COURSE MATERIAL

III Year B. Tech II- Semester

MECHANICAL ENGINEERING

FINITE ELEMENT METHODS

R17A0320

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MECHANICAL ENGINEERING

(Autonomous Institution-UGC, Govt. of India)
Secunderabad-500100,Telangana State, India.
www.mrcet.ac.in

CONTENTS

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

VISION

- To establish a pedestal for the integral innovation, team spirit, originality and competence in the students, expose them to face the global challenges and become technology leaders of Indian vision of modern society.

MISSION

- To become a model institution in the fields of Engineering, Technology and Management.
- To impart holistic education to the students to render them as industry ready engineers.
- To ensure synchronization of MRCET ideologies with challenging demands of International Pioneering Organizations.

QUALITY POLICY

- To implement best practices in Teaching and Learning process for both UG and PG courses meticulously.
- To provide state of art infrastructure and expertise to impart quality education.
- To groom the students to become intellectually creative and professionally competitive.
- To channelize the activities and tune them in heights of commitment and sincerity, the requisites to claim the never - ending ladder of **SUCCESS** year after year.

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Department of Mechanical Engineering

VISION

To become an innovative knowledge center in mechanical engineering through state-of-the-art teaching-learning and research practices, promoting creative thinking professionals.

MISSION

The Department of Mechanical Engineering is dedicated for transforming the students into highly competent Mechanical engineers to meet the needs of the industry, in a changing and challenging technical environment, by strongly focusing in the fundamentals of engineering sciences for achieving excellent results in their professional pursuits.

Quality Policy

- ✓ To pursuit global Standards of excellence in all our endeavors namely teaching, research and continuing education and to remain accountable in our core and support functions, through processes of self-evaluation and continuous improvement.
- ✓ To create a midst of excellence for imparting state of art education, industry-oriented training research in the field of technical education.

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Department of Mechanical Engineering

PROGRAM OUTCOMES

Engineering Graduates will be able to:

1. **Engineering knowledge:** Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and teamwork**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

12. **Life-long learning**: Recognize the need for and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSOs)

PSO1 Ability to analyze, design and develop Mechanical systems to solve the Engineering problems by integrating thermal, design and manufacturing Domains.

PSO2 Ability to succeed in competitive examinations or to pursue higher studies or research.

PSO3 Ability to apply the learned Mechanical Engineering knowledge for the Development of society and self.

Program Educational Objectives (PEOs)

The Program Educational Objectives of the program offered by the department are broadly listed below:

PEO1: PREPARATION

To provide sound foundation in mathematical, scientific and engineering fundamentals necessary to analyze, formulate and solve engineering problems.

PEO2: CORE COMPETANCE

To provide thorough knowledge in Mechanical Engineering subjects including theoretical knowledge and practical training for preparing physical models pertaining to Thermodynamics, Hydraulics, Heat and Mass Transfer, Dynamics of Machinery, Jet Propulsion, Automobile Engineering, Element Analysis, Production Technology, Mechatronics etc.

PEO3: INVENTION, INNOVATION AND CREATIVITY

To make the students to design, experiment, analyze, interpret in the core field with the help of other inter disciplinary concepts wherever applicable.

PEO4: CAREER DEVELOPMENT

To inculcate the habit of lifelong learning for career development through successful completion of advanced degrees, professional development courses, industrial training etc.

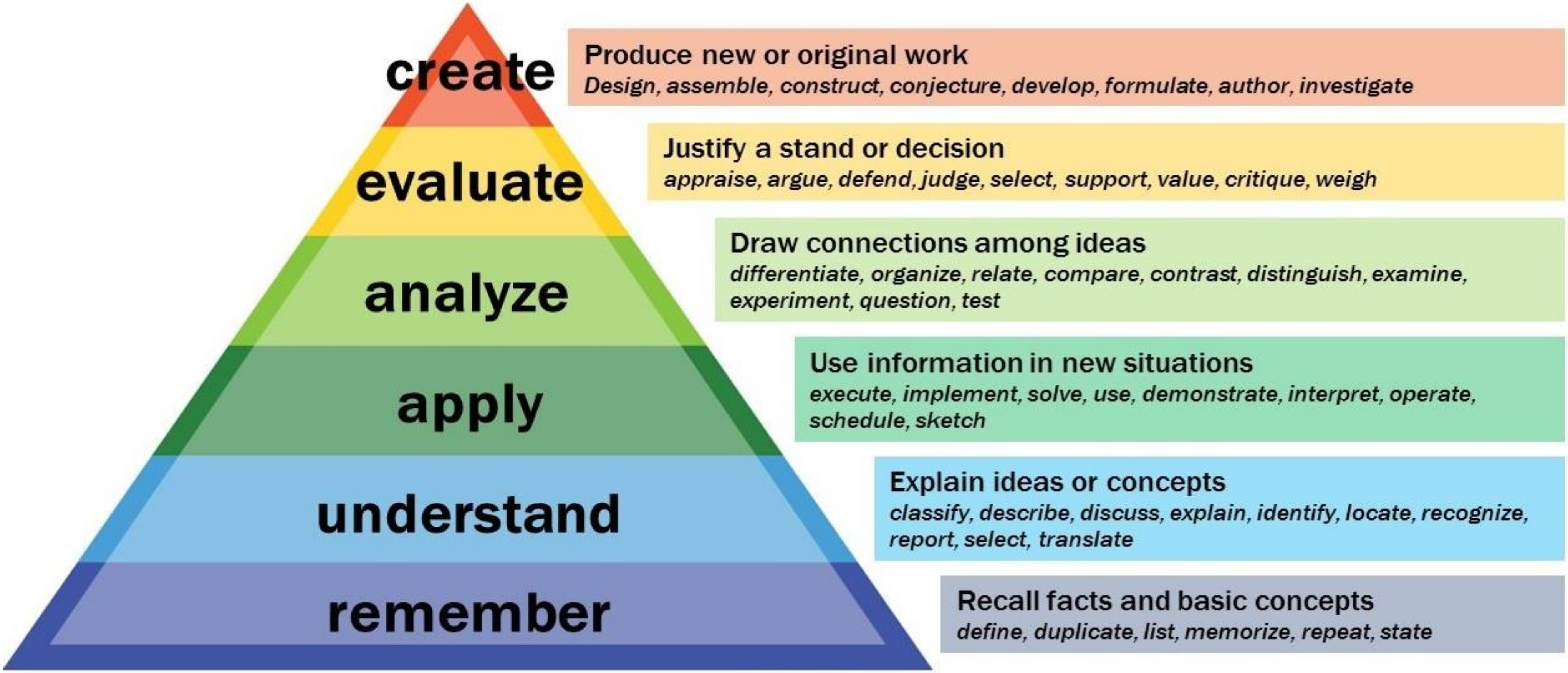
PEO5: PROFESSIONALISM

To impart technical knowledge, ethical values for professional development of the student to solve complex problems and to work in multi-disciplinary ambience, whose solutions lead to significant societal benefits.

Blooms Taxonomy

Bloom's Taxonomy is a classification of the different objectives and skills that educators set for their students (learning objectives). The terminology has been updated to include the following six levels of learning. These 6 levels can be used to structure the learning objectives, lessons, and assessments of a course.

1. **Remembering**: Retrieving, recognizing, and recalling relevant knowledge from long- term memory.
2. **Understanding**: Constructing meaning from oral, written, and graphic messages through interpreting, exemplifying, classifying, summarizing, inferring, comparing, and explaining.
3. **Applying**: Carrying out or using a procedure for executing or implementing.
4. **Analyzing**: Breaking material into constituent parts, determining how the parts relate to one another and to an overall structure or purpose through differentiating, organizing, and attributing.
5. **Evaluating**: Making judgments based on criteria and standard through checking and critiquing.
6. **Creating**: Putting elements together to form a coherent or functional whole; reorganizing elements into a new pattern or structure through generating, planning, or producing.



create
Produce new or original work
Design, assemble, construct, conjecture, develop, formulate, author, investigate
evaluate
Justify a stand or decision
appraise, argue, defend, judge, select, support, value, critique, weigh
analyze
Draw connections among ideas
differentiate, organize, relate, compare, contrast, distinguish, examine, experiment, question, test
apply
Use information in new situations
execute, implement, solve, use, demonstrate, interpret, operate, schedule, sketch
understand
Explain ideas or concepts
classify, describe, discuss, explain, identify, locate, recognize, report, select, translate
remember
Recall facts and basic concepts
define, duplicate, list, memorize, repeat, state

(R17A0320) FINITE ELEMENT METHODS

UNIT–I

Introduction to Finite Element Method for solving field problems. Stress and Equilibrium. Strain – Displacement relations. Stress – strain relations.
One Dimensional problem: Finite element modeling, local coordinates and shape functions. Potential Energy approach, Assembly of Global stiffness matrix and load vector. Finite element equations, Treatment of boundary conditions, Quadratic shape functions and its applications.

UNIT–II

Analysis of Trusses: Stiffness matrix for plane truss element, Stress calculations and problems.
Finite element modelling of two dimensional stress analyses with CST element and treatment of boundary conditions. Convergence requirements

UNIT–III

Finite element modeling of axi-symmetric solids subjected to axisymmetric loading with triangular elements. Two dimensional four node isoparametric elements and numerical integration.

UNIT–IV

Heat transfer analysis: One dimensional steady state analysis composite wall. One dimensional fin analysis and two dimensional analysis of thin plate.

BEAMS: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses.

UNIT-V

Dynamic Analysis: Formulation of finite element model, element matrices, evaluation of Eigen values and Eigen vectors for a stepped bar and a beam. Overview of commercial softwares like Ansys, Abaqus etc.

TEXT BOOKS:

1. Introduction to Finite Elements in Engineering / Chandraputla, Ashok and Belegundu /Prentice – Hall.
2. Introduction to Finite Element Analysis by S.Md.Jalaluddin, Anuradha Publishers.
3. The Finite Element Method for Engineers – Kenneth H. Huebner, Donald L. Dewhirst, Douglas E. Smith and Ted G. Byrom / John Wiley & sons (ASIA) Pte Ltd.

REFERENCE BOOKS:

1. An introduction to Finite Element Method / JN Reddy / Me Graw Hill
2. The Finite Element Methods in Engineering / SS Rao / Pergamon.
3. Finite Element Method and applications/R.D.Cook/WILEY publications

UNIT 1

BASIC CONCEPTS OF FEM

& 1-D STRUCTURAL ANALYSIS

Syllabus:

Introduction to Finite Element Method for solving field problems. Stress and Equilibrium. Strain – Displacement relations. Stress – strain relations. One Dimensional problem: Finite element modeling, local coordinates and shape functions. Potential Energy approach, Assembly of Global stiffness matrix and load vector. Finite element equations, Treatment of boundary conditions, Quadratic shape functions and its applications.

OBJECTIVE:

Enable the students to understand the fundamentals of FEA.

To learn the principles involved in the discretization of domains with various elements, polynomial and interpolation and assembly of global arrays.

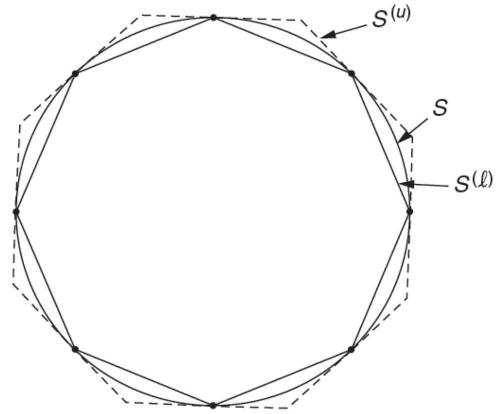
OUTCOME:

Identify the mathematical model to solve common engineering problems by applying FEM & formulate the elements for 1-D bar structures.

Unit-I

Historical Background

- Although the name of the finite element method was given recently, the concept dates back for several centuries. For example, ancient mathematicians found the circumference of a circle by approximating it by the perimeter of a polygon as shown in Figure 1.3.
- In terms of the present day notation, each side of the polygon can be called a "finite element." By considering the approximating polygon inscribed or circumscribed, one can obtain a lower bound $S^{(l)}$ or an upper bound $S^{(u)}$ for the true circumference S.



- Fig. 1.3 Lower and Upper Bounds to the Circumference of a Circle.

- Furthermore, as the number of sides of the polygon is increased, the approximate values converge to the true value. These characteristics, as will be seen later, will hold true in any general finite element application.
- To find the differential equation of a surface of minimum area bounded by a specified closed curve, Schellback discretized the surface into several triangles and used a finite difference expression to find the total discretized area in 1851.
- In the current finite element method, a differential equation is solved by replacing it by a set of algebraic equations. Since the early 1900s, the behavior of structural frameworks, composed of several bars arranged in a regular pattern, has been approximated by that of an isotropic elastic body.
- In 1943, Courant presented a method of determining the torsional rigidity of a hollow shaft by dividing the cross section into several triangles and using a linear variation of the stress function ϕ over each triangle in terms of the values of ϕ at net points (called nodes in the present day finite element terminology).
- This work is considered by some to be the origin of the present-day finite element method. Since mid-1950s, engineers in aircraft industry have worked on developing approximate methods for the prediction of stresses induced in aircraft wings.
- In 1956, Turner, Cough, Martin, and Topp presented a method for modeling the wing

skin using three-node triangles. At about the same time, Argyris and Kelsey presented several papers outlining matrix procedures, which contained some of the finite element ideas, for the solution of structural analysis problems. Reference is considered as one of the key contributions in the development of the finite element method.

- The name finite element was coined, for the first time, by Clough in 1960. Although the finite element method was originally developed mostly based on intuition and physical argument, the method was recognized as a form of the classical Rayleigh-Ritz method in the early 1960s.
- Once the mathematical basis of the method was recognized, the developments of new finite elements for different types of problems and the popularity of the method started to grow almost exponentially.
- The digital computer provided a rapid means of performing the many calculations involved in the finite element analysis and made the method practically viable. Along with the development of high-speed digital computers, the application of the finite element method also progressed at a very impressive rate.
- Zienkiewicz and Cheung presented the broad interpretation of the method and its applicability to any general field problem. The book by Przemienieckipresents the finite element method as applied to the solution of stress analysis problems.

Definition:

- In Finite Element Analysis, the structure or body is divided into finite numbers of elements, the solution is obtained for individual element and solution of all elements is assembled to give distribution of field variable over entire region.

- For example: Heat Analysis ⟶ Field variable is Temperature

  Stress & strain ⟶ Field variable is Displacement.

Engineering Applications of the Finite Element Method

The finite element method was developed originally for the analysis of aircraft structures.

I. Equilibrium problems or steady-state or time-independent problems
   - In an equilibrium problem, we need to find the steady-state displacement or stress distribution if it is a solid mechanics problem,
   - Temperature or heat flux distribution if it is a heat transfer problem and
   - Pressure or velocity distribution if it is a fluid mechanics problem.

II. Eigen-value problems
   - In eigenvalue problems also, time will not appear explicitly. They may be considered as extensions of equilibrium problems in which critical values of certain parameters are to be determined in addition to the corresponding steady-state configurations.
   - In these problems, we need to find the natural frequencies or buckling loads and mode shapes if it is a solid mechanics or structures problem.

– Stability of laminar flows if it is a fluid mechanics problem and
– Resonance characteristics if it is an electrical circuit problem.

III. Propagation or transient problems

– The propagation or transient problems are time-dependent problems. This type of problemarises, for example, whenever we are interested in finding the response of a body undertime-varying force in the area of a solid mechanics
– Under sudden heating or cooling inthe field of heat transfer.
– Crack propagation.

Engineering Applications of the Finite Element Method

Area of Study	Equilibrium Problems	Eigenvalue Problems	Propagation Problems
1.Civil engineering structures	Static analysis of trusses, frames, folded plates, shell roofs, shear walls, bridges, and prestressed concrete structures	Natural frequencies and modes of structures; stability of structures	Propagation of stress waves; response of structures to a periodic loads
2. Aircraft structures	Static analysis of aircraft wings, fuselages, fins, rockets, spacecraft, and missile structures	Natural frequencies, flutter, and stability of aircraft, rocket, spacecraft, and missile structures	Response of aircraft structures to Random loads; dynamic response of aircraft and spacecraft to a periodic loads
3.Heat conduction	Steady-state temperature distribution in solids and fluids	–	Transient heat flow in rocket nozzles, internal combustion engines, Turbine blades, fins, and building structures
4. Geomechanics	Analysis of excavations, retaining walls, undergo und openings, rock joints, and soil-structure interaction problem; stress analysis in soils, dams, layered piles, and machine foundations	Natural frequencies and modes of dam-reservoir systems and soil-structure interaction problems	Time-dependent soil-structure interaction problems; transient seepage in soils and rocks; stress wave propagation in soils and rocks
5.Hydraulic and water resources engineering; hydrodynamics	Analysis of potential flows, free surface flows, boundary layer flows, viscous flows, transonic aerodynamic problems; analysis of hydraulic structures and dams	Natural periods and modes of shallow basins, lakes, and harbors; sloshing of liquids in rigid and flexible containers	Analysis of unsteady fluid flow and wave propagation problems; transient seepage in aquifers and porous media; rarefied gas dynamics ;magnetohydrodynamic flows

6. Nuclear engineering	Analysis of nuclear pressure vessels and containment structures; steady – State Temperature distribution in reactor components	Natural frequencies and stability of containment structure s; neutron Flux distribution	Response of reactor containment structures to dynamic loads; unsteady temperature distribution in reactor components; thermal and viscoelastic analysis of reactor structures
7. Biomedical engineering	Stress analysis of eyeballs, bones, and teeth; load-bearing capacity of implant and prosthetic systems; mechanics of heart valves	–	Impact analysis of skull; dynamics of anatomical structures
8. Mechanical Design	Stress concentration problems; stress analysis of pressure vessels, pistons, Composite materials, linkages, and gears	Natural frequencies and stability of linkages, gears, and machine tools	Crack and fracture problems under dynamic loads
9. Electricalmachine s and electromagnetics	Steady-state analysis of synchronous and induction machines, eddy current, and core loss in electric machines, magnetostatics	–	Transient behavior of Electromechanical devices such as motors and actuators, magnetodynamics

3D Elasticity:

EXTERNAL FORCES ACTING ON THE BODY

Two basic types of external forces act on a body

Body force (force per unit volume) e.g., weight, inertia, etc

Surface traction (force per unit surface area) e.g., friction

Strains: 6 independent **strain components**

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

Consider the equilibrium of a differential volume element to obtain the 3 **equilibrium equations** of elasticity

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X_a = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + X_b = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + X_c = 0$$

Compactly;

EQUILIBRIUM EQUATIONS	$\underline{\partial}^T \underline{\sigma} + \underline{X} = \underline{0}$	(1)

where

$$\underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

3D elasticity problem is completely defined once we understand the following three concepts

Strong formulation (governing differential equation + boundary conditions)

Strain-displacement relationship

Stress-strain relationship

Equilibrium equations	$\underline{\partial}^T \underline{\sigma} + \underline{X} = \underline{0} \quad in \;\; V$	(1)

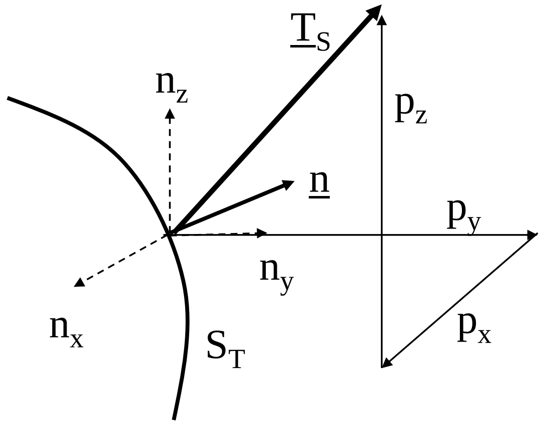
Boundary conditions

1. Displacement boundary conditions: Displacements are specified on portion S_u of the boundary

$$\underline{u} = \underline{u}^{specified} \quad on \;\; S_u$$

2. Traction (force) boundary conditions: **Tractions** are specified on portion S_T of the boundary

Now, how do I express this mathematically?



