a^b f(x)\,dx$

I.Trapezoidal Rule

Sub n=1 in Newton Cotes Quadrature formula and taking the curve $y = f(x)$ passing through $(x_0, y_0)and\ (x_1, y_1)$ as a straight line so that differences of order higher than first become zero(i.e., $\Delta^2, \Delta^3 etc\ become\ zero$) (n=number of intervals)

$\int_{x_0}^{x_1} f(x)dx = h[y_0 + \frac{1}{2}\Delta y_0] = \frac{h}{2}[y_0 + y_1]$……………………….(i)

Similarly we get

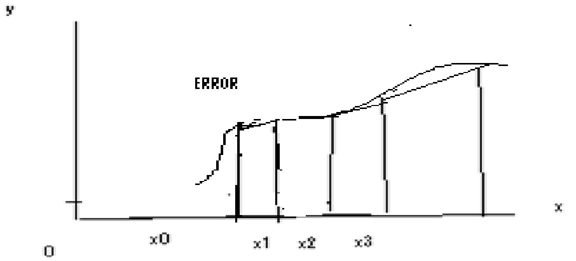
$\int_{x_1}^{x_2} f(x)dx = \frac{h}{2}[y_1 + y_2]$ …………………………………..(ii)

Adding above we get

$\int_{x_0}^{x_n} ydx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + ---- + y_{n-1})]$

$$\int_{x_0}^{x_n} ydx = \frac{h}{2}[(sum\ of\ the\ Ist\ \&\ last\ oridinates) + 2(sum\ of\ the\ remaining\ ord.)]$$

Geometrical interpretation of Trapezoidal Rule:



Here trapezoidal rule denotes sum of areas of above trapeziums.

Derive Simpson's 1/3 Rule for numerical integration of $\int_a^b f(x)dx$

II. Simpson's 1/3 Rule (n=2)

sub n=2 in Newton Cotes Quadrature_Formula and taking the curve $y = f(x)$ passing through $(x_0, y_0), (x_1, y_1)$ *and* (x_2, y_2) as a parabola so that differences of order higher than second become zero(i.e., $\Delta^3, \Delta^4 etc\ become\ zero$)

$\int_{x_0}^{x_2} f(x)dx = 2h[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0]$

We know E = 1+Δ

Δ

then $\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}[y_0 + 4y_1 + y_2]$

Similarly $\int_{x_2}^{x_4} f(x)\,dx = 2h[y_2 + 4y_3 + y_4]$

----

and so on $\int_{x_{n-2}}^{x_n} f(x)dx = \frac{h}{3}[y_{n-2} + 4y_{n-1} + y_n]$

Adding

$\int_{x_0}^{x_n} ydx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_2 + .. + y_{n-1}) + 2(y_2 + y_4 + .. + y_{n-2})]$

$$\int_{x_0}^{x_n} ydx = \frac{h}{3}[(sum\ of\ the\ first\ and\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates)$$

$$+ 2(sum\ of\ the\ remaining\ even\ ordinates)]$$

This is known as Simpson's $^1/_3$ Rule (or) Simply Simpson's Rule. .

III. Simpson's 3 / 8 Rule

$\int_{x_0}^{x_n} ydx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + .. + y_{n-1}) + 2(y_3 + y_6 + .. + y_{n-3})]$

Note: -

1. Trapezoidal Rule is applicable for any number of subintervals

2. Simpson's $^1/_3$ rule is applicable when the number of subintervals must be even

3. Simpson's $^3/_8$ rule is applicable when the number of subintervals must be multiple of 3

Compare Trapezoidal Rule and Simpson's 1/3 rule

In trapezoidal rule we take n=1(no of subintervals) between every two points we are taking a straight line(LINEAR) where as in simpsons rule n=2 means we are taking a parabola so error is less compare to trapezoidalrule.

Solved Problems

1. Evaluate $\int_0^{\pi} \frac{sinx}{x}$dx by using trapezoidal and simpson's 1/3 rules taking n=6

Sol: $h = \frac{b-a}{n} = \frac{\pi}{6}$

Here $\frac{\sin 0}{0}$=1 since $\lim_{x\to 0} \frac{sinx}{x}$=1

x	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
sinx	0	0.5	0.866	1	0.866	0.5	0
Sinx/x	1	0.9549	0.8270	0.6366	0.4135	0.1910	0

i) Trapezoidal rule :

$\int_0^1 \frac{1}{1+x}$dx = $\frac{h}{2}$ [(sum of first and last ordinates) + 2(sum of the remaining ordinates)]

= $\frac{\pi}{12}$ [(1+0) +2(0.827+0.4135+0.9549+0.6366+0.1910)] = 1.8446

ii) Simpson's 1/3 rule:

$\int_0^1 \frac{1}{1+x}$dx= $\frac{h}{3}[(sum\ of\ the\ Ist\ \&\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates) + 2(sum\ of\ the\ remaining\ even\ ordinates)]$

= $\frac{\pi}{18}$[(1+0)+2(0.827+0.4135)+4(0.9549+0.6366+0.1910)] = 1.852

2. Evaluate $\int_0^1 \frac{1}{1+x}dx$ by using trapezoidal , simpson's 1/3, Simpson's 3/8 rules.

Sol: We want to use above 3 rules so take $n = 6$

x	0	1/6	2/6	3/6	4/6	5/6	6/6
$y=\frac{1}{1+x}$	1 y_0	0.8571 y_1	0.75 y_2	0.6666 y_3	0.6 y_4	0.5454 y_5	0.5 y_6

$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

i) Trapezoidal rule :

$\int_0^1 \frac{1}{1+x}dx = \frac{h}{2}$ [(sum of first and last ordinates) + 2(sum of the remaining ordinates)]

$=\frac{1}{2}$ [(1+0.5) +2(0.8571+0.5454+0.75+0.6+0.6666)] = 0.69485

ii) Simpson's 1/3rd rule:

$\int_0^1 \frac{1}{1+x}dx= \frac{h}{3}[(sum\ of\ the\ first\ and\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates) +$

$2(sum\ of\ the\ remaining\ even\ ordinates)]$

$= \frac{1}{18}$ [(1+0.5)+2(0.75+0.6)+4(0.8571+0.6666+0.5454)] = 0.6931

iii) Simpson's 3/8th rule:

$\int_0^1 \frac{1}{1+x}dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + .. + y_{n-1}) + 2(y_3 + y_6 + .. + y_{n-3})]$

$=\frac{1}{16}$[(1+0.5)+2(0.6666)+3(0.8571+0.75+0.6+0.5454)] = 0.6932

3. Evaluate $\int_4^{5.2} \log x\, dx$ by using trapezoidal , simpson's 1/3,Simpsons 3/8 rules from

x	**4**	**4.2**	**4.4**	**4.6**	**4.8**	**5**	**5.2**
logx	**1.3863** y_0	**1.4351** y_1	**1.4816** y_2	**1.5261** y_3	**1.5686** y_4	**1.6094** y_5	**1.6487** y_6

Sol: Here h=4.2-4=0.2

i) **Trapezoidal rule :**

$\int_4^{5.2} \log x\, dx = \frac{h}{2}$[(sum of first and last ordinates) + 2(sum of the remaining ordinates)]

$=\frac{0.2}{2}$[(1.3863+1.6487)+2(1.4351+1.4816+1.5261+1.5686+1.6094)] = 1.8277

ii) Simpson's 1/3 rule:

$\int_0^1 \frac{1}{1+x}$dx=$\frac{h}{3}$$[(sum\ of\ the\ first\ and\ last\ oridinates) + 4(sum\ of\ the\ odd\ ordinates) + 2(sum\ of\ the\ remaining\ even\ ordinates)]$

=$\frac{0.2}{3}$[(1.3863+1.6487)+2(1.4816+1.5686)+4(1.4351+1.5261+1.6094)] =1.8279

iii) Simpson's 3/8 rule:

$\int_0^1 \frac{1}{1+x}$dx= $\frac{3h}{8}$$[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + .. + y_{n-1}) + 2(y_3 + y_6 + .. + y_{n-3})]$

=$\frac{0.6}{18}$[(1.3863+1.6487)+2(1.5261)+3(1.4351+1.4816+1.5686+1.6094)] = 1.8278

4. The velocity v (m/sec) of a particle at distance S(m) from a point on its path given by following table

S	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38

Estimate the time taken to travel 60 meters by Simpsons 1/3 and 3/8 rules.

Sol : Let $v = \frac{dv}{dt}$ be the velocity of particle at any time 't'

Then $dt = \frac{ds}{v}$ Integrating on both sides with limits 0 to 60

Then $t = \int_0^{60} \frac{1}{v}$ ds

S	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38
1/v	0.0212	0.0172	0.0156	0.0153	0.0163	0.0192	0.0263

i) Simpson's $\frac{1}{3}$rule:

$$\int_0^{60} \frac{1}{v}\ ds = \frac{h}{3}\begin{bmatrix} (sum\ of\ the\ first\ and\ last\ oridinates) \\ +4(sum\ of\ the\ odd\ ordinates) \\ +2(sum\ of\ the\ remaining\ even\ ordinates) \end{bmatrix}$$

=$\frac{10}{3}$[(0.0212+0.0263)+2(0.0156+0.0163)+4(0.0172+0.0153+0.0192)]= **1.0603 sec**

ii) Simpson's $\frac{3}{8}$ rule:

$t = \int_0^{60} \frac{1}{v}$ ds= $\frac{3h}{8}$$[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + . + y_{n-1}) + +2(y_3 + y_6 + . + y_{n-3})]$

=$\frac{30}{8}$[(0.0212+0.0263)+2(0.0153)+3(0.0172+0.0163+0.0192)]= **0.8857 sec**

5.Evaluate $\int_0^{\pi/2} e^{sinx}$dx correct to four decimals places by Simpsons 3/8rule

Sol: Here h=$\frac{\pi}{12}$

x	0	$\frac{\pi}{12}$	$\frac{2\pi}{12}$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
y	1	1.2954	1.6487	2.0281	2.3774	2.6272	2.718

Simpson's $\frac{3}{8}$ rule:

$t = \int_0^{60} \frac{1}{v}$ ds= $\frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 +. + y_{n-1}) + +2(y_3 + y_6 + . + y_{n-3})]$

= $\frac{3\pi}{96}$[(1+2.718)+2(2.0281)+3(1.2954+1.6487+2.3774+2.6272)=**3.1015**

6. Evaluate $\int_0^1 \frac{1}{1+x^2} dx$ using Simpson's 3/8 rule

Sol: Divide the interval into 6 sub intervals & tabulate the values **of $f(x_i) = \frac{1}{1+x^2}$** as follows

x_i	0	1/6	2/6	3/6	4/6	5/6	6/6
$f(x_i)$	1	0.9729	0.90	0.80	0.69231	0.59016	0.5

Here $h = 1/6$

Using Simpson's rule

$$I = \int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8}[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8.6}[(1.0 + 0.50) + 3(0.9729 + 0.90 + 0.69231 + 0.59016) + 2(0.80)]$$

$$= \frac{1}{16}(12.5662) = 0.785395 \cong 0.7854$$

7. **Find the value of $\int_0^1 \frac{1}{1+x^2} dx$, taking 5 sub internals & by using Trapezoidal rule.**

Sol:

$$f(x) = \frac{1}{1+x^2}, n = 5, a = 0, b = 1$$

$$\therefore h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

Construct a table of values of $x_i \& y_i = f(x_i)$ as follows

x_i	0.0	0.2	0.4	0.6	0.8	1.0
y_i	1.00	0.961538	0.832069	0.735294	0.609755	0.50

Using Trapezoidal rule we get

$$I=\int_0^1 \frac{1}{1+x^2}dx=\frac{0.2}{2}\left[(1.0+0.50)+2(0.961538+0.832069+0.735294+0.609759)\right]$$

$$=0.783734$$

8. Find the area bounded by the curve $f(x)=y$ and x-axis from $x=7.47\ to\ x=7.52$

x_i	7.47	7.48	7.49	7.50	7.51	7.52
y_i	1.93	1.95	1.98	2.01	2.03	2.06

Sol : Here h = 0.01

Area formed by the curve $y=f(x)$ and x – axis from $x=7.47\ to\ x=7.52$ is

$$Area=\int_{7.47}^{7.52} f(x)dx$$

Applying Trapezoidal rule we get

$$Area=\int_{7.47}^{7.52} f(x)dx=\frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$=\frac{0.01}{2}\left[(1.93+2.06)+2(1.95+1.98+2.01+2.03)\right]$$

$$=0.0996$$

9.Find $\int_0^1 x^3dx$ with 5 sub intervals by Trapezoidal rule

Sol: Here $a=0, b=1, n=5\ \&\ y=f(x)=x^3$

$$\therefore h=\frac{b-a}{n}=\frac{1-0}{5}=0.2$$

The values of x & y are tabulated below

x	0.2	0.4	0.6	0.8	1
y	0.008	0.064	0.216	0.512	1

By Trapezoidal rule

$$\int_0^1 x^3dx=\frac{h}{2}[(y_0+y_4)+2(y_1+y_2+\ y_3)]$$

$$=\frac{0.2}{2}[(0.008+1)+2(0.064+0.216+0.512)]$$

$$=0.2592\cong 0.26$$

10. Evaluate $\int_0^{\pi} t\sin t\,dt$ using Trapezoidal rule

Sol: Divide the interval $(0,\pi)$ in to 6 parts each of width $h=\frac{\pi}{6}$

The values of $f(t)=t\sin t$ are given below

t	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
$f(t)=y$	0 y_0	0.2618 y_1	0.9069 y_2	1.5708 y_3	1.8138 y_4	1.309 y_5	0 y_6

By Trapezoidal rule

$$\int_0^{\pi} t\sin t\,dt=\frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$=\frac{\pi}{12}\left[(0+0)+2(0.2618+0.9069+1.5708+1.8138+1.309)\right]$$

$$=\frac{\pi}{12}(11.7246)$$

$$=3.0695\cong 3.07$$

11. Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's 1/3 rule. Hence obtain approx. value of $log_e 2$

Sol: Divide the interval (1,2) in to 8(**even**) parts each of width $h=0.125$

x	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
$y=\frac{1}{x}$	1 y_0	0.8888 y_1	0.8 y_2	0.7272 y_3	0.6666 y_4	0.6153 y_5	0.5714 y_6	0.5333 y_7	0.5 y_8

By Simpson's 1/3 rule

$$\int_1^2 \frac{dx}{x}=\frac{h}{3}[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)]$$

$$=\frac{0.125}{3}\left[(1+0.5)+4(0.8888+0.7272+0.6153+0.5333)+2(0.8+0.6666+0.5714)\right]$$

$$=\frac{0.125}{3}[1.5+11.0584+4.076] \qquad =\frac{0.125}{3}[16.6344] \qquad =0.6931$$

By actual integration,

$$\int_1^2 \frac{dx}{x}=[\log x]_1^2 \;=\log 2-\log 1=\log 2$$

Hence $\log 2=0.6931$, correct to four decimal places

12. A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's 1/3 rule, find the velocity of the rocket at t = 80 seconds

t (sec)	0	10	20	30	40	50	60	70	80
$f\left(cm/\sec^2\right)$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Sol: We know that the rate of velocity is acceleration I.e., $f = \frac{\partial v}{\partial t}$

$\therefore$ Velocity of the rocket at $t = 80\sec$ is given

$$v = \int_0^{80} f\,dt$$

$$= \frac{10}{3}[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69)2(33.34 + 37.75 + 43.25)]$$

$$= \frac{10}{3}[80.67 + 616.48 + 228.68] = \frac{10}{3}(925.83) = 3086.1$$

13. A river is soft wide. The depth 'd' in feet at a distance x ft from one bank is given by the table

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find approximately the area of cross-section

Sol: Here $h = 10, y_0 = 0, y_1 = 4, y_2 = 7, y_3 = 9, y_4 = 12, y_5 = 15, y_6 = 14, y_7 = 8 \,\&\, y_8 = 3$

Area of cross section = $\int_0^{80} y\,dx$

Area = $\frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$

$$= \frac{10}{3}\left[(0+3) + 4(4+9+15+8) + 2(7+12+14)\right]$$

$$= \frac{10}{3}[3 + 144 + 66]$$

$$= 710 sq.ft$$

14. Evaluate $\int_0^{\pi} \sin x\,dx$ **by dividing the interval** $(0,\pi)$ **in to 8 sub intervals & using Simpson's 1/3 rule**

Sol:- Given $a = 0, b = \pi, n = 8 \,\&\, f(x) = \sin x$

$$\therefore h = \frac{b-a}{n} = \frac{\pi - 0}{8} = \pi/8$$

Tabulate the values of $\sin x$ as follows

x_i	0	$\pi/8$	$\pi/4$	$3\pi/8$	$\pi/2$	$5\pi/8$	$6\pi/8$	$7\pi/8$	π
$\sin x_i$	0	0.38	0.71	0.92	1	0.92	0.710	0.38	0

Simpson's 1/3 rule for n = 8 is

$$I=\int_a^b f(x)\,dx=\frac{h}{3}\left[(y_0+y_8)+4(y_1+y_3+y_5+y_7)+2(y_2+y_4+y_6)\right]$$

$$=\frac{\pi}{8.3}\left[(0+0)+4(0.38+0.92+0.92+0.38)+2(0.71+1.0+0.71)\right]$$

$$=1.99$$

15. Find the area bounded by the curve $y=e^{-x^2/2}$, x axis between $x=0\,\&\,x=3$ by using Simpson's 3/8 rule

Sol: Divide the interval $(0,3)$ in to 6 sub intervals $\therefore h=\frac{3-0}{6}=0.5$

The values of $y_i=e^{-x^2/2}$ are tabulated as follows

x_i	0.0	0.5	1.0	1.5	2.0	2.5	3.0
$y(x_i)$	1.0	1.33	1.649	3.080	7.389	22.760	90.017

By Simpson's 3/8 rule we get

$$I=\int_0^1 e^{-x^2/2}\,dx=\frac{3h}{8}[(y_0+y_6)+3(y_1+\ y_2+y_4+y_5)+2y_3]$$

$$=\frac{3(0.5)}{8}[(1.00+90.017)+3(1.33+1.649+7.389+22.760)+2(3.080)]$$

$$=36.8551$$

Numerical solutions of ordinary differential equations

The important methods of solving ordinary differential equations of first order numerically are as follows

1) Taylor's series method
2) Euler's method
3) Modified Euler's method of successive approximations
4) Runge- Kutta method

To describe various numerical methods for the solution of ordinary differential equations, we consider the general 1st order differential equation.

$\frac{dy}{dx}$ = f(x,y)-------(1) with the initial condition $y(x_0)=y_0$

The methods will yield the solution in one of the two forms:

i) A series for y in terms of powers of x, from which the values of y can be obtained by direct substitution.

ii) A set of tabulated values of y corresponding to different values of x

The methods of Taylor belong to class (i)

The methods of Euler, Runge - Kutta method, belong to class (ii)

Taylor's Series Method

To find the numerical solution of the differential equation $\frac{dy}{dx} = f(x,y) \rightarrow (1)$

With the initial condition $y(x_0) = y_0 \rightarrow (2)$

$y(x)$ Can be expanded about the point x_0 in a Taylor's series in powers of $(x - x_0)$ as

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + . + \frac{(x-x_0)^n}{n!}y^n(x_0) + .. \rightarrow (3)$$

In equation (3), $y(x_0)$ is known from initial condition equation. The remaining coefficients $y'(x_0), y''(x_0), ...y^n(x_0)$ etc are obtained by successively differentiating equation (1) and evaluating at x_0. Substituting these values in equation, $y(x)$ at any point can be calculated from equation. Provided $h = x - x_0$ is small.

When $x_0 = 0$, then Taylor's series equation can be written as

$$y(x) = y(0) + x.y'(0) + \frac{x^2}{2!}y''(0) + ... + \frac{x^n}{n!}y^n(0) + ... \rightarrow (4)$$

Note: We know that the Taylor's expansion of y(x) about the point x_0 in a power of

$(x - x_0)$ is.

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots \rightarrow (1) \quad \text{Or}$$

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y' + \frac{(x-x_0)^2}{2!}y'' + \frac{(x-x_0)^3}{3!}y''' + ...$$

If we let $x - x_0 = h$. (i.e. $x = x_0 + h = x_1$) we can write the Taylor's series as

$$y(x) = y(x_1) = y_0 + \frac{h}{1!}y' + \frac{h^2}{2!}y'' + \frac{h^3}{3!}y''' + \frac{h^4}{4!}y'''' + ...$$

$$\text{i.e. } y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{(4)} + ... \rightarrow (2)$$

Similarly expanding y(x) in a Taylor's series about $x = x_1$. We will get.

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{(4)} + ... \rightarrow (3)$$

Similarly expanding y(x) in a Taylor's series about $x = x_2$ We will get.

$$y_3 = y_2 + \frac{h}{1!}y_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \frac{h^4}{4!}y_2^{(4)} + ... \rightarrow (4)$$

In general, Taylor's expansion of y(x) at a point $x= x_n$ is

$$y_{n+1} = y_n + \frac{h}{1!}y_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \frac{h^4}{4!}y_n^{('v)} + ... \text{(I)}$$

Merits and Demerits of Taylor series method:

In this method taking h very small and taking upto order h^4 terms we get less error but finding derivatives may be complicate in some of the problems

Solved Problems:

1. **Solve** $\frac{dy}{dx} = xy + 1$ **and y(0) = 1 using Taylor's series method and compute y(0.1).**

Sol: Given that $\frac{dy}{dx} - 1 = xy$ and y(0) = 1

Here $\frac{dy}{dx} = 1 + xy$ and $y_0 = 1$, $x_0 = 0$.

Differentiating repeatedly w.r.t 'x' and evaluating at $x_0 = 0$

$y'(x) = 1 + xy$, $\quad y'(0) = 1+0(1) = 1$.

$y''(x) = x.y' + y$, $\quad y''(0) = 0+1=1$

$y'''(x) = x.y'' + y' + y'$ $\quad y'''(0) = 0.(1) + 2(1) =2$

The Taylor series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + x.\, y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) \quad \text{(Neglecting higher order terms)}$$

Substituting the values of $y(0)$, $y'(0)$, $y''(0)$, ….

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(2)$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} \qquad \rightarrow (1)$$

Now put x = 0.1 in equ (1),

$$y(0.1) = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3}.$$

$= 1 + 0.1 + 0.005 + 0.000333 =$ **1.105**

2. Solve the equation $\frac{dy}{dx} = x - y^2$ with the conditions y(0) = 1 and y′ (0) =1. Find y(0.2) and y(0.4) using Taylor's series method.

Sol: Given that $y' = x - y^2, y(0) = 1$ Here $y_0 = 1$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at x=0

$$y'(x) = x - y^2, y'(0) = 0 - y(0)^2 = 0 - 1 = -1$$
$$y''(x) = 1 - 2y.y', y''(0) = 1 - 2.y(0)y'(0) = 1 - 2(-1) = 3$$
$$y'''(x) = 1 - 2yy' - 2(y')^2, y'''(0) = -2.y(0).y''(0) - 2.(y'(0))^2 = -6 - 2 = -8$$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + \frac{x}{1!} y'(0) + \frac{x^2}{2!} y''(0) + \frac{x^3}{3!} y'''(0) \text{ (Neglecting higher order terms)}$$

Substituting the value of y (0), y' (0), y'' (0),…..

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{8}{6}x^3$$

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 \quad \rightarrow(1)$$

Now put x = 0.1 in (1)

$$y(0.1) = 1 - 0.1 + \frac{3}{2}(0.1)^2 + \frac{4}{3}(0.1)^3 = \mathbf{0.9138}$$

Similarly put x = 0.2 in (1)

$$y(0.2) = 1 - 0.2 + \frac{3}{2}(0.2)^2 - \frac{4}{3}(0.2)^3 = \mathbf{0.8516.}$$

3. Tabulate y(0.1), y(0.2) and y(0.3) using Taylor's series method given that $y' = y^2 + x$ and y(0) = 1.

Sol: Given $y' = y^2 + x$ ……….(1),

y(0) = 1 ……………………(2)

Here $x_0 = 0$, $y_0 = 1$. Take h = 0.1 then $x_1 = x_0 + h = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$

Differentiating (1) w.r.t 'x', we get

$$y'' = 2y \cdot y' + 1 \quad \rightarrow(3)$$

$$y''' = 2[y \cdot y' + (y')^2] \quad \rightarrow(4)$$

$$y^{(iv)} = 2[y \cdot y''' + y'y'' + 2y'\,y''] = 2[y \cdot y''' + 3\,y'\,y''] \quad \rightarrow(5)$$

Put $x_0 = 0$, $y_0 = 1$ in (1), (3), (4) and (5), we get

$$\boldsymbol{y_0'} = (1)^2 + 0 = 1$$

$$\boldsymbol{y_0''} = 2(1)\,(1) + 1 = 3,$$

$$\boldsymbol{y_0'''} = 2((1)\,(3) + (1)^2) = 8$$

$$\boldsymbol{y_0^{(iv)}} = 2[(1)(8) + 3(1)(3)] = 34$$

Take h = 0.1.

<u>**Step1**</u>: By Taylor's series expansion, we have

$$\mathbf{y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{(iv)} + \ldots. \quad \rightarrow(6)}$$

on substituting the values of y_o, y', y'' etc in (6),we get

$$y(0.1) = y_1 = 1 + (0.1)(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(34) + \ldots.$$

$$= 1 + 0.1 + 0.015 + 0.001333 + 0.000416 \quad \Rightarrow \; y_1 = 1.116749$$

<u>Step2</u>: Let us find y(0.2),we start with (x_1,y_1) as the starting values

Here $x_1 = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 1.116749$

Putting these values in (1),(3),(4) and (5), we get

$y_1' = y_1^2 + x_1 = (1.116749)^2 + 0.1 = 1.3471283$

$y_1'' = 2y_1y_1' + 1 = 2(10116749)\,(1.3471283) + 1 = 4.0088$

$y_1''' = 2(y_1y_1'' + (y_1')^2) = 2((1.116749)\,(4.0088) + (1.3471283)^2] = 12.5831$

$y_1^{(4)} = 2y_1y_1''' + 6y_1'y_1'' = 2(1.116749)\,(12.5831) + 6(1.3471283)\,(4.0088) = 60.50653$

By Taylor's expansion

$$\mathbf{y(x_2) = y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{(iv)}} + \ldots.$$

$$y(0.2) = y_2 = 1.116749 + (0.1)(1.3471283) + \frac{(0.1)^2}{2}(4.0088) + \frac{(0.1)^3}{6}(12.5831) + \frac{(0.1)^4}{24}(60.50653)$$

$y_2 = 1.116749 + 0.13471283 + 0.020044 + 0.002097 + 0.000252 = 1.27385$

$y(0.2) = 1.27385$

<u>Step3</u>: Let us find y(0.3),we start with (x_2,y_2) as the starting value.

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 1.27385$

Putting these values of x_2 and y_2 in eq (1), (3), (4) and (5), we get

$\boldsymbol{y_2'} = y_2^2 + x_2 = (1.27385)^2 + 0.2 = 1.82269$

$\boldsymbol{y_2''} = 2y_2\boldsymbol{y_2'} + 1 = 2(1.27385)\,(1.82269) + 1 = 5.64366$

$\boldsymbol{y_2'''} = 2[y_2\boldsymbol{y_2''} + (\boldsymbol{y_2'})^2\,] = 2[(1.27385)\,(5.64366) + (1.82269)^2]$

$= 14.37835 + 6.64439 = 21.02274$

$\boldsymbol{y_2^{(4)}} = 2y_2 + \boldsymbol{y_2'''} + 6\boldsymbol{y_2'y_2''} = 2(1.27385)\,(21.00274) + 6(1.82269)\,(5.64366)$

$= 53.559635 + 61.719856 \ = 115.27949$

By Taylor's expansion,

$$\mathbf{y(x_3) = y_3 = y_2 + \frac{h}{1!}y_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + \frac{h^4}{4!}y_2^{(iv)} + \ldots..}$$

$$y(0.3)=y_3=1.27385+(0.1)(1.82269)+\frac{(0.1)^2}{2}(5.64366)+\frac{(0.1)^3}{6}(21.02274)+\frac{(0.1)^4}{24}(115.27949)$$

$$= 1.27385 + 0.182269 + 0.02821 + 0.0035037 + 0.00048033 = 1.48831$$

$y(0.3) = 1.48831$

4. Solve $y' = x^2 - y$, $y(0) = 1$ using Taylor's series method and evaluate y(0.1),y(0.2),y(0.3) and y(0.4) (correct to 4 decimal places)

Sol: Given $y' = x^2 - y$ →(1) and $y(0) = 1$ →(2)

Here $x_0 = 0$, $y_0 = 1$

Differentiating (1) w.r.t 'x', we get

$y'' = 2x - y'$ → (3)

$y''' = 2 - y''$ → (4)

$y^{(iv)} = - y'''$ → (5)

put $x_0 = 0$, $y_0 = 1$ in (1),(3),(4) and (5), we get

$y_0' = x_0^2 - y_0 = 0-1 = -1$,

$y_0'' = 2x_0 - y_0' = 2(0) - (-1) = 1$

$y_0''' = 2 - y_0'' = 2-1 = 1$,

$y_0^{(iv)} = -y_0''' = -1$ Take h = 0.1

Step1: by Taylor's series expansion

$$\mathbf{y(x_1) = y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + + \frac{h^4}{4!}y_0^{(iv)} + \ldots.} \quad \rightarrow(6)$$

On substituting the values of y_0, y_0', y_0'' etc in (6), we get

$$y(0.1) = y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(1) + \frac{(0.1)^4}{24}(-1) + \ldots.$$

$$= 1-0.1 + 0.005 + 0.01666 - 0.0000416 + \ldots.$$

= 0.905125 ≃ 0.9051 (4 decimal place).

Step2: Let us find y(0.2) we start with (x_1,y_1) as the starting values

Here $x = x_0 + h = 0 + 0.1 = 0.1$ and $y_1 = 0.905125$,

Putting these values of x_1 and y_1 in (1), (3), (4) and (5), we get

$y_1' = x_1^2 - y_1 = (0.1)^2 - 0.905125 = -0.895125$

$y_1'' = 2x_1 - y_1' = 2(0.1) - (-0.895125) = 1.095125,$

$y_1''' = 2 - y_1'' = 2 - 1.095125 = 0.904875,$

$y_1^{(iv)} = -y_1''' = -0.904875,$

By Taylor's series expansion,

$y(x_2) = y_2 = \mathbf{y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + + \frac{h^4}{4!}y_1^{(iv)} + \ldots}$

$y(0.2)=y_2=0.905125+(0.1)(-0.895125)+\frac{(0.1)^2}{2}(1.09125)+\frac{(0.1)^3}{6}(0.904875)+\frac{(0.1)^4}{24}(-0.904875)+\ldots$

$y(0.2) = y_2 = 0.905125 - 0.0895125 + 0.00547562 + 0.000150812 - 0.00000377$

= 0.8212351 ≃ 0.8212 (4 decimal places)

Step3: Let us find y(0.3), we start with (x_2,y_2) as the starting value

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$ and $y_2 = 0.8212351$

Putting these values of x_2 and y_2 in (1),(3),(4), and (5) we get

$\boldsymbol{y_2'} = x_2^2 - y_2 = (0.2)^2 - 0.8212351 = 0.04 - 0.8212351 = -0.7812351$

$\boldsymbol{y_2''} = 2x_2 - \boldsymbol{y_2'} = 2(0.2) + (0.7812351) = 1.1812351,$

$\boldsymbol{y_2'''} = 2 - \boldsymbol{y_2''} = 2 - 1.1812351 = 0.818765,$

$\boldsymbol{y_2^{(4)}} = -\boldsymbol{y_2'''} = -0.818765,$

By Taylor's series expansion,

$y(x_3) = y_3 = \mathbf{y_2 + \frac{h}{1!}y_2' + \frac{h^2}{2!}y_2'' + \frac{h^3}{3!}y_2''' + + \frac{h^4}{4!}y_2^{(iv)} + \ldots}$

$y(0.3)= y_3 =0. 8212351+(0.1)(-0.7812351)+\frac{(0.1)^2}{2}(1.1812351)+\frac{(0.1)^3}{6}(0.818765)+\frac{(0.1)^4}{24}(-0.818765)+\ldots.$

$y(0.3) = y_3 = 0. 8212351– 0.07812351+ 0.005906 + 0.000136 – 0.0000034$

$= 0.749150 \simeq 0.7492$ (4 decimal places)

Step4: Let us find y(0.4), we start with (x_3,y_3) as the starting value

Here $x_3 = x_2 + h = 0.2+ 0.1 = 0.3$ and $y_3 = 0.749150$

Putting these values of x_3 and y_3 in (1),(3),(4), and (5) we get

$y_3' = x_3^2 - y_3 = (0.3)^2 – 0.749150= -0.65915,$

$y_3''= 2x_3 - y_3' = 2(0.3) + (0.65915) = 1.25915,$

$y_3'''= 2 - y_3'' = 2 – 1. 25915 = 0.74085,$

$y_3^{(iv)}= - y_3''' = -0.74085,$

By Taylor's series expansion,

$y(x_4) = y_4=y_3 + \frac{h}{1!}y_3' + \frac{h^2}{2!}y_3'' + \frac{h^3}{3!}y_3''' + + \frac{h^4}{4!}y_3^{(iv)} + \ldots.$

$y(0.4)= y_4=0.749150+(0.1)(-0.65915)+\frac{(0.1)^2}{2}(1.25915)+\frac{(0.1)^3}{6}(0.74085)+$

$\frac{(0.1)^4}{24}(0.74085)+\ldots.$

$y(0.4) = y_4 = 0.749150 – 0.065915+ 0.0062926+ 0.000123475 – 0.0000030$

$= 0.6896514 \simeq 0.6897$ (4 decimal places)

5. Using Taylor's expansion evaluate the integral of $y' - 2y = 3e^x, y(0) = 0$, at

a) $x = 0.1, 0.2, 0.3$ b) Compare the numerical solution obtained with exact solution.

Sol: Given equation can be written as $2y + 3e^x = y', y(0) = 0$

Differentiating repeatedly w.r.t to'x' and evaluating at $x = 0$

$y'(x) = 2y + 3e^x, y'(0) = 2y(0) + 3e^0 = 2(0) + 3(1) = 3$
$y''(x) = 2y' + 3e^x, y''(0) = 2y'(0) + 3e^0 = 2(3) + 3 = 9$
$y'''(x) = 2.y''(x) + 3e^x, y'''(0) = 2y''(0) + 3e^0 = 2(9) + 3 = 21$
$y^{iv}(x) = 2.y'''(x) + 3e^x, y^{iv}(0) = 2(21) + 3e^0 = 45$
$y^{v}(x) = 2.y^{iv} + 3e^x, y^{v}(0) = 2(45) + 3e^0 = 90 + 3 = 93$

In general, $y^{(n+1)}(x) = 2.y^{(n)}(x) + 3e^x$ or $y^{(n+1)}(0) = 2.y^{(n)}(0) + 3e^0$

The Taylor's series expansion of $y(x)$ about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \frac{x^5}{5!}y^{v}(0) + \cdots$$

Substituting the values of $y(0), y'(0), y''(0), y'''(0), \ldots\ldots\ldots$

$$y(x) = 0 + 3x + \frac{9}{2}x^2 + \frac{21}{6}x^3 + \frac{45}{24}x^4 + \frac{93}{120}x^5 + \ldots\ldots$$

$$y(x) = 3x + \frac{9}{2}x^2 + \frac{7}{2}x^3 + \frac{15}{8}x^4 + \frac{31}{40}x^5 + \ldots\ldots \rightarrow (1)$$

Now put $x = 0.1$ in equation

$$y(0.1) = 3(0.1) + \frac{9}{2}(0.1)^2 + \frac{7}{2}(0.1)^3 + \frac{15}{8}(0.1)^4 + \frac{31}{40}(0.1)^5 = 0.34869$$

Now put $x = 0.2$ in equation

$$y(0.2) = 3(0.2) + \frac{9}{2}(0.2)^2 + \frac{7}{2}(0.2)^3 + \frac{15}{8}(0.2)^4 + \frac{31}{40}(0.2)^5 = 0.811244$$

Now put $x = 0.3$ *in equation*(1)

$$y(0.3) = 3(0.3) + \frac{9}{2}(0.3)^2 + \frac{7}{2}(0.3)^3 + \frac{15}{8}(0.3)^4 + \frac{31}{40}(0.3)^5 = 1.41657075$$

Analytical Solution:

The exact solution of the equation $\frac{dy}{dx} = 2y + 3e^x$ with $y(0) = 0$ can be found as follows

$\frac{dy}{dx} - 2y = 3e^x$ This is a linear in y.

Here $P = -2, Q = 3e^x$

I.F $= e^{\int p(x)dx} = e^{\int -2x dx} = e^{-2x}$

General solution is $y.e^{-2x} = \int 3e^{x}.e^{-2x}dx + c = -3e^{-x} + c$

$\therefore y = -3e^{x} + ce^{2x}$ Where $x = 0, y = 0 \quad 0 = -3 + c \Rightarrow c = 3$

The particular solution is $y = 3e^{2x} - 3e^{x}$ or $y(x) = 3e^{2x} - 3e^{x}$

Put $x = 0.1$ in the above particular solution,

$y = 3.e^{0.2} - 3e^{0.1} = 0.34869$

Similarly put $x = 0.2$

$y = 3e^{0.4} - 3e^{0.2} = 0.811265$

put $x = 0.3$ $\quad y = 3e^{0.6} - 3e^{0.3} = 1.416577$

6. Using Taylor's series method, solve the equation $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given that y = 0 when x = 0

Sol: Given equation is $\frac{dy}{dx} = x^2 + y^2$ and $y = 0$ when $x = 0$ i.e. $y(0) = 0$

Here $y_0 = 0$, $x_0 = 0$

Differentiating repeatedly w.r.t 'x' and evaluating at $x = 0$

$$y'(x) = x^2 + y^2, y'(0) = 0 + y^2(0) = 0 + 0 = 0$$

$y''(x) = 2x + y'.2y, y''(0) = 2(0) + y'(0)2.y = 0$

$y'''(x) = 2 + 2yy'' + 2y'.y', y'''(0) = 2 + 2.y(0).y'(0) + 2.y'(0)^2 = 2$

$y^{(4)}(x) = 2.yy''' + 2.y'',y' + 4.y''.y', y'(0) = 0$

The Taylor's series for f(x) about $x_0 = 0$ is

$$y(x) = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y''''(0) + ...$$

Substituting the values of $y(0), y'(0), y''(0), ...$

$$y(x) = 0 + x(0) + 0 + \frac{2x^3}{3!} + 0 + ... = \frac{x^3}{3} + \text{(Higher order terms are neglected)}$$

$$\therefore y(0.4) = \frac{(0.4)^3}{3} = \frac{0.064}{3} = 0.02133$$

7. Find y (0.1),y (0.2), z(0.1), z(0.2) given $\frac{dy}{dx} = x + y, \frac{dz}{dx} = x - y^2$ $and\ y(0) = 2, z(0) = 1$**by**

Using Taylor's series method

Sol: Given $y' = x + z$, take $x_0 = 0$, $y_0 = 2$,h=0.1

We have to find $y_1 = y(0.1)$ and $y_2 = y(0.2)$

Now $y' = x+z$, $y'' = 1+ z'$, $y''' = z'$ ………………….(I)

Given $z' = x-y^2$

take $x_0 = 0$, $z_0 = 1$,h = 0.1

we have to find $z_1 = z(0.1)$ and $z_2 = z(0.2)$

now $z' = x-y^2$, $z'' = 1-2y.y'$, $z''' = -2[y.y'' + (y')^2]$ ……………………………….(II)

By Taylor's series for y_1 and z_1 ,we have

$y(x) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0'''$ (neglecting higher order terms)….(1)

$z(x) = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0'''$ (neglecting higher order terms)……………….(2)

From (I) and (II), we get

$y_0 = 2$ $\qquad z_0 = 1$

$y_0' = x_0 + z_0 = 0+1=1$ $\qquad z_0' = x_0 - y_0^2 = -4$

$y_0'' = 1+z_0' = 1+ x_0 - y_0^2 = 1+0-4 = -3$; $z_0'' = 1-2y_0.y_0' = 1-2(2)1 = -3$

$y_0''' = z_0'' = -3$; $z_0''' = -2[y_0.y_0'' + (y_0')^2] = 10$

Substituting these values in (1) and(2)

y_1=y(0.1)= $2+(0.1)1+ \frac{0.01}{2}(-3) + \frac{0.001}{6}(-3) = 2.0845.$

$z_1 = z(0.1) = 1+(0.1)(-4)+ \frac{0.01}{2}(-3)+ \frac{0.001}{6}(10) = 0.5867.$

Similarly

By Taylor's series for y_2, z_2 are

$y_2 = y_1 + h.\ y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1'''$ ………………..(3)

$z_2 = z_1 + h.\ z_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1'''$…………………..(4)

Now we have

$y_1 = 2.01845$; $\qquad z_1 = 1$;

$y_1' = x_1+z_1=0.1+0.5867=0.6867$ $\qquad z_1' = x_1-y_1^2 = -4.2451$

$y_1''=1+z_1' = 1+ x_1-y_1^2 = -3.2451$; $\qquad z_1'' = 1-2y_1.y_1' = -1.8628$

$y_1'''=z_1'' = -1.8628$; $\qquad z_1''' = -2\,[y_1.\,yy_1'' + (y_1)^2] = 12.5856$

Substituting in (3) and (4).We get

$y_2=y(0.2)$= $2.0845+(0.1)(0.6867)+\frac{0.01}{2}(-3.2451)+\frac{0.001}{6}(-1.8628)=2.1367.$

$z_2=z(0.2)= 0.5867+(0.1)(-4.2451)+\frac{0.01}{2}(-1.8628)+\frac{0.001}{6}(12.5856)=0.15497.$

Euler's Method

It is the simplest one-step method and it is less accurate. Hence it has a limited application.