<u>Traction:</u> Distributed force **per unit area**

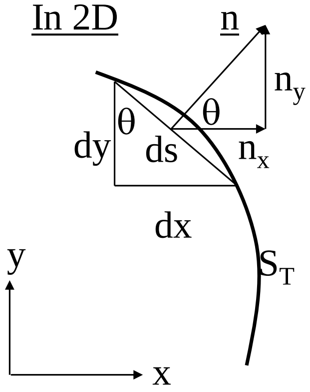
$$\underline{T}_S = \begin{Bmatrix} p_x \\ p_y \\ p_z \end{Bmatrix}$$

If the unit outward normal to S_T : $\underline{n} = \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix}$

Then

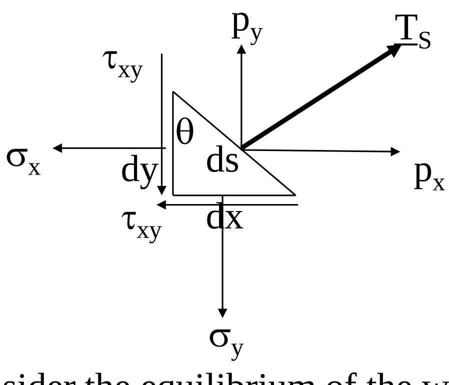
$$\begin{aligned} p_x &= \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z \\ p_y &= \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z \\ p_z &= \tau_{xz} n_x + \tau_{zy} n_y + \sigma_z n_z \end{aligned}$$

<u>In 2D</u>



$$\sin\theta = \frac{dx}{ds} = n_y$$

$$\cos\theta = \frac{dy}{ds} = n_x$$



Consider the equilibrium of the wedge in x-direction

$$p_x ds = \sigma_x dy + \tau_{xy} dx$$

$$\Rightarrow p_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$

$$\Rightarrow p_x = \sigma_x n_x + \tau_{xy} n_y$$

Similarly

$$p_y = \tau_{xy} n_x + \sigma_y n_y$$

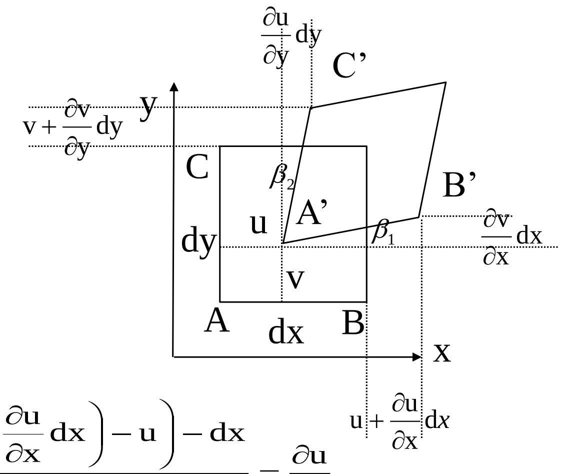
2. Strain-displacement relationships:

$$\varepsilon_x = \frac{\partial u}{\partial x}$$
$$\varepsilon_y = \frac{\partial v}{\partial y}$$
$$\varepsilon_z = \frac{\partial w}{\partial z}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$
$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

Compactly; $\boxed{\underline{\varepsilon} = \underline{\partial}\,\underline{u}}$ (2)

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \underline{\partial} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \qquad \underline{\mathrm{u}} = \begin{Bmatrix} \mathrm{u} \\ \mathrm{v} \\ \mathrm{w} \end{Bmatrix}$$

In 2D



$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\left(dx + \left(u + \frac{\partial u}{\partial x}dx\right) - u\right) - dx}{dx} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{A'C' - AC}{AC} = \frac{\left(dy + \left(v + \frac{\partial v}{\partial y}dy\right) - v\right) - dy}{dy} = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\pi}{2} - \text{angle } (C'A'B') = \beta_1 + \beta_2 \approx \tan\beta_1 + \tan\beta_2$$

$$\approx \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}$$

3. Stress-Strain relationship:

Linear elastic material (Hooke’s Law)

$$\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}} \qquad (3)$$

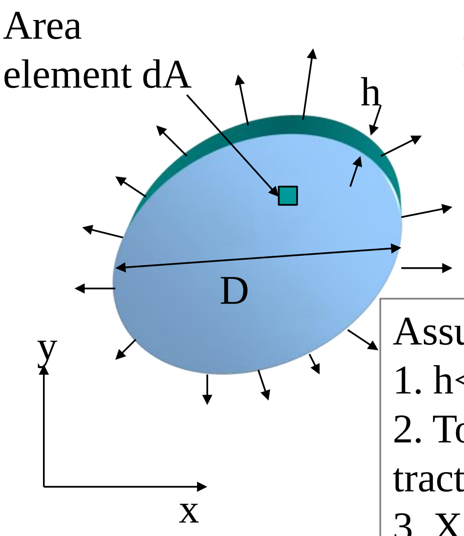
Linear elastic isotropic material

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

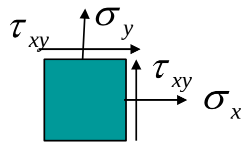
Special cases:

1. **1D elastic bar** (only 1 component of the stress (stress) is nonzero. All other stress (strain) components are zero)
Recall the (1) equilibrium, (2) strain-displacement and (3) stress-strain laws
2. **2D elastic problems:** 2 situations
   PLANE STRESS
   PLANE STRAIN
3. **3D elastic problem**: special case-**axisymmetric body with axisymmetric loading (we will skip this)**

PLANE STRESS: Only the in-plane stress components are nonzero



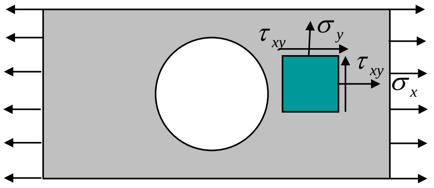
Nonzero stress components $\sigma_x, \sigma_y, \tau_{xy}$



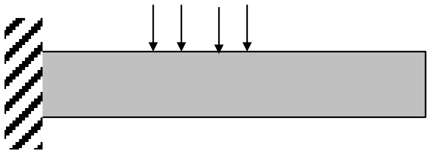
Assumptions:
1. h<<D
2. Top and bottom surfaces are free from traction
3. X_c=0 and p_z=0

PLANE STRESS Examples:

1. Thin plate with a hole



2. Thin cantilever plate



PLANE STRESS

Nonzero **stresses**: $\sigma_x, \sigma_y, \tau_{xy}$

Nonzero **strains**: $\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}$

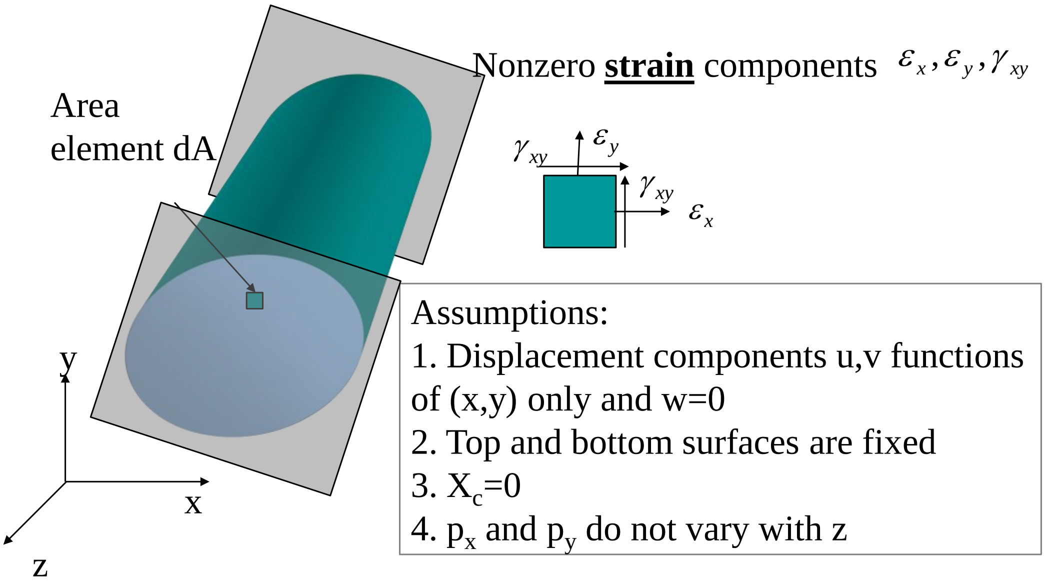
Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \varepsilon_z = -\frac{\nu}{1-\nu}\left(\varepsilon_x + \varepsilon_y\right)$$

Hence, the $\underline{D}$ matrix for the **plane stress case** is

$$\underline{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

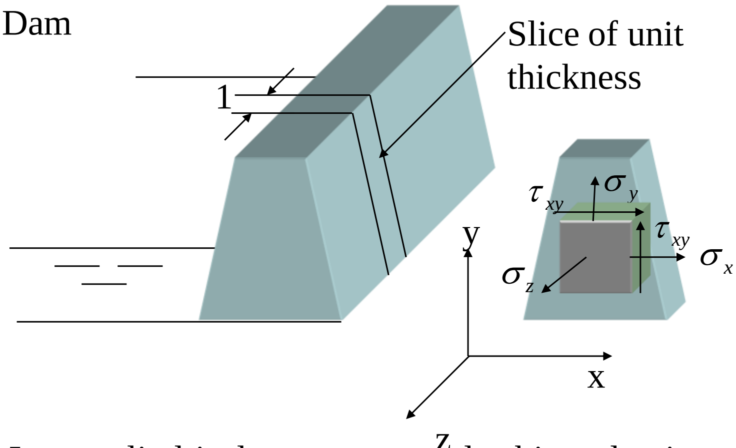
PLANE STRAIN: Only the in-plane **strain** components are nonzero



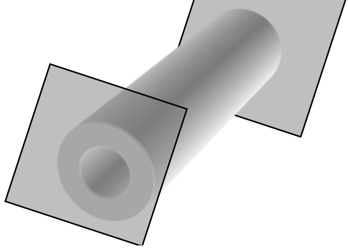
Assumptions:
1. Displacement components u,v functions of (x,y) only and w=0
2. Top and bottom surfaces are fixed
3. X_c=0
4. p_x and p_y do not vary with z

PLANE STRAIN Examples:

1. Dam



2. Long cylindrical pressure vessel subjected to internal/external pressure and constrained at the ends



PLANE STRAIN

Nonzero **stress**: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}$

Nonzero **strain** components: $\varepsilon_x, \varepsilon_y, \gamma_{xy}$

Isotropic linear elastic stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \qquad \sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

Hence, the D matrix for the **plane strain case** is

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Principle of Minimum Potential Energy

Definition: For a linear elastic body subjected to body forces $\underline{X}=[X_a,X_b,X_c]^T$ and surface tractions $\underline{T}_S=[p_x,p_y,p_z]^T$, causing displacements $u=[u,v,w]^T$ and strains $\underline{\varepsilon}$ and stresses $\underline{\sigma}$, the **potential energy** Π is defined as the strain energy minus the potential energy of the loads involving $\underline{X}$ and $\underline{T}_S$

$$\Pi = U - W$$

Strain energy of the elastic body

Using the stress-strain law $\boxed{\underline{\sigma} = \underline{D}\,\underline{\varepsilon}}$

$$U = \frac{1}{2}\int_V \underline{\sigma}^T \underline{\varepsilon}\, dV = \frac{1}{2}\int_V \underline{\varepsilon}^T \underline{D}\,\underline{\varepsilon}\, dV$$

In 1D

$$U = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_V E\varepsilon^2\, dV = \frac{1}{2}\int_{x=0}^{L} E\varepsilon^2\, Adx$$

In 2D **plane stress** and **plane strain**

$$U = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right) dV$$

Why?

Principle of minimum potential energy: Among all admissible displacement fields the one that satisfies the equilibrium equations also render the potential energy P a minimum.“admissible displacement field”:

1. first derivative of the displacement components exist

2. satisfies the boundary conditions on S_u

General procedure of Finite Element Method

Step 1: Divide structure into discrete elements (discretization).

- Divide the structure or solution region into subdivisions or elements. Hence, the structure is to be modeled with suitable finite elements.
- The number, type, size, and arrangement of the elements are to be decided.

Step 2: Select a proper interpolation or displacement model.

- Since the displacement solution of a complex structure under any specified loadconditions cannot be predicted exactly, we assume some suitable solution within anelement to approximate the unknown solution. The assumed solution must besimple from a computational standpoint, but it should satisfy certain convergencerequirements.
- In general, the solution or the interpolation model is taken in the formof a polynomial.

Step 3: Derive element stiffness matrices and load vectors.

- From the assumed displacement model finding,

Stiffness matrix – [K]e

Load vector. – [P]e

Step 4: Assemble element equations to obtain the overall equilibrium equations.

– Since the structure is composed of several finite elements, the individual element stiffness matrices and load vectors are to be assembled in a suitable manner and the overall equilibrium equations have to be formulated as

$$[K]\ [\phi] = [P]$$

Where [K] = the assembled stiffness matrix

[ϕ] = the vector of nodal displacements

[P] = Load vector

Step 5: Solve for the unknown nodal displacements.

The overall equilibrium equations have to be modified to account for the boundaryconditions of the problem. After the incorporation of the boundary conditions, theequilibrium equations can be expressed as

$$[K]\ [\phi] = [P]$$

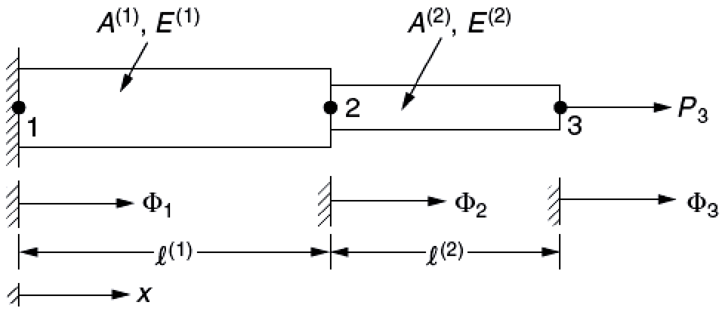
Step 6: Compute element strains and stresses.

From the known nodal displacementsϕ, if required, the element strains and stressescan be computed by using the necessary equations of solid or structural mechanics.

.

Example 1.1: $A_1 = 200\text{ mm}^2$, $A_2 = 100\text{ mm}^2$, $P_3 = 1000$ N.

$E_1 = E_2 = E = 2 \times 10^6\text{ N/mm}^2$
$l_1 = l_2 = 100$ mm
Find: Displacement and stress & strain.



$$[K^{(1)}] = \frac{A^{(1)}E^{(1)}}{l^{(1)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{matrix} \Phi_1 & \Phi_2 \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{matrix} \begin{matrix} \Phi_1 \\ \Phi_2 \end{matrix} \qquad \text{(E.16)}$$

$$[K^{(2)}] = \frac{A^{(2)}E^{(2)}}{l^{(2)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{matrix} \Phi_2 & \Phi_3 \\ \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{matrix} \begin{matrix} \Phi_2 \\ \Phi_3 \end{matrix} \qquad \text{(E.17)}$$

Let overall stiffness matrix $[K] = [K^{(1)}] + [K^{(2)}]$

$$[\underset{\sim}{K}] = 10^6 \begin{matrix} \Phi_1 & \Phi_2 & \Phi_3 \\ \begin{bmatrix} 4 & -4 & 0 \\ -4 & 4+2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{matrix} \begin{matrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{matrix} = 2 \times 10^6 \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

In the present case, external loads act only at the node points; as such, there is no need to assemblethe element load vectors. The overall or global load vector can be written as

$$\vec{P} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}$$

$$2 \times 10^6 = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ 0 \\ 1 \end{Bmatrix}$$

$$2 \times 10^6 \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \Phi_2 \\ \Phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

By solving the matrix
$\Phi_2 = 0.25 \times 10^{-6}$cm and
$\Phi_3 = 0.75 \times 10^{-6}$cm

Derive element strains and stresses.

Once the displacements are computed, the strains in the elements can be found as

$$\varepsilon^{(1)} = \frac{\partial \phi}{\partial x} \text{ for element 1} = \frac{\Phi_2^{(1)} - \Phi_1^{(1)}}{l^{(1)}} \equiv \frac{\Phi_2 - \Phi_1}{l^{(1)}} = 0.25 \times 10^{-7}$$

$$\varepsilon^{(2)} = \frac{\partial \phi}{\partial x} \text{ for element 2} = \frac{\Phi_2^{(2)} - \Phi_1^{(2)}}{l^{(2)}} \equiv \frac{\Phi_3 - \Phi_2}{l^{(2)}} = 0.50 \times 10^{-7}$$

The stresses in the elements are given by

$$\sigma^{(1)} = E^{(1)}\varepsilon^{(1)} = (2\times10^7)(0.25\times10^{-7}) = 0.5 \text{ N/cm}^2$$

$$\sigma^{(2)} = E^{(2)}\varepsilon^{(2)} = (2\times10^7)(0.50\times10^{-7}) = 1.0 \text{ N/cm}^2$$

Example 1.2:A thin plate as shown in Fig. 1.5(a) has uniform thickness of 2 cm and its modulus of elasticity is 200 x 10^3 N/mm^2 and density 7800 kg/m^3. In addition to its self weight the plate is subjected to a point load P of500 N is applied at its midpoint.

Solve the following:

(i) Finite element model with two finite elements.

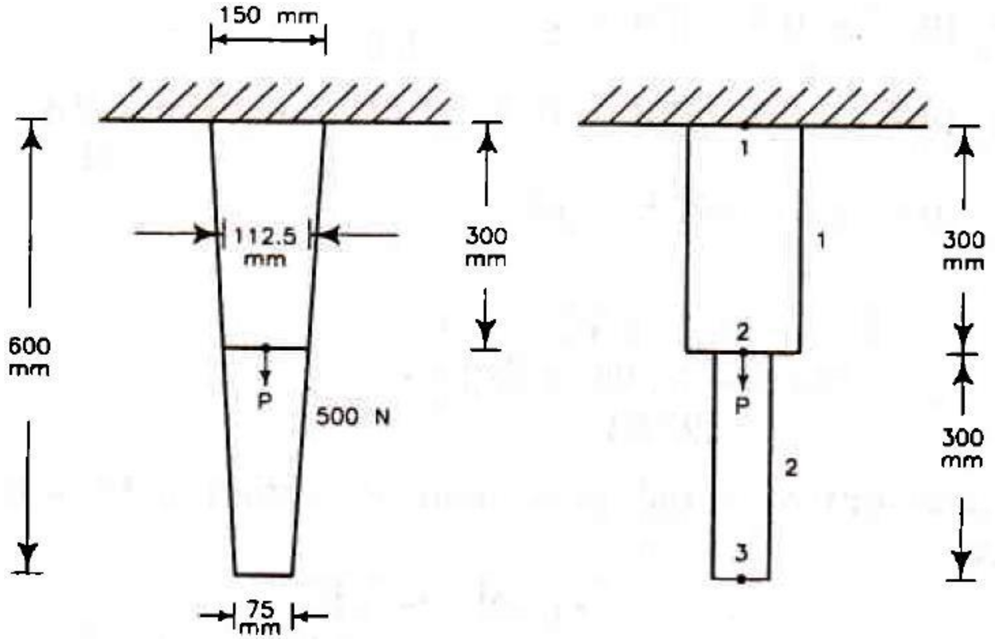
(ii) Global stiffness matrix.

(iii) Global load matrix.

(iv) Displacement at nodal point.

(v) Stresses in each element.

(vi) Reaction at support.



(a) Fig. 1.5 (b)

(i) The tapered plate can be idealized as two element model with the tapered area converted to the rectangular equivalent area Refer Fig. (b). The areas A_1 and A_2

are equivalent areas calculated as

$$A_1 = \frac{15 + 11.25}{2} \times 2 = 26.25\ cm^2$$

$$A_2 = \frac{11.25 + 7.5}{2} \times 2 = 18.75\ cm^2$$

(ii) Global stiffness matrix can be obtained as

$$[k] = \frac{EA_1}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{EA_2}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \frac{200 \times 10^3 \times 26.25 \times 10^2}{300}\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{200 \times 10^3 \times 18.75 \times 10^2}{300}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= 0.175 \times 10^7 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0.125 \times 10^7 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= 10^7 \begin{bmatrix} 0.175 & -0.175 & 0 \\ -0.175 & 0.3 & -0.125 \\ 0 & -0.125 & 0.125 \end{bmatrix}$$

(iii) The load matrix given by

$$F = \rho \begin{bmatrix} \frac{A_1 L_1}{2} \\ \frac{A_1 L_1}{2} + \frac{A_2 L_2}{2} \\ \frac{A_2 L_2}{2} \end{bmatrix} + \begin{bmatrix} -R_1 \\ P \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{26.25 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} - R_1 \\ \frac{26.25 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} + \frac{18.75 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} + P \\ \frac{18.75 \times 10^{-4} \times 0.3 \times 7.8 \times 10^4}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 30.75 - R_1 \\ 30.75 + 21.93 + 500 \\ 21.93 \end{bmatrix}$$

(iv) The displacement at nodal point can be obtained by writing the equation in global form as

$$[k]\ [\delta] = [F]$$

$$10^7 \begin{bmatrix} 0.175 & -0.175 & 0 \\ -0.175 & 0.3 & -0.125 \\ 0 & -0.125 & 0.125 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 30.75 - R_1 \\ 552.68 \\ 21.93 \end{bmatrix}.$$

Using elimination approach and eleminating first row and column in which reaction occurs.

$$10^7 \begin{bmatrix} 0.3 & -0.125 \\ -0.125 & 0.125 \end{bmatrix} \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \begin{bmatrix} 552.68 \\ 21.93 \end{bmatrix}$$

$$\delta_1 = 0, \quad \delta_2 = 3.28 \times 10^{-4} \text{ mm}, \ \delta_3 = 3.45 \times 10^{-4} \text{ mm}.$$

(v) The stress in the element 1

$$6_1 = \frac{E}{L_1} [-1,\ 1] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \frac{200 \times 10^3}{300} \times 3.28 \times 10^{-4}$$

$$= 2.18 \times 10^{-1} \text{ MPa}$$

stress in the element 2

$$6_2 = \frac{E}{L_2} [-1,\ 1] \begin{bmatrix} \delta_2 \\ \delta_3 \end{bmatrix} = \frac{200 \times 10^3}{300} [-\delta_2 + \delta_3] = 0.11 \times 10^{-1} \text{ MPa}$$

(vi) The reaction node 1

$$R_1 = \frac{E A_1}{L_1} [\delta_2 - 30.75]$$

$$= 0.175 \times 10^7 \times 3.28 \times 10^{-4} - 30.75 = 543.25.$$

PenaltyApproach

- In the preceding problems, the elimination approach was used to achieve simplified matrices. This method though simple, is not very easy to adapt in terms of algorithms written fix computer programs.
- An alternate method to achieve solutions is by the penalty approach. By this approach a rigid support is considered as a spring having infinite stiffness. Consider a system as shown in Fig. 1.7.

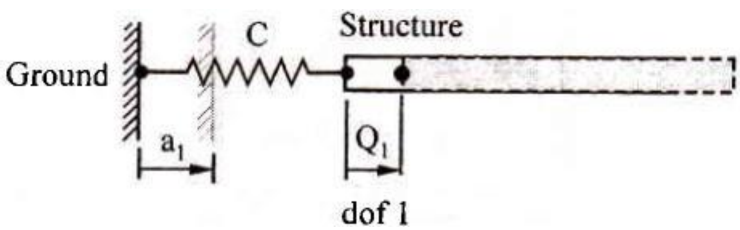


Fig. 1.6 Penalty Approach

- The support or the ground is modelled with a high stiffness spring, having a stiffness C. To represent a rigid ground, c must be infinity.
- However, instead of introducing an infinite value in the calculations, a substantially high value of stiffness constant is introduced for those nodes resting on rigid supports.

- The magnitude of the stiffness constant should be at least 10^4 times more than the maximum value in the global stiffness matrix.
- From Fig. 1.6, it is seen that one end of the spring will displace by a_1. The displacement Q_1 (for dof l) will be approximately equal to a_1 as the spring has a high stiffness.
- Consider a simple 1D element with node 1 fixed.

$$\mathbf{KQ} = \mathbf{F}$$

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

- At node l, the stiffness term is „C" is introduced to reflect the boundary condition related to a rigid support. To compensate this change, the force term will also be modified as:

$$\begin{bmatrix} k_{11}+C & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 + Ca_1 \\ F_2 \end{Bmatrix}$$

- The reaction force as per penalty approach would be found by multiplying the added stiffness with the net deflection of the node.

$$R = - C(Q-a)$$

- The penalty approach is an approximate method and the accuracy of the forces depends on the value of C.

Example 1.3:Consider the bar shown in Fig.1.7. An axial load $P = 200 \times 10^3$ N is applied as shown.Using the penalty approach for handling boundary conditions, do the following:

(a) Determine the nodal displacements

(b) Determine the stress in each material.

(c) Determine the reaction forces.

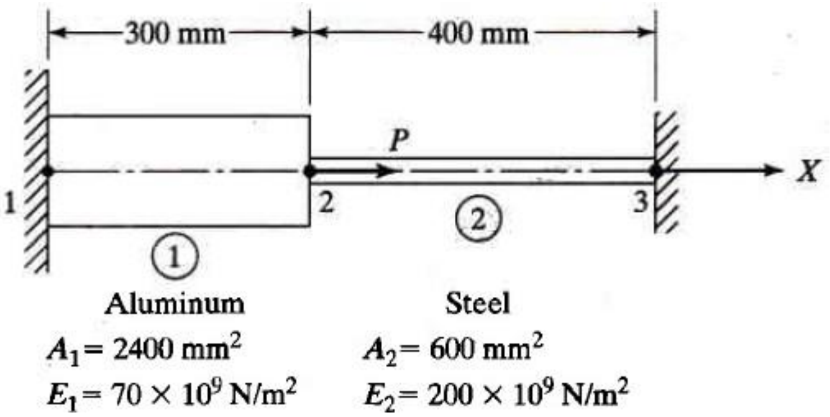


Fig. 1.7

(a) The element stiffness matrices are

$$\mathbf{k}^1 = \frac{70 \times 10^3 \times 2400}{300} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{matrix} \\ \end{matrix}$$

(Columns: 1, 2 ← Global dof)

and

$$\mathbf{k}^2 = \frac{200 \times 10^3 \times 600}{400} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(Columns: 2, 3)

The structural stiffness matrix that is assembled from k^1 and k^2 is

(Columns: 1, 2, 3)

$$\mathbf{K} = 10^6 \begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 0.30 \end{bmatrix}$$

The global load vector is

$$F = [0, 200 \times 10^3, 0]^T$$

Now dofs 1 and 3 are fixed. When using the penalty approach, therefore, a large numberC is added to thefirst and third diagonal elements of K. Choosing C

$$C = [0.86 \times 10^6] \times 10^4$$

Thus, the modified stiffness matrix is

$$\mathbf{K} = 10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}$$

The finite element equations are given by

$$10^6 \begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{Bmatrix}$$

which yields the solution

$$Q = [15.1432 \times 10^{-6}, 0.23257, 8.1127 \times 10^{-6}]\text{mm}$$

(b) The element stresses are

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{300}[-1 \quad 1] \begin{Bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{Bmatrix}$$

$$= 54.27 \text{ MPa}$$

where 1 MPa = 10^6 N/m^2= 1N/mm^2. Also,

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{400}[-1 \quad 1] \begin{Bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{Bmatrix}$$

$$= -116.29 \text{ MPa}$$

(c) The reaction forces are

$$R_1 = -CQ_1$$

$$= -[0.86 \times 10^{10}] \times 15.1432 \times 10^6$$

$$= -130.23 \times 10^3 \text{ N}$$

$$R_3 = -CQ_3$$

$$= -[0.86 \times 10^{10}] \times 8.1127 \times 10^6$$

$$= -69.77 \times 10^3 \text{ N}$$

Example 1.4:In Fig. 1.8(a), a load P = 60 x 10^3 N is applied as shown. Determine the displacement field,stress and support reactions in the body. Take E = 20 x 10^3N/mm^2.

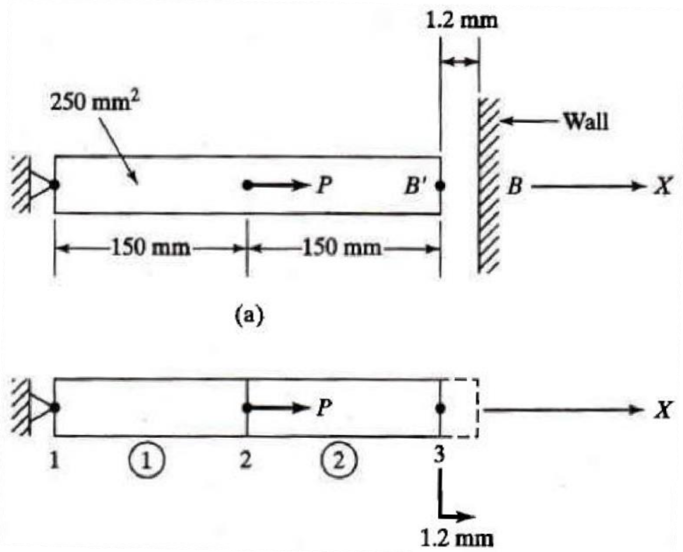


Fig. 1.8

The boundary conditions are Q_1= 0 and Q_3= 1.2 mm. The structural stiffness matrix K is

$$\mathbf{K} = \frac{20 \times 10^3 \times 250}{150}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

andthe global load vector F is

$$F = [0, 60 \times 10^3, 0]^T$$

In the penalty approach, the boundary conditions Q_1= 0 and Q_3=1.2imply the followingmodifications: A large number C chosen here as C = (2/3) x 10^{10}, is added on to the 1st and 3rd diagonal elements of K. Also, the number (C x 1.2) gets added on to the 3rd component of F. Thus, the modified equations are

The solution is

$$\frac{10^5}{3}\begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60.0 \times 10^3 \\ 80.0 \times 10^7 \end{Bmatrix}$$

$$Q = [7.49985 \times 10^{-5}, 1.500045, 1.200015]^T \text{mm}$$

The element stresses are

$$\sigma_1 = 200 \times 10^3 \times \frac{1}{150}[-1 \quad 1]\begin{Bmatrix} 7.49985 \times 10^{-5} \\ 1.500045 \end{Bmatrix}$$

$$= 199.996 \text{ MPa}$$

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{150}[-1 \quad 1]\begin{Bmatrix} 1.500045 \\ 1.200015 \end{Bmatrix}$$

$$= -40.004 \text{ MPa}$$

The reaction forces are

$$R_1 = -C \text{x } 7.49985 \text{ x } 10^{-5}$$

$$= -49.999 \text{ x } 10^3 \text{N}$$

$$R_3 = -C\text{x}(1.200015 - 1.2)$$

$=-10.001 \times 10^3 N$

Effect of Temperature on Elements:

When any material is subjected to a thermal stress, the thermal load is additional load acting on every element. This load can be calculated by using thermal expansion of the material due to the rise in. temperature.

Thermal stress in material can be given by

$\zeta_t = E\varepsilon_t$

Where ε_t= thermal strain

E = modulus of elasticity

$\varepsilon_t = \alpha\, \Delta t$

α= coefficient of linear expansion of material

Δt= change in temperature of material.

Then the thermal load is given by

Where, $Ft = \zeta_t A = AE\alpha\, \Delta t$

A = Area of the bar.

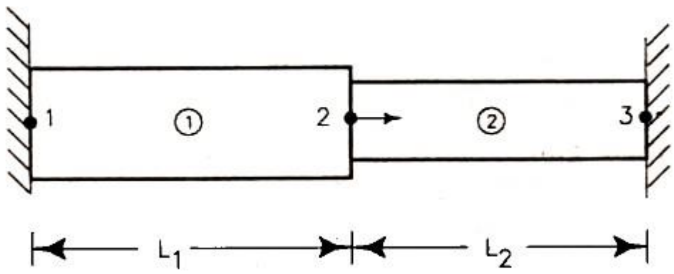


Fig. 1.9

Consider the horizontal step bar supported at two ends is subjected to athermal stress and load P at node 2 as shown in Fig. 5.9.

Thermal load in element 1

$$[F_1] = \begin{bmatrix} Ft_1 \\ Ft_{12} \\ 0 \end{bmatrix} = A_1\ E\ \alpha\ \Delta t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Thermal load in element 2

$$[F_2] = \begin{bmatrix} 0 \\ Ft_{21} \\ Ft_3 \end{bmatrix} = A_2\ E\ \alpha\ \Delta t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$[F] = [F_1] + [F_2] + \begin{bmatrix} 0 \\ P \\ 0 \end{bmatrix} = \begin{bmatrix} -A_1E\alpha\Delta t \\ A_1E\alpha\Delta t - A_2E\alpha\Delta t + P \\ A_2E\alpha\Delta t \end{bmatrix}$$

Example 1.5 : An axial load $P = 300 \times 10^3$ N is applied at 20°C to the rod as shown in Fig. 5.10.The temperature is then raised to 60°C.

(a) Assemble the K and F matrices.

(b) Determine the nodal displacements and element stresses.

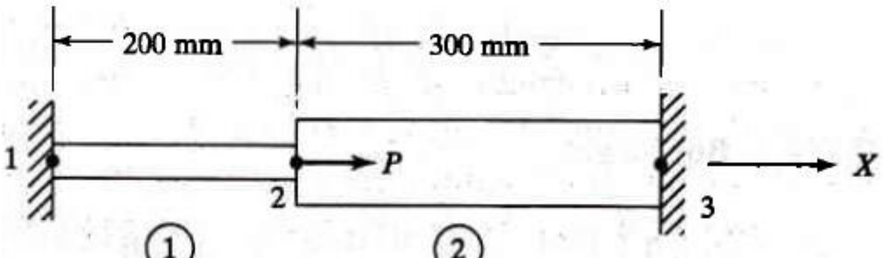


Fig. 1.10

(a) The element stiffness matrices are

$$\mathbf{k}^1 = \frac{70 \times 10^3 \times 900}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

$$\mathbf{k}^2 = \frac{200 \times 10^3 \times 1200}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{N/mm}$$

$$\mathbf{K} = 10^3\begin{bmatrix} 315 & -315 & 0 \\ -315 & 1115 & -800 \\ 0 & -800 & 800 \end{bmatrix} \text{N/mm}$$

Now, in assembling F, both temperature and point load effects have to be considered.