Consider the differential equation $\frac{dy}{dx} = f(x,y)$ →(1) With $y(x_0) = y_0$ →(2)

Consider the first two terms of the Taylor's expansion of y(x) at $x = x_0$

$y(x) = y(x_0) + (x - x_0)\, y'(x_0)$ →(3)

from equation (1) $y'(x_0) = f(x_0,y(x_0)) = f(x_0, y_0)$

Substituting in equation (3)

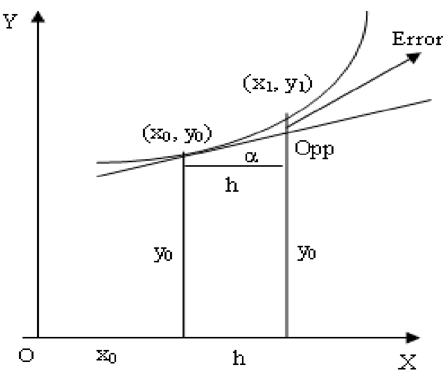
$\therefore y(x) = y(x_0) + (x - x_0)\, f(x_0, y_0)$ At $x = x_1$, $y(x_1) = y(x_0) + (x_1 - x_0)\, f(x_0, y_0)$

$\therefore y_1=y_0+ h\, f(x_0, y_0)$ where $h = x_1 - x_0$

Similarly at $x = x_2$, $y_2=y_1+ h\, f(x_1, y_1)$

Proceeding as above, $y_{n+1} = y_n + h\, f(x_n,y_n)$

This is known as Euler's Method



From the fig,

$\text{Tan}\alpha=\frac{opp}{adj} = \frac{opp}{h}$

Implies opp = h $\text{Tan}\alpha$

But $\text{Tan}\alpha$=slope at (x_0, y_0) $= \frac{dy}{dx}\, at(x_0, y_0) = f(x_0, y_0)$

$\therefore$opp=h f(x_0, y_0)

Hence $y_1=y_0$+opp implies $y_1=y_0+$ h f(x_0, y_0) [NEGLECTING ERROR]

we remove that error by using Euler's modified method.

Solved Problems:

1. Using Euler's method, solve for y at x = 2 from $\frac{dy}{dx}$ = $3x^2 + 1$,y(1) = 2,taking step size

(i) h = 0.5 and (ii) h=0.25

Sol: Here $f(x,y) = 3x^2 + 1$, $x_0 = 1, y_0 = 2$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$, n = 0,1,2,3,….. →(1)

(i) h = 0.5 $\therefore x_1 = x_0 + h = 1+0.5 = 1.5$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 1.5 + 0.5 = 2$

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y_1 = y(1.5) = 2 + (0.5)\ f(1,2) = 2 + (0.5)\ (3 + 1) = 2 + (0.5)(4) =4$

Here $x_1 = x_0 + h = 1 + 0.5 = 1.5$

$\therefore y(1.5) = 4 = y_1$

Taking n = 1 in (1),we have

$y_2 = y_1 + h\ f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 4 + (0.5)\ f(1.5,4) = 4 + (0.5)[3(1.5)^2 + 1] = 7.875$

Here $x_2 = x_4 + h = 1.5 + 0.5 = 2$

$\therefore y(2) = 7.875$

(ii) h = 0.25 $\therefore x_1 = 1.25, x_2 = 1.50, x_3 = 1.75, x_4 = 2$

Taking n = 0 in (1), we have

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y(x_1) = y_1 = 2 + (0.25)\ f(1,2) = 2 + (0.25)\ (3 + 1) = 3$

$y(x_2) = y_2 = y_1 + h\ f(x_1,y_1)$

i.e. $y(x_2) = y_2 = 3 + (0.25)\ f(1.25,3) = 3 + (0.25)[3(1.25)^2 + 1] = 5.42188$

Here $x_2 = x_1 + h = 1.25 + 0.25 = 1.5$

$y(1.5) = 5.42188$

Taking n = 2 in (1), we have

i.e. $y(x_3) = y_3 = y_2 + h\ f(x_2,y_2)$

$= 5.42188 + (0.25)\ f(1.5,5.42188)$

$= 5.42188 + (0.25)\ [3(1.5)^2 + 1]= 7.35938$

Here $x_3 = x_2 + h = 1.5 + 0.25 = 1.75$

$\therefore y(1.75) = 7.35938$

Taking n = 4 in (1),we have

$y(x_4) = y_4 = y_3 + h\ f(x_3,y_3)$

i.e. $y(x_4) = y_4 = 7.35938 + (0.25)\ f(1.75,7.35938)$

$= 7.35938 + (0.25)[3(1.75)^2 + 1] = 9.90626$

Note that the difference in values of y(2) in both cases (i.e. when h = 0.5 and when h = 0.25).The accuracy is improved significantly when h is reduced to 0.25 (Exact solution of the equation is $y = x^3 + x$ and with this $y(2) = y_2 = 10$.

2. Solve by Euler's method, $y'(x_0) = x + y$, $y(0) = 1$ and find y(0.3) taking step size h = 0.1. compare the result obtained by this method with the result obtained by analytical solution

Sol: Here $f(x,y) = x+y$, $x_0 = 0, y_0 = 1$

Euler's algorithm is $y_{n+1} = y_n + h\ f(x_n,y_n)$, n = 0,1,2,3,….. →(1)

Given h = 0.5 $\therefore x_1 = x_0 + h = 0+0.1= 0.1$

Taking n = 0 in (1) , we have $x_2 = x_1 + h = 0.1+ 0.1 = 0.2$

$y_1 = y_0 + h\ f(x_0,y_0)$

i.e. $y_1 = y(0.1) = 1 + (0.1)\ f(0,1) = 1.1$

$\therefore$ y(0.1) = 1.1

Here $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Taking n = 1 in (1),we have $y_2 = y_1 + h\, f(x_1, y_1)$

i.e. $y(x_2) = y_2 = 1.1 + (0.1)\, f(0.1.1.1) = 1.22$

Similarly we get $y_3 = y(0.3) = 1.362$

Analytical solution:

The exact solution of $\frac{dy}{dx} = x + y$, y(0)=1 can be found as follows.

The equation can be written as $\frac{dy}{dx} - y = x$

This is a linear equation in y [i.e, $\frac{dy}{dx} + p.y = Q$]

then p = -1, Q = x. $I.F = e^{\int p dx} = e^{\int (-1) dx} = e^{-x}$

General solution is y . I.F= $\int Q XI.F dx + c$

$$y.e^{-x} = \int x.e^{-x} dx + c$$

$$y.e^{-x} = - e^{-x}(x+1) + c. \text{ or } y = -(x+1) + ce^{+x}$$

when x = 0, y = 1 i.e, 1= - (0+1) + c or c = 2

Hence the particular solution of the equation is

$y = -(x+1) + 2e^{x} = 2e^{x} - x - 1.$

Particular solution is $y = 2e^{x} - (x + 1)$

Hence y(0.1) = 1.11034, y(0.2) = 1.3428, y(0.3) = 1.5997

We shall tabulate the result as follows

X	0	0.1	0.2	0.3
Euler y	1	1.1	1.22	1.362
Exact y	1	1.11034	1.3428	1.5997

The value of y deviate from the exact value as x increases. This indicate that the method is not accurate

3. Given $y' = x^2 - y, y(0) = 1$ **find correct to four decimal places the value of y (0,1), by using Euler's method.**

Sol: We have $f(x, y) = x^2 - y$ $\quad x_0 = 0; y_0$=1 and h=0.1

By Euler's algorithm

$y_{n+1} = y_n + h\, f(x_n,y_n)$ →(1)

∴ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0) = 1+(0.1)f(0,1) = 1+0.1(0-1)$ **= 0.9**

$\therefore \boldsymbol{y_1}$**=0.9**

4. Use Euler's method of find y(0.1),y(0.2) given $y' = \left(x^3 + xy^2\right)e^{-x}, y(0) = 1$

Sol: Given $y' = \left(x^3 + xy^2\right)e^{-x}, y(0) = 1$

Consider h = 0.1

Here $f(x,y) = (x^3+xy^2)e^{-x}$, $x_0 = 0$, $y_0 = 1$, $x_1=x_0+h=0.1$, $x_2=x_1+h=0.2$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ →(1)

∴ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0) = y_0+h({x_0}^3+x_0{y_0}^2)e^{-x_0} = 1+(0.1)(0) =1$

$\therefore y(0.1) = 1$

Again $x_2=x_1+h=0.2$

∴ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1) = y_1 + h({x_1}^3+x_1{y_1}^2)e^{-x_1}$

$= 1 + (0.1)[(0.1)^3 + (0.1)(1)^2] = 1.0091$

$\therefore y(0.2) = 1.0091$

5. Given that $\frac{dy}{dx}$ **= xy ,y(0) = 1 determine y(0.1),using Euler's method.**

Sol: The given differentiating equation is $\frac{dy}{dx}$ = xy, y(0) = 1

a=0, b=0.1

Here $f(x,y) = xy$, $x_0 = 0$ and $y_0 = 1$

Since h is not given much better accuracy is obtained by breaking up the interval (0,0.1) in to five steps.

i.e. $h = \frac{b-a}{5} = \frac{0.1}{5} = 0.02$

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ →(1)

∴ From (1) for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0) = 1 + (0.02)\, f(0,1) = 1 + (0.02)\,(0) = 1$

Next we have $x_1 = x_0 + h = 0 + 0.02 = 0.02$

∴ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1) = 1 + (0.02)\, f(0.02,1) = 1 + (0.02)\,(0.02) = 1.0004$

Next we have $x_2 = x_1 + h = 0.02 + 0.02 = 0.04$

∴ From (1), for n = 2,we have

$y_3 = y_2 + h\, f(x_2,y_2) = 1.004 + (0.02)\,(0.04)\,(1.000) = 1.0012$

Next we have $x_3 = x_2 + h = 0.04 + 0.02 = 0.06$

∴ From (1), for n = 3,we have

$y_4 = y_3 + h\, f(x_3,y_3) = 1.0012 + (0.02)\,(0.06)\,(1.00012) = 1.0024.$

Next we have $x_4 = x_3 + h = 0.06 + 0.02 = 0.08$

∴ From (1), for n = 4,we have

$y_5 = y_4 + h\, f(x_4,y_4) = 1.0024 + (0.02)\,(0.08)\,(1.00024) = 1.0040.$

Next we have $x_5 = x_4 + h = 0.08 + 0.02 = 0.1$

When $x = x_5$, $y \cong y_5$

∴ $y = 1.0040$ when $x = 0.1$

6. Given that $\frac{dy}{dx} = 3x^2 + y$, y(0) = 4.Find y(0.25) and y(0.5) using Euler's method

Sol: Given $\frac{dy}{dx} = 3x^2 + y$ and $y(1) = 2$.

Here $f(x,y) = 3x^2 + y$, $x_0 = (1)$, $y_0 = 4$

Consider h = 0.25

Euler's algorithm is $y_{n+1} = y_n + h\, f(x_n,y_n)$ → (1)

$\therefore$ From (1), for n = 0, we have

$y_1 = y_0 + h\, f(x_0,y_0) = 2 + (0.25)[0 + 4] = 2 + 1 = 3$

Next we have $x_1 = x_0 + h = 0 + 0.25 = 0.25$

When $x = x_1$, $y_1 \simeq y$

$\therefore$ y = 3 when x = 0.25

$\therefore$ From (1), for n = 1, we have

$y_2 = y_1 + h\, f(x_1,y_1) = 3 + (0.25)[3.(0.25)^2 + 3] = 3.7968$

Next we have $x_2 = x_1 + h = 0.25 + 0.25 = 0.5$

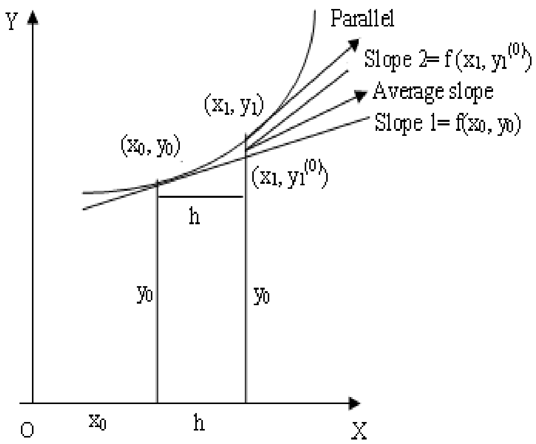
When $x = x_2$, $y \simeq y_2$ $\therefore$ y = 3.7968 when x = 0.5.

Modified Euler's Method

From fig

Avg slope = parallel line slope

$$=\frac{f(x0,y0)+f(x1,y_1^{(0)})}{2}$$



Hence

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{f(x0,y0)+f(x1,y_1^{(0)})}{2}$$

$$y_1^{(n+1)} = y_0 + \frac{f(x_0,y_0)+f(x_1,y_1^{(n)})}{2}$$

Continue till any two consecutive iterations nearly same upto three or four decimal places.

To find $y_2, y_3 \ldots..$

The formula is given by $y_{k+1}^{(i)} = y_k + h/2\left[f(x_k,y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right], i = 1,2 \ldots. and\ k = 0,1 \ldots \ldots$

Working rule for Modified Euler's method

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k, y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right], i = 1,2 \ldots . \, and \; k = 0,1 \ldots \ldots$$

ii) When $i=1$ y^{0}_{k+1} can be calculated from Euler's method

iii) k=0, 1……… gives number of iteration. $i = 1, 2...$

gives number of times, a particular iteration k is repeated

Suppose consider dy/dx=f(x, y) -------- (1) with y(x_0) =y_0----------- (2)

To find y (x_1) = y_1 at x = x_1 = x_0+h

Now take k=0 in modified Euler's method

……We get $y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(i-1)}\right)\right]$……………………… (3)

Taking i=1, 2, 3...k+1 in eqn (3), we get

$y_0^{(0)} = y_0 + h[f(x_0, y_0)]$ (By Euler's method)

$$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$y_1^{(k+1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(k)}\right)\right]$$

If two successive values of $y_1^{(k)}, y_1^{(k+1)}$ are sufficiently close to one another, we will take the common value ……….as $y_1 = y(x_2) = y(x_1 + h)$

Now we have $\frac{dy}{dx} = f(x, y) with\, y = y_1\, at\, x = x_1\, to\, get\, y_2 = y(x_2) = y(x_1 + h)$

Now we have $\frac{dy}{dx}$ =f(x,y) with y=y , at x=x To get y_2=y(x_2)=y($x_1 + h$)

We use the above procedure again

Solved Problems

1. Using modified Euler's method find the approximate value of x **when** $x = 0.3$

given that $dy/dx = x + y \; and \; y(0) = 1$

Sol: Given $dy/dx = x + y \ and \ y(0) = 1$

Here $f(x,y) = x + y, x_0 = 0, \ and \ y_0 = 1$

Take h = 0.1 which is sufficiently small

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \rightarrow (1)$$

Step1: To find $y_1 = y(x_1) = y(0.1)$

Taking k = 0 in eqn(1)

$$y_1^{(i)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f\left(x_1, y_1^{(i-1)}\right)\right] \quad \rightarrow (2)$$

when i = 1 in eqn (2) $y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0 + y_0) + f\left(x_1, y_1^{(0)}\right)\right]$

First apply Euler's method to calculate $y_1^{(0)} = y_1$

$$\therefore \quad y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

= 1+(0.1)f(0,1)=1+(0.1)(0+1)

= 1+(0.1) = 1.10

Now $[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1(0) = 1.10]$

$$\therefore \quad y_1^{(1)} = y_0 + 0.1/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

= 1+0.1/2[f(0,1) + f(0.1,1.10)

= 1+0.1/2[(0+1)+(0.1+1.10)]= 1.11

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[f(0.1)+f(0.1,1.11)]

= 1 + 0.1/2[(0+1)+(0.1+1.11)]= 1.1105

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[f(0,1)+f(0.1 , 1.1105)]

= 1+0.1/2[(0+1)+(0.1+1.1105)] = 1.1105

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ y_1 = 1.1105

Step:2 To find $y_2 = y(x_2) = y(0.2)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(i-1)}\right)\right] \rightarrow (3)$$ where i = 1,2,3,4,…..

For i = 1

$$y_2^{(1)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(0)}\right)\right]$$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h\ f\left(x_1, y_1\right)$$

= 1.1105 + (0.1) f(0.1 , 1.1105)

= 1.1105+(0.1)[0.1+1.1105]= 1.2316

$$\therefore\ y_2^{(1)} = 1.1105 + 0.1/2\left[f\left(0.1, 1.1105\right) + f\left(0.2, 1.2316\right)\right]$$

= 1.1105 +0.1/2[0.1+1.1105+0.2+1.2316]= 1.2426

$$y_2^{(2)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.1105 + 0.1/2[f(0.1 , 1.1105) , f(0.2 . 1.2426)]

= 1.1105 + 0.1/2[1.2105 + 1.4426]

= 1.1105 + 0.1(1.3266)= 1.2432

$$y_2^{(3)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.1105+0.1/2[f(0.1,1.1105)+f(0.2 , 1.2432)]

= 1.1105+0.1/2[1.2105+1.4432)]

= 1.1105 + 0.1(1.3268) = 1.2432

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.2432

Step:3 To find $y_3 = y(x_3)$ = y y(0.3)

Taking k =2 in eqn (1) we get

$$y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(i-1)}\right)\right] \to (4)$$

For i = 1 , $y_3^{(1)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(0)}\right)\right]$

$y_3^{(0)}$ is to be evaluated from Euler's method .

$$y_3^{(0)} = y_2 + h\, f\left(x_2, y_2\right)$$

= 1.2432 +(0.1) f(0.2 , 1.2432)

= 1.2432+(0.1)(1.4432) = 1.3875

$\therefore$ $y_3^{(1)}$ = 1.2432+ $\frac{0.1}{2}$ [f(0.2 , 1.2432)+f(0.3, 1.3875)]

= 1.2432 + 0.1/2[1.4432+1.6875]

= 1.2432+0.1(1.5654) = 1.3997

$$y_3^{(2)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 1.2432+0.1/2[1.4432+(0.3+1.3997)]

= 1.2432+ (0.1) (1.575) = 1.4003

$$y_3^{(3)} = y_2 + h/2\left[f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 1.2432+0.1/2[f(0.2 , 1.2432)+f(0.3 , 1.4003)]

= 1.2432 + 0.1(1.5718) = 1.4004

$$y_3^{(4)} = y_2 + h/2\left[\ f(x_2, y_2) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 1.2432 + 0.1/2[1.4432+1.7004]

= 1.2432+(0.1)(1.5718) = 1.4004

Since $y_3^{(3)} = y_3^{(4)}$

Hence $y_3 = 1.4004$

$\therefore$ The value of y at x = 0.3 is 1.4004

2. Using Modified Euler's method find y(0.2) y(0.4)with h=0.2,given that $\frac{dy}{dx}$=x + siny, y(0)=1

Sol: $f(x,y) = x + siny$ $x_0 = 0; y_0$=1 and h=0.2

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

$x_1=x_0+h=0.2$; $x_2=x_1+h=0.4$

The formula for modified Euler's method is given by

$$y_{k+1}^{(i)} = y_k + h/2\left[f(x_k + y_k) + f\left(x_{k+1}, y_{k+1}^{(i-1)}\right)\right] \rightarrow (1)$$

<u>**Step1:**</u> To find $y_1 = y(x_1)$ = y (0.2)

Euler's modified method is given by

$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$ (k=0 , i=1)

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$\therefore$ $y_1^{(0)} = y_0 + h\, f(x_0, y_0)$

= 1+(0.2)f(0,1)=1+(0.2)(0+sin1)

= 1.163

Now $\left[x_0 = 0, y_0 = 1, x_1 = 0.2, y_1^{(0)} = 1.163\right]$

$\therefore \quad y_1^{(1)}$ = 1+0.2/2[f(0,1) + f(0.2,1.163)

= 1+0.1/2[1+1.163]

= 1.1916

When i=2 in eqn (2)

$$y_1^{(2)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.2/2[f(0.1)+f(0.2,1.1916)]

=1.2038

$$y_1^{(3)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.2/2[f(0,1)+f(0.2 , 1.2038)]

=1.2045

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ **y_1 = 1.204**

<u>Step:2</u> To find $y_2 = y(x_2) = y(0.4)$

Taking k = 1 in eqn (1) , we get

$$y_2^{(i)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(i-1)}\right)\right] \to (3) \text{ where } \quad i = 1,2,3,4,\ldots..$$

For i = 1, $y_2^{(1)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h \; f(x_1, y_1)$$

= 1.204 + (0.2) f(0.2 , 1.204)

= 1.4313

$y_2^{(1)}$ = 1.204 +0.1[1.1337+1.4313]

= 1.4611

$$y_2^{(2)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 1.204 + 0.1/2[f(0.2 , 1.204) , f(0.4 . 1.416)

= 1.462

$$y_2^{(3)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 1.204+0.1/2[f(0.2,1.204)+f(0.4 , 1.462)] = 1.464

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 1.46

3. **Using modified Euler's method find the approximate value of** x **when** $x = 0.3$ **given that** $\boldsymbol{\frac{dy}{dx} = x - y \text{ and } y(0) = 1}$

Sol: Given $\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{x} - \mathrm{y}$ and $\mathrm{y}(0) = 1$

Here f(x,y) = x - y, x_0 = 0and y_0 = 1

Take h = 0.1

Here $x_0 = 0, x_1 = x_0 + h = 0.1, x_2 = x_1 + h = 0.2, x_3 = x_2 + h = 0.3$

Step1: To find $y_1 = y(x_1) = y(0.1)$

First apply Euler's method to calculate $y_1^{(0)}$ = y_1

$$y_1^{(0)} = y_0 + h\, f\left(x_0, y_0\right)$$

= 1+(0.1)(0-1)

= 1-(0.1)

= 0.9

Now $\left[x_0 = 0, y_0 = 1, x_1 = 0.1, y_1^{(0)} = 0.9\right]$

$$y_1^{(1)} = y_0 + h/2\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

= 1+0.1/2[-1 - 0.8]

= 1-0.09

= 0.91

$$y_1^{(2)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right]$$

= 1+0.1/2[-1 + (0.1-0.91)]

= 1+0.1/2[-1.81]

= 1-0.0905

= 0.9095

$$y_1^{(3)} = y_0 + h/2\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(2)}\right)\right]$$

= 1+0.1/2[-1+(0.1-0.9095)]

= 1+0.1/2[-1.8095]

= 1-0.090475

= 0.909525

Since $y_1^{(2)} = y_1^{(3)}$

$\therefore$ y_1 = 0.9095

Step:2 To find $y_2 = y(x_2) = y(0.2)$

$y_2^{(0)}$ is to be calculate from Euler's method

$$y_2^{(0)} = y_1 + h \ f\left(x_1, y_1\right)$$

= 0.9095+(0.1)(-0.8095)

= 0.82855

$$y_2^{(1)} = y_1 + h/2\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(0)}\right)\right]$$

= 0.9095+0.1/2[-0.8095-0.62855]

= 0.9095-0.0719

= 0.8376

$$y_2^{(2)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(1)}\right)\right]$$

= 0.9095+0.1/2[-0.8095-0.6376]

= 0.9095-0.075355

= 0.837145

$$y_2^{(3)} = y_1 + h/2\left[f(x_1, y_1) + f\left(x_2 y_2^{(2)}\right)\right]$$

= 0.9095+0.1/2[-10446645]

= 0.9095-0.07233

= 0.83716

Since $y_2^{(3)} = y_2^{(3)}$

Hence y_2 = 0.8371

Step:3 To find $y_3 = y(x_3) = y(0.3)$

$y_3^{(0)}$ is to be evaluated from Euler's method

$$y_3^{(0)} = y_2 + h\, f(x_2, y_2)$$

= 0.8371+0.1(-0.6371) = 0.7734

$$y_3^{(1)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(0)}\right)\right]$$

= 0.8371+ 0.1/2[-0.6371-0.4734]

= 0.8371-0.0555 = 0.7816

$$y_3^{(2)} = y_2 + h/2\left[f(x_2, y_2) + f\left(x_3, y_3^{(1)}\right)\right]$$

= 0.8371+ 0.1/2[-1.1187]

= 0.8371-0.056 = 0.7811

$$y_3^{(3)} = y_2 + h/2\left[\ f\left(x_2, y_2\right) + f\left(x_3, y_3^{(2)}\right)\right]$$

= 0.8371+ 0.1/2[-1.1182]

= 0.8371-0.05591 = 0.7812

$$y_3^{(4)} = y_2 + h/2\left[\ f\left(x_2, y_2\right) + f\left(x_3, y_3^{(3)}\right)\right]$$

= 0.8371-0.0559 = 0.7812

Since $y_3^{(3)} = y_3^{(4)}$

Hence y_3 = 0.7812

∴ The value of y at x = 0.3 is 0.7812

Runge-Kutta Methods

I.First order R-K Method

Euler's Method is the R-K method of the first order.

II. Second order R-K Method

$y_{i+1} = y_i + \frac{1}{2}(K_1+K_2)$,

Where $K_1 = h\,(x_i, y_i)$

$K_2 = h\,(x_i+h, y_i+k_1)$

For i= 0,1,2-------

Note: Euler's Modified Method Is R-K Method Of Second Order

III. Third order R-K Formula

$y_{i+1} = y_i + \frac{1}{6}\ (K_1+4K_2+ K_3)$, Where $K_1 = h\,(x_i, y_i)$

$K_2 = h\,(x_i+h/2,\ y_0+k_1/2)$

$K_3 = h\,(x_i+h,\ y_i+2k_2-k_1)$ For i= 0,1,2------

IV. Fourth order R-K Formula

$y_{i+1} = y_i+ \frac{1}{6}\,(K_1+2K_2+ 2K_3 +K_4)$,

Where $K_1 = h\,(x_i, y_i)$

$K_2 = h\,(x_i+h/2,\ y_i+k_1/2)$

$K_3 = h\,(x_i+h/2,\ y_i+k_2/2)$

$K_4 = h\,(x_i+h,\ y_i+k_3)$

For i= 0,1,2------

Advantages of Runge kutta method Over Taylor series method.

In R-K method no need to find derivatives where as we find derivatives in taylors method. Sometimes it may be complicate to find derivative of some function, sowe go for RK Method at that time.

Solved Problems:

1. solve $\frac{dy}{dx} = xy$ using R-K method for x=0.2,0.4 given y(0) =1, y`(0)-0 taking h = 0.2

Sol: Given $\frac{dy}{dx} = xy$ **;** $y(0) = 1$.

Here $f(x, y) = xy$, $x_0 = 0$, $y_0=1$ and $h = 0.2$

$\therefore$ $x_1 = x_0+h = 0+0.2 = 0.2.$, $x_2 = x_1+h =0.2+0.2 = 0.4$

By 4th order R-K method, we have

$y_1 = y_0+\frac{1}{6}\,(k_1+2k_2+2k_3+k_4)$

Where k_1=h $f(x_0,y_0)$=(0.2)f(0,1)= 0

k_2= h f $(x_0+\frac{h}{2}, y_0+\frac{k_1}{2})$ = $(0.2)[f(0.1,1)] = (0.2)(0.1) = 0.02$

k_3= h $f((x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$=(0.2)f (0.1,1.01) = 0.202

k_4= h $f(x_0+h, y_0+k_3)$=(0.2)f(0.2,1.202)=0.04808

Hence $y_1=1+\frac{1}{6}$ (0+0.04808+2(0.02+0.202)=1.08201

Step2: To find y(0.4)= y_2

Here x_1 = 0.2, y_1=1.08201 and h = 0.2

Again by 4th order R-K method, we have

$\therefore \quad y_2 = y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Where k_1=h $f(x_1,y_1)$=(0.2)[f(0.2,1.08201)] =0.04328

k_2= hf $(x_1+\frac{h}{2}, y_1+\frac{k_1}{2})$=0.2(f(0.3,1.10364)=0.0662

k_3=hf$(x_1+\frac{h}{2}, y_1+\frac{k_2}{2})$=(0.2)[f(0.3,1.1151)] =0.0669

k_4 =h $f(x_1+h, y_1+k_3)$=(0.2)[f(0.4,1.1489)]=0.0919

$y_2=1.082+\frac{1}{6}$(0.04328+0.0919+2(0.0662+0.0669)=1.14889

2. **Solve the following using R-K fourth method** $y' = y-x, y(0)=2, h=0.2$ **Find y(0.2).**

Sol: Given $\frac{dy}{dx} = y-x$; y(0) =2

Here f(x, y) = y-x , x_0 = 0, y_0=2 and h = 0.2

$\therefore \quad x_1 = x_0+h = 0+0.2 = 0.2.$

By 4th order R-K method, we have

$y_1 = y_0+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Where k_1=h $f(x_0,y_0)$=(0.2)f(0,2)= 0.2(0.2-0)=0.4

$k_2 = h\, f\,(x_0+\frac{h}{2},\ y_0+\frac{k_1}{2})$

$= (0.2)[f(0.1, 2.2)] = (0.2)(2.2 - 0.1) = 0.42$

$k_3 = h\, f((x_0+\frac{h}{2}, y_0+\frac{k_2}{2})$

$=(0.2)f\,(0.1,2.21)=0.2(2.21-0.1)= 0.422$

$k_4 = h\, f(x_0+h, y_0+k_3)$

$=(0.2)f(0.2,2.422)=0.4444$

Hence $y_1=2+\frac{1}{6}\,[0.4+0.4444+2(0.42+0.422)]$

$\therefore\ y(0.2)=2.4214$

3. Using Runge-Kutta method of second order, find $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{x}$, y(2)=2 , taking h = 0.25 .

Sol: Given $\frac{dy}{dx} = \frac{x+y}{x}$, y(2) = 2 .

Here $f(x, y) = \frac{x+y}{x}$, $x_0 = 2$, $y_0=2$ and $h = 0.25$

$\therefore\ x_1 = x_0+h = 2+0.25 = 2.25$, $x_2 = x_1+h =2.25+0.25 = 2.5$

By R-K method of second order,

$$y_{i+1} = y_i + 1/2\,(k_1 + k_2), k_1 = hf(x_i, y_i), k_2 = hf(x_i + h, y_i + k_1), i = 0,1 \ldots \longrightarrow (1)$$

Step -1:- To find $y(x_1)$i.e y(2.25) by second order R - K method taking i=0 in eqn(i)

We have $y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$

Where $k_1= hf\,(x_0,y_0\,)$, $k_2= hf\,(x_0+h,y_0+k_1)$

$f\,(x_0,y_0\,)=f(2,2)=2+2/2=2$

$k_1=hf\,(x_0,y_0\,)=0.25(2)=0.5$

$k_2= hf(x_0+h,y_0+k_1)=(0.25)f(2.25,2.5)$

$=(0.25)(2.25+2.5/2.25)=0.528$

$\therefore y_1=y(2.25)=2+1/2(0.5+0.528)=2.514$

Step2:

To find $y(x_2)$ i.e., $y(2.5)$

i=1 in (1)

$x_1=2.25, y_1=2.514$, and $h=0.25$

$y_2=y_1+1/2(k_1+k_2)$

where $k_1=h\, f((x_1,y_1)=(0.25)f(2.25,2.514)$

$=(0.25)[2.25+2.514/2.25]=0.5293$

$k_2 = hf(x_1 + h, y_1 + k_1)$

$=(0.25)[2.5+2.514+0.5293/2.5]=0.55433$

$y_2 = y\ (2.5)=2.514+1/2(0.5293+0.55433)=3.0558$

$\therefore y =3.0558$ when $x = 2.5$

4. Obtain the values of y at x=0.1,0.2 using R-K method of

(i)second order (ii)third order (iii)fourth order for the differential equation y'+ y = 0, y(0)=1

Sol: Given $\frac{dy}{dx}= -y$, $y(0)=1$

$f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$

Here $f(x,y) = -y$, $x_0 = 0$, $y_0 = 1$ take $h = 0.1$

$\therefore\ x_1 = x_0+h = 0.1$, $x_2 = x_1+h = 0.2$

Second order:

step1: To find $y(x_1)$ i.e $y(0.1)$ or y_1

by second-order R-K method,we have

$y_1 = y_0 + \frac{1}{2}(k_1+k_2)$

where $k_1=hf(x_0,y_0)=(0.1)\, f(0,1) = (0.1)(-1)= -\,0.1$

$k_2= hf\,(x_0+h,\, y_0+k_1)= (0.1)\, f\,(0.1,\, 1-0.1) = (0.1)(-0.9) = -0.09$

$y_1=y(0.1)=1+\frac{1}{2}\,(-0.1-0.09)=1-0.095=0.905$

$\therefore$ y =0.905 when x=0.1

Step2:

To find y_2 i.e $y(x_2)$ i.e $y(0.2)$

Here $x_1 = 0.1$, $y_1 = 0.905$ and $h=0.1$

By second-order R-K method, we have

$y_2 = y(x_2)= y_1+ \frac{1}{2}(k_1+k_2)$

Where $k_1 = h\ f\left(x_1, y_1\right)$=(0.1)f(0.1,0.905)=(0.1)(-0.905)=-0.0905

$$\begin{aligned} k_2 &= h\ f\left(x_1+h, y_1+k_1\right)=(0.1)\ f\left(0.2, 0.905-0.0905\right) \\ &=(0.1)\ f\left(0.2, 0.8145\right)=(0.1)(-0.8145) \\ &=-0.08145 \end{aligned}$$

$y_2 = y(0.2)=0.905+ \frac{1}{2}\,(-0.0905-0.08145)$

$= 0.905- 0.085975 = 0819025$

(ii) Third order

Step1: To find y_1 i.e $y(x_1)= y(0.1)$

By Third order Runge - Kutta method

$y_{1}= y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$

where $k_1 = h\, f(x_0, y_0) = (0.1)\, f(0,1) = (0.1)(-1) = -0.1$

$$k_2 = h\, f\left(x_0 + h/2, y_0 + k_1/2\right) = (0.1)\, f\left(0.1/2, 1 - 0.1/2\right) = (0.1)\, f\left(0.05, 0.95\right)$$
$$= (0.1)(-0.95) = -0.095$$

and $k_3 = h\, f((x_0+h, y_0+2k_2-k_1)$

$=(0.1)[f\,(0.1, 1 + 2(-0.095) + 0.1)] = -0.905$

Hence $y_1 = 1+\frac{1}{6}(-0.1+4(-0.095)-0.09) = 1+1/6\,(-0.57) = 0.905$

$y_1=0.905$ i.e $y(0.1)= 0.905$

Step2: To find y_2, i.e $y(x_2)= y(0.2)$

Here $x_1=0.1, y_1=0.905$ and $h = 0.1$

Again by 3rd order R-K method

$y_2 = y_1+\frac{1}{6}(k_1+4k_2+k_3)$

Where $k_1=h\, f(x_1, y_1) = (0.1)f\,(0.1, 0.905)= -0.0905$

$k_2 = h\, f\,(x_1+h/2, y_1+k_1/2)=(0.1)f(0.1+0.05, 0.905 - 0.04525)= (0.1)\, f\,(0.15, 0.85975)$

$= (0.1)\,(-0.85975)= -0.085975$

$k_3 = h\, f((x_1+h, y_1+2k_2-k_1)=(0.1)f(0.2, 0.905+2(0.085975)+0.0905= -0.082355$

$y_2 = 0.905+\frac{1}{6}(-0.0905+4(-0.085975)-0.082355)=0.818874$

$\therefore$ y = 0.905 when x = 0.1 and y =0.818874 when x =0.2

iii) Fourth order:

step1: $x_0=0, y_0=1, h=0.1$ To find y_1 i.e $y(x_1)=y(0.1)$

By 4^{th} order R-K method, we have

$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Where $k_1 = h\, f(x_0,y_0) = (0.1)f(0,1) = -0.1$

$k_2 = h\, f\left(x_0+h/2,\ y_0+\frac{k_1}{2}\right) = (0.1)[f(0.05, 0.95)] = (0.1)(-0.95) = -0.095$

$k_3 = h\, f((x_0+h/2, y_0+k_2/2) = (0.1)f(0.1/2, 1-0.095/2) = (0.1)(-0.9525) = -0.09525$

$k_4 = h\, f(x_0+h, y_0+k_3) = (0.1)[f(0.05, 1-0.09525)] = (0.1)f(0.05, 0.90475) = -0.090475$

Hence $y_1 = 1 + \frac{1}{6}(-0.1+2(-0.095)+2(0.09525)-0.090475)$

$= 1 + \frac{1}{6}(-0.570975) = 1-0.0951625 = 0.9048375$

Step2: To find $y_2, i.e., y(x_2) = y(0.2), y_1 = 0.9048375, i.e., y(0.1) = 0.9048375$

Here $x_1 = 0.1$, $y_1 = 0.9048375$ and $h = 0.1$

Again by 4^{th} order R-K method, we have

$y_2 = y_1 + 1/6(k_1+2k_2+2k_3+k_4)$

Where $k_1 = h\, f(x_1,y_1) = (0.1)[f(0.1, 0.9048375)] = -0.09048375$

$k_2 = hf(x_1+h/2, y_1+k_1/2) = (0.1)[f(0.1+0.1/2, 0.9048375 - 0.09048375/2)] = -0.08595956$

$k_3 = hf(x_1+h/2, y_1+k_2/2) = (0.1)[f(0.15, 0.8618577)] = -0.08618577$

$k_4 = h\, f(x_1+h, y_1+k_3) = (0.1)[f(0.2, 0.8186517)] = -0.08186517$

Hence $y_2 = 0.9048375 + \frac{1}{6}(-0.09048375-2(0.08595956)-2(0.08618577)-0.08186517$

$= 0.9048375-0.0861065 = 0.818731$

$y = 0.9048375$ when $x = 0.1$ and $y = 0.818731$ where $x = 0.2$

5. Apply the 4^{th} order R-K method to find an approximate value of y when x=0.2 in steps of 0.1, given that $y' = x^2+y^2$, y (1) = 1.5

Sol: Given $y' = x^2+y^2$, and $y(1) = 1.5$

Here $f(x,y)= x^2+y^2$, $y_0 =1.5$ and $x_0=1,h=0.1$

So that $x_1=1.1$ and $x_2=1.2$

Step1: To find y_1 i.e., $y(x_1)$

by 4th order R-K method we have

$y_1 = y_0+ \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$k_1 = hf(x_0,y_0)=(0.1)f(1,1.5)=(0.1)\,[1^2+(1.5)^2]=0.325$

$k_2 = hf\,(x_0+h/2,y_0+k_1/2)=(0.1)\left[f(1+0.05,1.5+\frac{0.325}{2})\right] =0.3866$

$k_3 = hf((x_0+h/2,y_0+k_2/2)=(0.1)f(1.05,1.5+0.\,3866/2)=(0.1)[(1.05)^2+(1.6933)^2]=0.39698$

$k_4 = hf(x_0+h,y_0+k_3)=(0.1)f(1.05,1.89698)=0.48085$

Hence

$$y_1 = 1.5+\frac{1}{6}\left[0.325+2(0.3866)+2(0.39698)+0.48085\right]$$
$$=1.8955$$

Step2: To find y_2, i.e., $y(x_2)=y(1.2)$

Here $x_1=0.1,y_1=1.8955$ and $h=0.1$

by 4th order R-K method we have

$y_2 = y_1+ \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$k_1= hf(x_1,y_1)=(0.1)f(1.10,1.8955)=(0.1)\,[(1.10)^2+(1.8955)^2]=0.48029$

$k_2 = hf\,(x_1+h/2,y_1+k_1/2)=(0.1)f(1.1+\frac{0.1}{2},1.8937+\frac{0.4796}{2}) =0.58834$

$k_3 = hf((x_1+h/2,y_1+k_2/2)=(0.1)f(1.15,1.8937+\frac{0.58834}{2}) =(0.1)[(1.15)^2+(2.189675)^2]=0.611715$

$k_4 = hf(x_1+h,y_1+k_3)=(0.1)f(1.2,1.8937+0.610728)=0.77261$

Hence y_2=1.8955+1/6(0.48029+2(0.58834)+2(0.611715)+0.7726) =2.5043

$\therefore$ y =2.5043 where x=0.2

6. Use R-K method, to approximate y when x=0.2 given that y' = x+y, y(0)=1

Sol: Here f(x,y) = x + y , y_0 =1, x_0 = 0

Since h is not given for better approximation of y

Take h=0.1

$\therefore$ x_1= 0.1, x_2= 0.2

Step1 To find y_1 i.e $y(x_1)$ = y(0.1)

By R-K method,we have

$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

Where k_1= $hf(x_0,y_0)$ = (0.1)f(0,1) = (0.1) (1) = 0.1

k_2= hf $(x_0+h/2, y_0+k_1/2)$ = (0.1)[f(0.05,1.05)] = 0.11

k_3= hf((x_0+h/2,y_0+k_2/2) = $(0.1)[f(0.05, 1+0.11/2)]$ = (0.1)[(0.05) +(1.055)]= 0.1105

k_4= h f (x_0+h,y_0+k_3) = (0.1)[f(0.1,1.1105)] = (0.1)[0.1+1.1105] = 0.12105

Hence $\therefore\ y_1 = y(0.1) = 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$

y = 1.11034

Step2: To find y_2 i.e $y(x_2)$ = y(0.2)

Here x_1=0.1, y_1=1.11034 and h=0.1

Again By R-K method,we have

$y_2=y_1+1/6(k_1+2k_2+2k_3+k_4)$

k_1= h f(x_1,y_1) = (0.1)[f(0.1,1.11034)] = (0.1) [1.21034] = 0.121034

k_2= h f (x_1+h/2, y_1+k_1/2) = (0.1)[$f(0.1 + 0.1/2, 1.11034 + 0.121034/2)$] = 0.1320857

k_3=h f((x_1+h/2,y_1+k_2/2) = (0.1)[$f(0.15, 1.11034 + 0.1320857/2)$] = 0.1326382

k_4=h f(x_1+h,y_1+k_3) = (0.1)[$f(0.2, 1.11034 + 0.1326382)$] =(0.1)(0.2+1.2429783)=0.1442978

Hence y_2=1.11034+ $\frac{1}{6}$ (0.121034+0.2641714+0.2652764+0.1442978)

=1.11034+0.1324631 =1.242803

y =1.242803 when x=0.2

7. Compute y(0.1) and y(0.2) by R-K method of 4th order for the D.E. y' = xy+y^2, y(0)=1

Sol: Given **y' = xy+y^2 and y(0)=1**

Here f(x,y)= xy+$y^{2,}$ y_0 =1 and x_0=0, h=0.1

So that x_1=0.1 and x_2=0.2

Step1: To find y_1 = y(x_1) = y(0.1)

by 4th order R-K method we have

y_1=y_0+ $\frac{1}{6}$ (k_1+2k_2+2k_3+k_4)

k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1) [0+1]=0.1

k_2= hf (x_0+h/2,y_0+k_1/2)=(0.1)[$f(0.05, 1.05)$] =0.1155

k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05, 1.05775)=0.11217

k_4=hf(x_0+h,y_0+k_3)=(0.1)f(0.1, 1.11217)=0.1248

Hence y_1 = y(0.1) = y_0 + $\frac{1}{6}$ [k_1+2k_2+2k_3+k_4]

= 1+ $\frac{1}{6}$ [0.1+0.0231+0.22434+0.1248]

= 1.1133

Step2: To find y_2 = y(x_2) = y(0.2)

Here x_1=0.1,y_1=1.1133 and h=0.1

by 4th order R-K method we have

$y_2 = y_1 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

k_1=hf(x_1,y_1)=(0.1)f(0.1, 1.1133)=0.1351

k_2= hf (x_1+h/2,y_1+k_1/2)=(0.1)f(0.15, 1.18085) =0.1571

k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15, 1.19185) =0.1599

k_4=hf(x_1+h,y_1+k_3)=(0.1)f(0.2, 1.2732) =1.1876

Hence $y_2 = y(0.2) = y_1 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

= 1.1133+1/6(0.1351+0.3142+0.3198+0.1876)

= 1.2728

8. Find y(0.1) and y(0.2) by R-K method of 4th order for the D.E. $y' = x^2 - y$ and y(0)=1

Sol: Given $y' = x^2 - y$ **and y(0)=1**

Here f(x,y)= $x^2 - y$, y_0 =1 and x_0=0, h=0.1

So that x_1=0.1 and x_2=0.2

Step1: To find $y_1 = y(x_1) = y(0.1)$

by 4th order R-K method we have

$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$

k_1=hf(x_0,y_0)=(0.1)f(0,1)=(0.1) [0-1]=-0.1

k_2= hf (x_0+h/2,y_0+k_1/2)=(0.1)[f(0.05, 0.95)] =-0.09475

k_3=hf((x_0+h/2,y_0+k_2/2)=(0.1)f(0.05, 0.952625)=-0.095

k_4=hf(x_0+h,y_0+k_3)=(0.1)f(0.1, 0.905)=-0.0895

Hence $y_1 = y(0.1) = y_0 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$$= 1+\frac{1}{6}[-0.1-0.1895-0.19-0.0895] = 0.9052$$

Step2: To find $y_2 = y(x_2) = y(0.2)$

Here $x_1=0.1, y_1=0.9052$ and $h=0.1$

by 4th order R-K method we have

$y_2 = y_1+\frac{1}{6}(k_1+2k_2+2k_3+k_4)$

$k_1=hf(x_1,y_1)=(0.1)f(0.1, 0.9052)=-0.08952$

$k_2= hf(x_1+h/2,y_1+k_1/2)=(0.1)f(0.15, 0.86044) =-0.08379$

$k_3=hf((x_1+h/2,y_1+k_2/2)=(0.1)f(0.15, 0.8633) =-0.0841$

$k_4=hf(x_1+h,y_1+k_3)=(0.1)f(0.2, 0.8211) =-0.07811$

Hence $y_2 = y(0.2) = y_1 + \frac{1}{6}[k_1+2k_2+2k_3+k_4]$

$$= 0.9052+\frac{1}{6}(-0.08952-0.16758-0.1682-0.07811) = 0.8213$$

CURVE FITTING

Method of Least Squares:

Suppose that a data is given in two variables x & y the problem of finding an analytical expression of the form $y = f(x)$ which fits the given data is called curve fitting.

Let $(x_1, y_1),(x_2, y_2)........(x_n, y_n)$ be the observed set of values in an experiment and $y = f(x)$ be the given relation $x \& y, Let\ E_1, E_2, ...E_x$ are the error of approximations then we have

$$E_1 = y_1 - f(x_1)$$
$$E_2 = y_2 - f(x_2)$$
$$E_3 = y_3 - f(x_3)$$

$E_n = y_n - f(x_n)$ Where $f(x_1), f(x_2)...........f(x_n)$ are called the expected values of y corresponding to $x = x_1, x = x_2........x = x_n$

$y_1, y_2......y_n$ are called the observed values of y corresponding to $x = x_1, x = x_2........x = x_n$ the differences $E_1, E_2.....E_n$ between expected values of y and observed values of y are called the errors, of all curves approximating a given set of points, the curve for which $E = E_1^{\ 2} + E_2^{\ 2} + ...E_n^{\ 2}$ is a minimum is called the best fitting curve (or) the least square curve, This is called the method of least squares (or) principles of least squares

I. Fitting of a Straight Line:-

Let the straight line be $y = a + bx \rightarrow (1)$

Let the straight line (1) passes through the data points $(x_1, y_1),(x_2, y_2)......(x_n, y_n) i.e., (x_i, y_i), i = 1, 2....n$

So we have $y_i = a + bx_i \quad \rightarrow (2)$

The error between the observed values and expected values of $y = y_i$ is defined as

$$E_i = y_i - (a + bx_i), i = 1,2 \ldots \ldots . . n \ \rightarrow (3)$$

The sum of squares of these errors is

$E = \sum_{i=1}^{n} E_1^2 = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$ Now for E to be minimum

$$\frac{\partial E}{\partial a} = 0; \frac{\partial E}{\partial b} = 0$$

These equations will give normal equations

$$\sum_{i=1}^{n} y_i = na + b\sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2$$

The normal equations can also be written as

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Solving these equation for a, b substituting in (1) we get required line of best fit to the given data.

II. Non Linear Curve Fitting

1. Parabola

Let the equation of the parabola is $y = a + bx + cx^2$ ——(1)

The parabola (1) passes through the data points $(x_1, y_1), (x_2, y_2) \ldots\ldots\ldots (x_n, y_n), i.e., (x_1, y_1); i = 1,2 \ldots\ldots n$

We have $y_i = a + bx_i + cx_i^2 \quad \longrightarrow (2)$

The error E_i between the observed an expected value of $y = y_i$ is defined as

$E_i = y_i - (a + bx_i + cx_i^2), i = 1,2,3 \ldots\ldots n \quad \to (3)$

The sum of the squares of these errors is

$$E = \sum_{i=1}^{n} E_1^2 = \sum_{i=1}^{n} (y_i - a - bx_i - cx_i^2)^2 \quad \to (4)$$

for E to be minimum, we have

$\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0, \frac{\partial E}{\partial c} = 0$

The normal equations can also be written as

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Solving these equations for a, b, c and satisfying (1) we get required parabola of best fit

2. Power Curve

The power curve is given by $y = ax^b \to (1)$

Taking logarithms on both sides $\log_{10}y = \log_{10}a + b\log_{10}x$

(or) Y = A+ bX → (2) $where\ Y = log_{10}y, A = log_{10}a\ and\ X = log_{10}x$

Equation (2) is a linear equation in X & Y

∴ The normal equations are given by

$$\Sigma Y = nA + b\Sigma X$$

$$\Sigma XY = A\Sigma X + b\Sigma x^2$$

From these equations, the values A and b can be calculated then a = antilog (A) substitute a & b in (1) to get the required curve of best fit.

3. Exponential Curve: $(1)\, y = ae^{bx}$ $(2)\, y = ab^{x}$

1. $y = ae^{bx} \rightarrow (1)$

Taking logarithms on both sides $log_{10}y = log_{10} + bxlog_{10}e$

$(or) Y = A + BX$ → (2) Where $Y = log_{10}y, A = log_{10}a, B = blog_{10}e$ & $X = x$

Equation (2) is a linear equation in X and Y

So the normal equation are given by

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Solving the equation for A & B, we can find

$$a = anti\log A\ \&\ b = \frac{B}{\log_{10} e}$$

Substituting the values of a and b so obtained in (1) we get

The curve of best fit to the given data.

2. $y = ab^{x} \rightarrow (1)$

Taking log on both sides $log_{10}y = log_{10}a + xlog_{10}b$

$$(or)\ Y = A + BX \quad \rightarrow (2)$$

Where $Y = log_{10}y, A = log_{10}a, B = blog_{10}b$ & $x = X$

The normal equation (2) are given by

$$\Sigma Y = nA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Solving these equations for A and B we can find $a = anti\log A, b = anti\log B$

Substituting a and b in (1)

Solved Problems:

1. By the method of least squares, find the straight line that best fits the following data

x	1	2	3	4	5
y	14	27	40	55	68

Sol: The values of $\Sigma x, \Sigma y, \Sigma x^2$ *and* Σxy are calculated as follows

x_i	y_i	x_i^2	$x_i y_i$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340

$\Sigma x_i = 15; \Sigma y_i = 204; \Sigma x_i^2 = 55 \ and \ \Sigma x_i \Sigma y_i = 748$

The normal equations are

$\Sigma y = na + b\Sigma x \ \longrightarrow (1) \quad \Sigma xy = a\Sigma x + b\Sigma x^2 \ \longrightarrow (2)$

Solving we get $a = 0, b = 13.6$

Substituting these values a & b we get

$y = 0 + 13.6x \Rightarrow y = 13.6x$

2. Fit a straight line y=a+bx from data

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

Sol: Let the required straight line be y=a+bx…(1)

x	y	x^2	xy
0	1	0	0
1	1.8	1	1.8
2	3.3	4	6.6
3	4.5	9	13.5
4	6.3	16	25.2
$\sum x = 10$	$\sum y = 16.9$	$\sum x^2 = 30$	$\sum xy = 47.1$

Normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

5a+10b=16.9

10a+30b=47.1

Solving we get a=0.72; b=1.33.

∴ The straight line is $y = 0.72 + 1.33x$

3. Fit a straight line y = a + bx from data

x	**0**	**5**	**10**	**15**	**20**
y	**7**	**-11**	**16**	**20**	**26**

Sol: Let the required straight line be y = a + bx…(1)

x	y	x^2	xy
0	7	0	0
5	-11	25	-55
10	16	100	160
15	20	225	300
20	26	400	520
$\sum x = 50$	$\sum y$ ==58	$\sum x^2$=750	$\sum xy$ =925

Normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

5a+50b=58

50a+750b=925

Solving we get a=-2; b=1.36.

∴ The straight line is $y = -2 + 1.36x$

4. Fit a straight line y=a+bx from data

x	**0**	**5**	**10**	**15**	**20**	**25**
y	**12**	**15**	**17**	**22**	**24**	**30**

Sol: Let the required straight line be y=a+bx…(1)

x	y	x^2	xy
0	12	0	0
5	15	25	75
10	17	100	170
15	22	225	330
20	24	400	480
25	30	625	750
$\sum x = 75$	$\sum y=120$	$\sum x^2=1375$	$\sum xy=1805$

Normal equations are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

6a+75b=58

x	**1**	**2**	**3**	**4**	**5**
y	**4**	**3**	**6**	**7**	**11**

75a+1375b=1805

Solving we get a=11.2862; b=0.6971.

$\therefore$ *The straight line is* $y = 11.2862 + 0.6971x$

5. Fit a straight line and a parabola to the following data and find out which one is most appropriate. Give your reason for the conclusion

Sol: Let the required straight line be y=a+bx…(1)

x	y	x^2	x^3	x^4	xy	x^2y
1	4	1	1	1	4	4
2	3	4	8	16	6	12
3	6	9	27	81	18	54
4	7	16	64	256	28	112
5	11	25	125	625	55	275
$\sum x=15$	$\sum y=31$	$\sum x^2=55$	$\sum x^3=225$	$\sum x^4=979$	$\sum xy=111$	$\sum x^2y=457$

Normal equations for fitting a straight line are

$$\sum y = na + b\sum x$$
$$\sum xy = a\sum x + b\sum x^2$$

Substitute in above we get

5a+15b=31

15a+55b=111

Solving we get a=0.8 ; b=1.8.

$$\therefore \text{The straight line is } y = 0.8 + 1.8x$$

Let the required parabola be $y=a+bx+cx^2$…(2)

Normal equations for fitting a parabola are

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$
$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Substituting values, we get

5a+15b+55c =31

15a+55b+225c = 111

55a+225b+979c =457

Solving we get a=4.7998;b=-1.6284;c=0.5714

∴The parabola fit is $4.7998x^2$-1.6284x+0.5714

Conclusion: Clearly parabola fit is best fit because error is near to zero than linear fit.

y	Error of linear fit E=y-f(x)	Error parabola fit E=y-g(x)
4	1.4	0.2572
3	-1.4	-0.8286
6	-0.2	0.9428
7	-1	-0.4286
11	1.2	0.0572

6. Fit a second degree parabola to the following data

x	**0**	**1**	**2**	**3**	**4**
y	**1**	**5**	**10**	**22**	**38**

Sol: Equation of parabola $y = a + bx + cx^2 \rightarrow (1)$

Normal equations $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \longrightarrow (2)$

x	y	xy	x^2	x^2y	x^3	x^4
0	1	0	0	0	0	0
1	5	5	1	5	1	1
2	10	20	4	40	8	16
3	22	66	9	198	27	81
4	38	152	16	608	64	256
$\sum x = 10$	$\sum y = 76$	$\sum xy=243$	$\sum x^2 = 30$	$\sum x^2 y = 851$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$

Normal equations

$$76 = 5a + 10b + 30c$$
$$243 = 10a + 30b + 100c$$
$$851 = 30a + 100b + 354c$$

Solving $a = 1.42, b = 0.26, c = 2.221$

Substitute in (1) $\Rightarrow y = 1.42 + 0.26x + 2.221x^2$

7. Fit a second degree parabola to the following data:

x	**0**	**1**	**2**	**3**	**4**
f(x)	**1**	**1.8**	**1.3**	**2.5**	**6.3**

Sol:

Let the equation of the parabola be Y= a + b x + c x^2 -----(1)

The normal equations are given by $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$-----(2)

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

x	y	x^2	x^3	x^4	xy	x^2y
0	1.0	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
$\sum x = 10$	$\sum y = 12.9$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 37.1$	$\sum x^2y = 130.3$

Since there are 5 pairs of values so n=5 substituting the above values in (2) we get

12.9 = 5a +10b +30c

37.1 = 10a+30b+100c

130.3 = 30a+100b+354c

Solving the above equations we get a = 14.2, b = -1.07, c = 0.55

Substituting the above values in (1) y = 14.2-1.07x + 0.55x^2

Which is the required equation of the parabola.

8. Fit a parabola $y = a + bx + cx^2$ to the data given below

x:	**1**	**2**	**3**	**4**	**5**
y:	**10**	**12**	**8**	**10**	**14**

Sol: Let the equation of the parabola be Y= a + b x + c x^2 -----(1)

The normal equations are given by $\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$-----(2)

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

x	y	x^2	x^3	x^4	xy	x^2y
1	10	1	1	1	10	10
2	12	4	8	16	24	48
3	8	9	27	81	24	72
4	10	16	64	256	40	160
5	14	25	125	625	70	350
$\sum x=15$	$\sum y = 54$	$\sum x^2 = 55$	$\sum x^3 = 225$	$\sum x^4 = 979$	$\sum xy = 168$	$\sum x^2y = 640$

Since there are 5 pairs of values so n=5 substituting the above values in (2) we get

$$54 = 5a + 15b + 55c$$

$$168 = 15a + 55b + 225c$$

$$640 = 55a + 225b + 979c$$

Solving the above equations we get a =14, b =-3.6857 , c = 0.7142

substituting the above values in (1) y = 14-3.6857x + 0.7142x^2

which is the required equation of the parabola.

9. Fit a parabola of the form $y = ax^2 + bx + c$

x:	1	2	3	4	5	6	7
y:	2.3	5.2	9.7	16.5	29.4	35.5	54.4

Sol: Let the equation of the parabola be $y = ax^2 + bx + c$ -----(1)

The normal equations are given by

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \text{-----(2)}$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Table for calculations:

x	y	x^2	x^3	x^4	xy	x^2y
1	2.3	1	1	1	2.3	2.3
2	5.2	4	8	16	10.4	20.8
3	9.7	9	27	81	29.1	87.3
4	16.5	16	64	256	66	264
5	29.4	25	125	625	147	735
6	35.5	36	216	1296	213	1278
7	54.4	49	343	2401	380.8	2665.6
28	153	140	784	4676	848.6	5053

Since there are 5 pairs of values so n=5 substituting the above values in (2) we get

153 = 7a +28b +140c

848.6 = 28a+140b+784c

5053 = 140a+784b+4676c

Solving the above equations we get a =2.3705, b =-1.0924, c = 1.1928

substituting the above values in (1) y = 1.1928 x^2-1.0924 x+2.3705

which is the required equation of the parabola.

10. Fit a curve $y = ax^b$ to the following data

x	**1**	**2**	**3**	**4**	**5**	**6**
y	**2.98**	**4.26**	**5.21**	**6.10**	**6.80**	**7.50**

Sol: Let the equation of the curve be $y = ax^b \rightarrow (1)$

Taking log on both sides $\log y = \log a + b \log x$

(or) $Y = A + bX \quad \longrightarrow (2)$ Where $Y = \log y, A = \log a, X = \log x$

The Normal Equations are $\Sigma Y = nA + b\Sigma X$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \quad \longrightarrow (3)$$

x	$X = \log x$	y	$Y = log\ y$	XY	X^2
1	0	2.98	0.4742	0	0
2	0.3010	4.26	0.6294	0.1894	0.0906
3	0.4771	5.21	0.7168	0.3420	0.2276
4	0.6021	6.10	0.7853	0.4728	0.3625
5	0.6990	6.80	0.8325	0.5819	0.4886
	$\Sigma X = 2.8574$		$\Sigma Y = 4.3133$	$\Sigma XY = 2.2671$	$\Sigma X^2 = 1.7749$

4.3313=6A+208574b and 2.2671=2.8574A+1.7749b

Solving A=0.4739, b=0.5143

A=anti log (A) =2.978

$$\therefore y = 2.978x^{0.5143}$$

11 . Fit a curve $y = ab^x$

x	**2**	**3**	**4**	**5**	**6**
y	**144**	**172.8**	**207.4**	**248.8**	**298.5**

Sol: Let the curve to be fitted is $y = ab^x$

Taking log on both sides $\log y = \log a + x \log b \quad \longrightarrow (1)$

$$Y = A + xB \quad \longrightarrow (2)$$

$$Y = \log y, A = \log a, B = \log b$$

$$\Sigma Y = nA + B\Sigma x$$

$$\Sigma xY = A\Sigma x + b\Sigma x^2 \quad \longrightarrow (3)$$

x	y	$x2$	$Y = \log y$	xy
2	144.0	4	2.1584	4.3168
3	172.8	9	2.2375	6.7125
4	207.4	16	2.3168	9.2672
5	248.8	25	2.3959	11.9795
6	298.5	36	2.4749	14.8494

Substituting these values the normal equations are

11.5835 = 5A + 20B

47.1254 = 20A+90B

Soving A and B, taking antilogarithms

a=100, b=1.2

Substituting in (1), the equation of the curve is y =100(1.2)^{x}

$= 36.744$ Square units

UNIT-III

PARTIAL DIFFERENTIAL EQUATIONS

Introduction

Partial differential equations are used to mathematically formulate ,and thus aid the solution of physical and other problems involving functions of several variables ,such as the propagation of heat or sound , fluid flow , elasticity , electro statistics, electro dynamics, etc.

Fluid mechanics, heat and mass transfer, and electromagnetic theory are all modeled by partial differential equations and all have plenty of real life applications.

For example,

- Fluid mechanics is used to understand how the circulatory system works, how to get rockets and planes to fly, and even to some extent how the weather behaves.
- Heat and mass transfer is used to understand how drug delivery devices work, how kidney dialysis works, and how to control heat for temperatute-sensitive things. It probably also explains why thermoses work!
- Electromagnetism is used for all electricity out there, and everything that involves light at all, from X rays to pulse oximetry and laser pointers.

Definition:

An equation which involves a dependent variable and its derivatives with respect to two or more independent variables is called partial differential equation.

Ex: $x\frac{\partial z}{\partial y} + 4y\frac{\partial z}{\partial x} = 2z + 3xy$

Notations: $p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}, s = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

Linear & non linear P.D.E:

If the partial derivatives of the dependent variable occur in first degree only and separately, Such a P.D.E is said to the linear P.D.E, otherwise it is said as non –linear P.D.E

Formation of partial differential equations:

Partial Differential equations can be formed by two methods

1.By the elimination of arbitrary constants

2.By the elimination of arbitrary functions

1.By elimination of arbitrary constants

Let the given function be $f(x,y,z,a,b)=0$ ………. (1) where a and b are arbitrary constants.

To eliminate a and b, differentiating (1) partially w.r.t. 'x' and 'y'

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0 \Rightarrow \frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.p=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(2) \text{ and}$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial y}=0 \Rightarrow \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}.q=0 \;\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Now eliminate the constants a and b from (1), (2) and (3). We get a partial differential equation of the first order of the form. $\phi(x,y,z,p,q)=0$

Note : 1. If the number of arbitrary constants is equal to the number of variables, a partial differential equation of first order can be obtained.

2.If the number of arbitrary constants is greater than the number of variables, a partial differential equation of order higher than one can be obtained.

Solved Problems

1. **Form the partial differential equation by eliminating the arbitrary constants a and b from (i) $z = ax + by + ab$**

Sol: we have $z = ax + by + ab$ ……. (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial z}{\partial x}=a \Rightarrow p=a \ldots\ldots\ldots (2) \text{ and } \frac{\partial z}{\partial y}=b \Rightarrow q=b \ldots\ldots\ldots\ldots(3)$$

Putting the values of a and b from equation (2) and (3) in (1), we get

$$\boxed{z = px + qy + pq}$$

Which is the required partial differential equation

2. **Form the partial differential equation by eliminating the arbitrary constants a and b from (a)** $z=ax+by+a^2+b^2$ (b) $z=ax+by+\dfrac{a}{b}-b$

Sol: (a) we have $z=ax+by+a^2+b^2$ ……. (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial z}{\partial x}=a \Rightarrow p=a \ldots\ldots\ldots (2) \text{ and } \frac{\partial z}{\partial y}=b \Rightarrow q=b \ldots\ldots\ldots\ldots(3)$$

Putting the values of a and b from equation (2) and (3) in (1), we get

$$\boxed{z = px + qy + p^2 + q^2}$$

Which is the required partial differential equations

(b) We have $z = ax + by + \frac{a}{b} - b$ ....(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$\frac{\partial z}{\partial x} = a \Rightarrow p = a$ ..... (2) and $\frac{\partial z}{\partial y} = b \Rightarrow q = b$ .... (3)

Putting the values of a and b from equation (2) and (3) in (1), we get

$$\boxed{z = px + qy + \frac{p}{q} - q}$$

Which is the required partial differential equation.

3. Form the partial differential equation by eliminating the arbitrary constants from

$(x-a)^2 + (y-b)^2 + z^2 = r^2$

(OR)

Find the differential equation of all spheres of fixed radius having their centre on the xy −plane.

Sol: The equation of sphere of radius r having their centers on xy-plane is

$(x-a)^2 + (y-b)^2 + z^2 = r^2$ ....(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get.

$2(x-a) + 2z.\frac{\partial z}{\partial x} = 0 \Rightarrow (x-a) + zp = 0 \, or \, x - a = -zp \rightarrow$ (2)

And $2(y-b) + 2z.\frac{\partial z}{2y} = 0 \, or \, (y-b) + zq = 0 \, or \, y - b = -zq \rightarrow$ (3)

Putting the values of (x-a) and (y-b) from (2) and (3) in (1), we get

$$\boxed{(-zp)^2 + (-zq)^2 + z^2 = r^2}$$

Which is the required partial differential equation.

4. **Form the partial differential equation by eliminating the arbitrary constants a and b from $z = (x + a)(y + b)$**

Sol: The given equation $z = (x + a)(y + b)$ −−−−−−−−−−(1)

Differentiating (1) w.r.t., x

P = $\frac{\partial z}{\partial x} = 1.(y + b)$ −−−−−−−−−−(2)

Differentiating (1) w.r.t., y

q = $\frac{\partial z}{\partial y} = 1.(x + a)$ −−−−−−−−− (3)

from (2) $P = (y + b)$

$$from\ (3)\ q = (x + a)$$

Substituting in (1) we get

$$\boxed{z = p.q}$$

Which is the required partial differential equations

5. Form the partial differential by eliminating the arbitrary constants from $\log(az-1) = x + ay + b$

Sol: We have $\log(az-1) = x + ay + b$ …………(1)

Differentiating (1) partially $w.r.t.$ 'x' and 'y', we get

$$\frac{1}{(az-1)}.a.\frac{\partial z}{\partial x} = 1\,or\,\frac{1}{(az-1)}ap = 1\,or\,ap = az-1 \text{ ……………(2)}$$

$$\text{and } \frac{1}{(az-1)}a.\frac{\partial z}{\partial y} = a \Rightarrow aq = (az-1)a \text{ …………(3)}$$

$$(3)\div(2), gives\,\frac{q}{p} = a \Rightarrow ap = q \text{ ………….(4)}$$

Putting (4) in (2), we get

$$\boxed{q = \frac{q}{p}z - 1\,or\,pq = qz - p\,or\,p(q+1) = q^2}$$

Which is the required partial differential equation.

6.**Form the differential equation by eliminating a and b from** $2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$

Sol: We have $2z = (x+a)^{\frac{1}{2}} + (y-a)^{\frac{1}{2}} + b$ …………..(1)

Differentiating (1) partially w.r.t. 'x' and 'y', we have,

$$2\frac{\partial z}{\partial x} = 2p = \frac{1}{2\sqrt{x+a}} \Rightarrow \frac{1}{\sqrt{x+a}} = 4p$$

$$\text{or } \sqrt{x+a} = \frac{1}{4p}$$

$$\text{or } x + a = \frac{1}{16p^2} \rightarrow (2)$$

$$\text{And } 2\frac{\partial z}{\partial y} = \frac{1}{2\sqrt{y-a}}\,or\,2q = \frac{1}{2\sqrt{y-a}}\,or\,\sqrt{y-a} = \frac{1}{4q}$$

$$\therefore y - q = \frac{1}{16q^2} \rightarrow \text{ ……….. (3)}$$

Adding (2) and (3), we get

$$x+y=\frac{1}{16}\left(\frac{1}{p^2}+\frac{1}{q^2}\right)$$

$$\boxed{or\,16(x+y)p^2q^2=p^2+q^2}$$

Which is the required partial differential equation.

7.Form the partial differential equation by eliminating the arbitrary constants a and b from $z=ax^3+by^3$

Sol: We have $z=ax^3+by^3 \to (1)$

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial z}{\partial x}=3ax^2 \text{ or } p=3ax^2 \Rightarrow a=\frac{p}{3x^2} \to (2)$$

And $\dfrac{\partial z}{\partial y}=3by^2 \text{ or } q=3by^2 \Rightarrow b=\dfrac{q}{3y^2} \to (3)$

Putting the values of 'a' and 'b' from (2) and (3) in (1), we get

$$z=\frac{p}{3}x+\frac{q}{3}y$$

Or

$$\boxed{3z=px+qy}$$

8.Form the partial differential equation by eliminating the arbitrary constants a and b from $\boldsymbol{z=(x^2+a)(y^2+b)}$

Sol: The given equation $z=(\boldsymbol{x^2+a})(\boldsymbol{y^2+b})$ −−−−−−−−−−(1)

Differentiating (1) w.r.t., x

$$p=\frac{\partial z}{\partial x}=2x(y^2+b) \text{ −−−−−−−−−−(2)}$$

$$\therefore (y^2+b)=\frac{p}{2x}$$

Differentiating (1) w.r.t., y,we get

$$q=\frac{\partial z}{\partial y}=2y\,(x^2+a) \text{ −−−−−−−−−(3)}$$

$$\therefore (x^2+a)=\frac{q}{2y}$$

Substituting in (1) we get $z=\frac{pq}{4xy}$ implies that

$$\boxed{pq-4xyz=0}$$

Which is the required partial differential equation.

9.Form the partial differential equation by eliminating the arbitrary constants from

$$(x-a)^2+(y-b)^2=z^2\cot^2\alpha$$

Sol: Given $(x-a)^2+(y-b)^2=z^2\cot^2\alpha$……..(1)

Differentiating (1) w.r.t., x

$(x-a)=z\,\mathrm{p}\cot^2\alpha$

Differentiating (1) w.r.t., y

$(y-b)=z\,\mathrm{q}\cot^2\alpha$

Substituting (2),(3) in (1),we get

$(z\,\mathrm{p}\cot^2\alpha)^2+(z\,\mathrm{q}\cot^2\alpha)^2=z^2\cot^2\alpha$

∴The required Partial differential equation is

$\boxed{p^2+q^2=\tan^2\alpha}$

10.Form the partial differential equation by eliminating a,b,c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol : Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ….. (1)

Differentiating (1) partially w.r.t. 'x' and 'y'.

$\frac{2x}{a^2}+\frac{2z}{c^2}.p=0$ or $\frac{x}{a^2}+\frac{z}{c^2}.p=0\to$ (2)

And $\frac{2y}{b^2}+\frac{2z}{c^2}.q=0$ or $\frac{y}{b^2}+\frac{z}{c^2}.q=0\to$ (3)

Since it is not possible to eliminate a,b,c from equation (1), (2) and (3). We require one more relation.

Differentiating (2), partially w.r.t. 'x', we get

$\frac{1}{a^2}+\frac{1}{c^2}\left(z.\frac{\partial p}{\partial x}+p.\frac{\partial z}{\partial x}\right)=0$ or $\frac{1}{a^2}+\frac{1}{c^2}.z.\frac{\partial^2 z}{\partial x^2}+\frac{1}{c^2}.p$

$\therefore \frac{1}{a^2}+\frac{1}{c^2}.zr+\frac{p^2}{c^2}=0\to$ (4)

Multiplying (4) by 'x' and then subtracting (2) from it, we get

$\frac{xz}{c^2}.r+\frac{xp^2}{c^2}-\frac{z}{c^2}.p=0$ or $\frac{1}{c^2}\left(xzr+xp^2-zp\right)=0$

$\boxed{\therefore pz=xp^2+xzr}$

Which is the required partial differential equation.

Formation of the partial differential equation by the elimination of arbitrary functions:

Derive a p.d.e by the elimination of the arbitrary function ϕ from $\phi(u,v)=0$ where u,v are functions of x,y and z.

$\phi(u,v)=0$ …. (1)

Differentaite partially equation (1) w.r.to. x,y

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}.\frac{\partial z}{\partial x}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}.\frac{\partial z}{\partial x}\right)=0$$

i.e., $\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)=0 \ldots\ldots(2)$

and $\frac{\partial \phi}{\partial 4}\left(\frac{\partial 4}{\partial y}+q\frac{\partial 4}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)=0 \ldots\ldots(3)$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$ from (2) and (3)

$$\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}p\right)\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)$$

i.e. $\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}+\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$

is the P.D.E after the elimination of ϕ from $\phi(u,v)=0$. Written in a simpler form

$$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}$$

Above equation is generally written as pP+qQ=R where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y},\ Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\ \text{and}\ R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

Solved Problems

1.Form a partial differential equation by eliminating the arbitrary function

$$\boldsymbol{z=f(x^2+y^2)}$$

Sol: We have $z=f(x^2+y^2)$ …..(1)

Put $u=x^2-y^2$, we have $z=f(u)\to$ (2)

Differentiating (2) partially w.r.t. 'x' and 'y',

$$\frac{\partial z}{\partial x}=f^1(u).\frac{\partial u}{\partial x}=f^1(u).2x$$

$\therefore p=f^1(u)2x\to$ (3)

And $\frac{\partial z}{\partial y}=f^1(u).\frac{\partial u}{\partial y}=f^1(u).2y$

$\therefore q = f^1(u)2y \rightarrow (4)$

$\therefore (3) \div (4)$, gives $\frac{p}{q} = \frac{f^1(u).2x}{f^1(u)2y} = \frac{x}{y}$

$\boxed{\therefore py - qx = 0}$

Which is the required partial differential equation.

2. Form a partial differential equation by eliminating the arbitrary function $\boldsymbol{\varphi(x^2 + y^2, z - xy) = 0}$

Sol: Given $\varphi(x^2 + y^2, z - xy) = 0$

This can be written as $z - xy = f(x^2 + y^2)$--------------(1)

Now we have to eliminate f from (1)

Differentiating (1) w.r.t., x

$$\frac{\partial z}{\partial x} - y = f'(x^2 + y^2)(2x)$$

$$p - y = f'(x^2 + y^2)(2x) \text{--------(2)}$$

Differentiating (2) w.r.t., y

$q - x = f'(x^2 + y^2)(2y)$--------(3)

Dividing (2) by (3)

$\boxed{py - qx = y^2 - x^2}$

Which is the required partial differential equation.

3. Form a partial differential equation by eliminating the arbitrary function from $z = f\left(x^2 - y^2\right)$

Sol : We have $z = f\left(x^2 - y^2\right) \rightarrow (1)$

Put $u = x^2 - y^2$, we have $z = f(u) \rightarrow (2)$

Differentiating (2) partially w.r.t. 'x' and 'y',

$$\frac{\partial z}{\partial x} = f^1(u).\frac{\partial u}{\partial x} = f^1(u).2x$$

$\therefore p = f^1(u)2x \rightarrow (3)$

Similarly we get

$q = -f^1(u)2y(4)$

$\therefore (3) \div (4)$, gives $\frac{p}{q} = \frac{x}{-y}$

$\boxed{\therefore px + qy = 0}$

Which is the required partial differential equation.

4. Form the partial differential equation by eliminating the arbitrary functions from

$$xyz = f\left(x^2+y^2+z^2\right)$$

Sol: We have $xyz = f\left(x^2+y^2+z^2\right) \rightarrow$ (1)

Differentiating (1) partially w.r.t. x and y

$$yz + xy.p = f^1\left(x^2+y^2+z^2\right).\left(2x+2z.\frac{\partial z}{\partial x}\right)$$

(or) $yz + xyp = f^1\left(x^2+y^2+z^2\right).(2x+2zp) \rightarrow$ (2)

And $xz + xy.q = f^1\left(x^2+y^2+z^2\right).(2y+2z.q) \rightarrow$ (3)

$\therefore (2)\div(3)$, gives

$$\frac{yz+xyp}{xz+xyq} = \frac{2x+2zp}{2y+2zq}$$

$$(yz+xyp)(y+zq) = (xz+xyq)(x+zp)$$

$$y^2z + z^2yq + xy^2p + xyzpq = x^2z + x^2zp + x^2yq + xyzpq$$

$$\boxed{x\left(y^2-z^2\right)p + y\left(z^2-x^2\right)q = \left(x^2-y^2\right)z}$$

Which is the required partial differential equation.

5. **Form the partial differential equation by eliminating the arbitrary functions from $xyz = f(x+y+z)$**

Sol: Given equations $xyz = f(x+y+z)$---------(1)

Differentiating (1) partially w.r.t. 'x'

y(xp+z) = $f'(x+y+z)(1+p)$----------(2)

Differentiating (1) partially w.r.t. 'x'

x(y q + z) = $f'(x+y+z)(1+q)$------------(3)

Dividing (2) by (3) $\frac{y(xp+z)}{x(yq+z)} = \frac{1+p}{1+q}$

$$Y(xp+z)(1+q) = x(yq+z)(1+p)$$

$$(xy-zx)p + (yz-xy)q = zx - yz$$

$$\boxed{x(y-z)p + y(z-x)q = z(x-y)}$$

Which is the required partial differential equation.

6. Form the partial differential equation by eliminating the arbitrary function

from $xy+yz+zx=f\left(\frac{z}{x+y}\right)$

Sol : We have $xy+yz+zx=f\left(\frac{z}{x+y}\right)\rightarrow$ (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$y+y.p+z+x.p=f^1\left(\frac{z}{x+y}\right)\frac{[(x+y).p-z]}{(x+y)^2}\rightarrow \text{ (2)}$$

And $x+z+yq+xq=f^1\left(\frac{z}{x+y}\right)\frac{[(x+y)q-z]}{(x+y)^2}\rightarrow$ (3)

Dividing (2) by (3), we get

$$\boxed{\frac{(x+y)p+y+z}{(x+y)q+x+z}=\frac{(x+y)p-z}{(x+y)q-z}}$$

is the required partial differential equation.

7.Form the partial differential equation by eliminating the arbitrary function

from $z=f(x)+e^y.g(x)$

Sol: We have $z=f(x)+e^y.g(x)\rightarrow$ (1)

Differentiating (1) partially w.r.t. 'x' and y, we get

$$\frac{\partial z}{\partial x}=f^1(x)+e^y.g^1(x) \text{ or } p=f^1(x)+e^y.g^1(x)\rightarrow \text{ (2)}$$

And $q=e^y.g(x)$ or $\frac{\partial z}{\partial y}=e^y.g(x)\rightarrow$ (3)

Differentiating (3), partially w.r.t. 'y', we get

$$\frac{\partial^2 z}{\partial y^2}=e^y.g(x)=\frac{\partial z}{\partial y} \text{ [using (3)]}$$

$$\therefore \frac{\partial^2 z}{\partial y^2}-\frac{\partial z}{\partial y}=0$$

$$\boxed{\therefore t-q=0}$$

Which is the required P.D.E.

8.Form a partial differential equation by eliminating the arbitrary

function $\varphi(x^2+y^2+z^2, ax+by+cz)=0$

Sol: Given function can be written as

$$x^2+y^2+z^2 = f(ax+by+cz) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Differentiating (1) partially w.r.t. 'x' and 'y',we get

$$2x+2zp = (a+cp)f^1(ax+by+cz) \ldots.(2)$$

$$and$$

$$2y+2zq = (b+cq)f^1(ax+by+cz) \ldots.(3)$$

$\frac{(2)}{(3)}$ implies

$$\boxed{\frac{x+zp}{y+zq} = \frac{(a+cp)}{(b+cq)}}$$

Which is the required complete solution of given Partial differential equation.

Solution Of Partial Differential Equations :

Complete integral:

A solution in which the number of arbitrary constants is equal to the number of independent variables is called complete integral or complete solution of the given equation.

Particular integral :

A solution obtained by giving particular values to the arbitrary constants in the complete integral is called a particular integral or particular solution.

Singular integral:

$Let\ f(x,y,z,p,q)=0 \rightarrow$ (1) be the partial differential equation.

$Let\ \phi(x,y,z,a,b)=0 \rightarrow$ (2)

be the complete integral of (1). Where a and b are arbitrary constants.

Now find $\frac{\partial \phi}{\partial a}=0 \rightarrow$ (3) $\frac{\partial \phi}{\partial b}=0 \rightarrow$ (4)

Eliminate a and b between the equations(2), (3) & (4) When it exists is called the singular integral of (1).

General integral : In the complete integral (2). Assume that one of the constant is a function of the other i.e. $b = f(a)$ Then (2), becomes $\phi(x,y,z,a,f(a))=0 \rightarrow$ (5)

Differentiating (5) partially w.r.t. 'a', we get $\frac{\partial \phi}{\partial a}+\frac{\partial \phi}{\partial f}.f^1(a)=0 \rightarrow$ (6)

Eliminate 'a' between (5) and (6), when it exists is called the general integral or general solution of (1).

Linear Partial Differential Equations Of The First Order:

A differential equation involving partial derivatives p and q only and no higher order derivatives is called a first order equation. If p and q occur in the first degree, it is called a linear partial differential equation of first order; otherwise it is called a non-linear partial differential equation of the first order.

For example: $px + qy^2 = z$ is a linear p.d.e of first order and $p^2 + q^2 = 1$ is non-linear

Lagrange's Linear Equation:

A linear partial differential equation of order one involving a dependent variable z and two independent variables x and y of the form $Pp + Qq = R$

Where P, Q, R are functions of x, y, z is called Lagrange's linear equation.

Lagrange's auxiliary equations or Lagrange's subsidiary equations

The equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ are called Lagrange's auxiliary equations.