The element temperature forces due to ΔT = 40°C are obtained as

$$\boldsymbol{\Theta}^1 = 70 \times 10^3 \times 900 \times 23 \times 10^{-6} \times 40\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\begin{matrix} 1 \\ 2 \end{matrix} \quad \text{N} \qquad \downarrow \text{Global dof}$$

$$\boldsymbol{\Theta}^2 = 200 \times 10^3 \times 1200 \times 11.7 \times 10^{-6} \times 40\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\begin{matrix} 2 \\ 3 \end{matrix} \quad \text{N}$$

Upon assembling $\boldsymbol{\Theta}^1$, $\boldsymbol{\Theta}^2$, and the point load, we get

$$\mathbf{F} = 10^3\begin{Bmatrix} -57.96 \\ 57.96 - 112.32 + 300 \\ 112.32 \end{Bmatrix}$$

$$F = 10^3[-57.96, 245.64, 112.32]^T \text{N}$$

(b) The elimination approach will now be used to solve for the displacements. Since dofs 1 and 3 are fixed, the first and third rows and columns of K, together with the first and third components of F, are deleted. This results in the scalar equation

$10^3[1115]\ Q_2 = 10^3 \times 245.64$

$Q_2 = 0.220$ mm

$Q = [0, 0.220, 0]^T$mm

In evaluating element stresses

$$\sigma_1 = \frac{70 \times 10^3}{200}[-1 \quad 1]\begin{Bmatrix} 0 \\ 0.220 \end{Bmatrix} - 70 \times 10^3 \times 23 \times 10^{-6} \times 40$$

$$= 12.60 \text{ MPa}$$

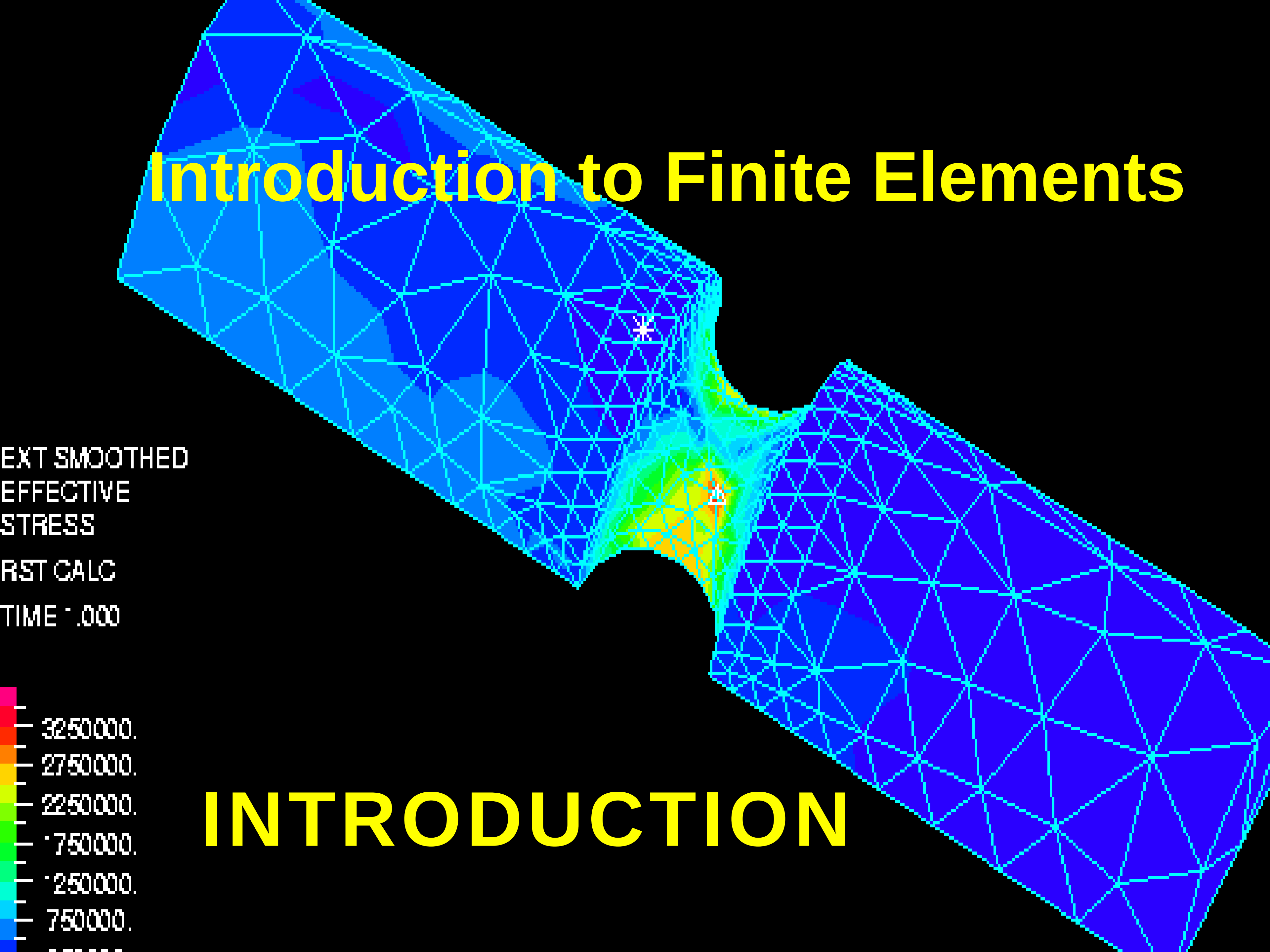
$$\sigma_2 = \frac{200 \times 10^3}{300}[-1 \quad 1]\begin{Bmatrix} 0.220 \\ 0 \end{Bmatrix} - 200 \times 10^3 \times 11.7 \times 10^{-6} \times 40$$

$$= -240.27 \text{ MPa}$$

INTRODUCTION
TO
FINITE ELEMENTS

DEPARTMENT OF MECHANICAL ENGINEERING

Introduction to Finite Elements



INTRODUCTION

Info

Course Instructor:

Professor

email:

room:

Tel:

Office hours:

Course website:

Course texts and references

Course text :
Title: Introduction to Finite Elements in Engineering
Author: Tirupathi R Chandrupatla & Ashok D Belegondu
Edition: Fourth
Publisher: Pearson

Relevant reference:

Finite Element Procedures, K. J. Bathe, Prentice Hall
A First Course in Finite Elements, J. Fish and T. Belytschko

Collaboration / academic integrity

1. Students are encouraged to collaborate in the solution of HW problems, but submit independent solutions that are NOT copies of each other.
   Funny solutions (that appear similar/same) will be given zero credit.
   Softwares may be used to verify the HW solutions. But submission of software solution will result in zero credit.

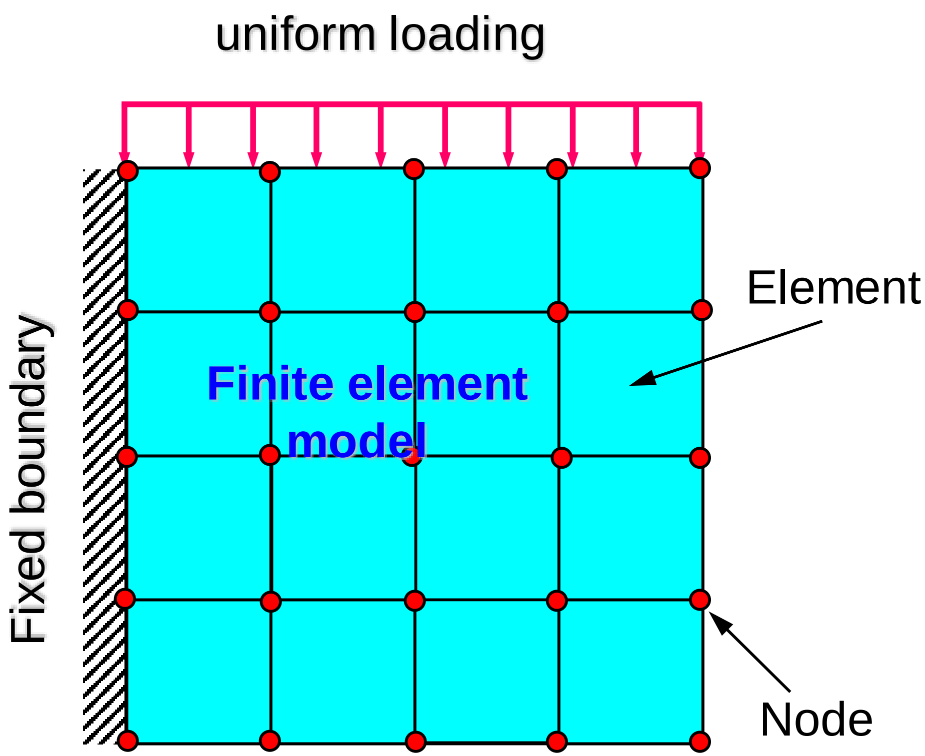
2. Groups of 4 for the projects
(no two projects to be the same/similar)
A single grade will be assigned to the group and not to the individuals.

Major project

Project examples:

1. Analysis of a rocker arm
2. Analysis of a bicycle crank-pedal assembly
3. Design and analysis of a "portable stair climber"
4. Analysis of a gear train
5. Gear tooth stress in a wind- up clock
6. Analysis of a gear box assembly
7. Analysis of an artificial knee
8. Forces acting on the elbow joint
9. Analysis of a soft tissue tumor system
10. Finite element analysis of a skateboard truck

Finite Element Analysis



Problem: Obtain the stresses/strains in the plate

- Approximate method
- Geometric model
- Node
- Element
- Mesh
- Discretization

Course content

1. “Direct Stiffness” approach for springs
2. Bar elements and truss analysis
3. Introduction to boundary value problems: strong form, principle of minimum potential energy and principle of virtual work.
4. Displacement-based finite element formulation in 1D: formation of stiffness matrix and load vector, numerical integration.
5. Displacement-based finite element formulation in 2D: formation of stiffness matrix and load vector for CST and quadrilateral elements.
6. Discussion on issues in practical FEM modeling
7. Convergence of finite element results
8. Higher order elements
9. Isoparametric formulation
10. Numerical integration in 2D
11. Solution of linear algebraic equations
12. FE analysis of Heat Transfer Problems like 1-D, Fin and 2-D problems
13. Finite Element analysis of Dynamic Problems(Eigen values and eigen Vectors)

Linear Algebra Recap

What is a matrix?

A rectangular array of numbers (we will concentrate on real numbers). A nxm matrix has 'n' rows and 'm' columns

$$M_{3x4} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \end{bmatrix}$$

First row → $M_{11}\ M_{12}\ M_{13}\ M_{14}$

Second row → $M_{21}\ M_{22}\ M_{23}\ M_{24}$

Third row → $M_{31}\ M_{32}\ M_{33}\ M_{34}$

First column, Second column, Third column, Fourth column

M_{12}: 1 = Row number, 2 = Column number

What is a vector?

A vector is an array of ‘n’ numbers

A **row vector of length ‘n’ is a 1xn matrix**

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}$$

A **column vector of length ‘m’ is a mx1 matrix**

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Special matrices

Zero matrix: **A matrix all of whose entries are zero**

$$\underline{0}_{3x4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Identity matrix: **A square matrix which has '1' s on the diagonal and zeros everywhere else.**

$$\underline{\mathbf{I}}_{3x3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix operations

Equality of matrices

If $\underline{A}$ and $\underline{B}$ are two matrices of the same size,
then they are "equal" if each and every entry of one matrix equals the corresponding entry of the other.

$$\underline{A} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \qquad \underline{B} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\underline{A} = \underline{B} \Leftrightarrow \begin{matrix} a = 1, & b = 2, & c = 4, \\ d = -3, & e = 0, & f = 7, \\ g = 9, & h = 1, & i = 5. \end{matrix}$$

Matrix operations

Addition of two matrices

If $\underline{A}$ and $\underline{B}$ are two matrices of the same size,
then the sum of the matrices is a matrix $\underline{C}=\underline{A}+\underline{B}$ whose entries are the sums of the corresponding entries of $\underline{A}$ and $\underline{B}$

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \qquad \underline{\mathbf{B}} = \begin{bmatrix} -1 & 3 & 10 \\ -3 & 1 & 0 \\ 1 & 0 & 6 \end{bmatrix}$$

$$\underline{\mathbf{C}} = \underline{\mathbf{A}} + \underline{\mathbf{B}} = \begin{bmatrix} 0 & 5 & 14 \\ -6 & 1 & 7 \\ 10 & 1 & 11 \end{bmatrix}$$

Matrix operations

Addition of of matrices

Properties

Properties of matrix addition:

1. Matrix addition is **commutative** (order of addition does not matter)

$$\underline{A} + \underline{B} = \underline{B} + \underline{A}$$

2. Matrix addition is **associative**

$$\underline{A} + \left(\underline{B} + \underline{C}\right) = \left(\underline{A} + \underline{B}\right) + \underline{C}$$

3. **Addition of the zero matrix**

$$\underline{A} + \underline{0} = \underline{0} + \underline{A} = \underline{A}$$

Matrix operations

Multiplication by a scalar

If $\underline{A}$ is a matrix and c is a scalar, then the product c$\underline{A}$ is a matrix whose entries are obtained by multiplying each of the entries of $\underline{A}$ by c

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \quad \mathbf{c} = 3$$

$$\mathbf{c}\underline{\mathbf{A}} = \begin{bmatrix} 3 & 6 & 12 \\ -9 & 0 & 21 \\ 27 & 3 & 15 \end{bmatrix}$$

Matrix operations

Multiplication by a scalar

Special case

If $\underline{A}$ is a matrix and c =-1 is a scalar, then the product

(-1)$\underline{A}$ =-$\underline{A}$ is a matrix whose entries are obtained by multiplying each of the entries of $\underline{A}$ by -1

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \quad \mathbf{c} = -1$$

$$\mathbf{c}\underline{\mathbf{A}} = -\underline{\mathbf{A}} = \begin{bmatrix} -1 & -2 & -4 \\ 3 & 0 & -7 \\ -9 & -1 & -5 \end{bmatrix}$$

Matrix operations

Subtraction

If $\underline{\mathbf{A}}$ and $\underline{\mathbf{B}}$ are two square matrices of the same size, then $\underline{\mathbf{A}}$-$\underline{\mathbf{B}}$ is defined as the sum $\underline{\mathbf{A}}$+(-1)$\underline{\mathbf{B}}$

$$\underline{\mathbf{A}} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \qquad \underline{\mathbf{B}} = \begin{bmatrix} -1 & 3 & 10 \\ -3 & 1 & 0 \\ 1 & 0 & 6 \end{bmatrix}$$

$$\underline{\mathbf{C}} = \underline{\mathbf{A}} - \underline{\mathbf{B}} = \begin{bmatrix} 2 & -1 & -6 \\ 0 & -1 & 7 \\ 8 & 1 & -1 \end{bmatrix}$$

Note that $\underline{\mathbf{A}} - \underline{\mathbf{A}} = \underline{\mathbf{0}}$ **and** $\underline{\mathbf{0}} - \underline{\mathbf{A}} = -\underline{\mathbf{A}}$

Special operations

Transpose

If $\underline{A}$ is a **mxn** matrix, then the **transpose** of $\underline{A}$ is the **nxm** matrix whose first column is the first row of $\underline{A}$, whose second column is the second column of $\underline{A}$ and so on.

$$\underline{A} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \Leftrightarrow \underline{A}^T = \begin{bmatrix} 1 & -3 & 9 \\ 2 & 0 & 1 \\ 4 & 7 & 5 \end{bmatrix}$$

Special operations

Transpose

If **A** is a square matrix (**mxm**), it is called **symmetric** if

$$\underline{A} = \underline{A}^T$$

Matrix operations

Scalar (dot) product of two vectors

If $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are two vectors **of the same size**

$$\underline{\mathbf{a}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}; \quad \underline{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The scalar (dot) product of $\underline{a}$ and $\underline{b}$ is a scalar obtained by adding the products of corresponding entries of the two vectors

$$\underline{\mathbf{a}}^T \underline{\mathbf{b}} = (a_1 b_1 + a_2 b_2 + a_3 b_3)$$

Matrix operations

Matrix multiplication

For a product to be defined, the number of columns of **A** must be equal to the number of rows of **B**.

A   **B** = **AB**

m x r   r x n   m x n

outside

Matrix operations

Matrix multiplication

If **A** is a mxr matrix and **B** is a rxn matrix, then the product **C**=**AB** is a mxn matrix whose entries are obtained as follows. The entry corresponding to row 'i' and column 'j' of **C** is the dot product of the vectors formed by the row 'i' of **A** and column 'j' of **B**

$$\underline{A}_{3x3} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 0 & 7 \\ 9 & 1 & 5 \end{bmatrix} \qquad \underline{B}_{3x2} = \begin{bmatrix} -1 & 3 \\ -3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\underline{C}_{3x2} = \underline{A}\underline{B} = \begin{bmatrix} -3 & 5 \\ 10 & -9 \\ -7 & 28 \end{bmatrix} \quad \text{notice} \quad \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}^T \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = -3$$

Matrix operations

Multiplication of matrices

Properties

Properties of matrix multiplication:

1. Matrix multiplication is **noncommutative** (order of addition **does** matter)

$$\underline{A}\,\underline{B} \neq \underline{B}\,\underline{A} \text{ in general}$$

- **It may be that the product <u>AB</u> exists but <u>BA</u> does not (e.g. in the previous example <u>C</u>=<u>AB</u> is a 3x2 matrix, but <u>BA</u> does not exist)**
- **Even if the product exists, the products <u>AB</u> and <u>BA</u> are not generally the same**

Matrix operations

Multiplication of matrices

Properties

2. Matrix multiplication is **associative**

$$\underline{A}\left(\underline{B}\,\underline{C}\right) = \left(\underline{A}\,\underline{B}\right)\underline{C}$$

3. Distributive law

$$\underline{A}\left(\underline{B} + \underline{C}\right) = \underline{A}\,\underline{B} + \underline{A}\,\underline{C}$$

$$\left(\underline{B} + \underline{C}\right)\underline{A} = \underline{B}\,\underline{A} + \underline{C}\,\underline{A}$$

4. Multiplication by identity matrix

$$\underline{A\,I} = \underline{A}\,;\ \underline{I}\,\underline{A} = \underline{A}$$

5. Multiplication by zero matrix $\underline{A}\,\underline{0} = \underline{0}\,;\ \underline{0}\,\underline{A} = \underline{0}$

6. $\left(\underline{A}\,\underline{B}\right)^T = \underline{B}^T\,\underline{A}^T$

Matrix operations

Miscellaneous properties

1. If **A** , **B** and **C** are square matrices of the same size, and $\underline{A} \neq \underline{0}$ then $\underline{A}\,\underline{B} = \underline{A}\,\underline{C}$ does not necessarily mean that $\underline{B} = \underline{C}$
2. $\underline{A}\,\underline{B} = \underline{0}$ does **not** necessarily imply that either **A** or **B** is zero

Inverse of a matrix

Definition

If **A** is any **square matrix** and **B** is another square matrix satisfying the conditions

$$\underline{A}\,\underline{B} = \underline{B}\,\underline{A} = \underline{I}$$

Then

(a) The matrix **A** is called **invertible**, and

(b) the matrix **B** is the inverse of **A** and is denoted as $\mathbf{A}^{-1}$.

The inverse of a matrix is **unique**

Inverse of a matrix

Uniqueness

The inverse of a matrix is **unique**
Assume that **B** and **C** both are inverses of **A**

$$AB = BA = I$$

$$AC = CA = I$$

$$(BA)C = IC = C$$

$$B(AC) = BI = B$$

$$\therefore\ B = C$$

Hence a matrix **cannot** have two or more inverses.

Inverse of a matrix

Some properties

Property 1: If **A** is any **invertible square matrix** the inverse of its inverse is the matrix **A** itself

$$\left(\underline{A}^{-1}\right)^{-1} = \underline{A}$$

Property 2: If **A** is any **invertible square matrix** and k is any scalar then

$$\left(k\underline{A}\right)^{-1} = \frac{1}{k}\underline{A}^{-1}$$

Inverse of a matrix

Properties

Property 3: If **A** and **B** are invertible square matrices then

$$\left(\underline{A}\,\underline{B}\right)^{-1} = \underline{B}^{-1}\,\underline{A}^{-1}$$

$$(AB)\left(AB\right)^{-1} = I$$

Premultiplying both sides by A^{-1}

$$A^{-1}(AB)\left(AB\right)^{-1} = A^{-1}$$

$$\left(A^{-1}A\right)B\left(AB\right)^{-1} = A^{-1}$$

$$B\left(AB\right)^{-1} = A^{-1}$$

Premultiplying both sides by B^{-1}

$$\left(AB\right)^{-1} = B^{-1}A^{-1}$$

What is a determinant?

The **determinant** of a **square matrix** is a **number** obtained in a specific manner from the matrix.