Working Rule To Solve Lagrange's Linear Equation $\boldsymbol{Pp + Qq = R}$

Step 1: Write down the auxiliary equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Step 2 : Solve the auxiliary equations by the method of grouping or the method of multipliers or both to get two independent solutions $u = a$ and $v = b$ where a, b are arbitrary constants

Step 3: Then $\emptyset(u, v) = 0$ or $u = f(v)$ is the general solution of the equation $Pp + Qq = R$

To solve $\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}$ …… (1)

(i) **Method of grouping** : In some problems, it is possible that two of the equations $\frac{dx}{P} = \frac{dy}{Q}$ or $\frac{dy}{Q} = \frac{dz}{R}$ or $\frac{dx}{P} = \frac{dz}{R}$ are directly solvable to get solutions $u(x, y) =$ constant or $v(y, z) =$constant or $w(x, z) =$constant. These give the complete solutions of (1)

Sometimes one of them, say $\frac{dx}{P} = \frac{dy}{Q}$ may give rise to solution $u(x, y) = c_1$

From this we may express y, as a function of x. Using this in $\frac{dy}{Q} = \frac{dz}{R}$ and integrating we get $v(y, z) = c_2$. These two relations $u = c_1, v = c_2$ give the complete solution of (1)

(ii). **Method of multipliers**: This is based on the following elementary result.

If $\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=......=\frac{a_n}{b_n}$ then each ratio is equal to $\frac{l_1a_1+l_2a_2+....+l_na_n}{l_1b_1+l_2b_2+....+l_nb_n}$

Consider $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

If possible identity multipliers l,m,n not necessarily constant, so that each ratio

$=\frac{ldx+mdy+ndz}{lP+mQ+nR}$

Where $l\,P\;+\;m\,Q\;+\;n\,R\;=\;0$ Then $l\,dx\;+\;m\,dy\;+\;n\,dz\;=\;0$

Integrating this we get $u(x,y,z)\;=c_1$.

Similarly we get another solution $v(x,y,z)\;=c_2$ independent of the earlier one.

We have the complete solution of (1) constituted $by\;u=c_1\;and\;v=c_2$

Solved Problems

1. **Solve $p\;tan\;x\;+q\;tan\;y\;=\;tan\;z$**

Sol : The given equations can be written as $\tan xp+\tan yq=\tan z\rightarrow$ (1)

Comparing with $Pp+Qq=R$, we have $P=\tan x, Q=\tan y, R=\tan z$

$\therefore$ The auxiliary equations are $\frac{dx}{\tan x}=\frac{dy}{\tan y}=\frac{dz}{\tan z}$

Taking the first two members, we have $\frac{dx}{\tan x}=\frac{dy}{\tan y}$

Integrating $\log\sin x=log\sin y+l\,og\,c_1$

or $\log\frac{\sin x}{\sin y}=\log c_1$ or $\frac{\sin x}{\sin y}=c_1\rightarrow$ (2)

Taking the last two members, we have $\frac{dy}{\tan y}=\frac{dt}{\tan z}$

Integrating, $log\;siny=log\;sinz+logc2$

or $\log\frac{\sin y}{\sin z}=\log c_2$ or $\frac{\sin y}{\sin z}=c_2\rightarrow$ (3)

From (2) and (3). The general solution of (1) is

$\phi(c_1,c_2)=0$

$i.e.\phi\left(\frac{\sin x}{\sin y},\frac{\sin y}{\sin z}\right)=0$ is the required Complete Solution.

2. **Find the general solution of** $y^2zp+x^2zq=y^2x$

Sol: We have $y^2zp+x^2zq=y^2x\rightarrow$ (1)

Comparing $with\ Pp+Qq=R, we\ have$

$P=y^2z, Q=x^2z,\ R=y^2x$

$\therefore$ The auxiliary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

Taking the first two members, we have

$\frac{dx}{y^2z}=\frac{dy}{x^2z}\Rightarrow\frac{dx}{y^2}=\frac{dy}{x^2} or\ x^2dx=y^2dy$

Integrating, $\frac{x^3}{3}=y^3/3+c_1\ or\ \frac{x^3}{3}-\frac{y^3}{3}=c_1\rightarrow$ (2)

Taking the first and last two members, we have

$\frac{dx}{y^2z}=\frac{dz}{y^2x} or\ xdx=zdz$

Integrating $\frac{x^2}{2}=\frac{z^2}{2}+c_2\ or\ \frac{x^2}{2}-\frac{z^2}{2}=c^2\rightarrow$ (3)

From (2) and (3) The general solution of (1) is

$\phi(c_1,c_2)=0\ i.e.$

$\phi\left(\frac{x^3}{3}-\frac{y^3}{3},\frac{x^2}{2}-\frac{z^2}{2}\right)=0$ is the required Complete Solution.

3. **Solve** $p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

Sol: The given equation can be written as

$\sqrt{x}p+\sqrt{y}q=\sqrt{z}\rightarrow(1)$

Comparing with Pp+Qq=R, we have

$P=\sqrt{x}, Q=\sqrt{y}, R=\sqrt{z}$

$\therefore$ The auxiliary equations are $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$

From the first two members, we have $\frac{dx}{\sqrt{x}}=\frac{dy}{\sqrt{y}}$

Integrating, $2\sqrt{x}=2\sqrt{y}+c_1\ or\ 2\sqrt{x}-2\sqrt{y}=c_1$ or $\sqrt{x}-\sqrt{y}=a\rightarrow$ (2)

From the last two members, we have $\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{z}}$

Integrating, $2\sqrt{y}=2\sqrt{z}+c_2\ or\ 2\sqrt{y}-2\sqrt{z}=c_2$

$or\ \sqrt{y}-\sqrt{z}=b\rightarrow$ (3)

From (2) and (3). The general solution of (1) is

$\phi(a,b)=0$ *i.e.*,

$\phi\left(\sqrt{x}-\sqrt{y},\sqrt{y}-\sqrt{z}\right)=0$ is the required Complete Solution.

4. **Solve** $x(y-z)p+y(z-x)q=z(x-y)$

Sol: We have $x(y-z)p+y(z-x)q=z(x-y)\to$ (1)

Comparing with $Pp+Qq=R$, we have

$P=x(y-z),Q=y(z-x),R=z(x-y)$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x(y-z)}=\dfrac{dy}{y(z-x)}=\dfrac{dz}{z(x-y)}$

Using $l=1,m=1,n=1$ as multipliers, we get

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}=\frac{dx+dy+dz}{0} \qquad [\therefore x(y-z)+y(z-x)+z(x-y)=0]$$

$\therefore dx+dy+dz=0$

Integrating, $x+y+z=a\to$ (2)

Again using $l=\dfrac{1}{x},m=\dfrac{1}{y},n=\dfrac{1}{z}$ as multipliers, we get

Each fraction $=\dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}=k(\text{say})$

$\therefore \dfrac{1}{x}dx+\dfrac{1}{y}dy+\dfrac{1}{z}dz=0$

Integrating, $log x+log y+log z=log b. or\ xyz=b\ldots\ldots$ (3)

From (2) and (3). The general solution of (1) is

$\phi(a,b)=0$ *i.e.*,

$\phi(x+y+z,xyz)=0$ is the required Complete Solution.

5. **Solve** $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

Sol: Given $x^2(y-z)p+y^2(z-x)q=z^2(x-y)\to$ (1)

Comparing with $Pp+Qq=R$, we have

$P=x^2(y-z),Q=y^2(z-x),R=z^2(x-y)$

$\therefore$ The auxiliary equations are $\dfrac{dx}{x^2(y-z)}=\dfrac{dy}{y^2(z-x)}=\dfrac{dz}{z^2(x-y)}$

Using $l=\frac{1}{x^2}, m=\frac{1}{y^2}, n=\frac{1}{z^2}$ as multipliers, we get

Each fraction $=\frac{\frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz}{0}$ =k(say)

$\therefore \frac{1}{x^2}dx+\frac{1}{y^2}dy+\frac{1}{z^2}dz=0$

Integrating , $-\frac{1}{x}-\frac{1}{y}-\frac{1}{z}=a \quad or \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_1 \rightarrow (2)$

Again using $l=\frac{1}{x}, m=\frac{1}{y}, n=\frac{1}{z}$ as multipliers, we get

Each fraction $=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}=k(\text{say})$

$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating $\log x+\log y+\log z=\log c_2$

$or\ xyz=c_2 \rightarrow (3)$

From (2) and (3), The general solution of (1) is .

$\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}, xyz\right)=0$ is the required Complete Solution.

6. **Solve** $(mz-ny)p+(nx-lz)q=ly-mx$

Sol: Given eqn is $(mz-ny)p+(nx-lz)q=ly-mx \rightarrow (1)$

Comparing with $Pp+Qq=R$, we have

$P=mz-ny, Q=nx-lz, R=ly-mx$

$\therefore$ The auxiliary equations are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using $l=x, m=y, n=z$ as multipliers, we get

Each fraction $=\frac{xdx+ydy+zdz}{0}$

$\therefore xdx+ydy+zdz=0$

Integrating, $\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=a\ or\ x^2+y^2+z^2=c_1 \rightarrow (2)$

Again using l, m, n as multipliers, we get

Each fraction $= \dfrac{ldx+mdy+nz}{0} = k(\text{say})$

$\therefore ldx + mdy + ndz = 0$

Integrating, $lx+my+nz=c_2 \rightarrow$ (3)

From (2) and (3), the general solution of (1) is

$\phi\left(x^2+y^2+z^2, lx+my+nz\right)=0$ is the required Complete Solution.

7. Solve $xp-yq=y^2-x^2$

Sol: Here $P=x, Q=y, R=y^2-x^2$

$\therefore$ The auxiliary eqn's are $\dfrac{dx}{x}=\dfrac{dy}{-y}=\dfrac{dz}{y^2-x^2}$

From the first two members, $\dfrac{dx}{x}=\dfrac{dy}{-y}$

Integrating, $\log x+\log y=\log c_1 \; or \; xy=c_1 \rightarrow (1)$

Using $l=x, m=y, n=1$ as multipliers, we get

Each fraction $=\dfrac{xdx+ydy+dz}{0}$

$\therefore xdx+ydy+dz=0$

Integrating, $\dfrac{1}{2}x^2+\dfrac{1}{2}y^2+z=c \; or \; x^2+y^2+2z=c_2 \rightarrow$ (2)

From (1) and (2), The general solution is

$\phi\left(xy, x^2+y^2+2z\right)=0$ is the required Complete Solution.

8. Find the integral surface of $x\left(y^2+z\right)p-y\left(x^2+z\right)q=\left(x^2-y^2\right)z$

Which contains the straight line x+y=0, z=1

Sol: Given that $x\left(y^2+z\right)p-y\left(x^2+z\right)q=\left(x^2-y^2\right)z$ ……..(1)

Comparing with $Pp+Qq=R$, we have

$P=x\left(y^2+z\right), Q=-y\left(x^2+z\right), R=\left(x^2-y^2\right)z$

$\therefore$ The auxiliary equations are

$$\frac{dx}{x\left(y^2+z\right)}=\frac{dy}{-y\left(x^2+z\right)}=\frac{dz}{\left(x^2-y^2\right)z}$$

Using $l=\dfrac{1}{x}, m=\dfrac{1}{y}, n=\dfrac{1}{z}$ as multipliers, we get

Each fraction = $\dfrac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$

$\therefore \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating, $\log x+\log y+\log z=\log a$

or $xyz=a \to$ (2)

Again using $l=x, m=y, n=-1$ as multipliers, we get

$\therefore$ Each fraction $=\dfrac{xdx+ydy-dz}{0}$ =k(say)

$\therefore xdx+ydy-dz=0$

Integrating, $\frac{x^2}{2}+\frac{y^2}{2}-z=c$ or $x^2+y^2-2z=b \to$ (3)

Given that $z=1$, using this (2) and (3), we get

$xy=a$ and $x^2+y^2-2=b$

Now b+2a= $x^2+y^2-2+2xy=(x+y)^2-2$ =0-2 $[\therefore x+y=0]$ =-2

$\therefore 2a+b+2=0$

Hence the required surface is

$x^2+y^2-2z+2xyz+2=0$ is the required Complete Solution.

9.Solve $px+qy=z$

Sol: Given $px+qy=z$ is a Lagrange's linear equation

The Auxillary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

By Consider first group, we get

$$\int\frac{dx}{x}=\int\frac{dy}{y}$$

$$\log x=\log y+\log c1$$

$c_1=\frac{x}{y}$.....(1)

By Consider second group, we get

$$\int \frac{dz}{z} = \int \frac{dy}{y}$$

$$\log y = \log z + \log c2$$

$c_2 = \frac{y}{z}$.....(2)

$\therefore f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ is the required solution.

10. Solve $(\boldsymbol{x^2 - y^2 - yz})\boldsymbol{p} + (\boldsymbol{x^2 - y^2 - xz})\boldsymbol{q} = \boldsymbol{z}(\boldsymbol{x - y})$

Sol: The auxiliary equations are

$$\frac{dx}{(x^2 - y^2 - yz)} = \frac{dy}{(x^2 - y^2 - xz)} = \frac{dz}{z(x - y)}$$

Taking 1,-1-1 multipliers,we get

$\frac{\boldsymbol{dx - dy - dz}}{(\boldsymbol{x^2 - y^2 - yz - x^2 + y^2 + xz - xz + yz})} = \frac{\boldsymbol{dx}}{(\boldsymbol{x^2 - y^2 - yz})}$

$$dx - dy - dz = 0$$

Integrating,we get

$x - y - z = c_1$.........(1)

Taking $x, -y, 0$ as multipliers,we get

$$\frac{xdx - ydy}{(x^3 - xy^2 - xyz - yx^2 + y^3 + xyz)} = \frac{dz}{z(x - y)}$$

$$\frac{xdx - ydy}{(x^2 - y^2)(x - y)} = \frac{dz}{z(x - y)}$$

$$\frac{1}{2}\log(x^2 - y^2) = \log z$$

$\frac{\boldsymbol{x^2 - y^2}}{\boldsymbol{z^2}} = \boldsymbol{c_2}$ ... ... **(2)**

$\therefore$Complete solution of given pde is $\varphi\left(x - y - z, \frac{x^2 - y^2}{z^2}\right) = 0$

11. Solve $\boldsymbol{x}(\boldsymbol{y^2 - z^2})\boldsymbol{p} - \boldsymbol{y}(\boldsymbol{x^2 + z^2})\boldsymbol{q} = \boldsymbol{z}(\boldsymbol{x^2 + y^2})$

Sol: The auxiliary equations are

$$\frac{dx}{x(y^2-z^2)}=\frac{dy}{-y(x^2+z^2)}=\frac{dz}{z(x^2+y^2)}$$

Taking $x, y, z,$ multipliers, we get

$$\frac{xdx+ydy+zdz}{(x^2y^2-x^2z^2-y^2x^2-z^2y^2+x^2z^2+y^2z^2)}=\frac{dx}{x(y^2-z^2)}$$

$$xdx+ydy+zdz=0$$

$$x^2+y^2+z^2=c_1 \dots\dots\dots\dots\dots\dots\dots.(1)$$

Taking $-\frac{1}{x},\frac{1}{y}\frac{1}{z},$ multipliers, we get

$-\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$

Integrating, we get

$\frac{yz}{x}=c_2$…….(2)

From (1),(2),

Complete solution of given pde $is\ \varphi\left(\frac{yz}{x},x^2+y^2+z^2\right)=0$

12. Solve $(y^2)p-xyq=x(z-2y)$

Sol: Comparing with $Pp+Qq=R$, we have

The auxiliary equations are

$\therefore\ \frac{dx}{y^2}=\frac{dy}{-yx}=\frac{dz}{x(z-2y)}$

From the first two members, we have

$$\frac{dx}{y}=\frac{dy}{-x}$$

Integrating,we get

$x^2+y^2=c_1$…..(2**)**

From the last two members, we have

$$\frac{dy}{-y}=\frac{dz}{(z-2y)}$$

$$-ydz=zdy-2ydy$$

$$d(yz)-2ydy=0$$

$yz-y^2=c_2$ **………..(3)**

From (2) and (3). The general solution of (1) is

i.e., $\emptyset(yz-y^2, x^2+y^2)$=0

13. Solve $(\boldsymbol{y}+\boldsymbol{z})\boldsymbol{p}+(\boldsymbol{z}+\boldsymbol{x})\boldsymbol{q}=(\boldsymbol{x}+\boldsymbol{y})$

Sol: Comparing with $Pp+Qq=R$, we have

The auxiliary equations are

$\therefore\ \frac{dx}{(y+z)}=\frac{dy}{(z+x)}=\frac{dz}{(x+y)}$

Taking 1,1,1 and 1,-1,0 and 0,1,-1 as multipliers , we have $\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{(y-x)}=\frac{dy-dz}{(z-y)}$

From the last two members, we have

$$\frac{dx-dy}{(y-x)}=\frac{dy-dz}{(z-y)}$$

Integrating,we get

$$\log\frac{(y-x)}{(z-y)}=\log C_2$$

$\frac{(y-x)}{(z-y)}=c_2$.....(1)

From the first two members, we have

$$\frac{dx+dy+dz}{2(x+y+z)}=\frac{dx-dy}{(y-x)}$$

Integrating,we get

$$\frac{1}{2}\log(x+y+z)=\log(y-x)+\log c_1$$

$(x+y+z)(y-x)^2=c_1$..........(2)

From (2) and (1). The general solution of given pde is

$\boxed{i.e.,\ \emptyset\left(\frac{(y-x)}{(z-y)}, (x+y+z)(y-x)^2\right)=0}$

14. Solve $\boldsymbol{x^2p-y^2q=z(x-y)}$

Sol: Comparing with $Pp+Qq=R$, we have

The auxiliary equations are

$\therefore\ \frac{dx}{x^2}=\frac{dy}{-y^2}=\frac{dz}{(z(x-y)}$

From the first two members, we have

$$\frac{dx}{x^2} = \frac{dy}{-y^2}$$

Integrating,we get

$$\frac{1}{x} + \frac{1}{y} = c_1 \ldots..(1)$$

Taking 1,1,0 as multipliers,we get

$$\frac{dx+dy}{x^2-y^2} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)(x-y)} = \frac{dz}{(z(x-y)}$$

$$\frac{dx+dy}{(x+y)} = \frac{dz}{z}$$

Integrating,we get

$$\frac{x+y}{z} = c_2 \quad \ldots.(2)$$

From (2) and (1). The general solution is

$\emptyset(\frac{x+y}{z}, \frac{1}{x} + \frac{1}{y})=0$

15. Solve $(x^2 - yz)p + (y^2 - xz)q = (z^2 - xy)$

Sol: The auxiliary equations are

$$\frac{dx}{(x^2-yz)} = \frac{dy}{(y^2-xz)} = \frac{dz}{(z^2-xy)}$$

Taking 1,-1,0 and 0,-1,-1 as multipliers,we get

$\frac{dx-dy}{(x^2-yz)-(y^2-xz)}$ and also $\frac{dy-dz}{((-z^2+yx)+(y^2-xz))}$

$\therefore \frac{dx-dy}{(x^2-yz)-(y^2-xz)} = \frac{dy-dz}{((-z^2+yx)+(y^2-xz))}$

$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z))}$

solving it,we get

$\frac{(x-y)}{y-z} = c_1 \ldots..(1)$

$Taking\ x, y, z\ and$ 1,1,1 as multipliers,we get

$\frac{(xdx+ydy+zdz)}{x^3+y^3+z^3-3xyz} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$

$\frac{(xdx+ydy+zdz)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{(dx+dy+dz)}{x^2+y^2+z^2-xy-yz-zx}$

$$(x + y + z)(dx + dy + dz) = (xdx + ydy + zdz)$$

$$(x+y+z)d(x+y+z) = (xdx + ydy + zdz)$$

Integrating,we get

$$\frac{(x+y+z)^2}{2} = \frac{x^2+y^2+z^2}{2} + c$$

$$\therefore (x+y+z)^2 = x^2+y^2+z^2+c_2$$

$xy + yz + zx = c_2$…..(2)

Complete solution of given pde is $\varphi\left(xy + yz + zx, \frac{(x-y)}{y-z}\right) = 0$

Non-Linear Partial Differential Equations Of First Order

A partial differential equation which involves first order partial derivatives p and q with degree higher than one and the products of p and q is called a non-linear partial differential equations.

Non linear PDE's can be classified in to 4 standard forms.

Standard Form I:

Equation of the form $f(p,q) = 0$ (i.e., equations containing p and q only) :

Given partial differential equation is of the form $f(p,q) = 0$ … … (1)

Procedure:

Given partial differential equation is $f(p,q) = 0$ … … (1)

Step1:Put $p = a$ in (1), then we get q value in terms of a then we can obtain 'p' value.

Step2:Substitute p, q values in $dz = p\,dx + q\,dy$

Step3:Integrating it ,we get required complete solution of (1) .

Solved Problems

1.Solve $pq = k$, where k is a constant.

Sol: Given that $pq = k$ … . (1)

Since (1) is of the form $f(p,q) = 0$

Put $p = a$ in (1),we get $q = \frac{k}{a}$

Now substitute p, q in

$dz = p\,dx + q\,dy$ then

$dz = a\,dx + \frac{k}{a}\,dy$

Integrating,we get

$$z = ax + \frac{k}{a}y + c$$

which contains two arbitrary constants a and c.

2.Solve $p^2 + q^2 = npq$

Sol : Given that $p^2+q^2=npq \ldots\ldots\ldots\ldots..(1)$

Since (1) is of the form $f(p,q)=0$

Put $p=a$ in (1),then we get $q=\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]$

Now substitute p,q in

$dz\ =p\,dx+q\,dy$

$=a\,dx+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]dy$

Integrating,we get , $dz=a\int dx+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]\int dy$

$$z=ax+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]y+c$$

This is the complete integral of (1), which contains two arbitrary constants a and c.

3.Find the complete integral of $p^2+\ q^2=m^2$

Sol: Given that $p^2+\ q^2=m^2\ \ldots\ldots\ldots.(1)$

Since (1) is of the form $f(p,q)=0$

Put $p=a\ in\ (1), we\ get\ q=\sqrt{m^2-a^2}$

Now substitute p,q in

$dz\ =p\,dx+q\,dy$ …….(2) then

$dz\ =adx+q=\sqrt{m^2-a^2}\,dy$

Now integrate on both sides

$$z=ax+\left(\sqrt{m^2-a^2}\right)y+c$$

Which is the complete integral of (1)

Standard Form II :

Equation of the form $f(p,q,z)=0$(i.e., not containing x and y)

Procedure :

Given partial differential equation is of the form $f(p,q,z)=0\ \ldots\ldots(1)$

Step1:Put $p=aq$ in (1),then we get q value in terms of a , z then

Step2:Substitute p,q values in $dz\ =\ p\,dx+q\,dy$

Step3:Integrating it ,we get required complete solution of (1) .

Solved Problems :

Solve the following partial differential equations

1. $z=p^2+q^2$ **2.** $p^2z^2+q^2=p^2q$ **3.** $zpq=p+q$

Sol : 1. We have $z = p^2+q^2 \ldots \ldots (1)$

Since (1) is of the form $f(z,p,q) = 0$

Put $p = aq \; in \; (1)$, then we get $q = \sqrt{\frac{z}{1+a^2}}$

$$\therefore p = a\sqrt{\frac{z}{1+a^2}}$$

Putting the values $of \; p \; and \; q \; in \; dz = p\,dx + q\,dy$, we get

$\frac{1}{\sqrt{z}} dz = \frac{1}{\sqrt{1+a^2}} (a\,dx + dy)$,

Integrating ,we get

$$\int \frac{1}{\sqrt{z}} dz = \frac{1}{\sqrt{1+a^2}} \int (a\,dx + dy)$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}} (ax + y)$$

which is the required solution of (1)

2. Given that $p^2z^2 + q^2 = p^2q \rightarrow$ (1)

Since (1) is of the form $f(z,p,q) = 0$

Put $p = aq$ in (1), then we get $q = \frac{(a^2z^2+1)}{a^2}$

$$\therefore q = \frac{(a^2z^2+1)}{a}$$

Putting the values of p and q in $dz = p\,dx + q\,dy$,we get

$\frac{\boldsymbol{dz}}{(a^2z^2+1)} = \frac{\mathbf{1}}{\boldsymbol{a^2}}(adx + dy)$

Integrating,we get

$\int \frac{\boldsymbol{dz}}{(a^2z^2+1)} = \frac{\mathbf{1}}{\boldsymbol{a^2}} \int (adx + dy)$

$$\therefore \text{a} \tan^{-1}(az) = ax + y + c$$

which is the required complete solution of (1)

3. Given that $zpq = p + q$…..(1)

Since (1) is of the form $f(z,p,q) = 0$

Put $p = aq \; in \; (1), then \; we \; get \; q = \frac{a+1}{az}$

$$\therefore p = \frac{a+1}{z}$$

Putting the values $of \; p \; and \; q \; in \; dz = p\,dx + q\,dy$, we get

$zdz = \frac{a+1}{a} (a\,dx + dy)$,

Integrating ,we get

$$\int z\,dz = \frac{a+1}{a}\int (a\,dx + dy)$$

$$\frac{aZ^2}{2(a+1)} = ax + y + c$$

This is the required solution of (1)

Standard Form III :

Equation of the form $f_1(x,p) = f_2(y,q)$ i.e. Equations not involving z and the terms containing x and p can be separated from those containing y and q.

We assume that these two functions should be equal to a constant say k.

$\therefore f_1(x,p) = f_2(y,q) = k$

Solve for p and q from the resulting equations

$\therefore f_1(x,p) = k \text{ and } f_2(y,q) = k$

Solve for p and q, we obtain

$p = F_1(x,k) \text{ and } q = F_2(y,k)$

Since z is a function of x and y

$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ [By total differentiation]

$dz = p\,dx + q\,dy$

$\therefore dz = F_1(x,k)dx + F_2(y,k)dy$

Integrating on both sides

$z = \int F_1(x,k)dx + \int F_2(y,k)dy + c$

Which is the complete solution of given equation

Solved Problems:

1. Solve $p^2 + q^2 = x + y$

Sol :. Given that $p^2 + q^2 = x + y$ ………..(1)

Separating p and x from q and y, the given equation can be written as

$$p^2 - x = -q^2 + y$$

Let $p^2 - x = -q^2 + y = k$ (constant)

$\therefore p^2 - x = k \text{ and } -q^2 + y = k$

$\Rightarrow p^2 = k + x \text{ and } q^2 = y - k$

$$\therefore p=\sqrt{k+x} \ \ and \ \ q=\sqrt{y-k}$$

Since $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$$\therefore dz=\sqrt{k+x}\,dx+\sqrt{y-k}\,dy$$

Integrating on both sides

$$z=\int(k+x)^{\frac{1}{2}}dx+\int(y-k)^{\frac{1}{2}}dy+c$$

$$\boxed{\therefore z=\frac{2}{3}(k+x)^{\frac{3}{2}}+\frac{2}{3}(y-k)^{\frac{3}{2}}+c}$$

Which is the complet solution of (1)

2. Solve $xp-yq=y^2-x^2$

Sol: Given that $xp-yq=y^2-x^2 \rightarrow (1)$

Separating p and x from q and y. The given equation can be written as.

$$xp+x^2=yq+y^2$$

Let $xp+x^2=yq+y^2=k$ (arbitrary constant)

$$\therefore xp+x^2=k \ \ and \ \ yq+y^2=k$$

$$\Rightarrow p=\frac{k-x^2}{x} \, and \, q=\frac{k-y^2}{y}$$

We have $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$$\therefore dz=\left(\frac{k}{x}-x\right)dx+\left(\frac{k}{y}-y\right)dy$$

Integrating on both sides

$$z=\int\left(\frac{k}{x}-x\right)dx+\int\left(\frac{k}{y}-y\right)dy+c$$

$$=k\log x-\frac{x^2}{2}+k\log y-\frac{y^2}{2}+c$$

$$\boxed{\therefore z=k\log(xy)-\frac{1}{2}\left(x^2+y^2\right)+c}$$

Which is the complete integral of (1)

3. Solve $\left(\frac{p}{2}+x\right)^2+\left(\frac{q}{2}+y\right)^2=1$

Sol: Separating p *and* x *from* q *and* y, the given equation can be written as.

$$\left(\frac{p}{2}+x\right)^2=1-\left(\frac{q}{2}+y\right)^2$$

Let $\left(\frac{p}{2}+x\right)^2=1-\left(\frac{q}{2}+y\right)^2=k^2$ (arbitrary constant)

$$\therefore \left(\frac{p}{2}+x\right)^2=k^2 \text{ and } 1-\left(\frac{q}{2}+y\right)^2=k^2$$

$$\Rightarrow \frac{p}{2}+x=k \text{ and } \left(\frac{q}{2}+y\right)^2=1-k^2 \text{ or } \frac{q}{2}+y=\sqrt{1-k^2}$$

$$\Rightarrow p=2(k-x) \text{ and } q=2\left[\sqrt{1-k^2}-y\right]$$

We have $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$

$$\therefore dz=2(k-x)dx+2\left[\sqrt{1-k^2}-y\right]dy$$

Integrating on both sides

$$z=2\int(k-x)dx+2\int\left[\sqrt{1-k^2}-y\right]dy+c$$

$$z=2(kx-\frac{x^2}{2})+2\left[\left(\sqrt{1-k^2}\right)y-\frac{y^2}{2}\right]+c$$

$$\boxed{\therefore z=2kx-x^2+2\left(\sqrt{1-k^2}\right)y-y^2+c}$$

This is the complete solution of (1)

4.Solve $\boldsymbol{p-x^2=q+y^2}$

Sol: Let $p-x^2=q+y^2=k^2(say)$

Then $p-x^2=k^2$ and $q+y^2=k^2$

$$\therefore p=k^2+x^2 \text{ and } q=k^2+y^2$$

But we have

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=pdx+qdy$$

Integrating,we get

$$\boxed{\boldsymbol{z = \frac{x^3}{3} + k^2 x + k^2 y + \frac{y^3}{3} + c}}$$

is the required complete solution.

5. Solve $q^2 - p = y - x$

Sol: Let $p - x = q^2 - y = k(say)$

Then $p = k + x \ and \ q = \sqrt{k + y}$

But

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$ Integrating,we get

$$\boxed{z = \frac{x^2}{2} + kx + \frac{2}{3}(k + y)^{\frac{3}{2}} + C}$$

is the required complete solution.

6.Solve $q = px + p^2$

Sol: Let $q = px + p^2 = k(say)$

Then we get

$$p^2 + px - k = 0 \ and \ q = k$$

Solving,we get

$p = \frac{-x \pm \sqrt{(x^2+4k}}{2}$ and $q = k$

But

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = pdx + qdy$$ Integrating,we get

$$\boxed{z = -\frac{x^2}{4} + \frac{1}{2}[\frac{x}{2}\sqrt{(x^2 + 4k} + 2k \sinh^{-1}(\frac{x}{2\sqrt{k}}) + ky + C}$$

is the required complete solution.

STANDARD FORM IV: $Z = px + qy + f(p, q)$

An equation analogous to the clairaut's equation it is complete solution is $Z = ax + by + f(a, b)$ which is obtained by writing $a \ for \ p \ and \ b \ for \ q$ The differential equation which satisfies some specified conditions known as the boundary conditions.The differential equation together with these boundary conditions, constitute a boundary value problem

Solved Problems:

1. **Solve $z = px + qy + pq$**

Sol : The given PDE is form IV

Therefore complete solution is given by

$$z = ax + by + ab$$

2. Find the solution of $(p+q)(z-px-qy)=1$

Sol: The given equation can be written as

$$z-px-qy=\frac{1}{p+q}$$

$$\therefore z = px+qy+\frac{1}{p+q} \rightarrow (1)$$

Hence the complete solution of (1) is given by

$$\boxed{z=ax+by+\frac{1}{a+b}}$$

3. Solve $pqz = p^2\left(qx+p^2\right)+q^2\left(py+q^2\right)$

Sol: The given equation can be written as

$$pqz = p^2q\left(x+\frac{p^2}{q}\right)+q^2p\left(y+\frac{q^2}{p}\right)$$

$$\therefore z = p\left(x+\frac{p^2}{q}\right)+q\left(y+\frac{q^2}{p}\right)$$

$$\therefore z = px+qy+\left(\frac{p^3}{q}+\frac{q^3}{p}\right) \rightarrow (1)$$

Since it is in the form $z = px+qy+f(p,q)$

Hence the complete solution of (1) is given by

$$\boxed{z=ax+by+\frac{a^3}{b}+\frac{b^3}{a}}$$

4. Solve $z = px + qy + pq + q^2$

Sol: We have $z = px + qy + pq + q^2$ … … … … … . (1)

Since (1) is of the form $z = px + qy + f(p,q)$.

Hence the complete solution of (1) is given by

$$z = ax + by + ab + b^2 \ldots\ldots(2)$$

For singular solution, differentiating (2) partially w.r.t. a and b, we get

$$\frac{\partial z}{\partial a} = 0, \frac{\partial z}{\partial b} = 0,$$

Implies that

$0 = x + b \ldots.(3)$ $\;and\;$ $0 = y + a + 2b \ldots\ldots\ldots(4)$

Eliminating a, b between (2), (3) and (4), we get

$$z = x(2x - y) - xy - (2x - y)x + x^2$$

$\boxed{\therefore z = x^2}$

is the singular solution

Equations Reducible To Standard Forms:

Equations of the form $f(x^m p, y^n q) = 0$ where m and n are constants

The above form of the equation of the type can be transformed to an equation of the form f(p,q)=0

By substitutions given below.

Case (i):- when $m \neq 1$ and $n \neq 1$

Put $X = x^{1-m}$ and $Y = y^{1-n}$ then $p = \frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}\frac{\partial X}{\partial x} = P(1-m)x^{-m}$ where $P = \frac{\partial Z}{\partial X}$

$$x^m p = P(1-m) \quad and\; q = \frac{\partial z}{\partial y} = \frac{\partial Z}{\partial Y}\frac{\partial Y}{\partial y} = Q(1-n)y^{-n}\; where Q = \frac{\partial Z}{\partial Y} \rightarrow y^n q = Q(1-n)$$

Now the given equation reduces to $f[(1-m)P, (1-n)Q] = 0$ which is of the form $f(P,Q) = 0$

Case(ii):- when $m = 1, n = 1$

Put $X = logx$ and $Y = logy$ then

p=$\frac{\partial z}{\partial x} = \frac{\partial Z}{\partial X}\frac{\partial X}{\partial x} = \frac{\partial Z}{\partial X}\frac{1}{x}$ implies $px = P$= where $P = \frac{\partial Z}{\partial X}$

$$similarly\quad qy = Q \quad where Q = \frac{\partial Z}{\partial Y}$$

now the given equation reduces to the form $f(P,Q) = 0$

Equations of the form $f(x^m p, y^n q, z) = 0$ *where m and n are constants*:

This can be reduced to an equation of the form $f(P, Q, z) = 0$ by the substitutions given for the equation $F(x^m p, y^n q, z) = 0$ as above.

Solved Problems:

1. **Solve the partial differential equation** $\frac{x^2}{p} + \frac{y^2}{q} = z$

Sol: Given equation can be written as

$x^2 p^{-1} + y^2 q^{-1} = z \; or \left(x^{-2}p\right)^{-1} + \left(y^{-2}q\right)^{-1} = z \rightarrow$ (1)

This is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = -2, and\; n = -2.$

Put $X = x^{1-m} = x^{1-(-2)} = x^3$ and $Y = y^{1-n} = y^{1+2} = y^3$

Then $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.3x^2$ where $P = \frac{\partial z}{\partial X}$

$\therefore x^{-2} p = 3P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.3y^2$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^{-2} q = 3Q$

Now equation (1), becomes.

$(3P)^{-1} + (3Q)^{-1} = z \rightarrow$ (2)

Since (2) is of the form $f\left(P, Q, z\right) = 0$

Put $P = aQ \; in \; (1), then\; we\; get\; Q = \frac{(a+1)}{3az}$

$$\therefore P = \frac{(a+1)}{3z}$$

Putting the values of Pand Q in $dz = P\, dX + Q\, dY$,we get

$\frac{3az}{a+1} dz = (adX + dY)$

Integrating,we get

$\int \frac{3az}{a+1} dz = (a \int dX + \int dY)$

$$\frac{3az^2}{2(a+1)} = (aX + Y) + c$$

$$\boxed{\therefore 3z^2 = 2\left(\frac{a+1}{a}\right)\left(x^3 + ay^3\right) + c_1}$$

,taking $c_1 = 2(\frac{a+1}{a})c$

Which is the required solution of (1)

2. **Solve the partial differential equation** $\frac{p}{x^2} + \frac{q}{y^2} = z$

Sol: The given equation can be written as

$px^{-2} + qy^{-2} = z \to (1)$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ With $m = -2, and\ n = -2$

Put $X = x^{1-m} = x^3$, and $Y = y^{1-n} = y^3$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.3x^2$ where $P = \frac{\partial z}{\partial X}$

$\therefore x^{-2}p = 3P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q3y^2$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^{-2}q = 3Q$

Equation (1) becomes, $3P + 3Q = z \to (2)$

Since (2) is of the form $f\left(P, Q, z\right) = 0$

Put $P = aQ\ in\ (1), then\ we\ get\ Q = \frac{z}{3(a+1)}$

$$\therefore P = \frac{az}{3(a+1)}$$

Putting the values of Pand Q in $dz = P\,dX + Q\,dY$,we get

$\frac{dz}{z} = \frac{1}{3(a+1)}(adX + dY)$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{3(a+1)}(a\int dX + \int dY)$

$$\log z = \frac{1}{3(a+1)}(aX + Y) + C$$

$$\boxed{\Rightarrow \log z = \frac{1}{3(1+a)}\left(x^3 + ay^3\right) + c}$$

This is the complete solution of (1)

3. **Solve** $q^2y^2 = z(z - px)$

Sol: Given equation can be written as

$q^2y^2 = z^2 - zpx \, or \, (xp)z + (qy)^2 = z^2 \rightarrow (1)$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ with $m = 1$ and $n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} . \frac{\partial X}{\partial x} = P.\frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y} . \frac{\partial Y}{\partial y} = Q.\frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes, $Pz + Q^2 = z^2 \rightarrow (2)$

Since (2) is of the form $f(P, Q, z) = 0$

Put $P = aQ$ in (1), then we get $Q = \frac{z}{2}[-a \pm \sqrt{a^2 + 4}]$

$$\therefore P = \frac{aZ}{2}[-a \pm \sqrt{a^2 + 4}]$$

Putting the values of Pand Q in $dz = P\,dX + Q\,dY$,we get

$\frac{dz}{z} = \frac{1}{2}[-a \pm \sqrt{a^2 + 4}](adX + dY)$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{2}[-a \pm \sqrt{a^2 + 4}](a \int dX + \int dY)$

$\log z = \frac{1}{2}[-a \pm \sqrt{a^2 + 4}](aX + Y) + c$

$$\boxed{\therefore \log z = \frac{1}{2}[-a \pm \sqrt{a^2 + 4}](ax^3 + y^3) + c}$$

Which is the complete integeral of (1)

4. **Solve the partial differential equation** $p^2x^4 + y^2zq = 2z^2$

Sol : Given that $p^2x^4 + y^2zq = 2z^2$

Then given equation can be written as

$(px^2)^2 + (qy^2)z = 2z^2 \rightarrow (1)$

Since (1) is of the form $f\left(x^m p, y^n q, z\right) = 0$ with m=2 and n=2

Put $X = x^{1-m} = x^{1-2} = x^{-1} = \frac{1}{x}$ and $Y = y^{-1} = \frac{1}{x}$

Now $P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\left(\frac{-1}{x^2}\right)$, where $P = \frac{\partial z}{\partial X}$

$\therefore x^2 p = -P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\left(\frac{-1}{y^2}\right)$, where $Q = \frac{\partial z}{\partial Y}$

$\therefore y^2 q = -Q$

Now equation (1) becomes, $P^2 - Qz = 2z^2$ or $P^2 - Qz = 2z^2 \rightarrow (2)$

Since (2) is of the form $f(P,Q,z) = 0$

Put $P = aQ$ in $(1), then\ we\ get\ Q = \frac{z}{2a^2}[1 \pm \sqrt{8a^2+1}]$

$$\therefore P = \frac{Z}{2a}[1 \pm \sqrt{8a^2+1}]$$

Putting the values of Pand Q in $dz = P\,dX + Q\,dY$,we get

$\frac{dz}{z} = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}](adX + dY)$

Integrating,we get

$\int \frac{dz}{z} = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}](a\int dX + \int dY)$

$\log z = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}]\,(aX + Y) + c$

$\boxed{\therefore \log z = \frac{1}{2a^2}[1 \pm \sqrt{8a^2+1}](ax^3 + y^3) + c}$

Which is the complete integral of (1).

5. Solve $x^2 p^2 + xpq = z^2$

Sol : The given equation can be written as

$(xp)^2 + (xp)q = z^2 \rightarrow (1)$

Since (1) is of the form $f\left(x^m p.y^n q, z\right) = 0$ with m=1 and n=0

Put $X = \log x$

Now $P = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P\frac{1}{x}$, where

$P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

Equation (1) becomes, $P^2 + Pq = z^2 \rightarrow$ (2)

Since (2) is of the form $f(P,q,z)=0$

Put $P=aq\ in\ (2), we\ get$

$q=\frac{z}{\sqrt{a(a+1)}}$, P=a$\frac{z}{\sqrt{a(a+1)}}$

But we have

$dz\ =\ P\ dX+q\ dy$

Substuting P,q ,we get

$\frac{dz}{z}\ =\ \frac{1}{\sqrt{a(a+1)}}(a\ dX+\ dy)$

Integrating on both sides

$$\int dz/z=\frac{1}{\sqrt{a(a+1)}}(a\int dX+d\,y)$$

$\boxed{\sqrt{a(a+1)}\log z=(aX+y)+C}$

be the complete integral of (1)

6.Solve $z=p^2x+q^2y$

Sol : Given that $z=p^2x+q^2y$

The given equation can be written as

$$\left(p\sqrt{x}\right)^2+\left(q\sqrt{y}\right)^2=z\,or\left(px^{\frac{1}{2}}\right)^2+\left(qy^{\frac{1}{2}}\right)^2=z\rightarrow(1)$$

This is of the form $f\left(x^m p,y^n q,z\right)=0\,with\,m=n=\frac{1}{2}$

Put $X=x^{1-m}=x^{1-\frac{1}{2}}=x^{\frac{1}{2}}\,and\,Y=y^{1-\frac{1}{2}}=y^{\frac{1}{2}}$

Now $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}.\frac{\partial X}{\partial x}=P\left(\frac{1}{2}x^{\frac{-1}{2}}\right),where\,P=\frac{\partial z}{\partial X}$

and $q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial x}=Q\left(\frac{1}{2}y^{\frac{-1}{2}}\right),where\,Q=\frac{\partial z}{\partial Y}$

$\therefore\ px^{\frac{1}{2}}=\frac{P}{2}\,and\,qy^{\frac{1}{2}}=\frac{Q}{2}$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2+\left(\frac{Q}{2}\right)^2=z\,i.e.P^2+Q^2=4z\rightarrow(2)$

This if of the form $f\left(P,Q,z\right)=0$

Put $P = aQ\ in\ (2), we\ get$

$$a^2Q^2 + Q^2 = 4z$$

$Q = \sqrt{\frac{4z}{a^2+1}}$, P=a$\sqrt{\frac{4z}{a^2+1}}$,

But we have

$dz = P\,dX + Q\,dY$

Substuting P,Q ,we get

$dz = \sqrt{\frac{4z}{a^2+1}}(a\,dX + dY)$

$$\frac{dz}{\sqrt{z}} = \frac{2}{\sqrt{a^2+1}}(a\,dX + dY)$$

Integrating on both sides

$$\int dz/\sqrt{z} = \frac{2}{\sqrt{a^2+1}}(a\int dX + \int dY)$$

$\sqrt{(a^2+1)}\sqrt{z} = (aX + Y) + C$

$\boxed{\sqrt{(a^2+1)}\sqrt{z} = \left(a\sqrt{x} + \sqrt{y}\right) + C}$

Which is the complete integral of (1)

7.Solve $x^2p^2 + y^2q^2 = z^2$

Sol: Given $x^2p^2 + y^2q^2 = z^2 \ldots\ldots.(1)$

$$(xp)^2 + (yq)^2 = z^2$$

Since (1) is of the form $f\left(x^m p, y^n q, z\right)=0$ with $m = 1\ and\ n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes

$P^2 + Q^2 = z^2 ......(2)$

$$Put\ P = aQ\ in\ (2), we\ get\ Q = \frac{z}{\sqrt{a^2+1}}; P = \frac{az}{\sqrt{a^2+1}}$$

But we have

$dz = P\,dX + Q\,dY$

Substuting P,Q ,we get

$dz = \frac{z}{\sqrt{a^2+1}}(a\,dX + dY)$

$$\frac{dz}{z} = \frac{1}{\sqrt{a^2+1}}(a\,dX + dY)$$

Integrating on both sides

$$\int dz/z = \frac{1}{\sqrt{a^2+1}}\left(a\int dX + \int dY\right)$$

$\sqrt{(a^2+1)}\log z = (aX + Y) + C$

$\boxed{\sqrt{(a^2+1)}\log z = (a\log x + \log y) + C}$

is the Complete solution of (1)

8. **Solve $x^2p^2 + y^2q^2 = 1$**

Sol: Given $\boldsymbol{x^2p^2 + y^2q^2 = 1 \ldots\ldots(1)}$

$$(xp)^2 + (yq)^2 = 1$$

Since (1) is of the form $f(x^m p, y^n q) = 0$ with $m = 1\ and\ n = 1$

Put $X = \log x$ and $Y = \log y$

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}.\frac{\partial X}{\partial x} = P.\frac{1}{x}$ where $P = \frac{\partial z}{\partial X}$

$\therefore xp = P$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial Y}.\frac{\partial Y}{\partial y} = Q.\frac{1}{y}$ where $Q = \frac{\partial z}{\partial Y}$

$\therefore qy = Q$

$\therefore$ Equation (1), becomes

$P^2 + Q^2 = 1 \ldots\ldots(2)$

$$Put\ P = a\ in\ (2), we\ get\ Q = \sqrt{1-a^2}$$

But we have

$dz = P\,dX + Q\,dY$

Substuting P,Q ,we get

$dz = (a\,dX + \sqrt{1-a^2}\,dY)$

Integrating on both sides

$$\int dz = (a\int dX + \sqrt{1-a^2}\int dY)$$

z $=\left(aX + \sqrt{1-a^2}Y\right) + C$

$\boxed{z=\left(a\log x + \sqrt{1-a^2}\log y\right) + C}$

is the Complete solution of (1)

Equations of the form $\boldsymbol{f(z^n p, z^n q) = 0\ where\ \ n\ is\ a\ \ constant}$:

Use the following substitution to reduce the above form to an equation of the form f(P,Q)=0

$$put \quad Z = \begin{cases} z^{n+1} & if\ n \neq -1 \\ \log z, & if\ n = -1 \end{cases}$$

Equations of the form $\boldsymbol{f(x, z^n p) = g(y, z^n q)\ \ where\ \ n\ is\ a\ \ constant}$**:**

An equation of the above form can be reduced to an equation of the form f(P,Q)=0

by the substitutions given for the equation $F(z^n p, z^n q) = 0$ as above

Solved Problems :

1. **Solve** $z^2\left(p^2+q^2\right) = x^2 + y^2$

Sol: Given that $z^2\left(p^2+q^2\right) = x^2 + y^2$

The given equation can be written as

$z^2p^2 + z^2q^2 = x^2 + y^2\ or\ z^2p^2 - x^2 = y^2 - z^2q^2$

Or $(zp)^2 - x^2 = y^2 - (zq)^2 \rightarrow (1)$

Since (1) is the of the form $f\left(x, pz^n\right) = g\left(y, qz^n\right)$. with n=1

$\therefore$ $put\ Z = z^{n+1} = z^{1+1} = z^2$

Then $\dfrac{\partial Z}{\partial x} = 2z.\dfrac{\partial z}{\partial x} \Rightarrow P = 2zp$ where $P = \dfrac{\partial z}{\partial x}$

$\therefore pz = \dfrac{P}{2}$

and $\dfrac{\partial Z}{\partial y} = 2z.\dfrac{\partial z}{\partial y} \Rightarrow Q = 2zq$ where $Q = \dfrac{\partial z}{\partial y}$ $\therefore qz = \dfrac{Q}{2}$

$\therefore$ Equation (1) becomes, $\dfrac{P^2}{4} - x^2 = y^2 - \dfrac{Q^2}{4}$

$i.e., P^2 - 4x^2 = 4y^2 - Q^2 \rightarrow (2)$

This is of the form $f_1(x, P) = f_2(y, Q)$

Let $P^2 - 4x^2 = 4y^2 - Q^2 = 4k^2$ (say)

$\therefore P^2 - 4x^2 = 4k^2\ and\ 4y^2 - Q^2 = 4k^2$

$\Rightarrow P^2 = 4x^2 + 4k^2\ and\ Q^2 = 4y^2 - 4k^2$

$\therefore P = 2\sqrt{x^2 + k^2}\ and\ Q = 2\sqrt{y^2 - k^2}$

We have $dZ = \dfrac{\partial Z}{\partial x}.dx + \dfrac{\partial Z}{\partial y}dy$

$= Pdx + Qdy$ [By total differentiation]

$\therefore dZ = 2\sqrt{x^2 + k^2}\,dx + 2\sqrt{y^2 - k^2}\,dy$

Integrating on both sides

$$Z = 2\int \sqrt{x^2 + k^2}\,dx + 2\int \sqrt{y^2 - k^2}\,dy$$

$$= 2\left[\frac{x}{2}\sqrt{x^2 + k^2} + \frac{k^2}{2}\sinh^{-1}\left(\frac{x}{k}\right)\right] + 2\left[\frac{x}{2}\sqrt{y^2 - k^2} - \frac{k^2}{2}\cosh^{-1}\left(\frac{y}{k}\right)\right] + c$$

$$= x\sqrt{x^2 + k^2} + k^2\sinh^{-1}\left(\frac{x}{k}\right) + x\sqrt{y^2 - k^2} + k^2\cosh^{-1}\left(\frac{y}{k}\right) + c$$

$$or\ z^2 = x\sqrt{x^2 + k^2} + y\sqrt{y^2 - k^2} + k^2\left[\sinh^{-1}\left(\frac{x}{k}\right) - \cosh^{-1}\left(\frac{y}{k}\right)\right] + c$$

$$\boxed{or\ z^2 = x\sqrt{x^2 + k^2} + y\sqrt{y^2 - k^2} + k^2\log\left(\frac{x + \sqrt{x^2 + k^2}}{y + \sqrt{y^2 - k^2}}\right) + c}$$

This is the complete solution of (1)

2. **Solve the partial differential equation.** $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

Sol: Given that $p^2z^2\sin^2 x+q^2z^2\cos^2 y=1$

The given equation can be written as

$$(pz)^2\sin^2 x+(qz)^2\cos^2 y=1 \; or \; (pz)^2\sin^2 x=1-(qz)^2\cos^2 y \rightarrow (1)$$

Since (1) is of the form $f\left(x, pz^n\right)=g\left(y, qz^n\right)$ with n=1.

Put $Z=z^{n+1}=z^2$

Now $\frac{\partial Z}{\partial x}=2z.\frac{\partial z}{\partial x}\Rightarrow P=2zp \; or \; pz=\frac{P}{2}$ where $P=\frac{\partial z}{\partial x}; Q=\frac{\partial z}{\partial y}$

and $\frac{\partial Z}{\partial y}=2z.\frac{\partial z}{\partial y}\Rightarrow Q=2zq \; or \; qz=\frac{Q}{2}$

Then equation (1) becomes, $\left(\frac{P}{2}\right)^2\sin^2 x=1-\left(\frac{Q}{2}\right)^2\cos^2 y$

$i.e. \frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y\rightarrow(2)$

This is of the form $f_1(x,p)=f_2(y,q)$

Let $\frac{P^2}{4}\sin^2 x=1-\frac{Q^2}{4}\cos^2 y=k^2$ (constant)

$\therefore \frac{P^2}{4}\sin^2 x=k^2 \; and \; 1-\frac{Q^2}{4}\cos^2 y=k^2$

$\Rightarrow P^2\sin^2 x=4k^2 \; and \; Q^2\cos^2 y=4\left(1-k^2\right)$

$\Rightarrow P=\frac{2k}{\sin x} \; and \; Q=\frac{2\sqrt{1-k^2}}{\cos y}$

We have $dZ=\frac{\partial Z}{\partial x}dx+\frac{\partial Z}{\partial y}dy$ [By total differential]

$\therefore dZ=Pdx+Qdy$

$dZ=\frac{2k}{\sin x}dx+\frac{2\sqrt{1-k^2}}{\cos y}dy$

Integrating on both sides

$$z=2k\int \csc x\,dx+2\sqrt{1-k^2}\int \sec y\,dy$$

$$=2k\log\left(\operatorname{cosec}x-\cot x\right)+2\sqrt{1-k^2}\log\left(\sec y+\tan y\right)+c$$

$$\boxed{\therefore z^2=2k\log\left(\operatorname{cosec}x-\cot x\right)+2\sqrt{1-k^2}\log\left(\sec y+\tan y\right)+c}$$

This is the required complete solution of (1)

3.Solve $(\boldsymbol{x+pz})^2+(\boldsymbol{y+qz})^2=\mathbf{1}$

Sol: Given $(x+pz)^2+(y+qz)^2=1$…….(1)

$since\ (1)\ is\ of\ the\ form\ F(z^n p, z^n q, x, y)=0 \quad n=1$

Put $Z=\ z^{n+1}=z^2$

Differentiating partially w.r.t 'x',we get $\frac{\partial Z}{\partial z}=2z\ implies\ that\ \frac{\partial z}{\partial Z}=\frac{1}{2z}$

But $p=\frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}=\frac{P}{2z}$ implies $\frac{\partial Z}{\partial x}=\frac{P}{2}=zp; Similarly\ we\ get\ qz=\frac{Q}{2}$

Substitute in (1),we get

$(x+\frac{P}{2})^2+(y+\frac{Q}{2})^2=1$

Separating $P\ and\ x\ from\ Q\ and\ y$, the given equation can be written as.

$(x+\frac{P}{2})^2=1-(y+\frac{Q}{2})^2=K^2$

$(x+\frac{P}{2})^2=K^2\ AND\ 1-(y+\frac{Q}{2})^2=K^2$

$\left(x+\frac{P}{2}\right)=K$

Implies that

$Q=2(\sqrt{(1-K^2)}-y)$

$P=2(K-x)$

We have $dZ=\frac{\partial Z}{\partial x}dx+\frac{\partial Z}{\partial y}dy$

$$\therefore dz=2(k-x)dx+2\left[\sqrt{1-k^2}-y\right]dy$$

Integrating on both sides

$$z=2\int(k-x)dx+2\int\left[\sqrt{1-k^2}-y\right]dy+c$$

$$z = 2(kx - \frac{x^2}{2}) + 2\left[\left(\sqrt{1-k^2}\right)y - \frac{y^2}{2}\right] + c$$

$$\boxed{\therefore z = 2kx - x^2 + 2\left(\sqrt{1-k^2}\right)y - y^2 + c}$$

This is the complete solution of (1).

4. Solve $\boldsymbol{z(p^2 - q^2) = x - y}$

Sol: Given

$$z(p^2 - q^2) = x - y \ldots\ldots\ldots\ldots (1)$$

$$(z^{\frac{1}{2}}p)^2 - (z^{\frac{1}{2}}q)^2 = x - y \ldots..(2)$$

$$since\ (2)\ is\ of\ the\ form\ F(z^n p, z^n q, x, y) = 0 \quad n = \frac{1}{2}$$

Put $Z = z^{n+1} = z^{\frac{3}{2}}$

Differentiating partially w.r.t 'x',we get $\frac{\partial Z}{\partial z} = \frac{3}{2} z^{\frac{1}{2}}$

$$implies\ that\ \frac{\partial z}{\partial Z} = \frac{2}{3z^{\frac{1}{2}}}$$

But $p = \frac{\partial z}{\partial Z}\frac{\partial Z}{\partial x}$=Pimplies $\frac{2}{3}P = z^{\frac{1}{2}}p; Similarly\ we\ get \frac{2}{3}Q = z^{\frac{1}{2}}q$ Substitute in (2),we get

$$(\frac{2}{3}P)^2 - (\frac{2}{3}Q)^2 = \boldsymbol{x - y}$$

Separating $P\ and\ x\ from\ Q\ and\ y,$ the given equation can be written as.

$$(\frac{2}{3}P)^2 - \boldsymbol{x} = -\boldsymbol{y} + (\frac{2}{3}Q)^2 = \boldsymbol{k}$$

Solving, we get

$P = \frac{3}{2}\sqrt{k + x}$ and $Q = \frac{3}{2}\sqrt{k + y}$

We have $dZ = \frac{\partial Z}{\partial x}dx + \frac{\partial Z}{\partial y}dy$ [By total differential]

$\therefore dZ = Pdx + Qdy$

$dZ = \frac{3}{2}[\sqrt{k + x}dx + \sqrt{k + y}dy]$

Integrating on both sides

$$Z = \frac{3}{2}[\int \sqrt{k+x}dx + \int \sqrt{k+y}\ dy$$

$$z^{\frac{3}{2}} = (\ k+x\ )^{\frac{3}{2}} + (k+y)^{\frac{3}{2}} + c$$

This is the required complete solution of (1)

Methods Of Separation Of Variables:

This method is used to reduce one partial differential equation to two or more ordinary differential equations,each one involving one of the independent variables.This will be done by separating these variables from the beginning. This method is explained through following examples.

1. **Solve by the method of separation of variables $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$ where U(x,0)=6e^{-3x}**

Sol : Given equation is $\frac{\partial U}{\partial x} = 2\frac{\partial U}{\partial t} + U$----------(1)

Let U(x,t)= X(x) T(t) =XT ---------------(2)

be a solution of (1)

Differentiating (2) partially w.r.t x ant t

$$\frac{\partial U}{\partial x} = X'T \quad , \quad \frac{\partial U}{\partial t} = T'X$$

Put these values in equation (1), we have

$X'T = 2\,T'X\ + XT$ Dividing by XT

$\frac{X'}{X} = 2\frac{T'}{T} + 1$--------(3)

Since L.H.S is a function of ‘x’ and the R.H.S is a function of ‘t’ where x and t are independent variables, the two sides of (3) can be equal to each other for all values of ‘x’ and ‘t’ if and only if both sides are equal to a constant.

Therefore $\frac{X'}{X} = 2\frac{T'}{T} + 1$=k--------(4) where k is a constant

Now from (4) $\frac{X'}{X}$ = k----(5) and $2\frac{T'}{T} + 1$=k-------(6)

Now consider (5) $\frac{X'}{X}$ = k $\Rightarrow$ $X' - kX = 0$ $\Rightarrow X = C_1e^{kx}$

Now consider (6) $2\frac{T'}{T} + 1$=k $\Rightarrow$ $T' - \left(\frac{k-1}{2}\right)T = 0$ $\Rightarrow T = C_2e^{\left(\frac{k-1}{2}\right)t}$ ---(8)

Substituting the values of X and T in (2) we get

$$U(x,t) = X = C_1e^{kx}C_2e^{\left(\frac{k-1}{2}\right)t}$$

$$U(x,t) = X = Ae^{kx}e^{\left(\frac{k-1}{2}\right)t} \quad (\textit{where } \mathrm{A=C_1\,C_2})$$

Put t=0 in the above equation ,we have U(x,0) = A e^{kx} -----(9)

but given that U(x,0)=$6e^{-3x}$--------(10)

from (9) and (10) we have A e^{kx} =$6e^{-3x}$

A=6 and k=-3 the solution of the given equation becomes

$$\boxed{U(x,t) = X = 6e^{-3x}e^{(-2)t} = 6e^{-(3x+2t)}}$$

2. Solve the equation by the method of separation of variables$\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$

Sol: Given equation is $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial y} + 2U$ ----------(1)

Let U(x,y)= X(x) Y(y) =X Y ---------------(2)

be a solution of (1)

Differentiating (2) partially w.r.t x ant y

$$\frac{\partial U}{\partial x} = X'Y \quad , \quad \frac{\partial U}{\partial y} = Y'X \quad \frac{\partial^2 U}{\partial x^2} = X''Y$$

Put these values in equation (1), we have

$$X''Y \quad = \; Y'X + 2XY$$

Dividing by XY on both sides we have $\frac{X''}{X} = \frac{Y''}{Y}$+2

$$\frac{X''}{X} - 2 = \frac{Y''}{Y}\text{-----------------(3)}$$

Since L.H.S is a function of 'x' and the R.H.S is a function of 'y' where x and y are independent variables, the two sides of (3) can be equal to each other for all values of 'x' and 'y' if and only if both sides are equal to a constant.

$$\frac{X''}{X} - 2 = \frac{Y''}{Y} = k\text{--------(4)}$$

Now from (4)

$$\frac{X''}{X} - 2 = k - - - - - - (5)$$

And $\frac{Y''}{Y} = k - - - - - - - (6)$

From (5) $X'' - 2X = kX$ $X'' - (2 + k)X = 0$

Which is second order differential equation

Auxiliary equation is $m^2 - (2 + k) = 0$ $\rightarrow$ m=$\pm\sqrt{(2 + k)}$

Solution of the given equation (5) is $X = C_1 e^{\sqrt{(2+k)}}$

Now consider equation (6) $Y' = kY$ $\rightarrow \frac{Y'}{Y} = k$

Integrating on both sides we get logy = ky + logC_3

$$\Rightarrow \log\left(\frac{Y}{C_3}\right) = ky \quad \Rightarrow Y = C_3 e^{ky} - - - (8)$$

Substituting the values of X and Y in (2) we have

$$U=\left[C_1e^{\sqrt{(2+k)}x}+C_2e^{-\sqrt{(2+k)}x}\right]C_3e^{ky}$$
$$U=\left[Ae^{\sqrt{(2+k)}x}+Be^{-\sqrt{(2+k)}x}\right]e^{ky}$$

Where $A=C_1C_3 \quad and \quad B=C_1C_2$

UNIT-IV

DOUBLE AND TRIPLE INTEGRALS

Introduction :

The multiple integral is a definite integral of a function of more than one real variable, for instance, *f*(*x*, *y*) or *f*(*x*, *y*, *z*). Integrals of a function of two variables over a region in R^2 are called double integrals, and integrals of a function of three variables over a region of R^3 are called triple integrals.

Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and the *x*-axis, the double integral of a positive function of two variables represents the volume of the region between the surface defined by the function (on the three-dimensional Cartesian plane where $z = f(x, y)$ and the plane which contains its domain. If there are more variables, a multiple integral will yield hyper olumes of multidimensional functions.Double integrals are used to calculate the area of a region, the volume under a surface, and the average value of a function of two variables over a rectangular region.

Definition of double integral : suppose we have a region in the plane R and a function $f(x, y)$, then the double integral $\iint_R f(x,y)dA$ is defined as follows

Divide the region R into small pieces, numbered from 1 to n.Let ΔA_i be the area of the i^{th} piece and also pick a point (x_i, y_i) in that piece.

Then the sum $\sum_{i=1}^{n} f\ (x_i, y_i)\Delta\, A_i$

Therefore $\iint_R f(x,y)dA = \lim_{\Delta A \to 0} \sum_{i=1}^{n} f\ (x_i, y_i)\Delta\, A_i$

Double Integral:

I. When y_1, y_2 are functions of x and x_1 and x_2 are constants. $f(x, y)$is first integrated with respect to y keeping 'x' fixed between limits y_1, y_2 and then the resulting expression is integrated with respect to 'x' within the limits x_1, x_2 i.e.,

$$\iint_R f(x,y)dx\,dy = \int_{x=x_1}^{x=x_2} \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)dydx$$

II. When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x, y)$ is first integrated with respect to 'x' keeping 'y' fixed, within the limits x_1, x_2 and then resulting expression is integrated with respect to y between the limits y_1, y_2 i.e.,

$$\iint_R f(x,y)dx\,dy = \int_{y=y_1}^{y=y_2} \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)dx\ dy$$

III. When x_1, x_2, y_1, y_2 are all constants. Then

$$\iint_R f(x,y)\,dx\,dy = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y)\,dx\,dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx$$

It can be used in any order

Solved Problems:

1. Evaluate $\int_1^2\int_1^3 xy^2\,dx\,dy$

Sol. $$\int_1^2\left[\int_1^3 xy^2\,dx\right]dy = \int_1^2\left[y^2.\frac{x^2}{2}\right]_1^3 dy = \int_1^2\frac{y^2}{2}\,dy\,[9-1]$$

$$=\frac{8}{2}\int_1^2 y^2dy = 4.\int_1^2 y^2dy$$

$$=4.\left[y^3/3\right]_1^2 = \frac{4}{3}[8-1] = \frac{4.7}{3} = \frac{28}{3}$$

2. Evaluate $\int_0^2\int_0^x y\,dy\,dx$

Sol : $$\int_{x=0}^{2}\int_{y=0}^{x} y\,dy\,dx = \int_{x=0}^{2}\left[\int_{y=0}^{x} y\,dy\right]dx$$

$$=\int_{x=0}^{2}\left[\frac{y^2}{2}\right]_0^x dx = \int_{x=0}^{2}\frac{1}{2}\left(x^2-0\right)dx = \frac{1}{2}\int_{x=0}^{2}x^2dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{1}{6}(8-0) = \frac{8}{6} = \frac{4}{3}$$

3. Evaluate $\int_0^5\int_0^{x^2} x\left(x^2+y^2\right)dx\,dy$

Sol. $$\int_{x=0}^{5}\int_{y=0}^{x^2} x\left(x^2+y^2\right)dy\,dx = \int_{x=0}^{5}\left[x^3y+\frac{xy^3}{3}\right]_{y=0}^{x^2} dx$$

$$=\int_{x=0}^{5}\left[x^3.x^2+\frac{x(x^2)^3}{3}\right]dx = \int_{x=0}^{5}\left(x^5+\frac{x^7}{3}\right)dx = \left[\frac{x^6}{6}+\frac{1}{3}.\frac{x^8}{8}\right]_0^5$$

$$=\frac{5^6}{6}+\frac{5^8}{24}$$

4. Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dydx}{1+x^2+y^2}$

Sol: $$\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dydx}{1+x^2+y^2} = \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}}\frac{1}{\left(1+x^2\right)+y^2}dy\right]dx$$

$$= \int_{x=0}^{1}\left[\int_{y=0}^{\sqrt{1+x^2}} \frac{1}{\left(\sqrt{1+x^2}\right)^2 + y^2}\, dy\right] dx$$

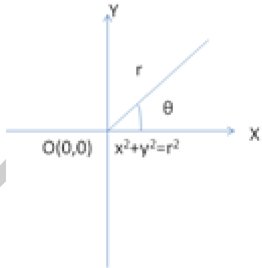
$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_{y=0}^{\sqrt{1+x^2}} dx \quad [\because \int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}(x/a)]$$

$$= \int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}}\left[Tan^{-1}1 - Tan^{-1}0\right] dx \;\; or \;\; \frac{\pi}{4}(\sinh^{-1}\mathrm{x})_0^1 = \frac{\pi}{4}(\sinh^{-1}1)$$

$$= \frac{\pi}{4}\int_{x=0}^{1} \frac{1}{\sqrt{1+x^2}} dx = \frac{\pi}{4}\left[\log(x+\sqrt{x^2+1})\right]_{x=0}^{1}$$

$$= \frac{\pi}{4}\log(1+\sqrt{2})$$

5. Evaluate $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy$



Sol: $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^\infty e^{-y^2}\left[\int_0^\infty e^{-x^2} dx\right] dy$

$$= \int_0^\infty e^{-y^2}\frac{\sqrt{\pi}}{2} dy \qquad \because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$= \frac{\sqrt{\pi}}{2}\int_0^\infty e^{-y^2} dy = \frac{\sqrt{\pi}}{2}.\frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

Alter: $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy = \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\infty} e^{-r^2} r dr d\theta \qquad (\because x^2+y^2 = r^2)$

(Changing to polar coordinates taking $x = r\cos\theta, y = r\sin\theta$)

$$= \int_0^{\pi/2}\left[\frac{e^{-r^2}}{-2}\right]_0^\infty d\theta = \int_0^{\pi/2}\left[\frac{0-1}{-2}\right] d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2} = \frac{1}{2}\left(\pi/2 - 0\right) = \frac{\pi}{4}$$

6. Evaluate $\iint xy(x+y)dxdy$ **over the region R bounded by** $y = x^2$ **and** $y = x$

Sol: $y = x^2$ is a parabola through (0, 0) symmetric about y-axis $y = x$ is a straight line through (0, 0) with slope1.

Let us find their points of intersection solving $y = x^2, y = x$ we get

$$x^2 = x \Rightarrow x = 0,1 \;\; Hence\; y = 0,1$$

$\therefore$ The points of intersection of the curves are (0, 0), (1, 1)

Consider $\iint_R xy(x+y)dxdy$

For the evaluation of the integral, we first integrate with respect to y from $y = x2\; to\; y = x$ and then with respect to x from x=0 to x=1

$$\int_{x=0}^{1}\left[\int_{y=x^2}^{x} xy\left(x+y\right)dy\right]dx = \int_{x=0}^{1}\left[\int_{y=x^2}^{x}\left(x^2y+xy^2\right)dy\right]dx$$

$$=\int_{x=0}^{1}\left(x^2\frac{y^2}{2}+\frac{xy^3}{3}\right)_{y=x^2}^{x} dx$$

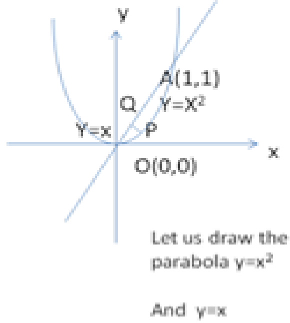
$$=\int_{x=0}^{1}\left(\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx$$

$$=\int_{x=0}^{1}\left(\frac{5x^4}{6}-\frac{x^6}{2}-\frac{x^7}{3}\right)dx = \left(\frac{5}{6}.\frac{x^5}{5}-\frac{x^7}{14}-\frac{x^8}{24}\right)_0^1$$

$$=\frac{1}{6}-\frac{1}{14}-\frac{1}{24}=\frac{28-12-7}{168}=\frac{28-19}{168}=\frac{9}{168}=\frac{3}{56}$$

7. Evaluate $\iint_R xydxdy$ where R is the region bounded by x-axis and $x = 2a$ and the curve $x^2 = 4ay$.

Sol: The line $x = 2a$ and the parabola $x^2 = 4ay$ intersect at $B(2a, a)$



$\therefore$The given integral = $\iint_R xy\; dx\; dy$

Let us fix 'y', for a fixed 'y', x varies from $2\sqrt{ay}\;\; to\; 2a$. Then y varies from $0\; to\; a$

Hence the given integral can also be written as

$$\int_{y=0}^{a}\int_{x=2\sqrt{ay}}^{x=2a} xy\,dx\,dy = \int_{y=0}^{a}\left[\int_{x=2\sqrt{ay}}^{x=2a} xdx\right]ydy$$

$$=\int_{y=0}^{a}\left[\frac{x^2}{2}\right]_{x=2\sqrt{ay}}^{2a} ydy$$

$$=\int_{y=0}^{a}\left[2a^2-2ay\right]y\,dy$$

$$=\left[\frac{2a^2y^2}{2}-\frac{2ay^3}{3}\right]_0^a$$

$$=a^4-\frac{2a^4}{3}=\frac{3a^4-2a^4}{3}=\frac{a^4}{3}$$

8. Evaluate $\int_0^1\int_0^{\pi/2} r\sin\theta\, d\theta\, dr$

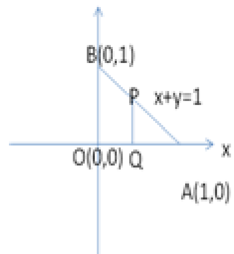
Sol. $\int_{r=0}^1 r\left[\int_{\theta=0}^{\pi/2}\sin\theta\, d\theta\right]dr=\int_{r=0}^1 r\left(-\cos\theta\right)_{\theta=0}^{\pi/2}dr$

$$=\int_{r=0}^1 -r\left(\cos\pi/2-\cos 0\right)dr$$

$$=\int_{r=0}^1 -r\left(0-1\right)dr=\int_0^1 r dr=\left(\frac{r^2}{2}\right)_0^1=\frac{1}{2}-0=\frac{1}{2}$$

9. Evaluate $\iint\left(x^2+y^2\right)dx\,dy$ **in the positive quadrant for which** $x+y\le 1$

Sol. $\iint_R\left(x^2+y^2\right)dx\,dy=\int_{x=0}^1 dx\int_{y=0}^{y=1-x}\left(x^2+y^2\right)dy$



$$=\int_{x=0}^1\left(x^2y+\frac{y^3}{3}\right)_0^{1-x}dx$$

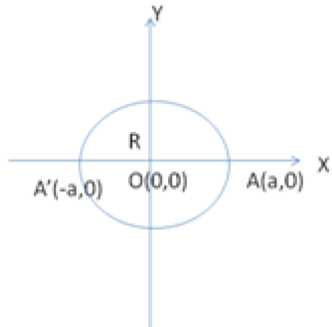
$$=\int_{x=0}^1\left(x^2-x^3+\frac{1}{3}(1-x)^3\right)dx$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{4}-\frac{1}{12}(1-x)^4\right]_0^1=\frac{1}{3}-\frac{1}{4}-0+\frac{1}{12}=\frac{1}{6}$$

10. Evaluate $\iint\left(x^2+y^2\right)dx\,dy$ **over the area bounded by the ellipse** $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Sol. Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$i.e.,\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{1}{a^2}\left(a^2-x^2\right)(or)\,y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$



$$\therefore y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

Hence the region of integration R can be expressed as

$$-a \le x \le a, \frac{-b}{a}\sqrt{a^2-x^2} \le y \le \frac{b}{a}\sqrt{a^2-x^2}$$

$$\therefore \iint_R \left(x^2+y^2\right)dx\,dy = \int_{x=-a}^{a}\int_{y=-b/a\sqrt{a^2-x^2}}^{b/a\sqrt{a^2-x^2}}\left(x^2+y^2\right)dx\,dy$$

$$=2\int_{x=-a}^{a}\int_{y=0}^{b/a\sqrt{a^2-x^2}}\left(x^2+y^2\right)dx\,dy = 2\int_{-a}^{a}\left(x^2y+y^3/3\right)_0^{b/a\sqrt{a^2-x^2}}$$

$$=2\int_{-a}^{a}\left[x^2.b/a\sqrt{a^2-x^2}+\frac{b^3}{3a^3}\left(a^2-x^2\right)^{3/2}\right]dx$$

$$=4\int_{0}^{a}\left[b/a\,x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}\left(a^2-x^2\right)^{3/2}\right]dx$$

Changing Cartesian to polar co-ordinates $\quad put \quad x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$

$$\frac{x}{a}=\sin\theta \Rightarrow \theta = \sin^{-1}\frac{x}{a}$$

$$If\;\; x\to 0, \text{Then}\,\theta\to 0 \quad and \quad if\;\; x\to a, \text{Then}\,\theta\to\frac{\pi}{2}$$

$$=4\int_0^{\pi/2}\left[b/a.a^2\sin^2\theta.a\cos\theta+\frac{b^3}{3a^3}.a^3\cos^3\theta\right]a\cos\theta d\theta$$

$$=4\int_0^{\pi/2}\left[a^3b\sin^2\theta\cos^2\theta+\frac{ab^3}{3}\cos^4\theta\right]d\theta = 4\left[a^3b.\frac{1}{4}\frac{1}{2}.\frac{\pi}{2}+\frac{ab^3}{3}.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$\left[\because \int_0^{\frac{\pi}{2}}\sin^m\theta\cos^n\theta d\theta = \frac{n-1}{m+n}.\frac{n-3}{m+n-2}\ldots\ldots\ldots\frac{\frac{1}{2}.\frac{\pi}{2}}{m}\right]$$

$$=\frac{4\pi}{16}\left(a^3b+ab^3\right)$$

$$=\frac{\pi ab}{4}\left(a^2+b^2\right)$$

Double integrals in polar co-ordinates:

1. Evaluate $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}$

Sol: $\int_0^{\pi/4}\int_0^{a\sin\theta}\frac{r\,dr\,d\theta}{\sqrt{a^2-r^2}}=\int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}dr\right\}d\theta=-\frac{1}{2}\int_0^{\pi/4}\left\{\int_0^{a\sin\theta}\frac{-2r}{\sqrt{a^2-r^2}}dr\right\}d\theta$

$$=\frac{-1}{2}\int_0^{\pi/4}2\left(\sqrt{a^2-r^2}\right)_0^{a\sin\theta}d\theta=(-1)\int_0^{\pi/4}2\left[\sqrt{a^2-a^2\sin^2\theta}-\sqrt{a^2-0}\right]d\theta$$

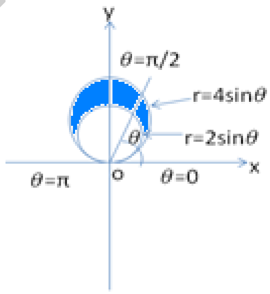
$$=(-a)\int_0^{\pi/4}(\cos\theta-1)\,d\theta=(-a)(\sin\theta-\theta)_0^{\pi/4}$$

$$=(-a)\left[\left[\sin\pi/4-\pi/4\right]-(0-0)\right]$$

$$=(-a)\left[1/\sqrt{2}-\pi/4\right]=2\left[\pi/4-1/\sqrt{2}\right]$$

2. Evaluate $\iint r^3 dr d\theta$ over the area included between the circles $r=2\sin\theta$ and $r=4\sin\theta$

Sol: The region of integration R is shown shaded .Here r varies from $P(r=2\sin\theta)$ to $Q(r=4\sin\theta)$ and to cover the whole region varies θ from 0 to π



$$\therefore \iint r^3 dr d\theta=\int_0^{\pi}\int_{r=2\sin\theta}^{4\sin\theta}r^3 dr d\theta$$

$$=\int_0^{\pi}\left\{\int_{r=2\sin\theta}^{4\sin\theta}r^3dr\right\}d\theta$$

$$=\int_0^{\pi}\left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta}d\theta$$

$$=\frac{1}{4}\int_0^{\pi}(256\sin^4\theta-16\sin^4\theta)\,d\theta$$

$$=60\int_0^{\pi}\sin^4\theta\ d\theta$$

$$\left[\because \int_0^{2a}f(x)dx=2\int_0^{a}f(x)dx, if\ f(2a-x)=f(x)\right]$$

$$=60\times 2\int_0^{\pi/2}\sin^4\theta\ d\theta=120\times\frac{3\times1}{4\times2}.\frac{\pi}{2}=\frac{45\pi}{2}$$

3. Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, dr d\theta$.

Sol: Let I = $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r^2\cos\theta\, dr d\theta = \int_0^{\pi}\cos\theta d\theta \int_0^{a(1+\cos\theta)} r^2\, dr =$ $\int_0^{\pi}\cos\theta d\theta \left(\frac{r^3}{3}\right)_0^{a(1+\cos\theta)}$

$$= \frac{a^3}{3}\int_0^{\pi}\cos\theta(1+\cos\theta)^3 d\theta \quad \ldots\ldots(1)$$

$$= \frac{a^3}{3}\int_0^{\pi}\cos(\pi-\theta)\big(1+\cos(\pi-\theta)\big)^3 d\theta \quad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \frac{a^3}{3}\int_0^{\pi} -\cos\theta(1-\cos\theta)^3 d\theta \ldots\ldots\ldots\ldots\ldots(2)$$

(1) + (2) gives

$$2I = \frac{a^3}{3}\int_0^{\pi}\cos\theta[(1+\cos\theta)^3-(1+\cos\theta)^3]d\theta$$

$$= \frac{a^3}{3}\int_0^{\pi}\cos\theta[2\cos\theta\{3+\cos^2\theta\}]d\theta \ [\because (a+b)^3-(a-b)^3 = 2b(3a^2+b^2)]$$

$$= \frac{2a^3}{3}\int_0^{\pi}(3\cos^2\theta+\cos^4\theta)\, d\theta = \frac{2a^3}{3}.2.\int_0^{\pi/2}(3\cos^2\theta+\cos^4\theta)\, d\theta$$

$$= \frac{4a^3}{3}\left[3.\frac{\pi}{4}+\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right]$$

$$or \quad I = 2a^3\left(\frac{\pi}{4}+\frac{\pi}{16}\right) = \frac{\pi a^3}{2}\left(1+\frac{1}{4}\right) = \frac{5\pi a^3}{8}$$

Change of order of Integration:

1. Change the order of Integration and evaluate $\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\, dx$

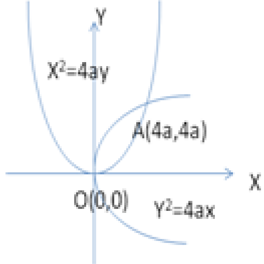
Sol: In the given integral for a fixed x, y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$ and then x varies from $0\ to\ 4a$.

$$Let\ us\ draw\ the\ curves\ y=\frac{x^2}{4a}\ and\ y=2\sqrt{ax}$$

The region of integration is the shaded region in diagram.

The given integral is $=\int_{x=0}^{4a}\int_{y=x^2/4a}^{2\sqrt{ax}} dy\, dx$

Changing the order of integration, we must fix y first, for a fixed y, x varies from $\frac{y^2}{4a}$ to $\sqrt{4ay}$
and then y varies from 0 to $4a$.Hence the integral is equal to



$$\int_{y=0}^{4a}\int_{x=y^2/4a}^{2\sqrt{ay}} dx\,dy = \int_{y=0}^{4a}\left[\int_{x=y^2/4a}^{2\sqrt{ay}} dx\right]dy$$

$$= \int_{y=0}^{4a}[x]_{x=y^2/4a}^{2\sqrt{ay}}\,dy = \int_{y=0}^{4a}\left[2\sqrt{ay}-y^2/4a\right]dy$$

$$= \left[2\sqrt{a}.\frac{y^{3/2}}{3/2}-\frac{1}{4a}.\frac{y^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}.\sqrt{a}.4a\sqrt{4a}-\frac{1}{12a}.64a^3 = \frac{32}{3}a^2-\frac{16}{3}a^2 = \frac{16}{3}a^2$$

2. Change the order of integration and evaluate $= \int_0^a\int_{x/a}^{\sqrt{x/a}}\left(x^2+y^2\right)dx\,dy$

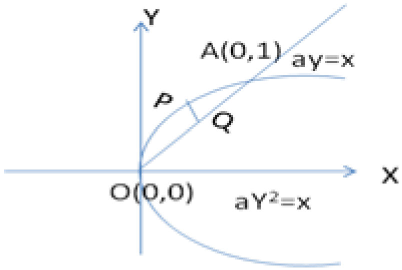
Sol: In the given integral for a fixed x, y varies from $\frac{x}{a}$ to $\sqrt{\frac{x}{a}}$ and then x varies from 0 to a

Hence we shall draw the curves $y=\frac{x}{a}$ *and* $y=\sqrt{\frac{x}{a}}$

$i.e.\ ay = x$ *and* $ay^2 = x$

We get $ay = ay^2$

$\Rightarrow ay - ay^2 = 0 \Rightarrow ay(1-y)=0 \Rightarrow y=0, y=1$



If y=0, x=0 if y=1, x=a

The shaded region is the region of integration. The given integral is

$$\int_{x=0}^{a}\int_{y=x/a}^{y=\sqrt{x/a}}\left(x^2+y^2\right)dx\,dy$$

Changing the order of integration, we must fix y first. For a fixed y, x varies from ay^2 to ay and then y varies from 0 to 1.