For a 1x1 matrix:

$$\underline{A} = [a_{11}] \; ; \quad \det(\underline{A}) = a_{11}$$

For a 2x2 matrix:

$$\underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \; ; \quad \det(\underline{A}) = a_{11}a_{22} - a_{12}a_{21}$$

Product along red arrow minus product along blue arrow

Example 1

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}$$

Notice (1) A matrix is an array of numbers
(2) A matrix is enclosed by square brackets

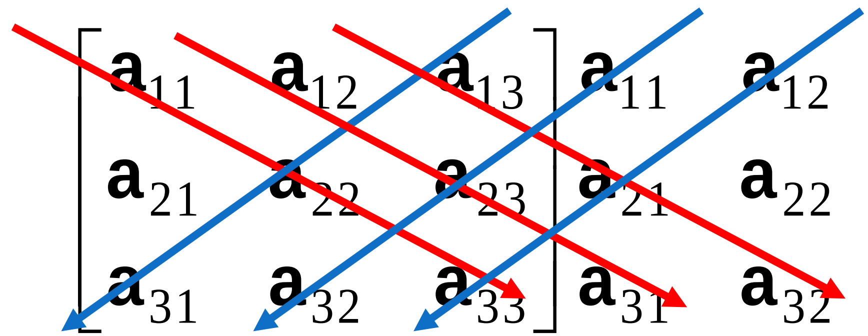
$$\det(A) = \begin{vmatrix} 1 & 3 \\ 5 & 7 \end{vmatrix} = 1 \times 7 - 3 \times 5 = -8$$

Notice (1) The determinant of a matrix is a number
(2) The symbol for the determinant of a matrix is a pair of parallel lines

Duplicate column method for 3x3 matrix

For **<u>ONLY a 3x3 matrix</u>** write down the first two columns after the third column

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$



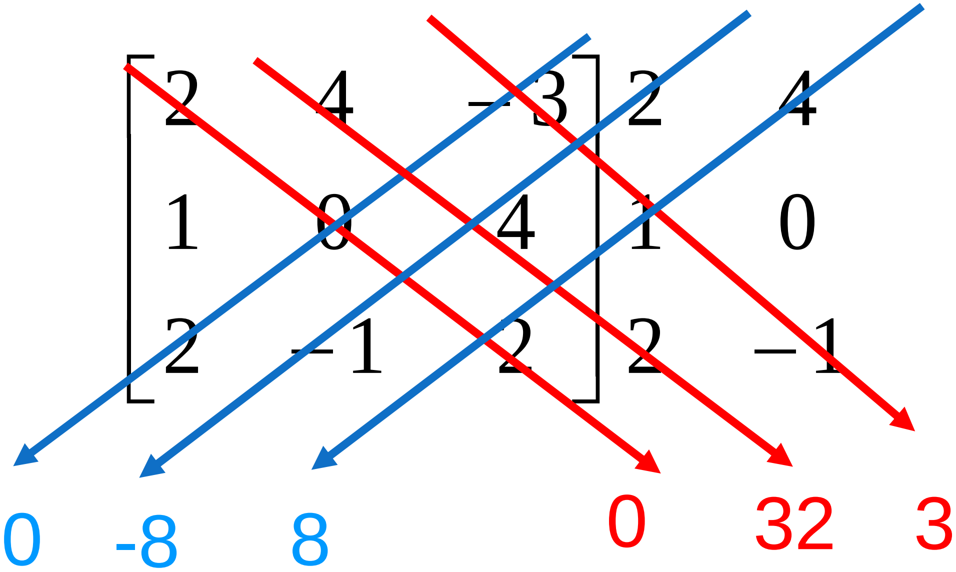
Sum of products along red arrow
minus sum of products along blue arrow

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

This technique works only for 3x3 matrices

Example

$$A = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 0 & 4 \\ 2 & -1 & 2 \end{bmatrix}$$



Sum of red terms = 0 + 32 + 3 = 35

Sum of blue terms = 0 – 8 + 8 = 0

Determinant of matrix A= det(A) = 35 – 0 = 35

Finding determinant using inspection

Special case. If **two rows** or **two columns** are **proportional** (i.e. multiples of each other), then the determinant of the matrix is zero

$$\begin{vmatrix} 2 & 7 & 8 \\ 3 & 2 & 4 \\ -2 & -7 & -8 \end{vmatrix} = 0$$

because rows 1 and 3 are proportional to each other

<u>If the **determinant of a matrix is zero**, it is called a **singular matrix**</u>

Cofactor method

What is a cofactor?

If A is a square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The **minor**, $\mathbf{M_{ij}}$, of entry a_{ij} is the determinant of the submatrix that remains after the i^{th} row and j^{th} column are deleted from A.
The **cofactor** of entry a_{ij} is $C_{ij}=(-1)^{(i+j)}\,\mathbf{M_{ij}}$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31} \quad C_{12} = -M_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

What is a cofactor?

Sign of cofactor

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Find the minor and cofactor of a_{33}

$$A = \begin{bmatrix} 2 & 4 & -3 \\ 1 & 0 & 4 \\ 2 & -1 & 2 \end{bmatrix}$$

Minor

$$M_{33} = \begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = 2 \times 0 - 4 \times 1 = -4$$

Cofactor

$$C_{33} = (-1)^{(3+3)} M_{33} = M_{33} = -4$$

Cofactor method of obtaining the determinant of a matrix

The determinant of a n x n matrix A can be computed by multiplying **ALL the entries in ANY row (or column)** by their **cofactors** and **adding** the resulting products. That is, for each $1 \le i \le n$ and $1 \le j \le n$

Cofactor expansion along the j[th] column

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Cofactor expansion along the i[th] row

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

EXAMPLE: EVALUATE DET(A) FOR:

$$A = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 3 & 4 & 0 & 1 \\ -1 & 5 & 2 & -2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$$

$$\det(A) = (1)\begin{vmatrix} 4 & 0 & 1 \\ 5 & 2 & -2 \\ 1 & 1 & 3 \end{vmatrix} - (0)\begin{vmatrix} 3 & 0 & 1 \\ -1 & 2 & -2 \\ 0 & 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 & 1 \\ -1 & 5 & -2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$- (-3)\begin{vmatrix} 3 & 4 & 0 \\ -1 & 5 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (1)(35) - 0 + (2)(62) - (-3)(13) = 198$$

Example : evaluate

$$\det(A)= \begin{vmatrix} 1 & 5 & -3 \\ 1 & 0 & 2 \\ 3 & -1 & 2 \end{vmatrix}$$

By a cofactor along the third column

$$\det(A)=a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

$$\det(A)= -3^*(-1)^4 \left\{ \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} \right\} +2^*(-1)^5 \left\{ \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} \right\} +2^*(-1)^6 \left\{ \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} \right\}$$

$= \det(A)= -3(-1-0)+2(-1)^5(-1-15)+2(0-5)=25$

Quadratic form

The scalar

$$U = \underline{d}^T \underline{k} \, \underline{d}$$

$$\underline{d} = vector$$

$$\underline{k} = square \;\; matrix$$

Is known as a **quadratic form**

If U>0: Matrix $\underline{k}$ is known as **positive definite**
If U≥0: Matrix $\underline{k}$ is known as **positive semidefinite**

Quadratic form

Let

$$\underline{d} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} \quad \underline{k} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

Symmetric matrix

Then

$$U = \underline{d}^T \underline{k}\, \underline{d} = [d_1 \quad d_2] \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$= [d_1 \quad d_2] \begin{Bmatrix} k_{11}d_1 + k_{12}d_2 \\ k_{12}d_1 + k_{22}d_2 \end{Bmatrix}$$

$$= d_1(k_{11}d_1 + k_{12}d_2) + d_2(k_{12}d_1 + k_{22}d_2)$$

$$= k_{11}d_1{}^2 + 2k_{12}d_1d_2 + k_{22}d_2{}^2$$

Differentiation of quadratic form

Differentiate U wrt d_1

$$\frac{\partial U}{\partial d_1} = 2k_{11}d_1 + 2k_{12}d_2$$

Differentiate U wrt d_2

$$\frac{\partial U}{\partial d_2} = 2k_{12}d_1 + 2k_{22}d_2$$

Differentiation of quadratic form

Hence

$$\frac{\partial U}{\partial \underline{d}} \equiv \begin{Bmatrix} \dfrac{\partial U}{\partial d_1} \\ \dfrac{\partial U}{\partial d_2} \end{Bmatrix} = 2\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}$$

$$= 2\,\underline{k}\,\underline{d}$$

Outline

- Role of FEM simulation in Engineering Design
- Course Philosophy

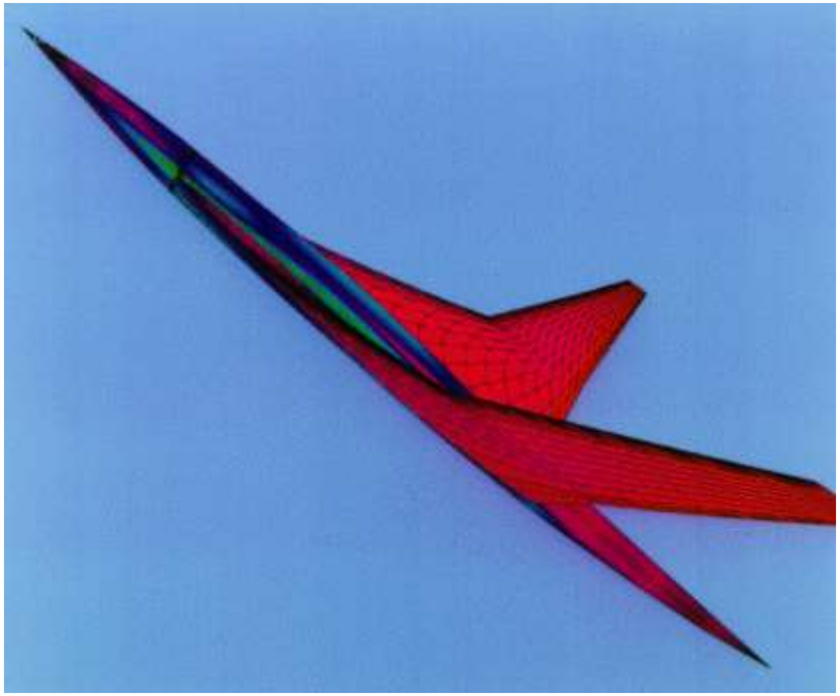
Role of simulation in design:Boeing 777

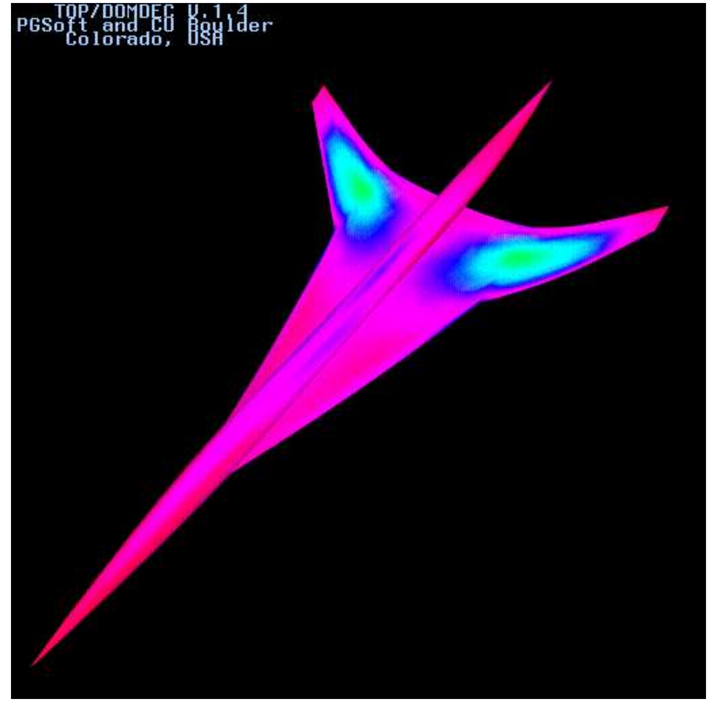
Source: Boeing Web site
(http://www.boeing.com/companyoffices/gallery/images/commercial/).

Another success ..in failure: Airbus A380

http://www.airbus.com/en/aircraftfamilies/a380/

Drag Force Analysis t of Aircraft





- Question

  What is the drag force distribution on the aircraft?

- Solve
  - Navier-Stokes Partial Differential Equations.
- Recent Developments
  - Multigrid Methods for Unstructured Grids

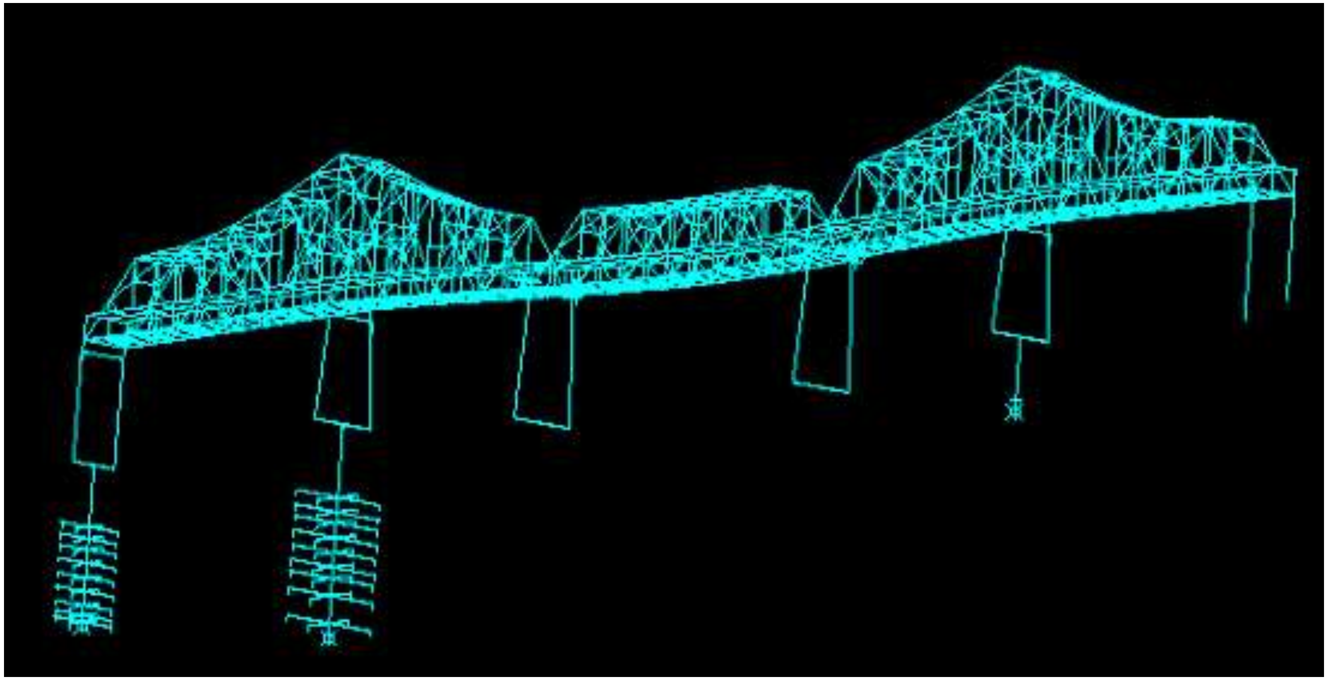
San Francisco Oakland Bay Bridge

Before the 1989 Loma Prieta earthquake

San Francisco Oakland Bay Bridge

- After the earthquake

San Francisco Oakland Bay Bridge

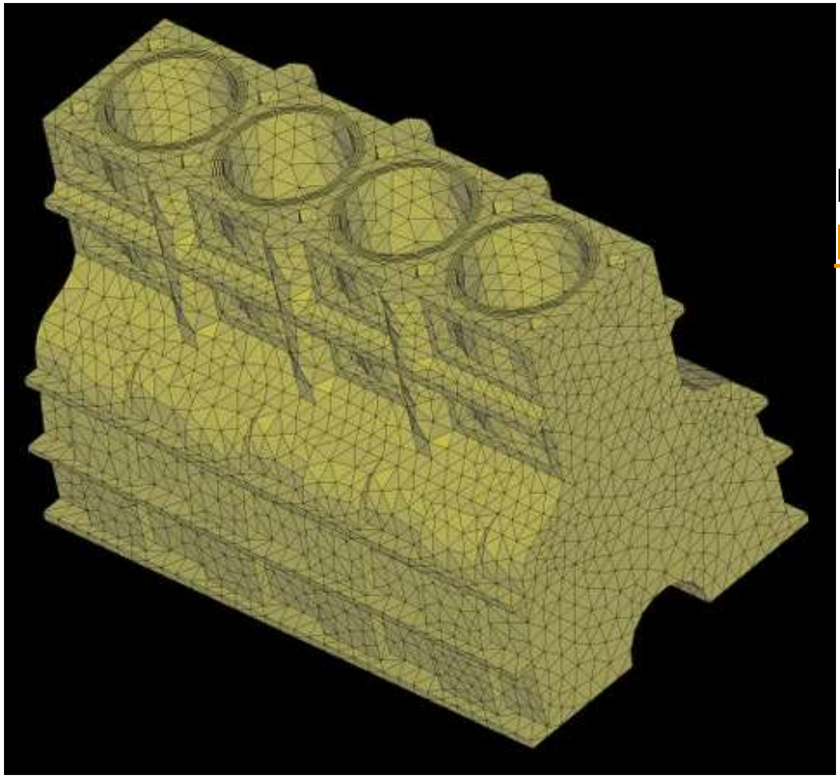


A finite element model to analyze the bridge under seismic loads
Courtesy: ADINA R&D

Crush Analysis of Ford Windstar

- Question
  - What is the load-deformation relation?
- Solve
  - Partial Differential Equations of Continuum Mechanics
- Recent Developments
  - Meshless Methods, Iterative methods, Automatic Error Control

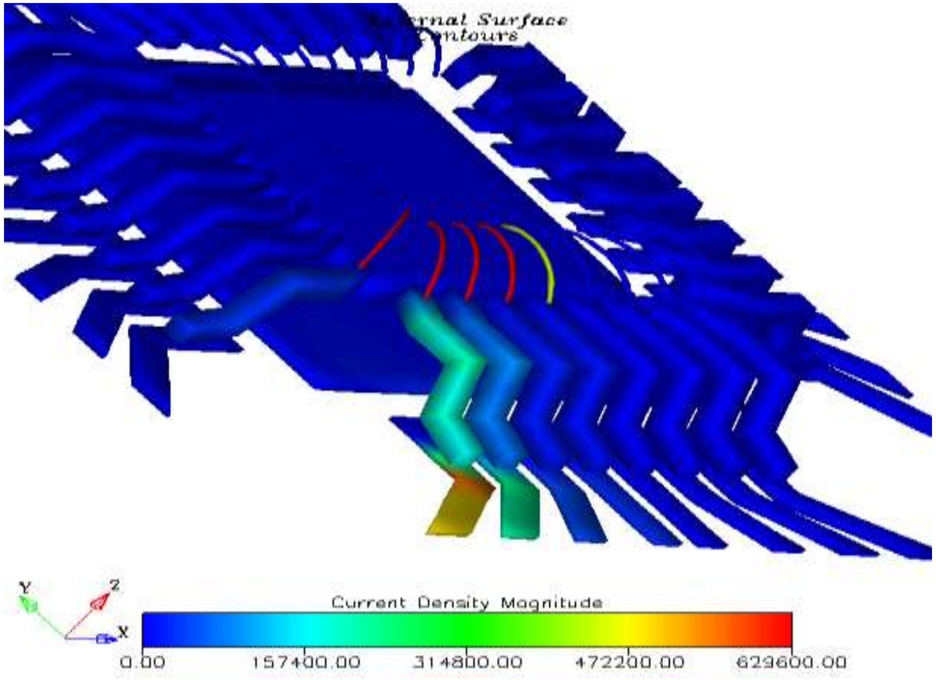
Engine Thermal Analysis



Picture from
http://www.adina.com

- Question
  - What is the temperature distribution in the engine block?
- Solve
  - Poisson Partial Differential Equation.
- Recent Developments
  - Fast Integral Equation Solvers, Monte-Carlo Methods