Hence the given integral, after change of the order of integration becomes

$$\int_{y=0}^{1}\int_{x=ay^2}^{ay}\left(x^2+y^2\right)dx\,dy=\int_{y=0}^{1}\left[\int_{x=ay^2}^{ay}\left(x^2+y^2\right)dx\right]dy$$

$$=\int_{y=0}^{1}\left(\frac{x^3}{3}+xy^2\right)_{x=ay^2}^{ay}dy$$

$$=\int_{y=0}^{1}\left(\frac{a^3y^3}{3}+ay^3-\frac{a^3y^6}{3}-ay^4\right)dy$$

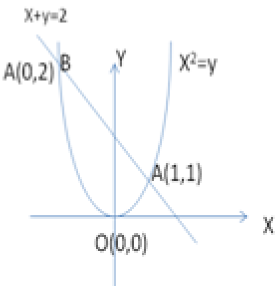
$$=\left(\frac{a^3y^4}{12}+\frac{ay^4}{4}-\frac{a^3y^7}{21}-\frac{ay^5}{5}\right)_{y=0}^{1}$$

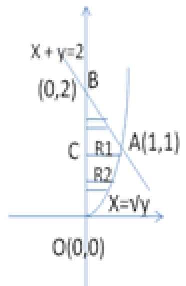
$$=\frac{a^3}{12}+\frac{a}{4}-\frac{a^3}{21}-\frac{a}{5}=\frac{a^3}{28}+\frac{a}{20}$$

3. Change the order of integration in $\int_0^1\int_{x^2}^{2-x} xydxdy$ **and hence evaluate the double integral.**

Sol: In the given integral for a fixed x, y varies from x^2 to $2-x$ and then x varies from 0 to 1. Hence we shall draw the curves $y = x^2$ and $y = 2-x$.

The line $y = 2-x$ passes through (0, 2), (2, 0)





Solving $y = x^2, y = 2-x$

Then we get, $x^2 = 2-x \Rightarrow x^2+x-2=0$

$\Rightarrow x^2+2x-x-2=0$

$\Rightarrow x(x+2)-1(x+2)=0 \Rightarrow (x-1)(x+2)=0$

$\Rightarrow x=1,-2$

$If\ x=1, y=1$

$If\ x=-2, y=4$

Hence the points of intersection of the curves are (-2, 4) (1, 1)

The Shaded region in the diagram is the region of intersection.

Changing the order of integration, we must fix y, for the region within OACO for a fixed y, x varies from 0 to $\sqrt{y}$

Then y varies from 0 to 1

For the region within CABC, for a fixed y, x varies from 0 to $2-y$, then y varies from 1 to 2

Hence $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \iint\limits_{OACO} xy\,dx\,dy + \iint\limits_{CABC} xy\,dx\,dy$

$$= \int_{y=0}^{1}\left[\int_{x=0}^{\sqrt{y}} xdx\right] y\,dy + \int_{y=1}^{2}\left[\int_{x=0}^{2-y} x\,dx\right] y\,dy$$

$$= \int_{y=0}^{1}\left(\frac{x^2}{2}\right)_{x=0}^{\sqrt{y}} y\,dy + \int_{y=1}^{2}\left(\frac{x^2}{2}\right)_{x=0}^{2-y} y\,dy$$

$$= \int_{y=0}^{1}\frac{y}{2}.y\,dy + \int_{y=1}^{2}\frac{(2-y)^2}{2}y\,dy$$

$$= \frac{1}{2}\int_{y=0}^{1} y^2 dy + \frac{1}{2}.\int_{y=1}^{2}\left(4y-4y^2+y^3\right)dy$$

$$= \frac{1}{2}.\left(\frac{y^3}{3}\right)_0^1 + \frac{1}{2}.\left[\frac{4y^2}{2}-\frac{4y^3}{3}+\frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}.\frac{1}{3}+\frac{1}{2}\left[2.4-2.1-\tfrac{4}{3}(8-1)+\tfrac{1}{4}(16-1)\right]$$

$$= \frac{1}{6}+\frac{1}{2}\left[6-\frac{28}{3}+\frac{15}{4}\right] = \frac{1}{6}+\frac{1}{2}\left[\frac{72-112+45}{12}\right]$$

$$= \frac{1}{6}+\frac{1}{2}\left[\frac{5}{12}\right] = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8}$$

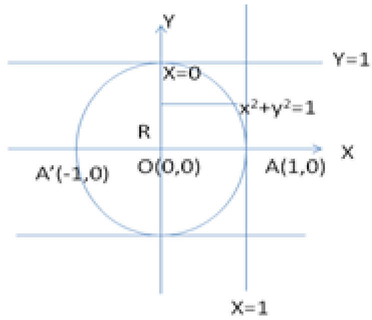
4. Change of the order of integration $\int_0^1\int_0^{\sqrt{1-x^2}} y^2 dx\,dy$

Sol: Now limits are $y = 0$ to 1 and $x = 0$ to $\sqrt{1-y^2}$

$put\ y = \sin\theta$

$\sqrt{1-y^2} = \cos\theta$

$dy = \cos\theta d\theta$



$$= \int_0^1 y^2\sqrt{1-y^2}\,dy$$

$$= \int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = \int_0^{\pi/2}\sin^2\theta d\theta - \int_0^{\pi/2}\sin^4\theta d\theta$$

$$=\frac{1}{2}\left(\pi/2\right)-\frac{3}{4}\cdot\frac{1}{2}\left(\pi/2\right)=\pi/16$$

Change of variables:

The variables x, y in $\iint_R f(x,y)\,dx\,dy$ are changed to u, v with the help of the relations $x = f_1(u,v), y = f_2(u,v)$ then the double integral is transferred into

$$\iint_{R^1} f\left[f_1(u,v), f_2(u,v)\right]\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Where R^1 is the region in the $u\,v$ plane, corresponding to the region R in the xy −plane.

Changing from Cartesian to polar co-ordinates:

$$x = r\cos\theta, y = r\sin\theta$$

$$\partial\left(\frac{(x,y)}{(r,\theta)}\right)=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$

$$=r\left(\cos^2\theta+\sin^2\theta\right)=r$$

$$\therefore \iint_R f(x,y)\,dx\,dy=\iint_{R_1} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Note: In polar form $dx\,dy$ is replaced by $r\,dr\,d\theta$

Solved Problems:

1 . Evaluate the integral by changing to polar co-ordinates $\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dx\,dy$

Sol: The limits of x and y are both from 0 to ∞.

$\therefore$ The region is in the first quadrant where r varies from 0 to ∞ and θ varies from 0 to $\pi/2$

Substituting $x = r\cos\theta, y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

Hence $\int_0^\infty\int_0^\infty e^{-\left(x^2+y^2\right)}dx\,dy=\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$

$$Put\, r^2 = t$$
$$\Rightarrow 2rdr = dt$$
$$\Rightarrow r\,dr = {dt}/{2}$$

$$Where\ r = 0 \Rightarrow t = 0\ and\ \ r = \infty \Rightarrow t = \infty$$

$$\therefore \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dx\,dy = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} \frac{1}{2} e^{-t} dt\,d\theta$$

$$= \int_0^{\pi/2} \frac{-1}{2} \left(e^{-t}\right)_0^\infty d\theta$$

$$= \frac{-1}{2} \int_0^{\pi/2} (0-1)\,d\theta \Rightarrow \frac{1}{2} (\theta)_0^{\pi/2} = \frac{1}{2} \pi/2 = \pi/4$$

2. Evaluate the integral by changing to polar co-ordinates

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dx\,dy$$

Sol: The limits for x are x=0 to $x = \sqrt{a^2-y^2}$

$$\Rightarrow x^2 + y^2 = a^2$$

$\therefore$ The given region is the first quadrant of the circle.

By changing to polar co-ordinates

$$x = r\cos\theta, y = r\sin\theta, dx\,dy = r\,dr\,d\theta$$

Here ‘r’ varies from 0 to a and 'θ' varies from 0 to $\pi/2$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2+y^2\right) dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 r dr d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^a d\theta = \frac{a^4}{4} (\theta)_0^{\pi/2}$$

$$= \pi/8\, a^4$$

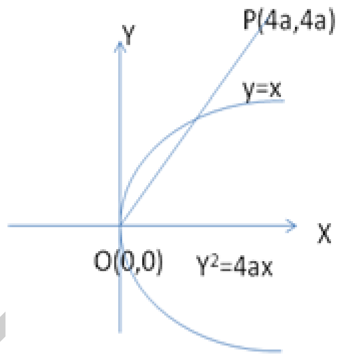
3. Show that $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2} dx\,dy = 8a^2 \left(\frac{\pi}{2} - \frac{5}{3}\right)$

Sol: The region of integration is given by $x = {y^2}/{4a}, x = y$ and $y = 0, y = 4a$.

i.e., The region is bounded by the parabola $y^2 = 4ax$ and the straight line $x = y$.

Let $x = r\cos\theta, y = r\sin\theta. Then\ dx\,dy = rdrd\theta$

The limits for r are $r = 0$ at O and for P on the parabola



$r^2\sin^2\theta = 4a(r\cos\theta) \Rightarrow r = \dfrac{4a\cos\theta}{\sin^2\theta}$

For the line y=x, slope m=1 i.e., $Tan\theta = 1, \theta = \pi/4$

The limits for $\theta : \pi/4 \to \pi/2$

Also $x^2 - y^2 = r^2(\cos^2\theta - \sin^2\theta)\, and\ x^2 + y^2 = r^2$

$$\therefore \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy = \int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{4a\cos\theta/\sin^2\theta} (\cos^2\theta - \sin^2\theta) r dr d\theta$$

$$= \int_{\theta=\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\left(\frac{r^2}{2}\right)_0^{4a\cos\theta/\sin^2\theta} d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} (\cos^2\theta - \sin^2\theta)\frac{\cos^2\theta}{\sin^4\theta}d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} (\cos^4\theta - \cot^2\theta)\,d\theta = 8a^2\left[\frac{3\pi - 8}{12} + \frac{\pi}{4} - 1\right] = 8a^2\left(\frac{\pi}{2} - \frac{5}{3}\right)$$

4.Evaluate $\iint\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dxdy$ over the region R of the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using the transforms $x = au\ and\ y = bv$.

Sol: Given the transformation is $x = au\ and\ y = bv$ ………. (1)

$\therefore$ The region R is transformed to R' where R' is the area of the circle $u^2 + v^2 = 1$ in the first quardrant.

The Jacobian of transformation is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$$

Hence $\iint_R \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dxdy = \iint_{R'} (1 - u^2 - v^2)|J|du\,dv$ $[using\ (1)]$

$$= \iint_{R'} (1 - u^2 - v^2)\, ab\, du\, dv$$

$$= ab \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} (1 - r^2) r\, dr\, d\theta \; [Putting\; u = r cos\theta, v = r sin\theta]$$

$$= ab \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} (r - r^3) dr\, d\theta$$

$$= ab \int_0^{\pi/2} \left(\frac{r^2}{2} - \frac{r^4}{4}\right)_0^1 d\theta = ab \int_0^{\pi/2} \left(\frac{1}{2} - \frac{1}{4}\right) d\theta$$

$$= \frac{ab}{4} \int_0^{\pi/2} d\theta = \frac{ab}{4} (\theta)_0^{\pi/2} = \frac{\pi ab}{8}$$

5. **Evaluate the following integral by transforming into polar coordinates.**

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} y\sqrt{x^2 + y^2}\, dxdy$$

Sol: The region of integration given by

$$y = 0, y = \sqrt{a^2 - x^2}, x = 0 \; and \; x = a$$

i.e., $y = 0, x^2 + y^2 = a^2, x = 0 \; and \; x = a$

i.e., the given region is a quadrant of the circle $x^2 + y^2 = a^2$.

Changing to polar coordinates by putting $x = rcos\theta, y = rsin\theta$ we have $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$

The limits for $r :\; 0\; to\; a$ and $\theta: 0\; to\; \frac{\pi}{2}$.

$$\therefore \int_0^a \int_0^{\sqrt{a^2 - x^2}} y\sqrt{x^2 + y^2}\, dxdy = \int_0^{\pi/2} \int_0^a rsin\theta . r . rdrd\theta$$

$$= \int_0^{\pi/2} \int_0^a r^3 sin\theta drd\theta$$

$$= \int_0^{\pi/2} sin\theta d\theta \int_0^a r^3 dr$$

$$= \int_0^{\pi/2} sin\theta \left(\frac{r^4}{4}\right)_0^a d\theta = \frac{a^4}{4} \int_0^{\pi/2} sin\theta d\theta$$

$$= \frac{a^4}{4} (-cos\theta)_0^{\pi/2} = \frac{a^4}{4} (0 - 1) = \frac{a^4}{4}$$

Triple integrals

If x_1, x_2 are constants. y_1, y_2 are functions of x and z_1, z_2 are functions of x and y, then $f(x,y,z)$ is first integrated with respect to 'z' between the limits z_1 and z_2 keeping x and y fixed. The resulting expression is integrated with respect to 'y' between the limits y_1 and y_2 keeping x constant. The resulting expression is integrated with respect to 'x' from x_1 to x_2

$$i.e. \iiint_v f(x,y,z)\,dx\,dy\,dz = \int_{x=a}^{b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\;dy\;dx$$

Solved Problems:

1. Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

Sol. $\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\,dx\,dy\,dz$

$$=\int_{x=0}^{1}dx\int_{y=0}^{\sqrt{1-x^2}}dy\int_{z=0}^{\sqrt{1-x^2-y^2}}xyz\,dz$$

$$=\int_{x=0}^{1}dx\int_{y=0}^{\sqrt{1-x^2}}xy\left(\frac{z^2}{2}\right)_{z=0}^{\sqrt{1-x^2-y^2}}dy$$

$$=\frac{1}{2}\int_{x=0}^{1}dx\int_{y=0}^{\sqrt{1-x^2}}xy\left(1-x^2-y^2\right)dy$$

$$=\frac{1}{2}\int_{x=0}^{1}dx\int_{y=0}^{\sqrt{1-x^2}}x\left[\left(1-x^2\right)y-y^3\right]dy$$

$$=\frac{1}{2}\int_{x=0}^{1}x\left[\left(1-x^2\right)\frac{y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}}dx$$

$$=\frac{1}{2}.\int_{x=0}^{1}x\left[\frac{y^2}{2}-\frac{x^2y^2}{2}-\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}}dx$$

$$=\frac{1}{8}.\int_{x=0}^{1}x\left[2\left(1-x^2\right)-2x^2\left(1-x^2\right)-\left(1-x^2\right)^2\right]dx$$

$$=\frac{1}{8}\int_{x=0}^{1}\left(x-2x^3+x^5\right)dx=\frac{1}{8}\left[\frac{x^2}{2}-\frac{2x^4}{4}+\frac{x^6}{6}\right]_0^1$$

$$=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)=\frac{1}{48}$$

2. Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)dx\,dy\,dz$

Sol: $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+y)\,dx\,dydz$

$$=\int_{-1}^{1}\int_{0}^{z}\left[\left(xy+{y^2}/{2}+zy\right)_{x-z}^{x+z}\right]dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}x(x+z)-x(x-z)+\left[\frac{x+z}{2}\right]^2-\left[\frac{x-z}{2}\right]^2+z(x+z)-z(x-z)dx\,dz$$

$$=\int_{-1}^{1}\int_{0}^{z}\left[2z(x+z)+\frac{1}{2}4xz\right]dx\,dz$$

$$=2\int_{-1}^{1}\left[z.\frac{x^2}{2}+z^2x+z.\frac{x^2}{2}\right]_0^z dz$$

$$=2.\int_{-1}^{1}\left[\frac{z^3}{2}+z^3+\frac{z^3}{2}\right]dz=4.\left(\frac{z^4}{4}\right)_{-1}^{1}=0$$

3. Evaluate $\int\int_V\int(\boldsymbol{xy+yz+zx})\boldsymbol{dxdydz}$**, where V is the region of space bounded by planes** $\boldsymbol{x=0,x=1,y=0,y=2\ and\ z=0,z=3}$.

Sol: $\int\int_V\int(xy+yz+zx)dxdydz=\int_{z=0}^{3}\int_{y=0}^{2}\int_{x=0}^{1}(xy+yz+zx)dxdydz$

$$=\int_{z=0}^{3}\int_{y=0}^{2}\left(\frac{x^2}{2}y+xyz+\frac{x^2}{2}z\right)_0^1 dydz$$

$$=\int_{z=0}^{3}\int_{0}^{2}\left(\frac{y}{2}+yz+\frac{z}{2}\right)dydz$$

$$=\int_{z=0}^{3}\left(\frac{y^2}{4}+\frac{y^2}{2}z+\frac{zy}{2}\right)_0^2 dz$$

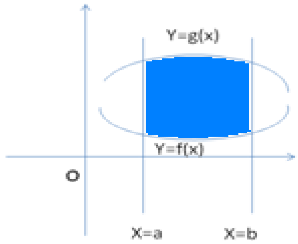
$$=\int_{z=0}^{3}(1+2z+z)dz$$

$$=\int_{z=0}^{3}(1+3z)dz$$

$$=\left(z+\frac{3z^2}{2}\right)_0^3=3+\frac{27}{2}\ =\frac{33}{2}$$

Applications of Multiple integrals:

Finding the area of a region using double integration:

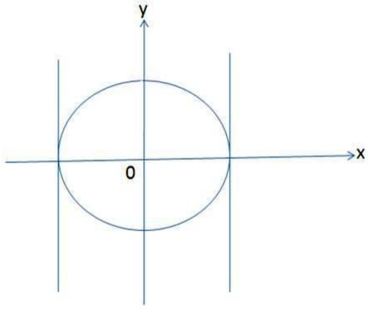


The area of the region R bounded by given curves is given by

$$\iint_R dxdy \text{ or } \iint_R dydx = \int_{x=a}^{b}\int_{y=f(x)}^{g(x)} dydx$$

In polar form the area of the region R is $\iint_R r\,dr\,d\theta$

1. Find the area of the circle $x^2+y^2=a^2$



Sol: Area of the plane region = $\iint_R dxdy$

For the region R

X-various from $-a$ to a

Y-various from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

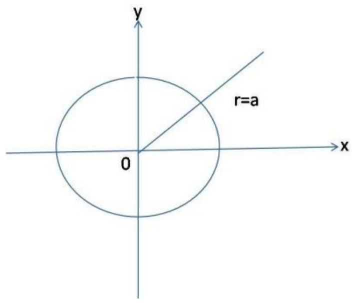
$$\therefore \text{Area of the circle} = \iint_R dxdy$$

$$= 4\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} dydx$$

$$=4\int_0^a \sqrt{a^2-x^2}\,dx$$

$$=4\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right] = 4\left[0+\frac{a^2}{2}.\frac{\pi}{2}\right] = \pi a^2$$

2. **Find the area of the circle $r = a$**

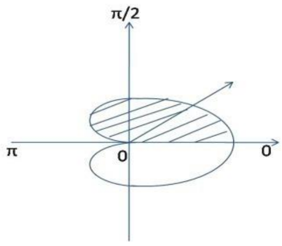


Sol: Area of the circle = $\iint_R r\,dr\,d\theta$

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} r\,dr\,d\theta = \left(\frac{r^2}{2}\right)_0^a (\theta)_0^{2\pi}$$

$$= \frac{a^2}{2}.2\pi = \pi a^2$$

3. **Find the area of the cardiod $r = a(1+cos\theta)$**



Sol: Area=2 $\iint_R r\,dr\,d\theta = 2\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta$

$$= \frac{2}{2}.\int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$= 4a^2.2\int_0^{\frac{\pi}{2}} \cos^4\phi d\phi \quad (\frac{\theta}{2} = \phi)$

$= 8a^2.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{3\pi a^2}{2}$

Finding the volume of a region using Triple integration:

Volume of the solid = $\iiint_v dv$

$= \iiint_v dxdydz$

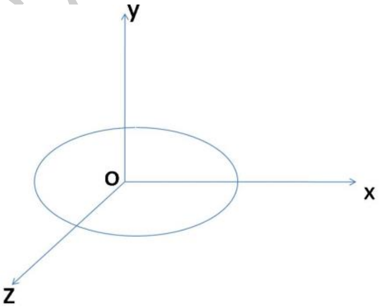
1. Find the volume of the Sphere $x^2+y^2+z^2=a^2$

Sol: The sphere $x^2+y^2+z^2=a^2$ is cut into 8 equal parts by three co-ordinates Planes .Hence the volume of the sphere is equal to 8 times the volume of the solid bounded $by\ x=0, y=0, z=0$ and $x^2+y^2+z^2=a^2$.

Z- varies from 0 to $\sqrt{a^2-x^2-y^2}$

Y- varies from 0 to $\sqrt{a^2-x^2}$

X-varies from 0 $to\ a$



$\therefore$ Required volume v=8 $\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\int_{z=0}^{\sqrt{a^2-x^2-y^2}} dzdydx$

$=8\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dydx$

$=8\int_{x=0}^{a}\left[\frac{y}{2}\sqrt{a^2-x^2-y^2}+\left(\frac{a^2-x^2}{2}\right)\sin^{-1}\left(\frac{y}{\sqrt{a^2-x^2}}\right)\right]_0^{\sqrt{a^2-x^2}} dx$

$=8\int_{x=0}^{a}\left(\frac{a^2-x^2}{2}\right).\frac{\pi}{2}dx$

$=8\frac{\pi}{4}\left(a^2x-\frac{x^3}{3}\right)_0^a$

$=8.\frac{\pi}{3}.\frac{2a^3}{3}$ $=8.\frac{\pi a^3}{6}$ $=4.\frac{\pi a^3}{3}$

Change of variable in Triple integrals:

Let $x = \Phi 1(u,v,w), y = \Phi 2(u,v,w), z = \Phi 3(u,v,w)$ be the relation between old variables (x,y,z) with the new variables (u,v,w) of the new coordinates system .

Then $\iiint_v F(x,y,z)dxdydz = \iiint_v F(\phi_1,\phi_2,\phi_3)Jdudvdw$

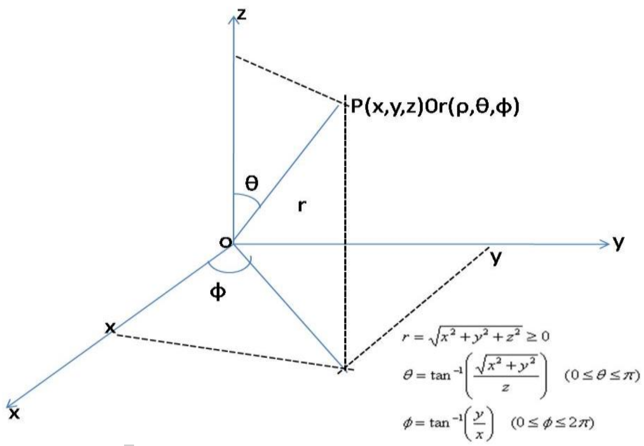
1. Change of variables from Cartesian to spherical co-ordinate system:

The relation between Cartesian co-ordinates x,y,z and spherical co-ordinates, r,θ,ϕ are given by

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$



When J= $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$
Then

$$\iiint_v F(x,y,z)dxdydz = \iiint_v F(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)r^2\sin\theta drd\theta d\phi$$

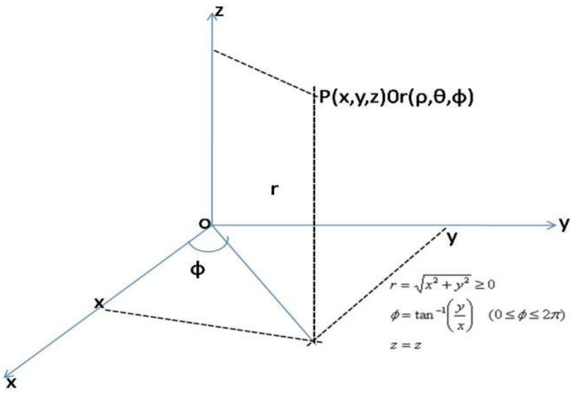
Cartesion to cylindrical co-ordinate system:

$$x,y,z \to r,\phi,z(or)\rho,\phi,z$$

$$x = p cos\phi, y = sin\phi, z = z$$

J= $\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}$ =ρ.

$$\iiint_v F(x,y,z)dxdydz = \iiint_v F(\rho\cos\phi,\rho\sin\phi,z)d\rho d$$



1. Using spherical polar coordinates then the volume of the sphere $x^2+y^2+z^2 = a^2$

Sol: By changing into polar coordinates

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta \qquad dxdydz = r^2\sin\theta dr d\theta d\phi$

$r \to 0 \; to \; a$

$\theta \to 0 \; to \; \pi$

$\phi \to 0 \; to \; 2\pi$

Required volume = $\iiint dxdydz$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 \sin\theta dr d\theta d\phi$$

$$= \frac{a^3}{3} (-\cos\theta)_0^{\pi} (\phi)_0^{2\pi}$$

$$= \frac{4\pi a^3}{3}$$

2. Evaluate $\iiint (x^2+y^2+z^2)dxdydz$ **taken over the volume enclosed by the sphere** $x^2+y^2+z^2=1$**, by transforming into spherical polar coordinates**

Sol:

$$\iiint (x^2+y^2+z^2)dxdydz = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} r^2 r^2 \sin\theta dr d\theta d\phi$$

$$= \frac{a^5}{5}.2.2\pi = \frac{4\pi}{5}$$

3. Using cylindrical co-ordinates $\iiint (x^2+y^2)dxdydz$ **taken over the volume bounded by the xy -plane and the parabolid** $z = 9 - x^2 - y^2$.

Sol:

$$\int_{r=0}^{3} \int_{\theta=0}^{2\pi} \int_{z=0}^{9-r^2} r^2 r dr d\theta dz = 2\pi \int_{r=0}^{3} (9-r^2) r^3 dr$$

$$= 2\pi \left[\frac{9r^4}{4} - \frac{r^6}{6} \right]_0^3$$

$$= 2\pi \left[\frac{729}{4} - \frac{243}{2} \right] = \frac{243\pi}{2}$$

4 Evaluate $\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2}}$ **by changing to spherical polar coordinates.**

Sol: Given region of integration is the volume of the sphere $x^2+y^2+z^2=1$ in the first octant

$r \to 0\ to\ 1$

$\theta \to 0\ to \frac{\pi}{2}$

For which $\phi \to 0\ to\ \frac{\pi}{2}$

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{\sqrt{1-x^2-y^2}} = \int_{r=0}^{1}\int_{\theta=0}^{\frac{\pi}{2}}\int_{\phi=0}^{\frac{\pi}{2}} \frac{r^2 \sin\theta}{\sqrt{1-r^2}} dr d\theta d\phi$$

$$= (-\cos)_0^{\frac{\pi}{2}} (\phi)_0^{\frac{\pi}{2}} \int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}} dr =$$

$$1.\frac{\pi}{2}\left[\sin^{-1} r - \left\{\frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1}\right\}\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{\pi}{2} - \frac{1}{2}\frac{\pi}{2}\right] = \frac{\pi^2}{8}$$

5. Using cylindrical co-ordinates, find the volume of the cylindrical with base radius a and height h.

Sol: $x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$

$r \to 0\ to\ a$

$\theta \to 0\ to 2\pi$

J= r $\quad z \to 0\ to h$

Required volume= $\iiint dxdydz = \int_{r=0}^{a}\int_{\theta=0}^{2\pi}\int_{z=0}^{h} r dr d\theta dz = \frac{a^2}{2}.2\pi.h = \pi a^2 h$

6.Using cylindrical coordinates evaluate $\iiint (x^2+y^2+z^2)dxdydz$ **taken over the region** $0 \le z \le x^2+y^2 \le 1$

Sol:

$r \to 0\ to\ 1$

$\theta \to 0\ to 2\pi$

$z \to 0\ to 1$

$x^2+y^2 = r^2$, J=r

Given integration= $\int_{r=0}^{1}\int_{\theta=0}^{2\pi}\int_{z=0}^{1} (r^2+z^2) r dr d\theta dz$

$$= \frac{r^4}{4}.2\pi.1 + \frac{1}{3}.\frac{1}{2}.2\pi = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$$

UNIT-V

VECTOR CALCULUS

Introduction :

Vector calculus is concerned with differentiation and integration of vector fields primarily in 3-dimensional Euclidean space R^3. The term "vector calculus" is sometimes used as a synonym for the broader subject of multivariable calculus. Vector calculus plays an important role in differential geometry and in the study of partial differential equations. It is used extensively in physics and engineering, especially in the description of electromagnetic fields, gravitational fields and fluid flow. Vector analysis is very important in many fields of engineering such as mechanical, civil, computer, structural and electrical engineering. Scalar values, such as mass and temperature convey only a magnitude, but vectors such as velocity employ both a magnitude and a direction. In physics, the term work is used to describe the energy that is added to or removed from an object or system when a force is applied to it. The work done by a force can be described by the dot product of the force vector and the displacement vector.

Vector finds many applications in Electrical Engineering: The generator that generates Electrical Energy or the Motor that Generates mechanical power work on the principles of physics which are based on vector manipulation. Since vectors and matrices are used in linear algebra, anything that requires the use of arrays that are linear dependent requires vectors. A few well-known examples in Computer engineering are Internet search, Graph analysis, Machine learning, Graphics, Bioinformatics, Data mining, Computer vision, Speech recognition, Compilers, Parallel computing and Scientific computing. Robotics also have Vector Calculus applications. Vectors can be used by air-traffic controllers when tracking planes, by meteorologists when describing wind conditions, and by computer programmers when they are designing virtual worlds.

Definitions :

Scalar : A quantity which is completely specify by its magnitude only.

Ex: Time, Temperature.

Vector : A quantity which is completely specify by its magnitude and direction.

Ex: Force ,Velocity.

Position Vector: Let A and B are two vectors then the position vector of AB is $\overline{AB} = \overline{OB} - \overline{OA}$.

If $\bar{a} = a_1 i + a_2 j + a_3\, k$ then $|\bar{a}|$=$\sqrt{{a_1}^2 + {a_2}^2 + a_3^2}$

If $\bar{a}$ is any vector then its unit vector is given by $\frac{\bar{a}}{|\bar{a}|}$

Dot Product

$\bar{a}.\bar{b}$=$|\bar{a}||\bar{b}|$ cosθ where θ is angle between two vectors

We know $i.i = j.j = k.k = 1$ and $i.j = j.k = k.i = 0$

if $\bar{a} = a_1 i + a_2 j + a_3\, k$, $\bar{b} = b_1 i + b_2 j + b\, k$ then $\bar{a}.\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Cross Product

$\bar{a}\ x\ \bar{b}$ =$|\bar{a}||\bar{b}|$ $\hat{n}$ sinθ

$$=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{since } i\ x\ i = j\ x\ j = k\ x\ k = 0$$

i x j =k; j x k = i; k x i = j; j x i = - k; i x k =- j; k x j = i

Scalar and Vector Point Functions

Consider a region in three dimensional space. To each point $P(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $P(x,y,z)$ we associate a unique vector $\bar{f}\,(x,y,z)$, $\bar{f}$ is called vector point functions**.**

Examples:

For example take a heated solid. At each point P(x,y,z)of the solid, there will be temperature $T(x,y,z)$. This T is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position $P(x,y,z)$ in space, it will be having some speed, say, v. This **speed** v is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

Tangent vector to a curve in space

Consider an interval [a,b].

Let $x = x(t), y = y(t), z = z(t)$be continuous and derivable for $a \le t \le b$.

Then the set of all points $(x(t), y(t), z(t))$ is called a curve in a space.

Let A = $(x(a), y(a), z(a))$ and B = $(x(b), y(b), z(b))$. These A,B are called the end points of the curve. If A =B, the curve in said to be a closed curve.

Let P and Q be two neighbouring points on the curve.

Let $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t + \delta t) = \bar{r} + \delta\bar{r}. Then\ \delta\bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then $\frac{\delta\bar{r}}{\delta t}$ is along the vector $\overline{PQ}$. As Q→P, $\overline{PQ}$ and hence $\frac{\overline{PQ}}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\underset{\delta t \to 0}{lt} \frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

Suppose arc length AP = s. If we take the parameter as the arc length parameter, we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.

Vector Differential Operator

Def. The vector differential operator ∇(read as del) is defined as

$$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}.$$

This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator.

We will define now some quantities known as "**gradient", "divergence"** and **"curl"** involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably

Gradient of a Scalar Point Function

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$$

Directional Derivative

Let $\phi(x,y,z)$ be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is $\overline{OP} = \bar{r}$. Let $\phi+\Delta\phi$ be the value of the function at neighbouring point Q. If $\overline{OQ} = \bar{r} + \Delta\bar{r}$. Let Δr be the length of $\Delta\bar{r}$. $\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q.

The limiting value of $\frac{\Delta\phi}{\Delta r}$ as $\Delta r \to 0$ is called the derivative of ϕ in the direction of $\overline{PQ}$ or simply directional derivative of ϕ at P and is denoted by $d\phi/dr$.

The physical interpretation of ∇ϕ

The gradient of a scalar function $\phi(x,y,z)$ at a point $P(x,y,z)$ is a vector along the normal to the level surface $\phi(x,y,z) = c$ at P and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ .

Greatest value of directional derivative of $\bar{\Phi}$ **at a point P = |grad ϕ|** at that point.

NOTE:

1.Let $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$. Then $d\bar{r}=dx\,\bar{i}+dy\,\bar{j}+dz\,\bar{k}$ if ϕ is any scalar point function, then

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=\left(\bar{i}\frac{\partial\Phi}{\partial x}+\bar{j}\frac{\partial\Phi}{\partial y}+\bar{k}\frac{\partial\Phi}{\partial z}\right).\left(\bar{i}dx+\bar{j}dy+\bar{k}dz\right)=\nabla\Phi.d\bar{r}$$

2. $grad\Phi$ at any point is a vector normal to the surface $\Phi(x,y,z)=c$ through that point w $P(x,y,z)$ where c is a constant.

3. The directional derivative of a scalar point function ϕ at a point $P(x,y,z)$ in the direction of a unit vector $\bar{e}$ is equal to $\bar{e}$. grad ϕ= $\bar{e}$. $\nabla\phi$.

4.If θ is angle between two surfaces $\emptyset_1,\emptyset_2$ then

Cos $\theta = \frac{\nabla\emptyset_1 . \nabla\emptyset_2}{|\nabla\emptyset_1||\nabla\emptyset_1|}$

5.Unit Normal vector of a surface $\emptyset$ is $\frac{\nabla\phi.}{|\nabla\emptyset|}$

Solved Problems

1.Show that ∇[f(r)] = $\frac{f'(r)}{r}r$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: Since $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, we have $r^2 = x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.Similarly \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)f(r) = \sum \bar{i}f^1(r)\frac{\partial r}{\partial x} = \sum \bar{i}f^1(r)\frac{x}{r}$$

$$= \frac{f^1(r)}{r}\sum \bar{i}x = \frac{f^1(r)}{r}.\bar{r}$$

Note : From the above result, $\nabla(\log r) = \frac{1}{r^2}\bar{r}$, $\nabla(r^n) = nr^{n-2}\,\bar{r}$.

2.Find the directional derivative of $f = xy + yz + zx$ in the direction of vector $\bar{i} + 2\bar{j} + 2\bar{k}$ at the point (1,2,0).

Sol: Given $f = xy + yz + zx$.

Grad f = $\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{z}\frac{\partial f}{\partial z} = (y+z)\bar{i} + (z+x)\bar{j} + (x+y)\bar{k}$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i} + 2\bar{j} + 2\bar{k}$, then

$$\bar{e} = \frac{\bar{i} + 2\bar{j} + 2\bar{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}(\bar{i} + 2\bar{j} + 2\bar{k})$$

Directional derivative of f along the given direction $= \bar{e}.\nabla f$

$$= \frac{1}{3}\left(\bar{i}+2\bar{j}+2\bar{k}\right).\left[(y+z)\bar{i}+(z+x)\bar{j}+\left(x+y\bar{k}\right)\right] at\,(1,2,0)$$

$$= \frac{1}{3}\left[(y+z)+2(z+x)+2(x+y)\right](1,2,0) = \frac{10}{3}$$

3. Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x = t, y = t^2, z = t^3$ at the point (1,1,1).

Sol: Here $f = xy^2+yz^2+zx^2$

$$\nabla f = \bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z} = \left(y^2+2xz\right)\bar{i}+\left(z^2+2xy\right)\bar{j}+\left(x^2+2yz\right)\bar{k}$$

At (1,1,1) , $\nabla f = 3\bar{i}+3\bar{j}+3\bar{k}$

Let $\bar{r}$ be the position vector of any point on the curve $x = t, y = t^2, z = t^3$. then

$$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k} = t\bar{i}+t^2\bar{j}+t^3\bar{k}$$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i}+2t\bar{j}+3t^2\bar{k} = (\bar{i}+2\bar{j}+3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along the tangent $= \bar{e} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+2^2+3^2}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent $= \nabla f.\bar{e} = \frac{1}{\sqrt{14}}(\bar{i}+2\bar{j}+3\bar{k}).3(\bar{i}+\bar{j}+\bar{k})$

$$\frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}$$

4. Find the directional derivative of the function f = $x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol: The position vectors of P and Q with respect to the origin are $\overline{OP} = \bar{i}+2\bar{j}+3\bar{k}$ and

$\overline{OQ} = 5\bar{i} + 4\bar{k}$; $\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$

Let $\bar{e}$ be the unit vector in the direction of $\overline{PQ}$. Then $\bar{e} = \dfrac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

grad $f = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of $\overline{PQ} = \bar{e}.\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}).(2x\bar{i} - 2y\bar{j} + 4z\bar{k}) \frac{1}{\sqrt{21}}(8x + 4y + 4z)_{at(1,2,3)} = \frac{1}{\sqrt{21}}(28)$$

5. Find the greatest value of the directional derivative of the function f = x^2yz^3 at (2,1,-1).

Sol: we have

grad $f = \bar{i}\dfrac{\partial f}{\partial x} + \bar{j}\dfrac{\partial f}{\partial y} + \bar{k}\dfrac{\partial f}{\partial z} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k} = -4\bar{i} - 4\bar{j} + 12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of $f = |\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

6.Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a direction of the normal to the surface $3xy2 + y = z$ at (0,1,1).

Sol: Let f(x, y, z) ≡ $3xy^2+y - z = 0$

Let us find the unit normal e to this surface at (0,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2, \frac{\partial f}{\partial y} = 6xy+1, \frac{\partial f}{\partial z} = -1.$$

$\nabla f = 3y^2$i+($6xy$+1)j-k

$(\nabla f)_{(0,1,1)}$ = 3i+j-k = $\bar{n}$

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i + j - k}{\sqrt{9+1+1}} = \frac{3i + j - k}{\sqrt{11}}$$

Let $g(x, y, z) = xyz^2 + xz$, then

$$\frac{\partial g}{\partial x} = yz^2 + z, \frac{\partial g}{\partial y} = xz^2, \frac{\partial g}{\partial z} = 2xy + x$$

$\nabla g = (yz^2 + z)i + xz2j + (2xyz + x)k$

And $[\nabla g]_{(1,1,1)} = 2i+j+3k$

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$$=(2i+j+3k).\left(\frac{3i+j-k}{\sqrt{11}}\right) = \frac{6+1-3}{\sqrt{11}} = \frac{4}{\sqrt{11}}$$

7.Evaluate the angle between the normal to the surface $xy = z^2$ at the points (4,1,2) and (3,3,-3).

Sol: Given surface is $f(x, y, z) = xy - z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normal to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x} = y, \frac{\partial f}{\partial y} = x, \frac{\partial f}{\partial z} = -2z.$$

grad f = $y\bar{i} + x\bar{j} - 2z\bar{k}$

$\bar{n}_1$ = (grad f) at (4,1,2) $= \bar{i} + 4\bar{j} - 4\bar{k}$

$$\cos\theta = \frac{\overline{n_1}.\overline{n_2}}{|\overline{n_1}||\overline{n_2}|} = \frac{(i+4j-4k)}{\sqrt{1+16+16}}.\frac{(3i+3j+6k)}{\sqrt{9+9+36}}$$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

8. Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol: Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26 = 0$. Then

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i}\frac{\partial f}{\partial x} = 2xi + 2yj + 4zk$

Normal vector at (2,2,3) = $[\nabla f]_{(2,2,3)} = 4\,\bar{i} + 4\,\bar{j} + 12\,\bar{k}$

Unit normal vector = $\frac{\nabla f}{|\nabla f|} = \frac{4(\bar{i}+\bar{j}+3\bar{k})}{4\sqrt{11}} = \frac{\bar{i}+\bar{j}+3\bar{k}}{\sqrt{11}}$

9. Find the values of a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface $ax^2 - byz = (a+2)x$ will orthogonal to $4x^2y + z^3 = 4$ at the point (1,-1,2).

Sol: Let the given surfaces be $f(x,y,z) = \boldsymbol{ax^2 - byz = (a+2)x}$ -------------(1)

And $g(x,y,z) = \boldsymbol{4x^2y + z^3 = 4}$ ------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

$$a + 2b - (a+2) = 0 \Rightarrow b = 1$$

Now $\frac{\partial f}{\partial x} = 2ax - (a+2), \frac{\partial f}{\partial y} = -bz \text{ and } \frac{\partial f}{\partial z} = -by.$

$$\nabla f = \sum \bar{i}\frac{\partial f}{\partial x} = [(2ax - (a+2)]i - bz + bk = (a-2)i - 2bj + bk$$

$= (a-2)i - 2j + k = \bar{n}_1$, normal vector to surface 1.

Also $\frac{\partial g}{\partial x} = 8xy, \frac{\partial g}{\partial y} = 4x^2, \frac{\partial g}{\partial z} = 3z^2.$

$\nabla g = \sum \bar{i}\frac{\partial g}{\partial x} = 8xyi + 4x^2j + 3z^2k$

$(\nabla g)_{(1,-1,2)} = -8i + 4j + 12k = \bar{n}_2$, normal vector to surface 2.

Given the surfaces $f(x,y,z), g(x,y,z)$ are orthogonal at the point (1,-1,2).

$$[\nabla f].[\nabla g] = 0 \Rightarrow ((a-2)i - 2j + k).(-8i + 4j + 12k) = 0$$

$$\Rightarrow -8a + 16 - 8 + 12 \Rightarrow a = 5/2$$

$$Hence\ a = 5/2\ and\ b = 1.$$

Divergence of a vector

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e., div $\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

Hence we can write div $\bar{f}$ as

div $\bar{f} = \nabla.\,\bar{f}$

This is a scalar point function.

NOTE: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then *div* $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Solenoidal Vector

A vector point function $\bar{f}$ is said to be solenoidal if $div\ \bar{f} = 0$.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ ($\nabla.\,\bar{f}$).

Suppose $\bar{F}(x, y, z, t)$ is the velocity of a fluid at a point(x, y, z) and time 't'. Though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagine a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\bar{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Solved Problems

1. Find $div\ \bar{f}$ when $grad(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$

Then $\frac{\partial \phi}{\partial x} = 3x^2 - 3yz, \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$

grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)] + \frac{\partial}{\partial y}[3(y^2 - zx)] + \frac{\partial}{\partial z}[3(z^2 - xy)]$

$= 3(2x)+3(2y)+3(2z) = 6(x+y+z)$

2. If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is Solenoidal, find *P*.

Sol: Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1, \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 1 + 1 + p = 2 + p$

since $\bar{f}$ is solenoidal, we have div $\bar{f} = 0 \Rightarrow 2 + p = 0 \Rightarrow p = -2$

3. Find div $\bar{f} = r^n\bar{r}$. Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n\bar{r}$. where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ *and* $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially with respect to x , we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n\ (x\bar{i} + y\bar{j} + z\bar{k})$

div $\bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$

$= nr^{n-1}\frac{\partial r}{\partial x}x + r^n + nr^{n-1}\frac{\partial r}{\partial y}y + r^n + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$

$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$

Let $\bar{f} = r^n\bar{r}$ be solenoidal. Then $div\ \bar{f} = 0$

$(n+3)\, r^n = 0 \Rightarrow n = -3$

4. Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$ **.**

Sol: We have $\bar{r} = xi + yj + zk$ and $r = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$$

$\therefore \frac{\bar{r}}{r^3} = \bar{r}.$

$r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1 i + f_2 j + f_3 k$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

We have $f_1 = r^{-3}x \Rightarrow \frac{\partial f_1}{\partial x} = r^{-3}.1 + x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$$\therefore \frac{\partial f_1}{\partial x} = r^{-3} - 3xr^{-4}\frac{x}{y} = r^{-3} - 3x^2 r^{-5}$$

$$\nabla.\left(\frac{\bar{r}}{r^3}\right) = \sum \frac{\partial f_1}{\partial x} = 3r^{-3} - 3r^{-5}\sum x^2 = 3r^{-3} - 3r^{-5}r^2 = 0$$

Curl of a Vector

Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x $\bar{f}$).

$$\text{Curl } \bar{f} = \bar{i}\times\frac{\partial \bar{f}}{\partial x} + \bar{j}\times\frac{\partial \bar{f}}{\partial y} + \bar{k}\times\frac{\partial \bar{f}}{\partial z} = \sum\left(\bar{i}\times\frac{\partial \bar{f}}{\partial x}\right)$$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $= \bar{f}$ $f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then

$$\text{curl } \bar{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

Note : **curl** $\bar{f}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$ **=∇x** $\bar{f}$

Note (2) : If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Physical Interpretation of curl

If $\bar{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point $P(x, y, z)$ on the body, then $\bar{w}$ = ½ $curl\ \bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word “curl of a vector”.

Any motion in which curl of the velocity vector is a null vector i.e $curl\ \bar{v} = \bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if $curl\ \bar{f} = \bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function $\varphi(x,y,z)$ such that $\bar{f} = grad\ \phi$. This ϕ is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f} = grad\ \phi$, then curl $\bar{f} = 0$.

Hence $\nabla x\ \bar{f} = 0 \Leftrightarrow$ there exists a scalar function ϕ such that $\bar{f} = \nabla\phi$.

This idea is useful when we study the "work done by a force later.

Solved Problems

1. Find $curl\ \bar{f}$ where $\bar{f} = grad(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$ Then

$$grad\ \phi = \sum \bar{i}\frac{\partial\phi}{\partial x} = 3(x^2-yz)\bar{i} + 3(y^2-zx)\bar{j} + 3(z^2-xy)\bar{k}$$

$$curl\ grad\ \phi = \nabla x\ grad\ \phi = 3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix} =$$

$$3[\bar{i}(-x+x) - \bar{j}(-y+y) + \bar{k}(-z+z)] = \bar{0}$$

$$\therefore curl\ \bar{f} = \bar{0}.$$

Note: We can prove in general that $curl\ (grad\ \phi) = \bar{0}$. $(i.e)$ $grad\ \phi$ is always irrotational.

2.Show that the vector $(x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2-yz)\bar{i} + (y^2-zx)\bar{j} + (z^2-xy)\bar{k}$

Then curl $\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum \bar{i}(-x + x) = \bar{0}$

$\therefore$ $\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x} = x^2 - yz \Rightarrow \phi = \int (x^2 - yz)dx = \frac{x^3}{3} - xyz + f_1(y,z) \ldots\ldots(1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \Rightarrow \phi = \frac{y^3}{3} - xyz + f_2(z,x) \ldots\ldots(2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \Rightarrow \phi = \frac{z^3}{3} - xyz + f_3(x,y) \ldots\ldots(3)$$

From (1), (2),(3), $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz$

$$\therefore \quad \phi = \frac{1}{3}(x^3 + y^3 + z^3) - xyz + constant$$

Which is the required scalar potential.

3. Find constants *a*, *b* and *c* if the vector

$\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$ **is Irrotational.**

Sol: Given $\bar{f} = (2x + 3y + az)\bar{i} + (bx + 2y + 3z)\bar{j} + (2x + cy + 3z)\bar{k}$

Curl $\bar{f}$ = $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+3y+az & bx+2y+3z & 2x+cy+3z \end{vmatrix}$ =

$(c-3)\bar{i}-(2-a)\,\bar{j}+(b-3)\bar{k}$

If the vector is Irrotational then curl $\bar{f} = \bar{0}$

$\therefore 2-a=0 \Rightarrow a=2, b-3=0 \Rightarrow b=3, c-3=0 \Rightarrow c=3$

4.If $f(r)$ is differentiable, show that $curl\{\,\bar{r}\,f(r)\} = \bar{0}$ where

$\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ **.**

Sol: r = $\bar{r} = \sqrt{x^2+y^2+z^2}$

$$\therefore r^2 = x^2+y^2+z^2$$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$$curl\{\,\bar{r}\,f(r)\} = curl\{f(r)(\ x\bar{i}+y\bar{j}+z\bar{k}\ )\} = curl\ \ (x.f(r)\bar{i}+y.f(r)\,\bar{j}+z.f(r)\bar{k})$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y} - yf^1(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^1(r)\frac{y}{r} - yf^1(r)\frac{z}{r}\right] = \bar{0}\,.$$

5.Find constants a,b,c so that the vector $\bar{A}$ = $(x+2y+az)\bar{i}+(bx-3y-z)\,\bar{j}+(4x+cy+2z)\bar{k}$ **is Irrotational. Also find ϕ such that** $\bar{A}$ = **∇ϕ.**

Sol: Given vector is $\bar{A} = (x+2y+az)\bar{i} + (bx-3y-z)\,\bar{j} + (4x+cy+2z)\,\bar{k}$

Vector $\bar{A}$ is Irrotational $\Rightarrow curl\ \bar{A} = \bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\,\bar{k} = \bar{0}$

$\Rightarrow (c+1)\bar{i} + (a-4)\,\bar{j} + (b-2)\,\bar{k} = 0\bar{i} + 0\,\bar{j} + 0\,\bar{k}$

Comparing both sides,

$c+1 = 0, a-4 = 0, b-2 = 0$

$c = -1, a = 4, b = 2$

Now $\bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\,\bar{k}$, on substituting the values of a, b, c

we have $\bar{A} = \nabla\phi$.

$\Rightarrow \bar{A} = (x+2y+4z)\bar{i} + (2x-3y-z)\,\bar{j} + (4x-y+2z)\,\bar{k} = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x} = x + 2y + 4z \Rightarrow \phi = \frac{x^2}{2} + 2xy + 4zx + f_1(y,z)$

$\frac{\partial\phi}{\partial y} = 2x - 3y - z \Rightarrow \phi = 2xy - 3y^2/2 - yz + f_2(x,z)$

$\frac{\partial\phi}{\partial z} = 4x - y + 2z \Rightarrow \phi = 4xz - yz + z^2 + f_3(y,x)$

Hence $\phi = x^2/2 - 3y^2/2 + z^2 + 2xy + 4zx - yz + c$

Laplacian Operator

$$\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right) = \sum\frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi) = div(grad\ \phi)$

(ii). If $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

Solved Problems

1.Prove that $div.(grad\ r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m1)r^{m-2}$ (or)

$\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ then $r^2 = x^2 + y^2 + z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now $grad(r^m) = \sum \bar{i}\frac{\partial}{\partial x}(r^m) = \sum \bar{i} m r^{m-1}\frac{\partial r}{\partial x} = \sum \bar{i} m r^{m-1}\frac{x}{r} = \sum \bar{i} m r^{m-2} x$

$\therefore div\ (grad\ r^m) = \sum \frac{\partial}{\partial x}[mr^{m-2}x] = m\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$

$= m\sum\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] = m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$

$= m[(m-2)r^{m-4}(r^2) + 3r^{m-2}]$

$= m[(m-2)\ r^{m-2} + 3r^{m-2}]$

$= m[(m-2+3)r^{m-2}] = m(m+1)r^{m-2}.$

$Hence\ \nabla^2(r^m) = m(m+1)r^{m-2}$

2. Show that $\nabla^2[f(r)] = \frac{d^2f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r}f^1(r)$ where r = $|\bar{r}|$.

Sol: $grad\ [f(r)] = \nabla f(r) = \sum i\frac{\partial}{\partial x}[f(r)] = \sum i\, f^1(r)\frac{\partial r}{\partial x} = \sum i\, f^1(r)\frac{x}{r}$

$\therefore div\ [grad\ f(r)] = \nabla^2[f(r)] = \nabla.\nabla f(r) = \sum \frac{\partial}{\partial x}\left[f^1(r)\frac{x}{r}\right]$

$$= \sum \frac{r\frac{\partial}{\partial x}[f^1(r)x] - f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^1(r)\right) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \sum \frac{rf^{11}(r)\frac{x}{r}x + rf^1(r) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{\sum rf^{11}(r)\frac{x}{r}x + rf^1(r) - x^2}{r^2}\cdot\frac{f^1(r)}{r}$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}\sum f^1(r) - \frac{1}{r^3}f^1(r)\sum x^2$$

$$= \frac{f^{11}(r)}{r^2}(r^2) + \frac{3}{r}f^1(r) - \frac{1}{r^3}f^1(r)r^2$$

$$= f^{11}(r) + \frac{2}{r}f^1(r)$$

3. If ϕ satisfies Laplacian equation, show that $\nabla\phi$ is both solenoidal and irrotational.

Sol: Given $\nabla^2\phi = 0 \Rightarrow div(grad\ \phi) = 0 \Rightarrow grad\ \phi$ is solenoidal

We know that $curl\ (grad\ \phi) = \bar{0} \Rightarrow grad\ \phi$ is always irrotational

4. Prove that curl grad ϕ = 0.

Sol: Let ϕ be any scalar point function. Then

$$.grad\ \phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$curl(grad\phi) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) - \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \bar{0}$$

Note : Since $Curl(grad\phi) = \bar{0}$, we have $grad\ \phi$ is always irrotational.

5. Prove that $\boldsymbol{div}$ $curl\,\bar{f} = 0$

$Proof$: $Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore curl\ \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore\quad div\ curl\ \bar{f} = \nabla.(\nabla\times\bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_1}{\partial y\partial z} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y} = 0$$

Note : Since $div(curl\,\bar{f}) = 0$, we have $curl\,\bar{f}$ is always solenoidal.

VECTOR INTEGRATION

Line Integral

Any integral which is to be evaluated over a Curve C is called Line integral of $\bar{F}$.

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

Solved Problems

1. If $\bar{F} = (x^2-27)\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Sol : Given $\bar{F} = (x^2-27)\bar{i} - 6yz\,\bar{j} + 8xz^2\,\bar{k}$

Now $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$\therefore \quad \bar{F}.d\bar{r} = (x^2-27)dx - (6yz)dy + 8xz^2dz$$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here $y = 0 = z$ and $dy = dz = 0$. Also x changes from 0 to 1.

$$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_o^1 (x^2-27)dx = \left[\frac{x^3}{3} - 27x\right]_0^1 = \frac{1}{3} - 27 = \frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here $x = 1, z = 0 \Rightarrow dx = 0, dz = 0.$ y changes from 0 to 1.

$$\therefore \int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{1} (-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

$x = 1 = y$ ⟶ $dx = dy = 0$ *and* z changes from 0 to 1.

$$\therefore \int_{BC} \bar{F}\,.d\bar{r} = \int_{z=0}^{1} 8xz^2 dz = \int_{z=0}^{1} 8xz^2 dz = \left[\frac{8z^3}{3}\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F}\,.d\bar{r} = \frac{88}{3}$$

2. If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$, evaluate $\int_C \bar{F}.d\bar{r}$ along the curve C in xy-plane $y = x^3$ from (1,1) to (2,8).

Sol: Given $\bar{F} = (5xy - 6x^2)\,\bar{i} + (2y - 4x)\,\bar{j}$,--------(1)

Along the curve $y = x^3, dy = 3x^2\,dx$

$$\therefore \quad \bar{F} = (5x^4 - 6x^2)\,\bar{i} + (2x^3 - 4x)\,\bar{j},\ [Putting\ y = x^3\ in\ (1)]$$

$$d\bar{r} = dx\bar{i} + dy\bar{j} = dx\bar{i} + 3x^2 dx\,\bar{j}$$

$$\therefore \quad \bar{F}.d\bar{r} = [(5x^4 - 6x^2)\bar{i} + (2x^3\text{-}4x)\,\bar{j}].\left[dx\,\bar{i} + 3x^2 dx\,\bar{j}\right]$$

$$= (5x^4 - 6x^2)\,dx + (2x^3 - 4x)3x^3 dx$$

$$= (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$$

$$\text{Hence } \int_{y=x^3} \bar{F}\,.d\bar{r} = \int_1^2 (6x^5 + 5x^4 - 12x^3 - 6x^2)dx$$

$$= \left(6.\frac{x^6}{6} + 5.\frac{x^5}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^5 - 3x^4 - 2x^3\right)_1^2$$

= 16(4+2-3-1) – (1+1-3-2) = 32+3 = 35

3. Find the work done by the force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\,\bar{k}$ from $t = 0$ to $t = 2\pi$

Sol : Given force $\bar{F} = z\bar{i} + x\bar{j} + y\bar{k}$ and the arc is $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} - t\bar{k}$

$$i.e., x = \cos t, y = \sin t, z = -t$$

$$\therefore d\bar{r} = (-\sin t\,\bar{i} + \cos t\,\bar{j} - \bar{k})dt$$

$\therefore \bar{F}.d\bar{r} = (-t\,\bar{i} + \cos t\,\bar{j} + \sin t\,\bar{k}).(-\sin t\,\bar{i} + \cos t\,\bar{j} - \bar{k})dt = (t\sin t + \cos^2 t - \sin t)dt$

Hence work done $= \int_0^{2\pi} \bar{F}.d\bar{r} = \int_0^{2\pi} (t\sin t + \cos^2 t - \sin t)\,dt$

$$= [t(-\cos t)]_0^{2\pi} - \int_0^{2\pi} (-\sin t)dt + \int_0^{2\pi} \frac{1+\cos 2t}{2}dt - \int_0^{2\pi} \sin t\,dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

Surface Integral

Any integral which is to be evaluated over a surface S is called surface integral and it is denoted by $\int_S \bar{F}.\bar{n}\,ds$

Let $\bar{F} = F_1\,i + F_2\,j + F_3\,k$, where *F1 ,F2 ,F3* are continuous and differentiable functions of *x,y,z.*

Then $\int_S \bar{F}.ndS = \iint_s F_1 dydz + F_2 dxdz + F_3 dxdy$

Note: 1.Let R be the projection of S on xy plane.then $\int_S \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{k}|}dxdy$

2. Let R be the projection of S on yz plane.then $\int_S \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{i}|}dydz$

3. Let R be the projection of S on zx plane.then $\int_S \bar{F}.\bar{n}dS = \iint_R \frac{\bar{F}.\bar{n}}{|\bar{n}.\bar{j}|}dxdz$

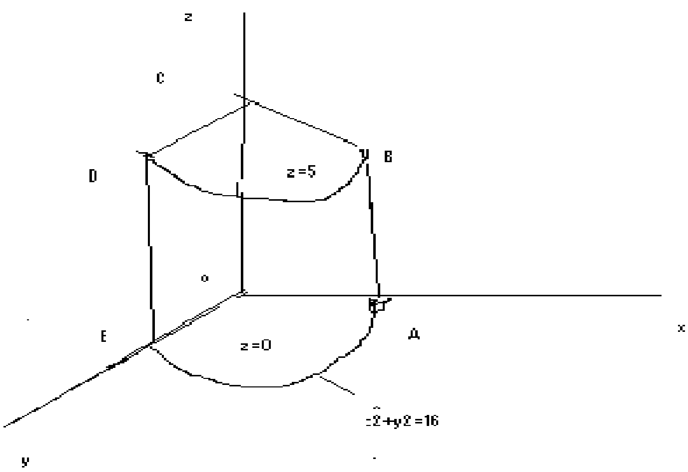
Solved Problems

1.Evaluate $\int \bar{F}.\bar{n}dS$ where $\bar{F} = zi + xj - 3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Sol: The surface S is $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Let $\varphi = x^2 + y^2 = 16$

Then $$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2x\bar{i} + 2y\bar{j}$$



unit normal

$$\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{x\bar{i} + y\bar{j}}{4} \quad (\because x^2 + y^2 = 16)$$

Let R be the projection of S on yz-planeThen

$$\int_S \overline{F}.\overline{n}dS = \iint_R \overline{F}.\overline{n}\frac{dydz}{|\overline{n}\,.\,\overline{i}|} \;\ldots\ldots\ldots\ldots\ldots\; *$$

Given $\overline{F} = zi + xj - 3y^2zk$

$$\overline{F}\,.\,\overline{n} = \frac{1}{4}(xz + xy)$$

and $\overline{n}\,.\,\overline{i} = \dfrac{x}{4}$

In *yz*-plane, $x = 0, y = 4$

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

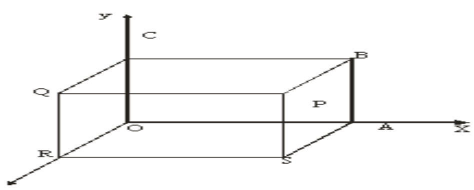
$$\int_S \overline{F}.\overline{n}dS$$

$$= \int_{y=0}^{4}\int_{z=0}^{5}(y+z)dz\,dy = 90.$$

2 : If $\overline{F} = zi + xj - 3y^2zk$, evaluate $\int_S \overline{F}.\overline{n}dS$ where S is the surface of the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0, z = a$

Sol: Given that S is the surface of the $x = 0, x = a, y = 0, y = a, z = 0, z = a$, and

$$\overline{F} = zi + xj - 3y2zk$$



we need to evaluate $\int_S \overline{F}.\overline{n}dS$.

(I)For *OABC*

Equation is z = 0 and $dS = dxdy$

$$\bar{n} = -\bar{k}$$

$$\int_{S_1} \bar{F}.\bar{n}dS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\, dxdy = 0$$

(II)For *PQRS*

Equation is $z = a$ and $dS = dxdy$

$$\bar{n} = \bar{k}$$

$$\int_{S_2} \bar{F}.ndS = \int_{x=0}^{a} \left(\int_{y=0}^{a} y(a)dy \right) dx = \frac{a^4}{2}$$

(III)For *OCQR*

Equation is $x = 0$, and $\bar{n} = -\bar{i}$, $dS = dydz$

$$\int_{S_3} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} 4xzdydz = 0$$

(IV)For *ABPS*

Equation is $x = a$, and $\bar{n} = -\bar{i}$, $dS = dydz$

$$\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a} \left(\int_{z=0}^{a} 4azdz \right) dy = 2a^4$$

(V)For *OASR* Equation is $y = 0$, and $\bar{n} = -\bar{j}$, $dS = dxdz$

$$\int_{S_5} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

(VI)For *PBCQ* Equation is $y = a$, and $\bar{n} = -\bar{j}$, $dS = dxdz$

$$\int_{S_6} \bar{F}.\bar{n}dS = -\int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

Adding (i) to (vi)

we get $\int_{S_6} \bar{F}.\bar{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$

Volume Integrals

Let V be the volume bounded by a surface $\bar{r} = \bar{f}$ (u,v). Let $\bar{F}$ ($\bar{r}$) be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, ...\delta V_p ....\delta V_m$

Let P_i ($\bar{r}_i$) be a point in δV_i .Then form the sum $I_m = \sum_{i=1}^{m} \bar{F}(r_i)\delta V_i$. Let $m \to \infty$ in such a way that δV_i shrinks to a point,. The limit of I_m if it exists, is called the volume integral of $\bar{F}$ ($\bar{r}$) in the region V is denoted by $\int_V \bar{F}(\bar{r})\, dv$ or $\int_V \bar{F} dv$.

Cartesian Form : Let $\bar{F}(r) = F_1\bar{i} + F_2\bar{i} + F_3\bar{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that $dv = dx\, dy\, dz$. The volume integral given by $\int_v \bar{F} dv = \iiint (F_1\bar{i} + F_2\bar{i} + F_3\bar{k})\, dx\, dydz$

$= \bar{i}\iiint F_1\, dxdydz + \bar{j}\iiint F_2\, dxdydz + \bar{k}\iiint F_3\, dxdydz$

Solved Problems

1.If $\bar{F} = 2xz\, i - x\, j + y^2 k$ evaluate $\int \bar{F}\, dv$ over V where V is the region bounded by the surfaces $x = 0, x = 2, y = 0, y = 6, z = x^2, z = 4$.

Given $\bar{F} = 2xz\, i - x\, j + y^2 k$.

The volume integral is given by

$$\int \bar{F}\, dv = \int_0^2 \int_{y=0}^6 \int_{z=x^2}^4 (2xz\, i - x\, j + y^2 k)\, dxdydz$$

$= \quad i\int_0^2\int_{y=0}^6\int_{z=x^2}^4(2xz\,)dxdydz - j\int_0^2\int_{y=0}^6\int_{z=x^2}^4(x\,)dxdydz + k\int_0^2\int_{y=0}^6\int_{z=x^2}^4(y^2)dxdydz$

$= i\int_0^2\int_{y=0}^6 x(16-x^4)\,dxdy - j\int_0^2\int_{y=0}^6 x(4-x^2)\,dxdy + k\int_0^2\int_{y=0}^6 y^2(x^2 - 4)\,dxdy$

$= i\int_0^2\int_{y=0}^6(16x-x^5)\,dxdy - j\int_0^2\int_{y=0}^6(4x-x^3)\,dxdy + k\int_0^2\int_{y=0}^6 y^2(x^2 - 4)\,dxdy$

$= i\int_0^2 6(16x-x^5)\,dx - j\int_0^2 6(4x-x^3)\,dx + k\int_0^2 72(x^2-4)\,dx$

$= i\int_0^2(96x-6x^5)\,dx - j\int_0^2(24x-6x^3)\,dx + k\int_0^2(72x^2-218)\,dx$

$= 128i - 24j - 384k$

Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i)

$\int_S \bar{F}.\bar{n}\, ds$ into a volume integral where S is a closed surface.

(ii)

$\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii)

$\int_S (\nabla \times \bar{A}) . \bar{n}\, ds$ into a line integral around the boundary of an open two sided surface.

Gauss Divergence Theorem

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\, F dv = \int_s \bar{F}.\bar{n}\, dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

Solved Problems

1. Verify Gauss Divergence theorem for $\bar{F} = (x^3 - yz)\bar{\imath} - 2x^2 y\bar{\jmath} + z\bar{k}$ taken over the surface of the cube bounded by the planes $x = y = z = a$ and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int_S \bar{F}.\bar{n}dS = \int_V div\bar{F}dv$$

Now $div\ \bar{f} = \sum \bar{i}.\left(\frac{\partial \bar{f}}{\partial x}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$= 3x^2 - 2x^2 + 1$$

Here the cube bounded by the planes $x = y = z = a$ and coordinate planes.

Hence

$x \to 0\ to\ a$

$y \to 0\ to\ a$

$z \to 0\ to\ a$

$$RHS = \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)dx\ dy\ dz = \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\ dy\ dz = \int_0^a\int_0^a \left(\frac{x^3}{3} + x\right)_0^a dy\ dz$$

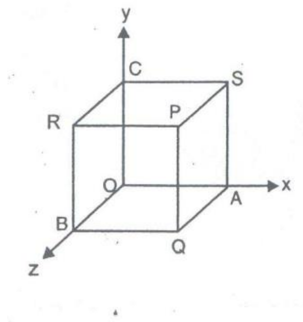
$$\int_0^a\int_0^a\left[\frac{a^3}{3} + a\right]dydz = \int_0^a\left[\frac{a^3}{3} + a\right](y)_0^a dz = \left(\frac{a^3}{3} + a\right)a\int_0^a dz = \left(\frac{a^3}{3} + a\right)(a^2) = \frac{a^5}{3} + a^3 \text{......(1)}$$

Verification: We will calculate the value of $\int_S \bar{F}.\bar{n}dS$ over the six faces of the cube.

(i)

For $S_1 = PQAS$; unit outward drawn normal $\bar{n} = \bar{\imath}$

$x = a;\ ds = dy\,dz;\ 0 \le y \le a, 0 \le z \le a$



$\therefore \overline{F}.\overline{n} = x^3 - yz = a^3 - yz \sin cex = a$

$$\therefore \int_{S_1}\int \overline{F}.\overline{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} (a^3 - yz)dydz$$

$$= \int_{z=0}^{a} \left[a^3 y - \frac{y^2}{2} z\right]_{y=0}^{a} dz$$

$$= \int_{z=0}^{a} \left(a^4 - \frac{a^2}{2} z\right) dz$$

$$= a^5 - \frac{a^4}{4} \ldots (2)$$

(ii)

For $S_2 = OCRB$; unit outward drawn normal $\bar{n} = -\bar{i}$ $x = 0;\ ds = dy\,dz;\ 0 \le y \le a, y \le z \le a$

$\bar{F}.\bar{n} = -(x^3 - yz) = yz\ since\ x = 0$

$$\int_{S_2}\int \bar{F}.\bar{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\,dy\,dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} zdz = \frac{a^4}{4} \ldots (3)$$

(iii)

For $S_3 = RBQP$; z = a; $ds = dxdy$; $\bar{n} = \bar{k}$

$0 \le x \le a, 0 \le y \le a$

$\bar{F}.\bar{n} = z = a\ \ since\, z = a$

$$\therefore \int_{S_3}\int \overline{F}\overline{n}dS = \int_{y=0}^{a}\int_{x=0}^{a} adxdy = a^3 \ldots..(4)$$

(iv)

For $S_4 = OASC$; z = 0; $\bar{n} = -\bar{k}$, $ds = dxdy$;

$0 \le x \le a, 0 \le y \le a$

$\bar{F}.\bar{n} = -z = 0 \ \ since\, z = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}dS = 0 \ldots (5)$$

(v)

For $S_5 = PSCR$; y = a; $\bar{n} = \bar{j}, ds = dzdx;$

$$0 \le x \le a, 0 \le z \le a$$

$\bar{F}.\bar{n} = -2x^2y = -2ax^2 \ \ since\, y = a$

$$\int_{S_5}\int \bar{F}.\bar{n}dS = \int_{x=0}^{a}\int_{z=0}^{a}(-2ax^2)dzdx$$

$$\int_{x=0}^{a}(-2ax^2z)_{z=0}^{a}dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \ldots (6)$$

(vi)

For $S_6 = OBQA$; y = 0; $\bar{n} = -\bar{j}, ds = dzdx;$

$$0 \le x \le a, 0 \le y \le a$$

$\bar{F}.\bar{n} = 2x^2y = 0 \ since\, y = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}dS = 0$$

$$\int_{S}\int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{\nabla}.\bar{F}\,dv\, using\ (1)$$

Hence Gauss Divergence theorem is verified

2. Use divergence theorem to evaluate $\int\int_S \bar{F}.d\bar{S}$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$.

Sol: We have

$$div\overline{F} = \nabla.\overline{F} = \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2) = 4 - 4y + 2z$$

By divergence theorm,

$$\int\int_S \bar{F}.dS = \int\int_V\int \bar{\nabla}.\bar{F}dV$$

$$= \int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{z=0}^{3}(4-4y+2z)dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[(4-4y)z+z^2]_0^3\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[12(1-y)+9]\,dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21-12y)dx\,dy$$

$$= \int_{-2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}21\,dy - 12\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}ydy\right]dx$$

$$= \int_{-2}^{2}\left[21\times 2\int_{0}^{\sqrt{4-x^2}}dy - 12(0)\right]dx$$

[**Since the integrans in first integral is even and in 2nd integral it is on add function**]

$$= 42\int_{-2}^{2}(y)_0^{\sqrt{4-x^2}}\,dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}dx = 42\times 2\int_{0}^{2}\sqrt{4-x^2}dx$$

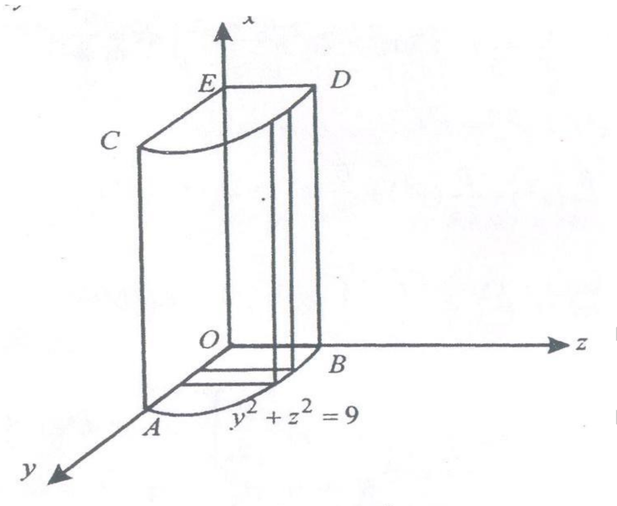
$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0 + 2.\frac{\pi}{2} - 0\right] = 84\pi$$

3. Verify divergence theorem for $2x^2y\,\bar{i} - y^2\,\bar{j} + 4xz^2\,\bar{k}$ taken over the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$.

(or) Evaluate $\iint_S \overline{F}.\overline{n}dS$, where $\overline{F} = 2x^2y\,\overline{i} - y^2\,\overline{j} + 4xz^2\,\overline{k}$ and S is the closed surface of the region in the first octant bounded by the cylinder $y^2+z^2 = 9$ and the planes $x = 0, x = 2, y = 0, z = 0$

Sol: Let $\overline{F} = 2x^2y\,\overline{i} - y^2\,\overline{j} + 4xz^2\,\overline{k}$ $\quad \therefore \nabla.\overline{F} = \frac{\partial}{\partial x}(2x^2) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2) = 4xy - 2y + 8xz$



$$\iiint_V \nabla.\overline{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy - 2y + 8xz)\,dz\,dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy - 2y)z + 8x\frac{z^2}{2}\right]_{z=0}^{\sqrt{9-y^2}} dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy - 2y)\sqrt{9-y^2} + 4x(9-y^2)\right] dy\,dx$$

$$= \int_0^2\int_0^3 [(1-2x)(-2y)\sqrt{9-y^2} + 4x(9-y^2)]\,dy\,dx$$

$$= \int_0^2 \left\{\left[(1-2x)\frac{(9-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3 + 4x\left(9y - \frac{y^3}{3}\right)_0^3\right\} dx$$

↓

{ Since $\int f'(x)[f(x)]^n \, dx = \frac{[f(x)]^{n+1}}{n+1}$ }

$$= \int_0^2 \{\frac{2}{3}(1-2x)[0-27]+4x[27-9]\}\,dx = \int_0^2 [-18(1-2x)+72x]dx$$

$$\left[-18(x-x^2)+72\frac{x^2}{2}\right]_0^2 = -18(2-4)+36(4)=36+144=180\ldots(1)$$

Now we sall calculate $\int_S \bar{F}.\bar{n}\,ds$ *for all the five faces.*

$$\int_s \bar{F}.\bar{n}dS = \int_{s_1}\bar{F}.\bar{n}dS + \int_{s_2}\bar{F}.\bar{n}dS + \ldots\ldots + \int_{s_5}\bar{F}.\bar{n}dS$$

Where S_1 is the face *OAB*, S_2 is the face *CED*, S_3 is the face *OBDE*, S_4 is the face *OACE* and S_5 is the curved surface *ABDC*.

(i)

On $S_1 : x=0, \bar{n}=-i$ $\quad \therefore \bar{F}.\bar{n}=0$ *Hence* $\int_{s_1}\bar{F}.\bar{n}dS$

(ii) On $S_2 : x=2, \bar{n}=i$ $\quad \therefore \bar{F}.\bar{n}=8y$

$$\therefore \int_{s_2}\bar{F}.\bar{n}dS = \int_0^3\int_0^{\sqrt{9-z^2}} 8y\,dy\,dz = \int_0^3 8\left(\frac{y^2}{2}\right)_0^{\sqrt{9-z^2}} dz$$

$$= 4\int_0^3 (9-z^2)dz = 4\left(9z-\frac{z^3}{3}\right)_0^3 = 4(27-9)=72$$

(iii) On $S_3 : y=0, \bar{n}=-j. \therefore \bar{F}.\bar{n}=0$ *Hence* $\int_{s_3}\bar{F}.\bar{n}dS$

$(iv) On\ S_4 : z=0, \bar{n}=-k. \quad \bar{F}.\bar{n}=0. \quad Hence \int_{S_4}\bar{F}.\bar{n}ds = 0$

(v) On $S_5 : y^2+z^2=9, \bar{n}=\frac{\nabla(y^2+z^2)}{|\nabla(y^2+z^2)|}=\frac{2y\bar{j}+2z\bar{k}}{\sqrt{4y^2+4z^2}}=\frac{y\bar{j}+z\bar{k}}{\sqrt{4\times 9}}=\frac{y\bar{j}+z\bar{k}}{3}$

$\bar{F}.\bar{n} = \frac{-y^3+4xz^3}{3}$ and $\bar{n}.\bar{k} = \frac{z}{3} = \frac{1}{3}\sqrt{9-y^2}$

Hence $\int_{S_5} \bar{F}.\bar{n}ds = \int\int_R \bar{F}.\bar{n}\frac{dx\,dy}{|\bar{n}.\bar{k}|}$ *Where R is the projection of* S_5 *on* $xy-plane.$

$$= \int_R\int \frac{4xz^3 - y^3}{\sqrt{9-y^2}} dx\,dy = \int_{x=0}^{2}\int_{y=0}^{3} \left[4x(9-y^2) - y^3\left(9-y^2\right)^{-\frac{1}{2}}\right] dy\,dx$$

$Tofind \int_0^3 y^3(\sqrt{9-y^2})\,\mathrm{dy}$

sub

$\mathrm{y} = 3\sin\theta$

$\mathrm{dy} = 3\cos\theta$

$$\int_0^3 y^3(\sqrt{9-y^2})\,\mathrm{dy} = \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta$$

sub

$\sin^3\theta = 3\sin\theta - \sin 3\theta$

We get

$$\int_0^3 y^3(\sqrt{9-y^2})\,\mathrm{dy} = \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta = -18$$

Hence

$$\int_{S3} \bar{F}.\bar{n}ds$$

$$= \int_0^2 72x\,dx - 18\int_0^2 dx = 72\left(\frac{x^2}{2}\right)_0^2 - 18(x)_0^2 = 144 - 36 = 108$$

$Thus \int_S \bar{F}.\bar{n}ds = 0 + 72 + 0 + 0 + 108 = 180 \ldots \ldots (2)$

Hence the Divergence theorem is verified from the equality of (1) and (2).

4. Verify Gauss divergence theorem for $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$ taken over the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0, z = a.$

Sol: We have $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$

$$\nabla.\overline{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\int\int_V\int \bar{V}.\bar{F}\,dv = \int\int_V\int (3x^2 + 3y^2 + 3z^2)dx\,dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a}(x^2 + y^2 + z^2)dx\,dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{x^3}{3} + xy^2 + z^2x\right)_0^a dy\,dz$$

$$= 3\int_{z=0}^{a}\int_{y=0}^{a}\left(\frac{a^3}{a} + ay^2 + az^2\right)dy\,dz$$

$$= 3\int_{z=0}^{a}\left(\frac{a^3}{3}y + a\frac{y^3}{3} + az^2y\right)_0^a dz$$

$$= 3\int_0^a\left(\frac{a^4}{3} + \frac{a^4}{3} + a^2z^2\right)dz = 3\int_0^a\left(\frac{2}{3}a^4 + a^2z^2\right)dz$$

$$= 3\left(\frac{2}{3}a^4z + a^2.\frac{z^3}{3}\right)_0^a = 3\left(\frac{2}{3}a^5 + \frac{1}{3}a^5\right)$$

$$= 3a^5$$

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.

i.e.,

S_1 : The face *DEFA* ; S_4 : The face *OBDC*

S_2 : The face *AGCO* ; S_5 : The face *GCDE*

S_3 : The face *AGEF* ; S_6: The face *AFBO*

$$\int_S\int \bar{F}.\bar{n}ds = \int_{S_1}\int \bar{F}.\bar{n}ds + \int_{S_2}\int \bar{F}.\bar{n}ds + \cdots + \int_{S_6}\int \bar{F}.\bar{n}ds$$

On S_1, we have $\bar{n} = \bar{\imath}, x = a$

$$\therefore \iint_{S_1} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(a^3\bar{i} + y^3\bar{j} + z^3\bar{k}\right).\bar{i}dydz$$

$$\int_{S_1}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}(a^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}).\bar{\imath}\,dy\,dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3 dy\,dz = a^3\int_0^a (y)_0^a\,dz$$

$$= a^4(z)_0^a = a^5$$

On S_2, we have $\bar{n} = -\bar{\imath}, x = 0$

$$\iint_{S_2} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a}\left(y^3\bar{j} + z^3\bar{k}\right).\left(-\bar{i}\right)dydz = 0$$

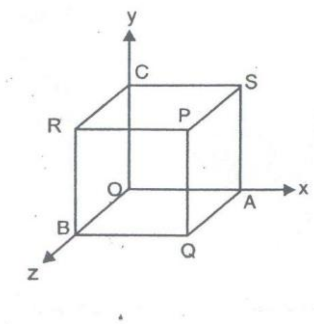
On S_3, we have $\bar{n} = \bar{\jmath}, y = a$

$$\iint_{S_3} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}\left(x^3\bar{i} + a^3\bar{j} + z^3\bar{k}\right).\bar{j}dxdz = a^3\int_{z=0}^{a}\int_{x=0}^{a}dxdz = a^3\int_0^a adz = a^4\left(z\right)_0^a$$

$$= a^5$$

On S_4, we have $\bar{n} = -\bar{\jmath}, y = 0$

$$\int_{S_4}\int \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a}(x^3\bar{\imath} + z^3\bar{k}).(-\bar{\jmath})dx\,dz = 0$$



On S_5, we have $\bar{n} = \bar{k}, z = a$

$$\int_{S_5}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}(x^3\bar{\imath} + y^3\bar{\jmath} + a^3\bar{k}).\bar{k}\,dx\,dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a^3 dx\,dy = a^3\int_0^a (x)_0^a dy = a^4(y)_0^a = a^5$$

On S_6, we have $\bar{n} = -\bar{k}, z = 0$

$$\int_{S_6}\int \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a}(x^3\bar{\imath}+y^3\bar{\jmath}).(-\bar{k})dx\,dy = 0$$

$$Thus \int_S\int \bar{F}.\bar{n}ds = a^5+0+a^5+0+a^5+0 = 3a^5$$

$$Hence \int_S\int \bar{F}.\bar{n}ds = \int_V\int \bar{V}.\bar{F}\,dv$$

$\therefore$ *The* Gauss divergence theorem is verified.