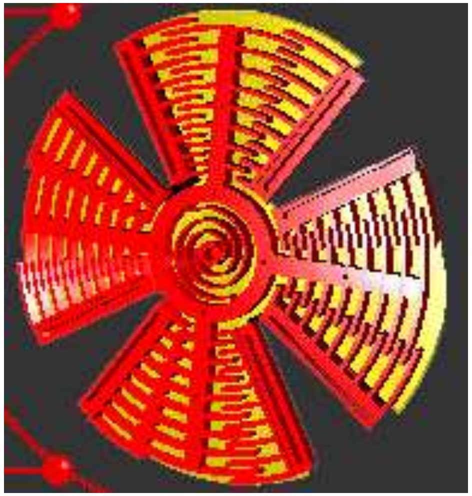
Electromagnetic Analysis of Packages



Thanks to Coventor
http://www.coventor.com

- Solve
  - Maxwell's Partial Differential Equations
- Recent Developments
  - Fast Solvers for Integral Formulations

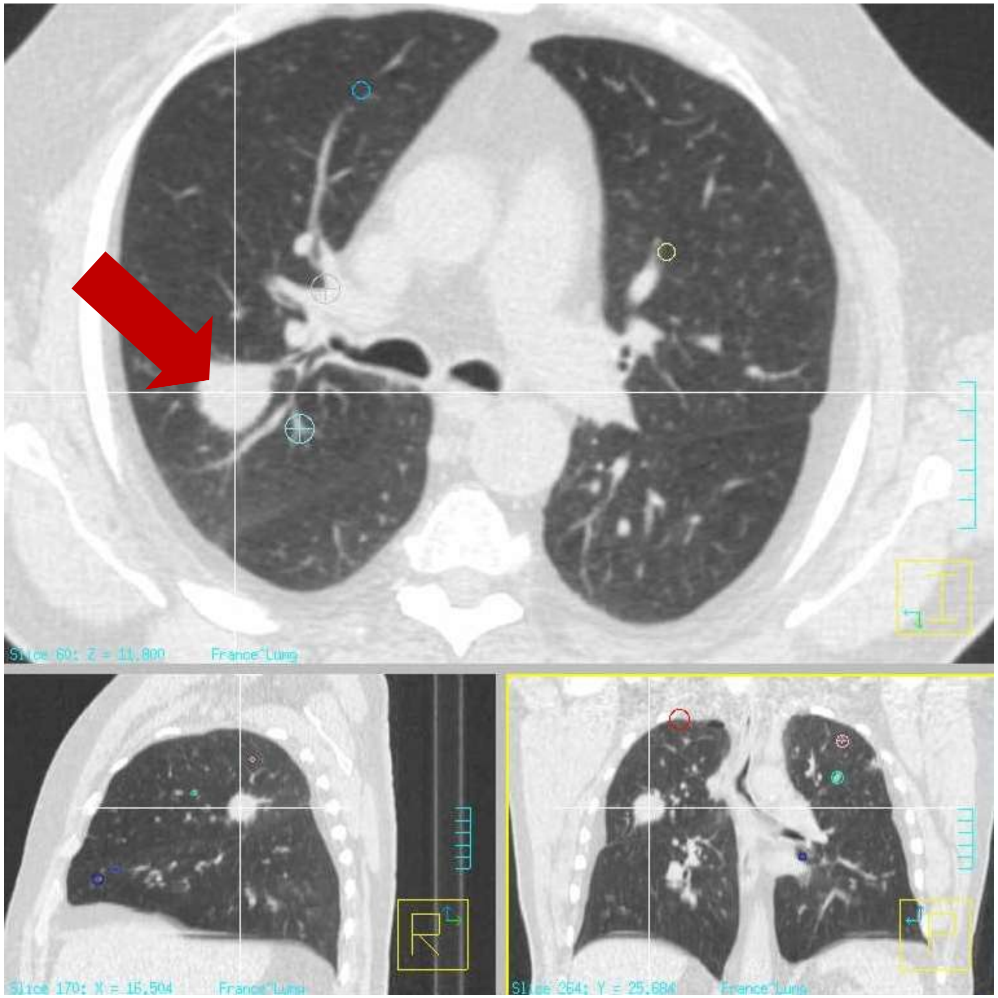
Micromachine Device   Performance Analysis

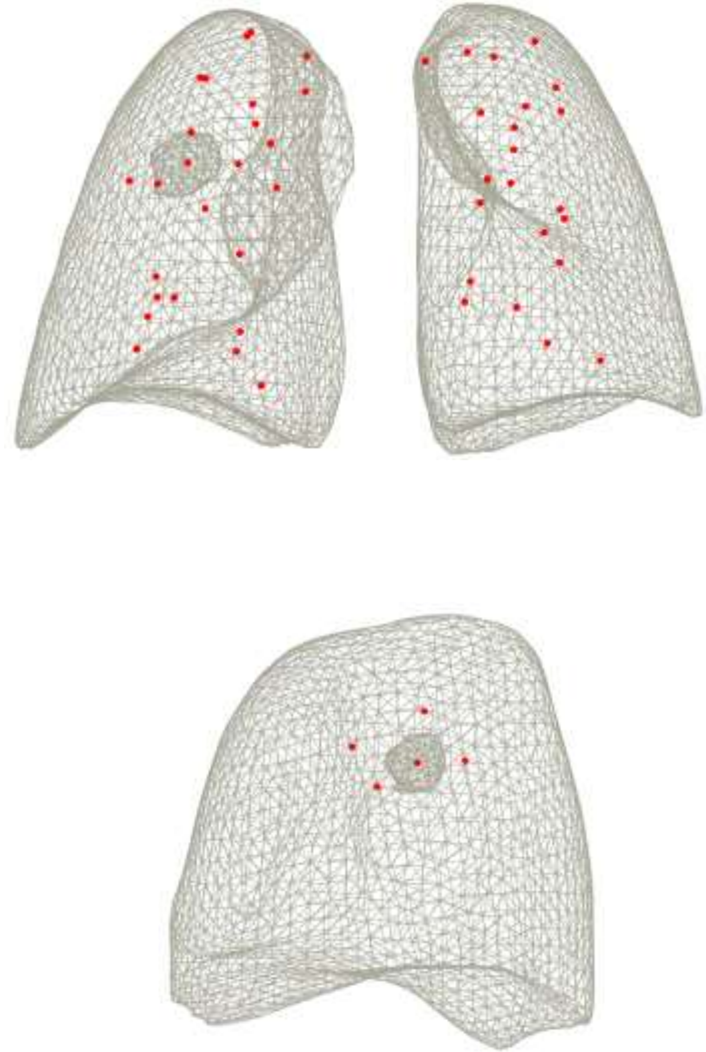


From www.memscap.com

- Equations
  - Elastomechanics, Electrostatics, Stokes Flow.
- Recent Developments
  - Fast Integral Equation Solvers, Matrix-Implicit Multi-level Newton Methods for coupled domain problems.

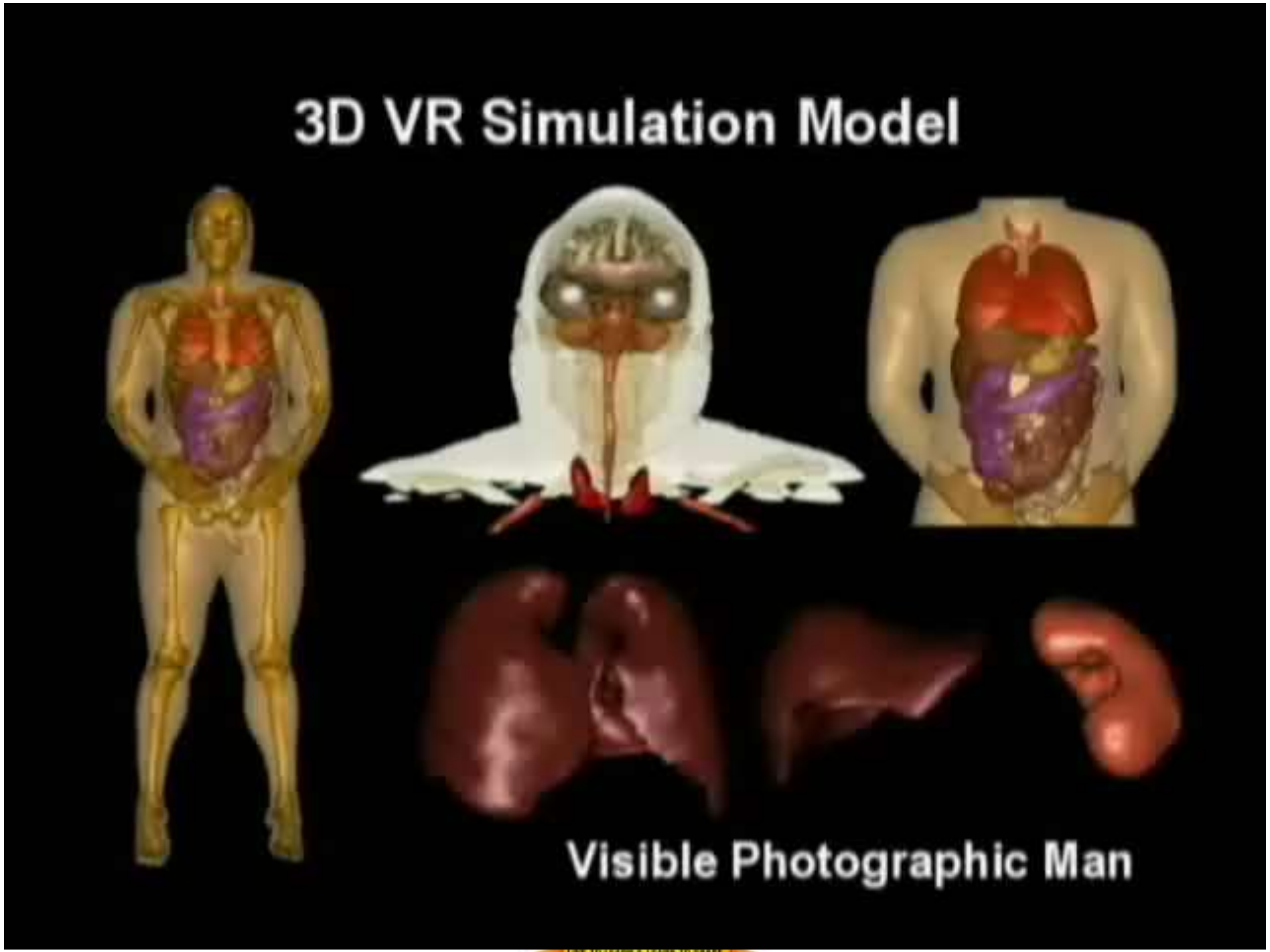
Radiation Therapy of Lung Cancer





http://www.simulia.com/academics/research_lung.html

Virtual Surgery



Engineering design

General scenario..

Physical Problem

↓

Question regarding the problem
...how large are the deformations?
...how much is the heat transfer?

Mathematical model

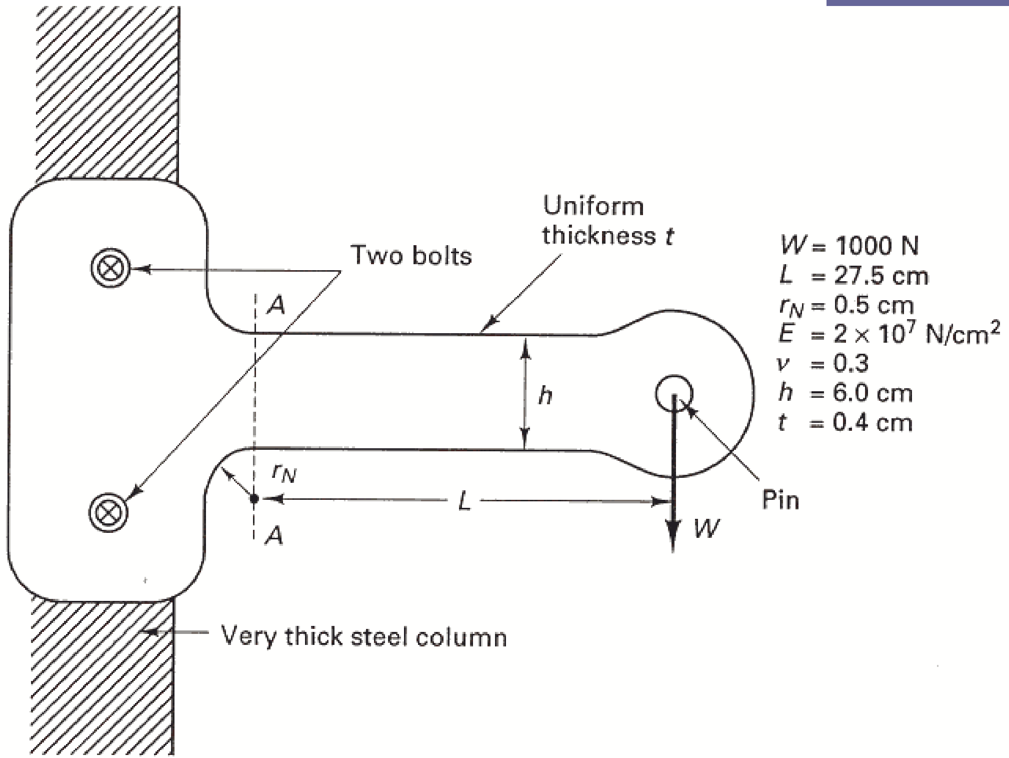
Governed by differential equations

Assumptions regarding
Geometry
Kinematics
Material law
Loading
Boundary conditions
Etc.

Engineering design

Example: A bracket

Physical problem



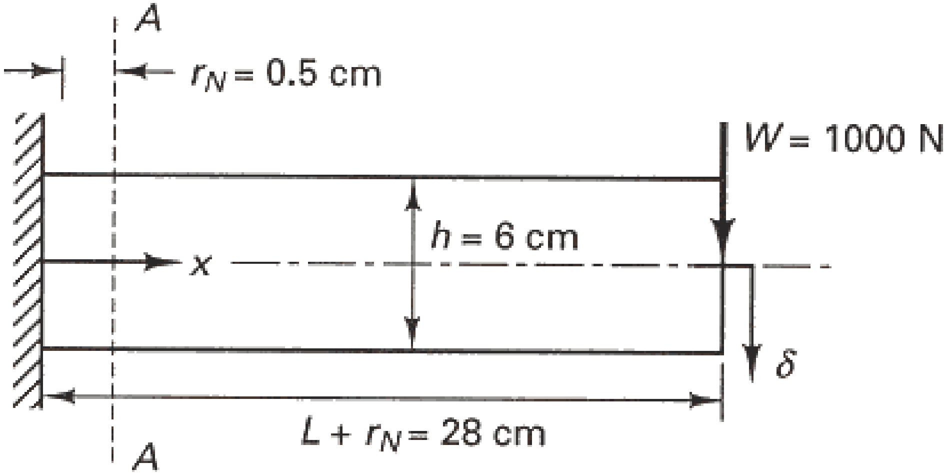
Questions:

1. What is the bending moment at section AA?
2. What is the deflection at the pin?

Finite Element Procedures, K J Bathe

Example: A bracket

Engineering design



Mathematical model 1: beam

How **reliable** is this model?

How **effective** is this model?

Moment at section AA

$$M = WL$$
$$= 27{,}500 \text{ N cm}$$

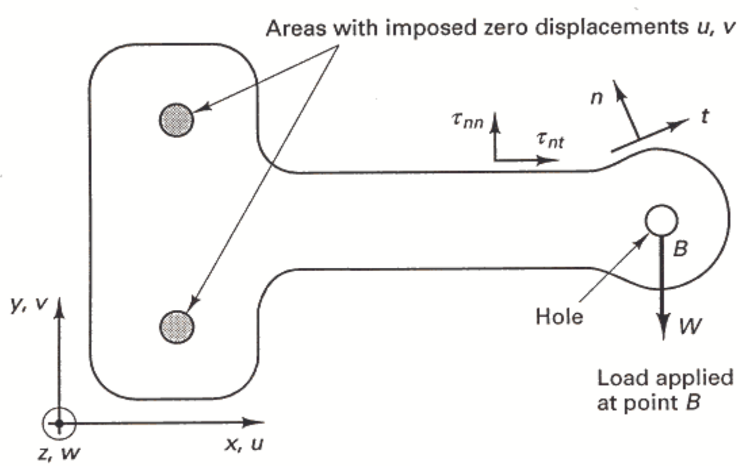
Deflection at load

$$\delta_{\text{at load W}} = \frac{1}{3}\frac{W(L + r_N)^3}{EI} + \frac{W(L + r_N)}{\frac{5}{6}AG}$$
$$= 0.053 \text{ cm}$$

Engineering design

Example: A bracket

Mathematical model 2: plane stress



Equilibrium equations

$$\left.\begin{aligned}\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} = 0\end{aligned}\right\} \text{ in domain of bracket}$$

$\tau_{nn} = 0$, $\tau_{nt} = 0$ on surfaces except at point B and at imposed zero displacements

Stress-strain relation

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

E = Young's modulus, ν = Poisson's ratio

Strain-displacement relations

$$\epsilon_{xx} = \frac{\partial u}{\partial x}; \qquad \epsilon_{yy} = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Difficult to solve by hand!

..General scenario..

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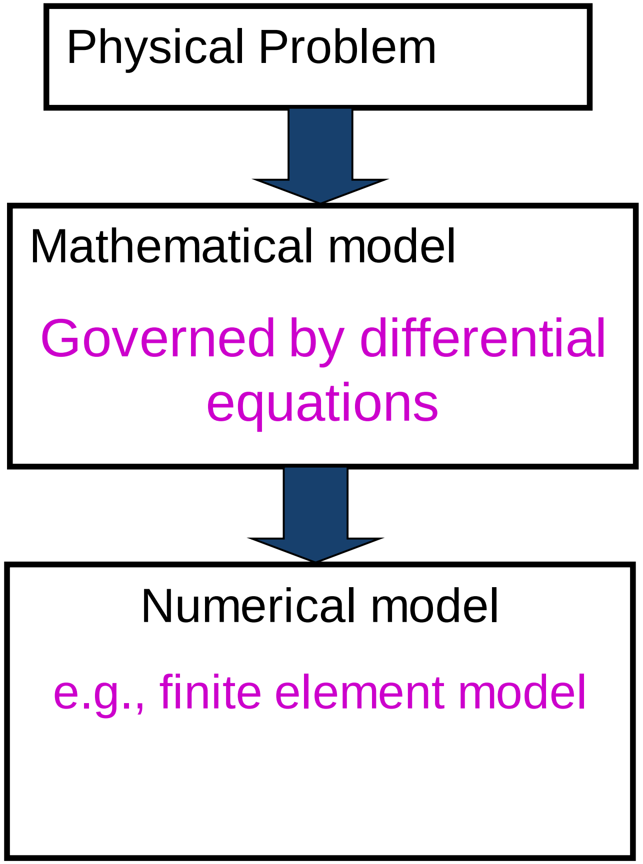

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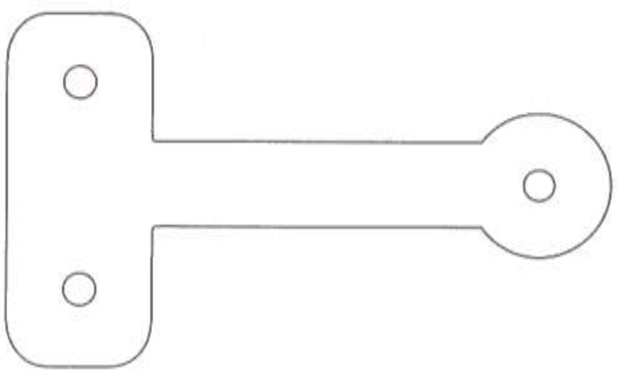
..General scenario..