5. Compute $\int(ax^2+by^2+cz^2)dS$ over the surface of the sphere $x^2+y^2+z^2=1$

Sol: By divergence theorem $\int_S \bar{F}.\bar{n}dS = \int_V \bar{V}.\bar{F}\ dv$

$Given\ \bar{F}.\bar{n} = ax^2+by^2+cz^2. Let\ \phi = x^2+y^2+z^2-1$

$\therefore$ *Normal vector* $\bar{n}$ *to the surface* ϕ *is*

$$\bar{V}\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial y}\right)(x^2+y^2+z^2-1) = 2(x\bar{i}+y\bar{j}+z\bar{k})$$

$\therefore$ Unit normal vector $=\bar{n}=\dfrac{2(x\bar{i}+y\bar{j}+z\bar{k})}{2\sqrt{x^2+y^2+z^2}} = x\bar{i}+y\bar{j}+z\bar{k}$ Since $x^2+y^2+z^2=1$

$$\therefore\ \bar{F}.\bar{n} = \bar{F}.(x\bar{i}+y\bar{j}+z\bar{k}) = (ax^2+by^2+cz^2) = (a\,x\bar{i}+by\bar{j}+cz\bar{k}).(x\bar{i}+y\bar{j}+z\bar{k})$$

i.e., $\bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$ $\quad \nabla.\bar{F} = a+b+c$

Hence by Gauss Divergence theorem,

$$\int_S (ax^2+by^2+cz^2)dS = \int_V (a+b+c)dv = (a+b+c)V = \frac{4\pi}{3}(a+b+c)$$

$$\left[SInce\ V = \frac{4\pi}{3}\ is\ the\ volume\ of\ the\ sphere\ of\ unit\ radius\right]$$

6. Use divergence theorem to evaluate $\int_s\int \bar{F}.d\bar{S}$ where $\bar{F}=x^3i+y^3j+z^3k$ and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have $\overline{V}.\overline{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(y^3) + \frac{\partial}{\partial z}(z^3) = 3(x^2 + y^2 + z^2)$

∴ By divergence theorem,

$\bar{V}.\bar{F}\,dV = \int\int_V \int \bar{V}.\bar{F}\,dV$ $= \iiint_v 3(x^2+y^2+z^2)dxdydz$

$$= 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 (r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

Applying spherical coordinates,

$$\int_S \int \bar{F}.dS = 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi} r^4 \sin\theta \left[\int_{\phi=0}^{2\pi} d\phi \right] dr \, d\theta$$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4 \sin\theta(2\pi - 0) dr d\theta = 6\pi \int_{r=0}^{a} r^4 \left[\int_0^{\pi} \sin\theta d\theta \right] dr$$

$$= 6\pi \int_{r=0}^{a} r^4 (-\cos\theta)_0^{\pi} \, dr = -6\pi \int_0^{a} r^4 (\cos\pi - \cos 0) \, dr$$

$$= 12\pi \int_0^{a} r^4 dr = 12\pi \left[\frac{r^5}{5} \right]_0^{a} = \frac{12\pi a^5}{5}$$

7. **Verify divergence theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane $x+y+z=a$ in the first octant.**

Sol: By Gauss theorem, $\int_s \overline{F}.\overline{n}dS = \int_v div\overline{F}dv$

$$\frac{\partial\phi}{\partial x} = 1, \frac{\partial\phi}{\partial y} = 1, \frac{\partial\phi}{\partial z} = 1$$

$$\therefore grad\phi = \sum \bar{i}\frac{\partial\phi}{\partial x} = \bar{i} + \bar{j} + \bar{k}$$

$$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{i} + \bar{j} + \bar{k}}{\sqrt{3}}$$

Let R be the projection of S on *xy*-plane

Then the equation of the given plane will be $x+y=a \Rightarrow y=a-x$

Also when $y=0, x=a$

$$\therefore \int_s \overline{F}.\overline{n}dS = \iint_R \frac{\overline{F}.\overline{n}dxdy}{|\overline{n}.\overline{k}|}$$

$$= \int_0^a \int_0^{a-x} [2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2]dx\, dy$$

$$= \int_{x=0}^{a} \left[2x^2y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x)dx$$

$$\therefore \int_s \overline{F}.\overline{n}dS = \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2x + \frac{2}{3}a^3\right)dx = \frac{a^4}{4}, \text{ on simplification…(1)}$$

Given $\overline{F} = x^2\overline{i} + y^2\overline{j} + z^2\overline{k}$

$$\therefore div\ \overline{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2(x+y+z)$$

$$Now \iiint div\overline{F}.dv = 2\int_{x=0}^{a}\int_{y=0}^{a-x}\int_{z=0}^{a-x-y}(x+y+z)dxdydz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}\left[z(x+y) + \frac{z^2}{2}\right]_0^{a-x-y} dx\, dy$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)\left[x+y+\frac{a-x-y}{2}\right]dx\, dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x}(a-x-y)[a+x+y]dx\, dy$$

$$= \int_0^a\int_0^{a-x}[a^2-(x+y)^2]\ dy\, dx = \int_0^a\int_0^{a-x}(a^2-x^2-y^2-2xy)dx\, dy$$

$$= \int_0^a [a^2y - x^2y - \frac{y^3}{3} - xy^2]_0^{a-x}\ dx$$

$$= \int_0^a (a-x)(2a^2 - x^2 - ax)dx = \frac{a^4}{4} \dots\dots (2)$$

Hence from (1) and (2), the Gauss Divergence theorem is verified.

8. Use Gauss Divergence theorem to evaluate $\int\int_S (yz^2\bar{i} + zx^2\bar{j} + 2z^2\bar{k}).ds,$ where S is the closed surface bounded by the *xy*-plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane.

Sol: Divergence theorem states that

$$\int\int_S \bar{F}.ds = \int\int_V\int \bar{\nabla}.\bar{F}\ dv$$

Here $\nabla.\overline{F} = \frac{\partial}{\partial x}(yz^2) + \frac{\partial}{\partial y}(zx^2) + \frac{\partial}{\partial z}(2z^2) = 4z$

$$\therefore \int\int_s \overline{F}.ds = \int\int_V\int 4z dxdydz$$

Introducing spherical polar coordinates $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi,$

$z = r\cos\theta$ then $dxdydz = r^2 dr d\theta d\phi$

$$\therefore \int\int_s \overline{F}.ds = 4\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\cos\theta)(r^2\sin\theta dr d\theta d\phi)$$

$$= 4\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\left[\int_{\phi=0}^{2\pi} d\phi\right] dr\ d\theta$$

$$= 4.\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3\sin\theta\cos\theta\ (2\pi - 0)dr\ d\theta$$

$$= 4\pi\int_{r=0}^{a} r^3\left[\int_0^{\pi}\sin 2\theta\ d\theta\right]dr = 4\pi\int_{r=0}^{a} r^3\left(-\frac{\cos 2\theta}{2}\right)_0^{\pi} dr$$

$$= (-2\pi)\int_0^a r^3(1-1)dr = 0$$

9. Use Divergence theorem to evaluate $\iint \left(x\bar{i} + y\bar{j} + z^2\bar{k}\right).\bar{n}ds$. Where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z = 4$.

Sol: Given $\int\int (x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds$ *Where* S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane $z = 4$. Let $\bar{F} = x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}$

By Gauss Divergence theorem, we have

$$\int\int (x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds = \int\int_V\int \bar{\nabla}.\bar{F}\, dv$$

$$\nabla.\bar{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2) = 1+1+2z = 2(1+z)$$

On the cone, $x^2 + y^2 = z^2$ and z=4 $\Rightarrow x^2 + y^2 = 16$

The limits are $z = 0$ *to* $4, y = o$ *to* $\sqrt{16-x^2}, x = 0$ *to* 4.

$$\int\int_V\int \bar{\nabla}.\bar{F}\, dv = \int_0^4\int_0^{\sqrt{16-x^2}}\int_0^4 2(1+z)dx\, dy\, dz$$

$$= 2\int_0^4\int_0^{\sqrt{16-x^2}} \left\{[z]_0^4 + \left[\frac{z^2}{2}\right]_0^4\right\} dx\, dy$$

$$= 2\int_0^4\int_0^{\sqrt{16-x^2}} [4+8]dxdy = 2\times 12\int_0^4 [y]_0^{\sqrt{16-x^2}} dx$$

$$= 24\int_0^4 \sqrt{16-x^2}\, dx = 24\int_0^{\frac{\pi}{2}} \sqrt{16-16\sin^2\theta}\,.4\cos\theta d\theta$$

$[put x = 4\sin\theta \Rightarrow dx = 4\cos\theta d\theta.$ *Also* $x = 0 \Rightarrow \theta = 0$ *and* $x = 4 \Rightarrow \theta = \frac{\pi}{2}]$

$$\therefore \int\int_V\int \nabla.\bar{F}dv = 96\times 4\int_0^{\frac{\pi}{2}} 4\sqrt{1-\sin^2\theta}\cos\theta d\theta = 96\times 4\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta$$

$$\int\int_V\int \bar{\nabla}.\bar{F}\, dv = 96\ X\ 4\int_0^{\frac{\pi}{2}} 4\sqrt{1-\sin^2\theta}\ \cos\theta d\theta = 96\ X\ 4\int_0^{\frac{\pi}{2}} \cos^2\theta\ d\theta$$

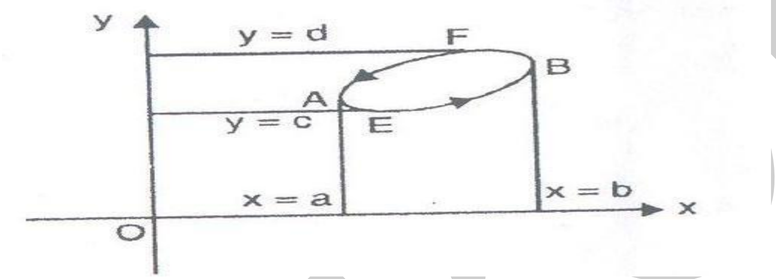
$$= 96\,X\,4\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta = 96X4\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}+\frac{\cos 2\theta}{2}\right]d\theta$$

$$= 384\left[\frac{1}{2}\theta+\frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 96\pi$$

Green’s Theorem in a Plane(Transformation b/w Line Integral and Surface Integral)

If S is Closed region in *xy* plane bounded by a simple closed curve C and if M and N are continuous functions of *x and y* having continuous derivatives in R, then

$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy.$ Where C is traversed in the anti clock-wise direction

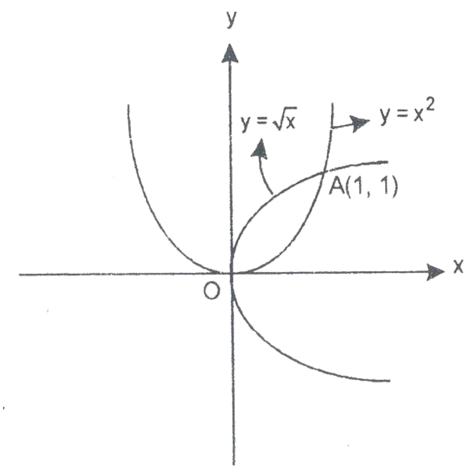


Solved Problems

1.Verify Green’s theorem in plane for $\oint(3x^2-8y^2)dx+(4y-6xy)dy$ **where *C* is the region bounded by y=**$\sqrt{x}$ **and y=**x^2 .

Sol: Let $M=3x^2-8y^2$ and $N=4y-6xy$. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$$

Now $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_R (16y - 6y) dxdy$

$$= 10\iint_R ydxdy = 10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} ydydx = 10\int_{x=0}^{1}\left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (x - x^4)dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$$

….(1)

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2=x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1}\{[3x^2 - 8(x^2)^2]dx + [4x^2 - 6x(x^2)]2xdx\}$ $\left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$$= \int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$$

And $I_2 = \int_1^0 \left[\left(3x^2 - 8x\right)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}}dx\right] = \int_1^0 \left(3x^2 - 11x + 2\right)dx = \frac{5}{2}$

$\therefore\ I_1 + I_2 = -1+5/2 = 3/2$.

From(1) and (2), we have $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$

Hence the verification of the Green's theorem.

2.Evaluate $\oint (3x^2 - 8y^2)dx + (4y - 6xy)dy$ over triangle enclosed by the $lines$ $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$ using Green's theorem.

Sol : Let M=y-$\sin x$ and $N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-$\sin x$

$\therefore$ By Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$

$$\Rightarrow \int_c (y - \sin x)dx + \cos x dy = \iint_R (-1 - \sin x)dxdy$$

$$= -\int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2x}{\pi}} (1 + \sin x)\, dxdy$$

$$= -\int_{x=0}^{\pi/2} (\sin x + 1)\, [y]_0^{2x/\pi}\, dx$$

$$= \frac{-2}{\pi} \int_{x=0}^{\pi/2} x(\sin x + 1)dx$$

$$= \frac{-2}{\pi}\left[x(-\cos x + x)\right]_0^{\pi} - \int_0^{\pi/2} 1(-\cos x + x)dx$$

$$= \frac{-2}{\pi}\left[x(-\cos x + x) + \sin x - \frac{x^2}{2}\right]_0^{\pi/2}$$

$$= \frac{-2}{\pi}\left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

3.A Vector field is given by $\bar{F} = (\sin y)\bar{i} + x(1 + \cos y)\bar{j}$

Evaluate the line integral over the circular path x^2+$y^2 = a^2$, z=0

(i) Directly (ii) By using Green's theorem

Sol: (i) Using the line integral

$$\oint_c \overline{F}.d\overline{r} = \oint_c F_1\,dx + F_2\,dy = \oint_c \sin y\,dx + x(1+\cos y)dy$$

$$= \oint_c \sin y\,dx + x\cos y\,dy + x\,dy = \oint_c d(x\sin y) + x\,dy$$

Given Circle is $x^2+y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta\ d\theta$ and dy=a $\cos\theta$ dθ and $\theta = 0 \to 2\pi$

$$\therefore \oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a\ \cos\theta \sin(a\ \sin\theta)] + \int_0^{2\pi} a(\cos\theta)a\ \cos\theta\, d\theta$$

$$=[a\ \cos\theta \sin(a\ \sin\theta)]_0^{2x} + 4a^2 \int_0^{\pi/2} cos^2\theta\, d\theta$$

$$=0+4a^2.\frac{1}{2}.\frac{\pi}{2} = \pi a^2$$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1+\cos y)$. Then

$\frac{\partial M}{\partial y}$=$\cos y$ and $\frac{\partial N}{\partial x}$=$(1+\cos y)$

By Green's theorem,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\therefore \oint_c \sin y dx + x(1+\cos y)dy = \iint_R (-\cos y + 1 + \cos y)dxdy == \iint dxdy$$

$$= \iint_R dA = A = \pi a^2 (\because area\ \ of\ \ circle = \pi a^2)$$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

4.Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x= $a\cos\theta, y = b sin\theta \quad (i.e)\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(ii)The Circle x=$a\cos\theta, y = a\sin\theta \ (i.e)x^2 + y^2 = a^2$

Sol: We have by Green's theorem $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1 \ and \ \frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2\int_R dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=$a\cos\theta$ and y=$b\sin\theta$ and $\theta = 0 \to 2\pi$

$\therefore Area, A = \frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(a\ \cos\theta)(b\cos\theta) - (b\ \sin\theta\ (-a\ \sin\theta))]d\ \theta$

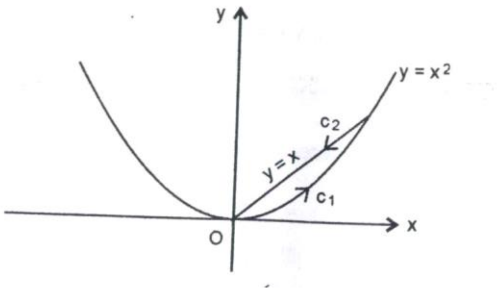
$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta + sin^2\theta)\ d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$

(ii)Put a=b to get area of the circle A=πa^2

5. Verify Green's theorem for $\int_c[(xy + y^2)dx + x^2dy]$, where *C* is bounded by *y*=*x* and *y*=x^2

Sol: By Green's theorem, we have $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here M=xy +y^2 and N=x^2



The line y=x and the parabola y=x^2 intersect at O$(0,0)$ and A$(1,1)$

Now $\oint_c Mdx+Ndy=\int_{c_1} Mdx+Ndy+\int_{c_2} Mdx+Ndy$......(1) .....(1)

Along C_1 $(i.e.\, y=x^2)$, the line integral is

$$\int_{c_1} Mdx+Ndy=\int_{c_1}[x(x^2)+x^4]dx+x^2d(x^2)\int_c (x^3+x^4+2x^3)dx=\int_0^1(3x^3+x^4)dx$$

$$=\left(3.\frac{x^4}{4}+\frac{x^5}{5}\right)_0^1 = \frac{3}{4}+\frac{1}{5}=\frac{19}{20}$$ ....(2)

Along C_2 $(i.e.\, y=x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} Mdx+Ndy=\int_{c_2}(x.x+x^2)dx+x^2dx \quad [\because dy = dx]$$

$$=\int_{c_2} 3x^2dx = 3\int_1^0 x^2dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 = 0-1=-1$$ ....(3)

From (1), (2) and (3), we have

$\int_c Mdx+Ndy = \frac{19}{20}-1=\frac{-1}{20}$

...(4)

Now

$$\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy = \iint_R (2x-x-2y)dxdy$$

$$=\int_0^1[(x^2-x^2)-(x^3-x^4)]dx=\int_0^1(x^4-x^3)dx$$

$$=\left(\frac{x^5}{5}+\frac{x^4}{4}\right)_0^1=\frac{1}{5}-\frac{1}{4}=\frac{-1}{20} \quad \ldots.(5)$$

From$(4) and (5), We\ have$ $\oint_c Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

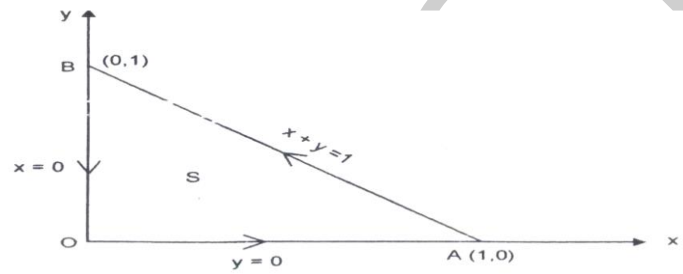
Hence the Green's Theorem is verified.

6. Verify Green's theorem for $\int_c[(3x^2-8y^2)dx+(4y-6xy)dy]$ where c is the region bounded by *x=0, y=0 and x+y=1*.

Sol : By Green's theorem, we have

$$\int_c Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

Here M=$3x^2-8y^2$ and N=*4y-6xy*



$$\therefore \frac{\partial M}{\partial y}=-16y \text{ and } \frac{\partial N}{\partial x}=-6y$$

Now $\int_c Mdx+Ndy=\int_{OA} Mdx+Ndy+\int_{AB} Mdx+Ndy+\int_{BC} Mdx+Ndy$...(1)

Along OA, y=0 $\therefore dy=0$

$$\int_{OA} Mdx+Ndy=\int_0^1 3x^2dx=\left(\frac{x^3}{3}\right)_0^1=1$$

Along AB, x+y=1 $\therefore dy=-dx$ and x=1-y and y varies from 0 to 1.

$$\int_{AB} Mdx+Ndy\ =\int_0^1[3(y-1)^2-8y^2](-dy)+[4y+6y(y-1)]dy$$

$$=\int_0^1 (-5y^2-6y+3)(-dy)+(6y^2-2y)dy$$

$$=\int_0^1 (11y^2+4y-3)dy=\left(11\frac{y^3}{3}+4\frac{y^2}{2}-3y\right)_0^1$$

$$=\frac{11}{3}+2-3=\frac{8}{3}$$

Along BO, x=0 $\therefore\ dx=0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx+Ndy=\int_1^0 4ydy=\left(4\frac{y^2}{2}\right)_1^0=(2y^2)_0^1=-2$

from (1), we have $\int_c Mdx+Ndy=1+\frac{8}{3}-2=\frac{5}{3}$

Now $\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy=\int_{x=0}^{1}\int_{y=0}^{1-x}(-6y+16y)dxdy$

$$=10\int_{x=0}^1\left[\int_{y=0}^{1-x} ydy\right]dx=10\int_0^1\left(\frac{y^2}{2}\right)_0^{1-x}dx$$

$$=5\int_0^1 (1-x)^2dx=5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

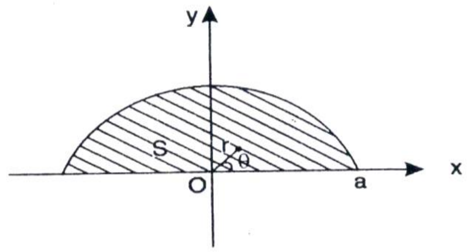
$$=-\frac{5}{3}[(1-1)^3-(1-0)^3]=\frac{5}{3}$$

From (2) and (3), we have $\int_c Mdx+Ndy=\iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$

Hence the Green's Theorem is verified.

7. **Apply Green's theorem to evaluate** $\oint_c (2x^2-y^2)dx+(x^2+y^2)\,dy, where\ c\ is$ **the boundary of the area enclosed by the *x*-axis and upper half of the circle** $x^2+y^2=a^2$

Sol: Let M=$2x^2 - y^2$ and N=$x^2 + y^2$ Then



Figure

$$\frac{\partial M}{\partial y} = -2y \text{ and } \frac{\partial N}{\partial x} = 2x$$

$$\therefore \text{ By Green's Theorem, } \int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$\oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = \iint_R (2x + 2y)dxdy$$

$$=2\iint_R (x+y)dy$$

$$=2\int_0^a \int_0^\pi r(\cos\theta + \sin\theta).rd\theta dr$$

[Changing to polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π]

$$\therefore \oint_c [(2x^2 - y^2)dx + (x^2 + y^2)dy] = 2\int_0^a r^2 dr \int_0^\pi (\cos\theta + \sin\theta)d\theta$$

$$=2.\frac{a^3}{3}(1+1) = \frac{4a^3}{3}$$

8. Verify Green's theorem in the plane for $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$

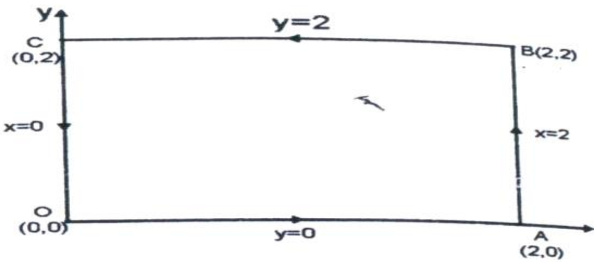
Where *C* is square with vertices (0,0), (2,0), (2,2), (0,).

Sol: The Cartesian form of Green's theorem in the plane is

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Here M=$x^2 - xy^3$ and N=$y^2 - 2xy$

$\therefore \quad \frac{\partial M}{\partial y} = -3\,xy^2$ and $\frac{\partial N}{\partial x} = -2y$



Evaluation of $\int_C (M\,dx + N\,dy)$

To Evaluate $\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy$, we shall take C in four different segments viz (i) along *OA(y=0)* (ii) along *AB(x=2)* (iii) along *BC(y=2)* (iv) along *CO(x=0).*

(i)Along *OA(y=0)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 =$$

…..(1)

(ii)Along *AB(x=2)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 (y^2 - 4y)\,dy \quad [\because x = 2, dx = 0]$$

$$= \left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3}$$

….(2)

(iii)Along *BC(y=2)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 (x^2 - 8x)\,dx \quad [\because y = 2, dy = 0]$$

$$= \left(\frac{x^3}{3} - 4x^2\right)_0^2 = -\left(\frac{8}{3} - 16\right) = \frac{40}{3} \text{......(3)}$$

(iv)Along *CO(x=0)*

$$\int_C (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 y^2\,dx \quad [\because x = 0, dx = 0] = \left(\frac{y^3}{3}\right)_2^0 = -\frac{8}{3}$$

…..(4)

Adding(1),(2),(3) and (4), we get

$$\int_c \left(x^2 - xy^3\right)dx + \left(y^2 - 2xy\right)dy = \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3} = \frac{24}{3} = 8 \qquad \ldots(5)$$

Evaluation of $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$

Here x ranges from 0 to 2 and *y* ranges from 0 to 2 .

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \int_0^2\int_0^2 (-2y + 3xy^2)dxdy$$

$$= \int_0^2 \left(-2xy + \frac{3x^2}{2}y^2\right)_0^2 dy$$

$$= \int_0^2 (-4y + 6y^2)dy = \left(-2y^2 + 2y^3\right)_0^2$$

$$= -8+16=8 \qquad \ldots(6)$$

From (5) and (6), we have

$$\int_c M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

Hence the Green's theorem is verified.

Stoke's Theorem (Transformation between Line Integral and Surface Integral)

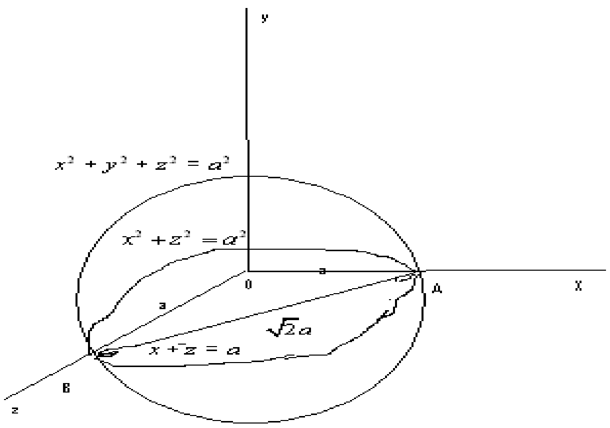
Let S be a open surface bounded by a closed, non intersecting curve C. If $\bar{F}$ is any differentiable vector point function then

$\oint_c \bar{F}.d\,\bar{r} = \int_S curl\ \bar{F}.\bar{n}\,ds$ *where c is traversed in the positive* direction and $\bar{n}$ *is unit outward drawn normal at any point of the surface.*

Solved Problems

1. Apply Stokes theorem, to evaluate $\oint_c (ydx + zdy + xdz)$ **where** c **is the curve of intersection of the sphere** $x^2 + y^2 + z^2 = a^2$ **and** $x+z=a$**.**

Sol: The intersection of the sphere $x^2 + y^2 + z^2 = a^2$ *and* the plane $x+z=a$ is a circle in the plane $x+z=a$ with AB as diameter.



Equation of the plane is $x+z=a \Longrightarrow \frac{x}{a} + \frac{z}{a} = 1$

$\therefore OA = OB = a$ i.e., $A = (a, 0, 0)$ and B=(0,0,a)

$\therefore$ Length of the diameter AB $= \sqrt{a^2 + a^2 + 0} = a\sqrt{2}$

Radius of the circle, $r = \frac{a}{\sqrt{2}}$

Let $\bar{F}.d\bar{r} = ydx + zdy + xdz \Longrightarrow \bar{F}.d\bar{r} = \bar{F}.\left(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz\right) = ydx + zdy + xdz$

$\Longrightarrow \bar{F} = y\,\bar{\imath} + z\,\bar{\jmath} + x\,\bar{k}$

$$\therefore curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\left(\bar{\imath} + \bar{\jmath} + \bar{k}\right)$$

Let $\bar{n}$ be the unit normal to this surface. $\bar{n} = \frac{\nabla S}{|\nabla S|}$

Then $s=x+z-a$, $\nabla S = \bar{i} + \bar{k}$ $\therefore$ $\bar{n} = \frac{\nabla S}{|\nabla S|} = \frac{\bar{\imath}+\bar{k}}{\sqrt{2}}$

Hence $\oint_c \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}\, ds\ (by\ Stokes\ Theorem)$

$$=-\int(\bar{\imath}+\bar{\jmath}+\bar{k}).\left(\frac{\bar{\imath}+\bar{k}}{\sqrt{2}}\right)ds = -\int\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)ds$$

$$=-\sqrt{2}\int_S ds = -\sqrt{2}S = -\sqrt{2}\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^2}{\sqrt{2}}$$

2.Prove by Stokes theorem, $Curl\ grad\ \phi = \bar{O}$

Sol: Let S be the surface enclosed by a simple closed curve C.

$\therefore By\ Stokes\ theorem$

$$\int_s (curl\ grand\phi).\bar{n}\ ds = \int_s (\nabla x \nabla\phi).\bar{n}\ dS = \oint_C \nabla\phi.d\bar{r} = \oint_C \nabla\phi.d\bar{r}$$

$$=\oint_c \left(\frac{\bar{i}\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}\right).\left(\bar{i}dx+\bar{j}dy+\bar{k}dz\right)$$

$$=\oint_c \left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right) = \int d\phi = [\phi]_p$$ where P is any point on C.

$$\therefore \int curl\ grad\phi.\bar{n}\ ds = \bar{0} \Rightarrow curl\ grad\phi = \bar{0}$$

3. Verify Stokes theorem for $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$, Where S is the circular disc

$x^2 + y^2 \leq 1, z = 0.$

Sol: Given that $\bar{F} = -y^3\bar{\imath} + x^3\bar{\jmath}$. The boundary of C of S is a circle in xy plane.

$x^2 + y^2 \leq 1, z = 0.$ We use the parametric co-ordinates x=cos $\theta, y = sin\theta, z = 0, 0 \leq \theta \leq 2\pi;$

dx=-sin$\theta\ d\theta$ and dy =cos$\theta\ d\theta$

$$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3 dx + x^3 dy$$

$$=\int_0^{2\pi}[-sin^3\theta(-sin\theta) + cos^3\theta cos\theta]d\theta = \int_0^{2\pi}(cos^4\theta + sin^4\theta)d\theta$$

$$=\int_0^{2\pi}(1 - 2sin^2\theta\ cos^2\theta)d\theta = \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi}(2sin\theta\ cos\theta)^2\, d\theta$$

$$=\int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi}(1 - cos4\theta)d\theta$$

$$=2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16}sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$$

We have $(\bar{k}.\bar{n})ds = dxdy$ and R is the region on xy-plane

$$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2)\, dx\, dy$$

Put x=r cos$\emptyset$, $y = r\ sin\emptyset \quad \therefore dxdy = rdr\ d\emptyset$

r is varying from 0 to 1 and $0 \le \emptyset \le 2\pi$.

$$\therefore \int(\nabla \times \bar{F}).\,\bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2.rdr\, d\emptyset = \frac{3\pi}{2}$$

L.H.S=R.H.S. Hence the theorem is verified.

4.Verify Stokes theorem for $\bar{F} = (2x - y)\bar{\imath} - yz^2\bar{\jmath} - y^2z\bar{k}$ **over the upper half surface of the sphere** $x^2 + y^2 + z^2 = 1$ **bounded by the projection of the** ***xy*****-plane.**

Sol: The boundary C of S is a circle *in xy* plane i.e $x^2 + y^2$=1, *z=0*

The parametric equations are x=$cos\theta, y = sin\theta, \theta = 0 \to 2\pi$

$$\therefore \quad dx = -sin\theta\ d\theta, dy = cos\theta\ d\theta$$

$$\int_c \bar{F}.d\bar{r} = \int_c \bar{F_1}dx + \bar{F_2}dy + \bar{F_3}dz = \int_c (2x - y)dx - yz^2dy - y^2zdz$$

$$= \int_c (2x - y)dx (since\ z = 0\ and\ dz = 0)$$

$$= -\int_0^{2\pi}(2\cos\theta - \sin\theta)\sin\theta d\theta = \int_0^{2\pi}\sin^2\theta d\theta - \int_0^{2\pi}\sin 2\theta d\theta$$

$$= \int_{\theta=0}^{2\pi}\frac{1 - cos2\theta}{2}d\theta - \int_0^{2\pi} sin2\theta\ d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}sin2\theta + \frac{1}{2}.cos2\theta\right]_0^{2\pi}$$

$=\frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(cos4\pi - cos0) = \pi$

Again $\nabla \times \bar{F}=\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} = \bar{\imath}(-2yz + 2yz) - \bar{\jmath}(0 - 0) + \bar{k}(0 + 1) = \bar{k}$

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds=\int_S \bar{k}.\bar{n}ds = \int_R \int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds = dxdy$

Now $\int \int_R dxdy = 4\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} dydx = 4\int_{x=0}^{1} \sqrt{1-x^2}\, dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^1$

$=4\left[\frac{1}{2}\sin^{-1} 1\right] =2\frac{\pi}{2}=\pi$

$\therefore$ *The* Stokes theorem is verified.

5.Evaluate by Stokes theorem $\oint_c (x + y)dx + (2x - z)dy + (y + z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,0,0) and (1,1,0).

Sol: Let $\bar{F}.d\bar{r} = \bar{F}.\left(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz\right) = (x + y)dx + (2x - z)dy + (y + z)dz$

Then $\bar{F} = (x + y)\bar{\imath} + (2x - z)\bar{\jmath} + (y + z)\bar{k}$

By Stokes theorem, $\oint_c \bar{F}.d\bar{r} = \int \int_S curl\ \bar{F}.\bar{n}\, ds$

Where S is the surface of the triangle *OAB* which lies in the *xy* plane. Since the z Co-ordinates of $O, A\ and\ B$

Are zero. Therefore $\bar{n} = \bar{k}$. Equation of *OA* is *y=0* and

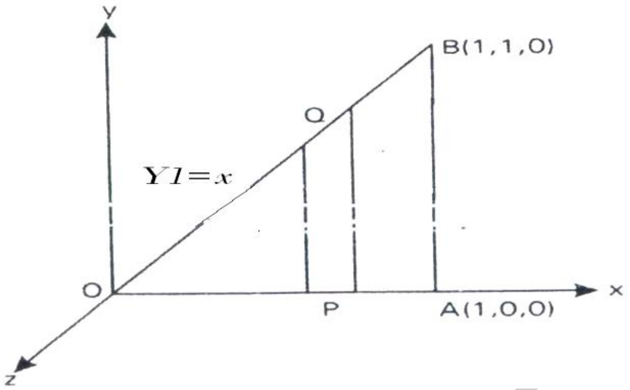
that of *OB*, *y=x* in the *xy* plane.

$\therefore\ curl\ \bar{F}=\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + y & 2x - z & y + z \end{vmatrix} =2\bar{\imath} + \bar{k}$

$\therefore\ curl\ \bar{F}.\bar{n}$ds=curl $\bar{F}.\bar{K}\ dx\ dy = dx\ dy$

$\therefore \oint_c \bar{F}.d\bar{r} = \int \int_s dx\ dy = \int \int_s dA = A = area\ of$ the $\Delta\ OAB$

$= OA \times AB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$



6: Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{\imath} - 2xy\bar{\jmath}$ taken round the rectangle bounded by the lines x=$\pm a, y = 0, y = b$.

Sol: Let ABCD be the rectangle whose vertices are (a,0), (a,b), (-a,b) and (-a,0).

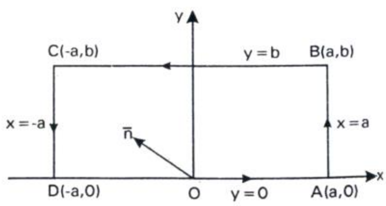
Equations of AB, BC, CD and DA are x=a, y=b, x=-a and y=0.

We have to prove that $\oint_c \bar{F}.d\bar{r} = \int_S curl\ \bar{F}.\bar{n}ds$

$\oint_c \bar{F}.d\bar{r} = \oint_c \{(x^2 + y^2)\bar{\imath} - 2xy\bar{\jmath}\}.\{\bar{\imath}dx + \bar{\jmath}dy\}$

$= \oint_C (x^2 + y^2)\,dx - 2xydy$

$= \int_{AB} + \int_{Bc} + \int_{CD} + \int_{DA}$ …..(1)



(i) Along AB, x=a, dx=0

from (1), $\int_{AB} = \int_{y=0}^{b} -2ay\,dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$

(ii)Along BC, y=b, dy=0

from (1), $\int_{BC} = \int_{x=a}^{x=-a} (x^2+b^2)dx = \left[\frac{x^3}{3}+b^2x\right]_{x=a}^{-a} = \frac{-2a^3}{3} - 2ab^2$

(iii) Along CD, x=-a, dx=0

from (1), $\int_{CD} = \int_{y=b}^{0} 2aydy = \cancel{2}a\left[\frac{y^2}{\cancel{2}}\right]_{y=b}^{0} = -ab^2$

(iv)Along DA, y=0, dy=0

from (1), $\int_{DA} = \int_{x=-a}^{x=a} x^2dx = \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{2a^3}{3}$

(i)+(ii)+(iii)+(iv) gives

$\therefore \oint_c \bar{F}.d\bar{r} = -ab^2 - -\frac{-2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2$ ….(2)

Consider $\int_S curl\ \bar{F}.\bar{n}\,dS$

Vector Perpendicular to *the xy*-plane is $\bar{n} = k$

$$\therefore\ curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2+y^2) & -2xy & 0 \end{vmatrix} = 4y\bar{k}$$

Since the rectangle lies in the *xy* plane,

$\bar{n} = \bar{k}$ and *ds* =*dx dy*

$\int_S curl\ \bar{F}.\bar{n}dS = \int_S -4y\bar{k}.\bar{k}dx\,dy = \int_{x=-a}^{a}\int_{y=0}^{b} -4y\,dx\,dy$

$= \int_{y=0}^{b}\int_{x=-a}^{a} -4y\,dx\,dy = 4\int_{y=0}^{b} y[x]_{-a}^{a}\,dy = -4\int_{y=0}^{b} 2aydy$

$= -4a[y^2]_{y=0}^{b} = -4ab^2$ ...(3).Hence from (2) and (3),

Stoke's theorem verified.

7.Verify Stoke's theorem for $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the *cube x =0, y=0, z=0, x=2, y=2,z=2* above the *xy* plane.

Sol: Given $\bar{F} = (y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}$ where S is the surface of the cube.

x=0, y=0, z=0, x=2, y=2, z=2 above the xy plane.

By Stoke's theorem, we have $\int curl\ \bar{F}.\bar{n}ds = \int \bar{F}.d\bar{r}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - z + 2 & y + 4 & -xz \end{vmatrix} = \bar{\imath}(0 + y) - \bar{\jmath}(-z + 1) + \bar{k}(0 - 1) = y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}$$

$$\therefore\ \nabla \times \bar{F}.\bar{n} = \nabla \times \bar{F}.k = \left(y\bar{\imath} - (1 - z)\bar{\jmath} - \bar{k}\right).k = -1$$

$$\therefore \int \nabla \times \bar{F}.\bar{n}.ds = \int_0^2 \int_0^2 -1\ dx\ dy \quad (\because z = 0, dz = 0) = -4$$

…..(1)

To find $\int \bar{F}.d\bar{r}$

$$\int \bar{F}.d\bar{r} = \int \left((y - z + 2)\bar{\imath} + (yz + 4)\bar{\jmath} - xz\bar{k}\right).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k})$$

$$= \int[(y - z + 2)dx + (yz + 4)dy - (xz)dz]$$

Sis the surface of the cube above the *xy*-plane

$$\therefore z = 0 \Rightarrow dz = 0$$

$$\therefore \int \bar{F}.d\bar{r} = \int(y + 2)dx + \int 4dy$$

Along $\overline{OA}, y = 0, z = 0, dy = 0, dz = 0, x\ change\ from\ 0\ to\ 2.$

$\int_0^2 2dx = 2[x]_0^2 = 4$ ……..(2)

Along $\overline{BC}, y = 2, z = 0, dy = 0, dz = 0, x\ change\ from\ 2\ to\ 0.$

$\int_2^0 4dx = 4[x]_2^0 = -8$ ……. .(3)

Along $\overline{AB}, x = 2, z = 0, dx = 0, dz = 0, y\ change\ from\ 0\ to\ 2.$

$\int \bar{F}.d\bar{r} = \int_0^2 4dy = [4y]_0^2 = 8$ ……(4)

Along $\overline{CO}, x = 0, z = 0, dx = 0, dz = 0, y\ change\ from\ 2\ to\ 0.$

$.\int_2^0 4dy = -8$ …..(5)

Above the surface When $z=2$

Along $0'A',\ \int_0^2 \bar{F}\,\overline{dr} = 0$ ….(6)

Along $A'B', x = 2, z = 2, dx = 0, dz = 0,$ y changes from 0 to 2

$\int_0^2 \bar{F}.d\bar{r} = \int_0^2 (2y+4)dy = 2\left[\frac{y^2}{2}\right]_0^2 + 4[y]_0^2 = 4+8 = 12$ ….(7)

Along $B'C', y = 2, z = 2, dy = 0, dz = 0,$ x changes from 2 to 0

$\int_0^2 \bar{F}.\overline{dr} = 0$ ….(8)

Along $C'D', x = 0, z = 2, dx = 0, dz = 0,$ y changes from 2 to 0.

$\int_2^0 (2y+4) = 2\left[\frac{y^2}{2}\right]_2^0 + 4[y]_2^0 = -12$ …..(9)

(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9) gives

$\int_C \bar{F}.d\bar{r} = 4 - 8 + 8 - 8 + 0 + 12 + 0 - 12 = -4$ …..(10)

By Stokes theorem, We have

$\int \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}$ds=-4

Hence Stoke's theorem is verified.