Finite element analysis

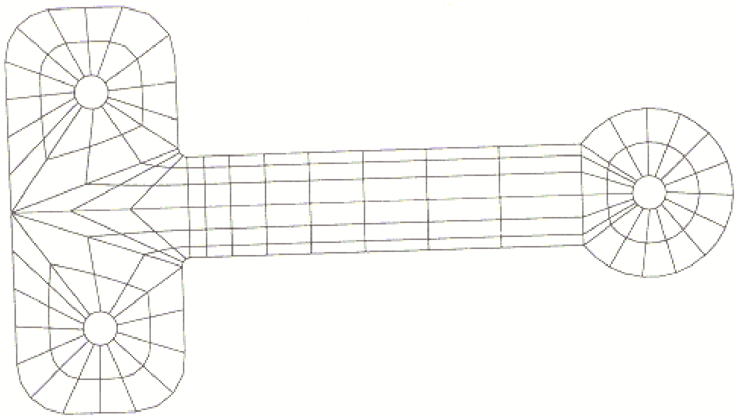
PREPROCESSING

1. Create a geometric model

2. Develop the finite element model



Solid model



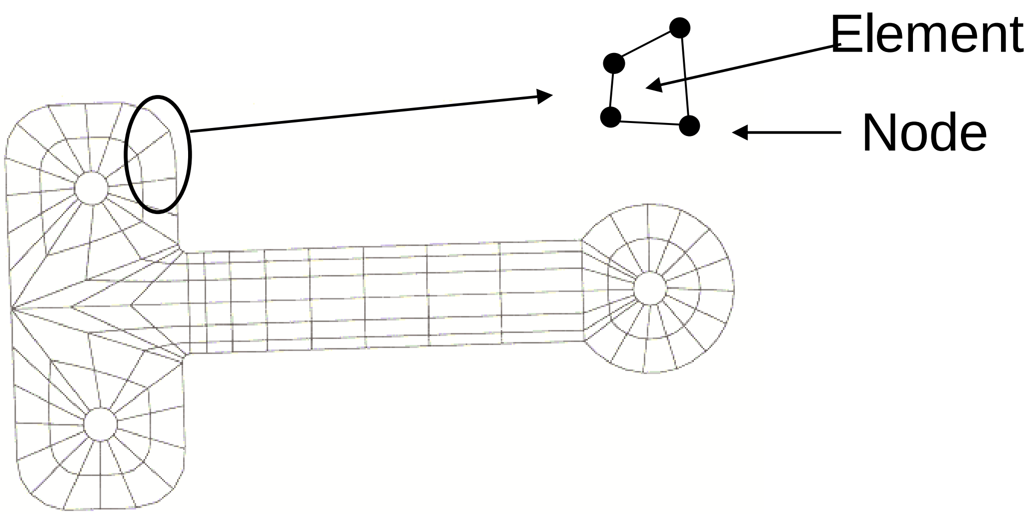
Finite element model

Engineering design

Finite element analysis

FEM analysis scheme

Step 1: Divide the problem domain into non overlapping regions ("**elements**") connected to each other through special points ("**nodes**")



Finite element model

Engineering design

FEM analysis scheme

Step 2: Describe the behavior of each element

Step 3: Describe the behavior of the entire body by putting together the behavior of each of the elements (this is a process known as “**assembly**”)

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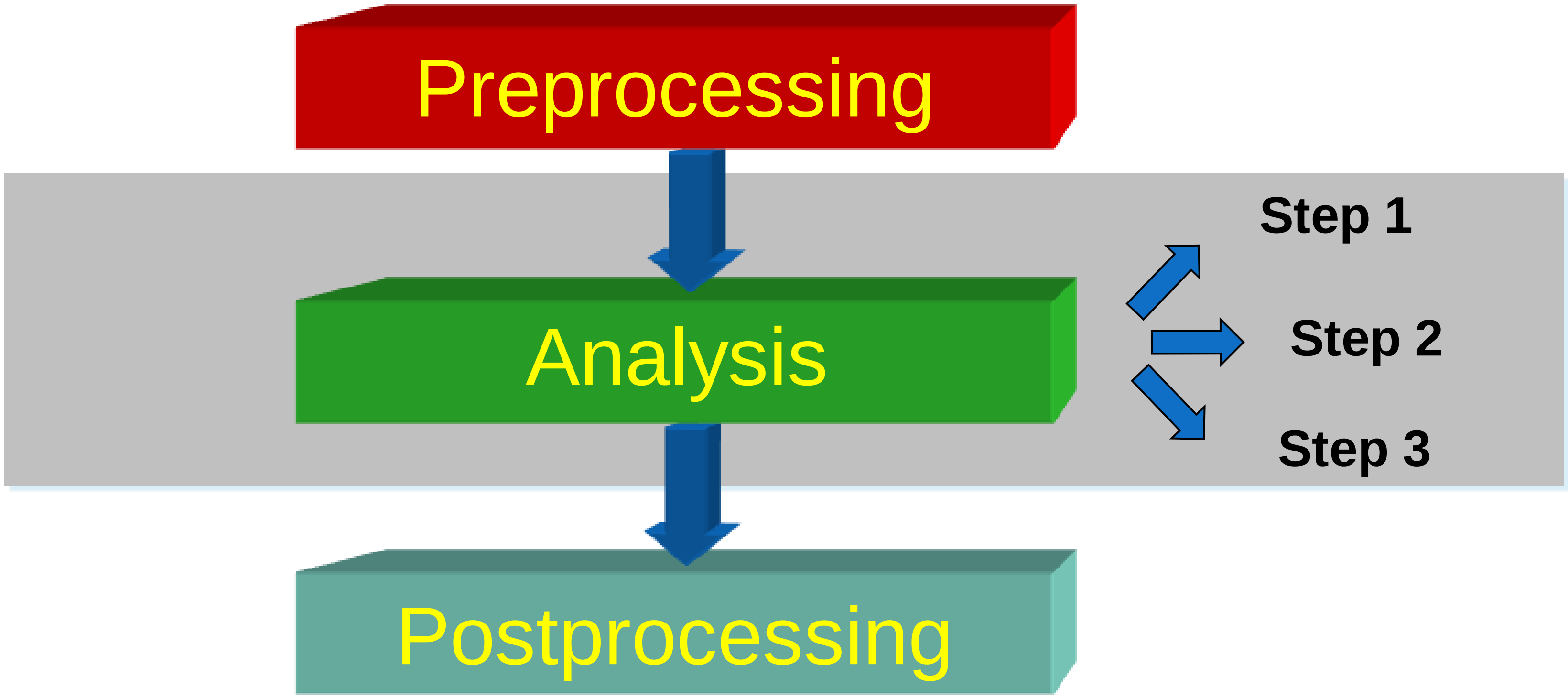
POSTPROCESSING

Compute moment at section AA

..General scenario..

Engineering design

Finite element analysis



Engineering design

Example: A bracket

Mathematical model 2: plane stress

FEM solution to mathematical model 2 (plane stress)

Moment at section AA

$$M = 27{,}500 \text{ N cm}$$

Deflection at load

$$\delta_{\text{at load W}} = 0.064 \text{ cm}$$

Conclusion: With respect to the questions we posed, the beam model is ***reliable*** if the required bending moment is to be predicted within 1% and the deflection is to be predicted within 20%. The beam model is also highly ***effective*** since it can be solved easily (by hand).

What if we asked: what is the maximum stress in the bracket?

would the beam model be of any use?

Example: A bracket

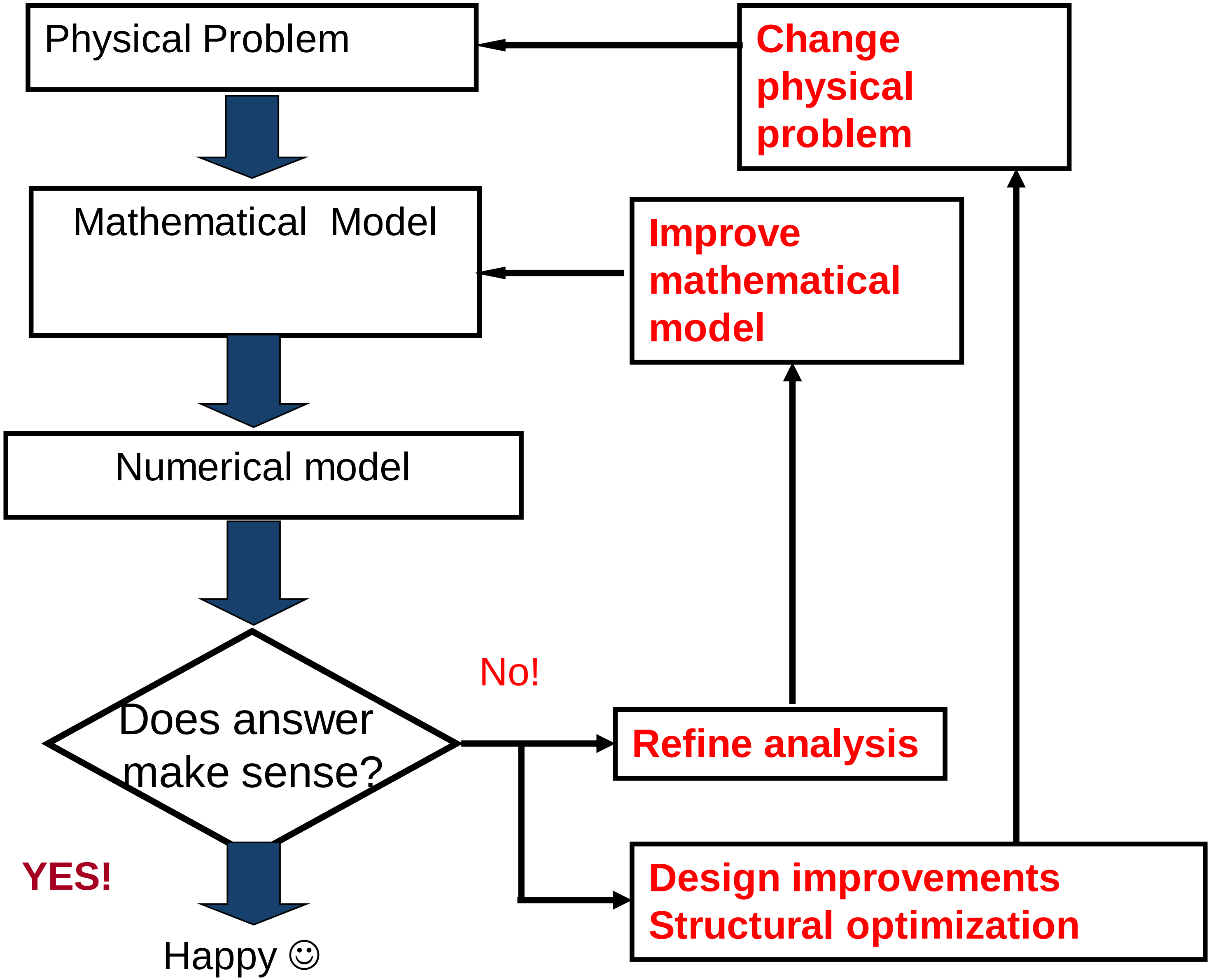
Engineering design

Summary

1. The **selection** of the mathematical model depends on the response to be predicted.
2. The **<u>most effective</u>** mathematical model is the one that delivers the answers to the questions in reliable manner with least effort.
3. **<u>The numerical solution is only as accurate as the mathematical model.</u>**

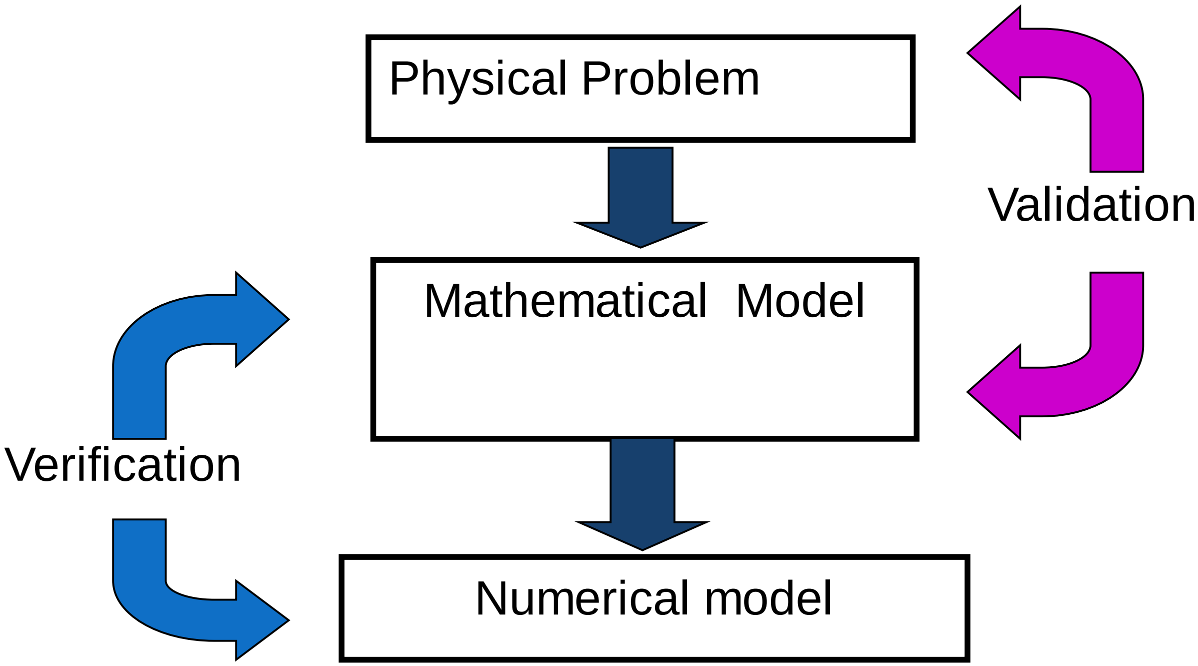
Modeling a physical problem

...General scenario



Modeling a physical problem

Verification and validation



Critical assessment of the FEM

Reliability:

For a well-posed mathematical problem the numerical technique should ***always***, for a reasonable discretization, give a reasonable solution which ***must*** converge to the accurate solution as the discretization is refined.
e.g., use of reduced integration in FEM results in an unreliable analysis procedure.

Robustness:

The performance of the numerical method should not be unduly sensitive to the material data, the boundary conditions, and the loading conditions used.
e.g., displacement based formulation for incompressible problems in elasticity

Efficiency:

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DEPARTMENT OF MECHANICAL ENGINEERING

INTRODUCTION TO FINITE ELEMENTS

PRINCIPLES OF MINIMUM POTENTIAL ENERGY AND RAYLEIGH-RITZ

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Summary:

- Potential energy of a system
    - Elastic bar
    - String in tension
- Principle of Minimum Potential Energy
- Rayleigh-Ritz Principle

A generic problem in 1D

$$\frac{d^2u}{dx^2} + x = 0; \qquad 0 < x < 1$$

$$u = 0 \quad at \ x = 0$$

$$u = 1 \quad at \ x = 1$$

Approximate solution strategy:

Guess $u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are “known” functions and a_o, a_1, etc are constants chosen such that the approximate solution

1. Satisfies the boundary conditions
2. Satisfies the differential equation

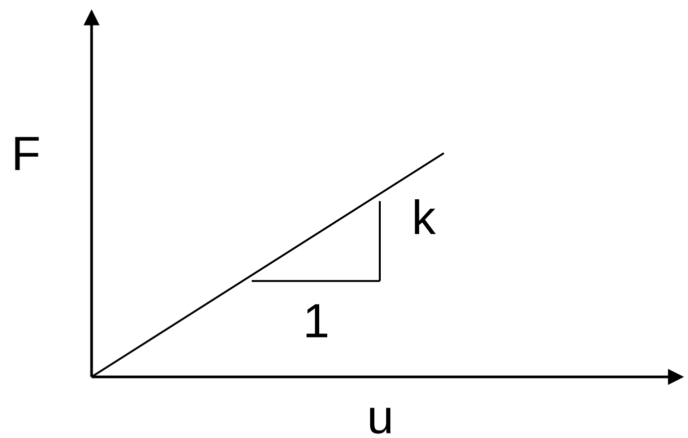
Too difficult to satisfy for general problems!!

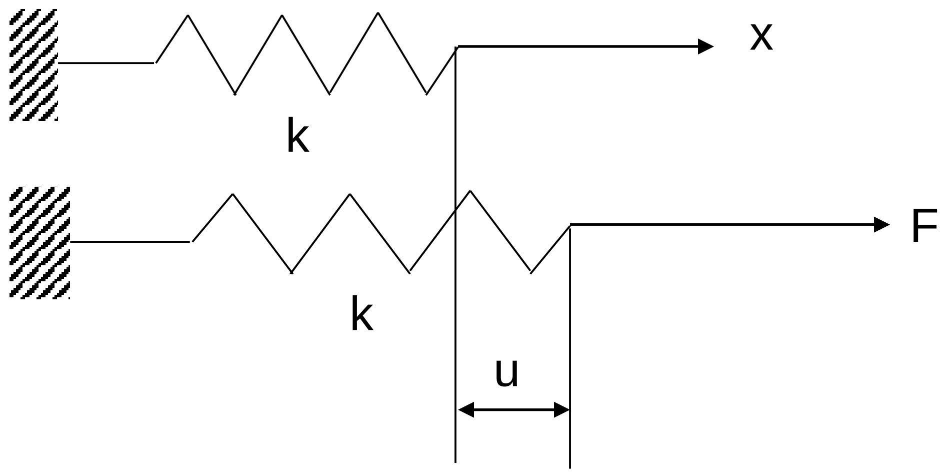
Potential energy

The **potential energy** of an elastic body is defined as

$$\Pi = \text{Strain energy (U)} - \text{potential energy of loading } (W)$$

Strain energy of a linear spring





F = Force in the spring
u = deflection of the spring
k = "stiffness" of the spring

Hooke's Law
F = ku

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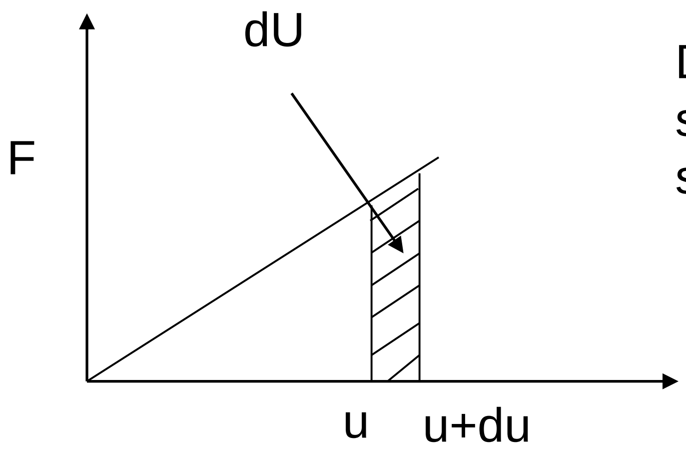
Strain energy of a linear spring



Differential strain energy of the spring for a small change in displacement (du) of the spring

$$dU = F du$$

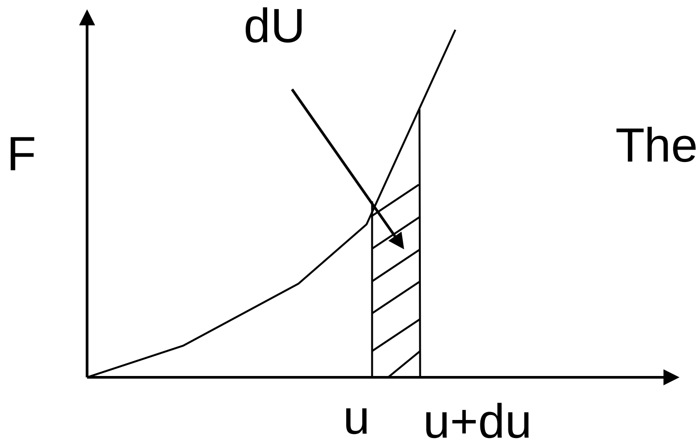
For a linear spring

$$dU = k u du$$

The total strain energy of the spring

$$U = \int_0^u k\, u\, du = \frac{1}{2} k\, u^2$$

Strain energy of a nonlinear spring



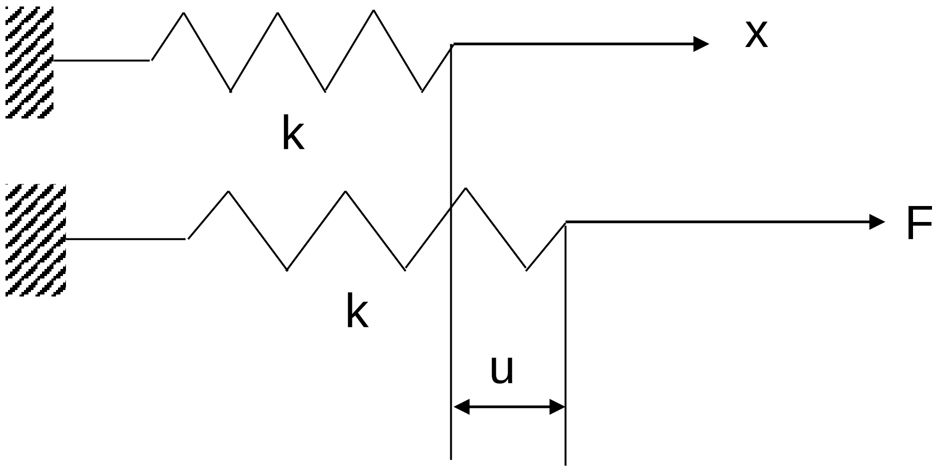
$$dU = F du$$

The total strain energy of the spring

$$U = \int_0^u F\, du = \text{Area under the force} - \text{dispalcement curve}$$

Potential energy of the loading (for a single spring as in the figure)

$$W = Fu$$



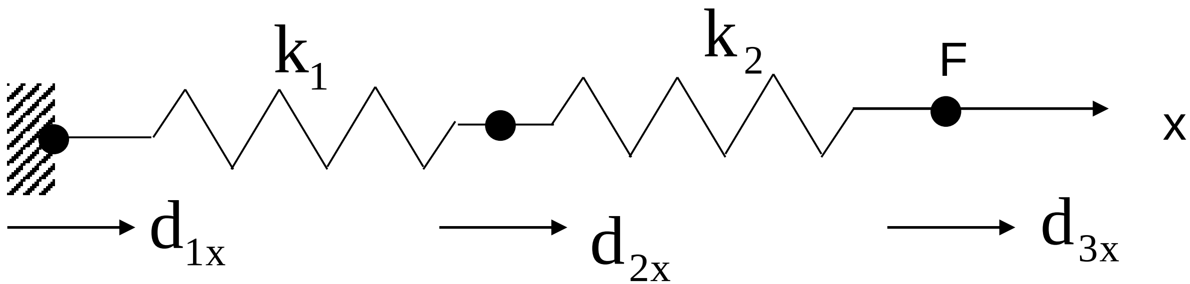
Potential energy of a linear spring

$$\Pi = \text{Strain energy (U)} - \text{potential energy of loading (W)}$$

$$\Pi = \frac{1}{2}ku^2 - Fu$$

Example of how to obtain the equilibr

Principle of minimum potential energy for a system of springs



For this system of spring, first write down the total potential energy of the system as:

$$\Pi_{system} = \left[\frac{1}{2}k_1(d_{2x})^2 + \frac{1}{2}k_2(d_{3x} - d_{2x})^2\right] - Fd_{3x}$$

Obtain the equilibrium equations by minimizing the potential energy

$$\frac{\partial \Pi_{system}}{\partial d_{2x}} = k_1 d_{2x} - k_2(d_{3x} - d_{2x}) = 0 \quad Equation\,(1)$$

$$\frac{\partial \Pi_{system}}{\partial d_{3x}} = k_2(d_{3x} - d_{2x}) - F = 0 \quad Equation\,(2)$$

Principle of **minimum potential energy** for a system of springs

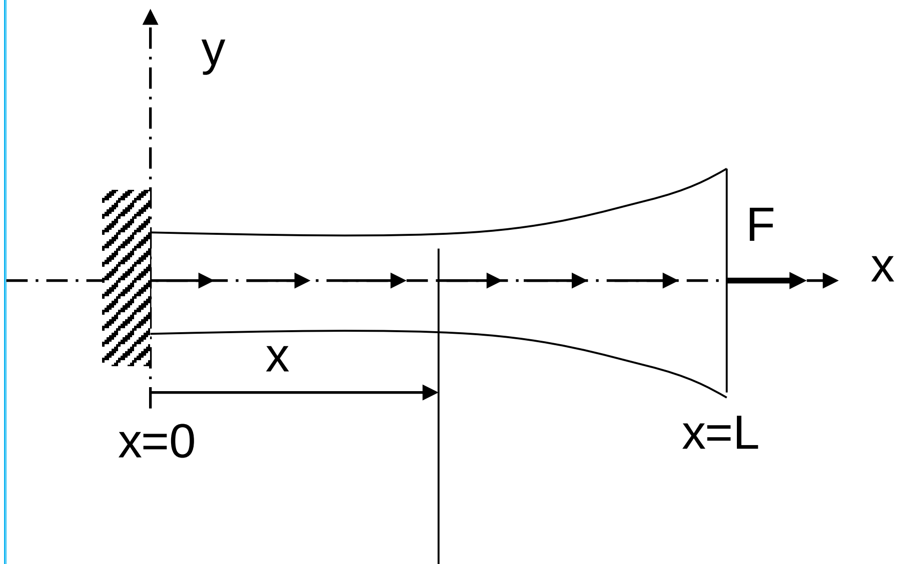
In matrix form, equations 1 and 2 look like

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

Does this equation look familiar?

Also look at example problem **worked out in class**

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus
u(x) = displacement of the bar at x

Axial strain $$\varepsilon = \frac{du}{dx}$$

Axial stress $$\sigma = E\varepsilon = E\frac{du}{dx}$$

Strain energy per unit volume of the bar $$dU = \frac{1}{2}\sigma\varepsilon = \frac{1}{2}E\left(\frac{du}{dx}\right)^2$$

Strain energy of the bar

$$U = \int dU = \int \frac{1}{2}\sigma\varepsilon\, dV = \int_{x=0}^{L} \frac{1}{2}\sigma\varepsilon\, A dx$$

since dV=Adx

Axially loaded elastic bar

Strain energy of the bar

$$U = \int_0^L \frac{1}{2}\sigma\varepsilon A\, dx = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$$

Potential energy of the loading

$$W = \int_0^L bu\, dx + Fu(x = L)$$

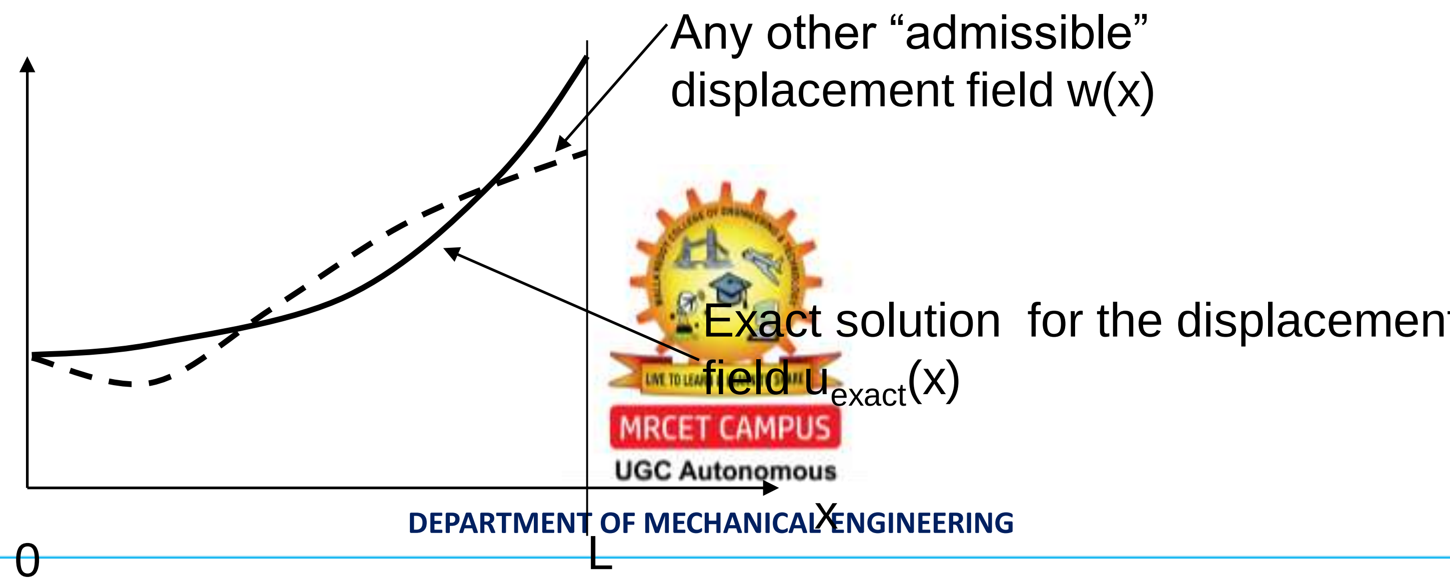
Potential energy of the axially loaded bar

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\, dx - Fu(x = L)$$

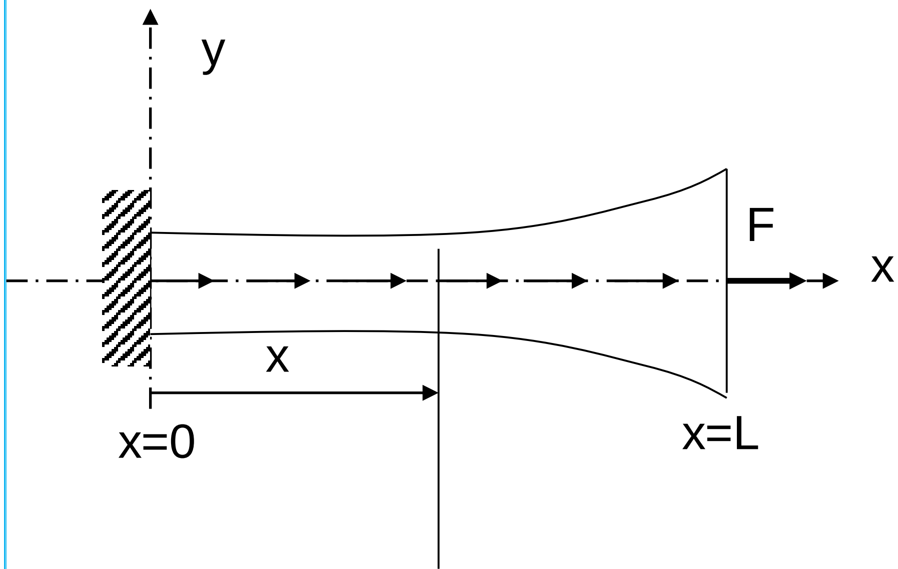
<u>Principle of Minimum Potential Energy</u>

Among all admissible displacements that a body can have, the one that minimizes the total potential energy of the body satisfies the strong formulation

Admissible displacements: these are any reasonable displacement that you can think of that satisfy the ***displacement boundary conditions of the original problem*** (and of course certain minimum ***continuity requirements***). Example:



Lets see what this means for an axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution $u_{exact}(x)$

$$\Pi(u_{exact}) = \frac{1}{2}\int_0^L EA\left(\frac{du_{exact}}{dx}\right)^2 dx - \int_0^L b u_{exact}\ dx - F u_{exact}(x=L)$$

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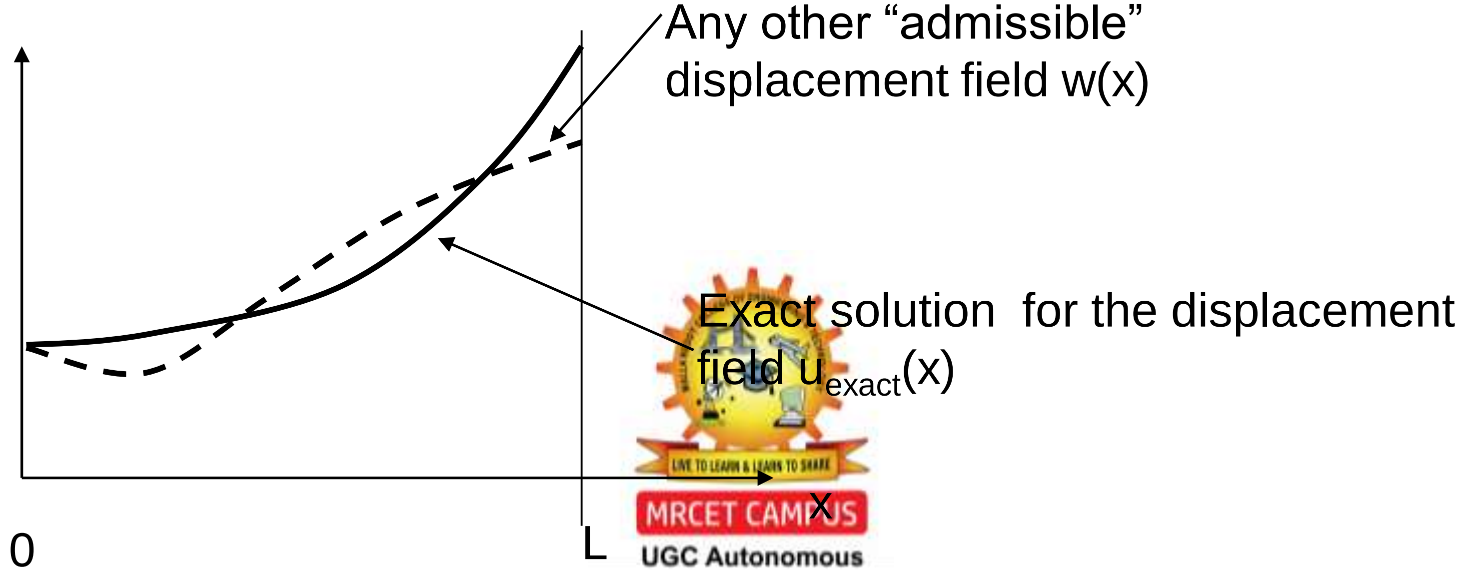
Potential energy of the axially loaded bar corresponding to the "**admissible**" displacement w(x)

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$



Example:

$$AE\frac{d^2u}{dx^2}+b=0; \qquad 0<x<L$$

$$u=0 \quad at \quad x=0$$

$$EA\frac{du}{dx}=F \quad at \ x=L$$

Assume EA=1; b=1; L=1; F=1
Analytical solution is

$$u_{exact}=2x-\frac{x^2}{2}$$

Potential energy corresponding to this analytical solution

$$\Pi(u_{exact})=\frac{1}{2}\int_0^1\left(\frac{du_{exact}}{dx}\right)^2 dx-\int_0^1 u_{exact}\,dx-u_{exact}(x=1)=-\frac{7}{6}$$

Now assume an admissible displacement

$$w = x$$

Why is this an "**admissible**" displacement? This displacement is quite arbitrary. But, it satisfies the given **displacement boundary condition** w(x=0)=0. Also, its first derivate does not blow up.

Potential energy corresponding to this admissible displacement

$$\Pi(w) = \frac{1}{2}\int_0^1 \left(\frac{dw}{dx}\right)^2 dx - \int_0^1 w\ dx - w(x=1) = -1$$

Notice

since $\quad -\frac{7}{6} < -1$

$$\boxed{\Pi(u_{exact}) < \Pi(w)}$$

Principle of Minimum Potential Energy

Among all admissible displacements that a body can have, the one that minimizes the total potential energy of the body satisfies the strong formulation

Mathematical statement: If 'u_{exact}' is the exact solution (which satisfies the differential equation together with the boundary conditions), and 'w' is an admissible displacement (that is quite arbitrary except for the fact that it **satisfies the displacement boundary conditions** and its **first derivative does not blow up**), then

$$\Pi(u_{exact}) < \Pi(w)$$

unless $w=u_{exact}$ (i.e. **the exact solution minimizes the potential energy**)

The Principle of Minimum Potential Energy and the strong formulation are exactly equivalent statements of the same problem.

The exact solution (u_{exact}) that satisfies the strong form, renders the potential energy of the system a minimum.

So, why use the Principle of Minimum Potential Energy?

The short answer is that it is much less demanding than the strong formulation. The long answer is, it

1. requires only the first derivative to be finite
2. incorporates the force boundary condition automatically. The admissible displacement (which is the function that you need to choose) needs to satisfy only the displacement boundary condition

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Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

Task is to find the function 'w' that minimizes the potential energy of the system

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.

Rayleigh-Ritz Principle

The minimization of the potential energy is difficult to perform exactly. The Rayleigh-Ritz principle is an **approximate** way of doing this.

Step 1. Assume a solution

$$w(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "***admissible***" functions and a_o, a_1, etc are constants to be determined from the solution.

Rayleigh-Ritz Principle

Step 2. Plug the approximate solution into the potential energy

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw}\ \mathrm{dx} - \mathrm{Fw}(\mathrm{x} = \mathrm{L})$$

$$\Rightarrow \Pi(\mathrm{a}_0, \mathrm{a}_1, \ldots) = \frac{1}{2}\int_0^L \mathrm{EA}\left(a_0 \frac{\mathrm{d}\varphi_0}{\mathrm{dx}} + a_1 \frac{\mathrm{d}\varphi_1}{\mathrm{dx}} + \ldots\right)^2 \mathrm{dx}$$

$$- \int_0^L \mathrm{b}\left(a_0\varphi_0 + a_1\varphi_1 + \ldots\right)\mathrm{dx}$$

$$- \mathrm{F}\left(a_0\varphi_0(x = L) + a_1\varphi_1(x = L) + \ldots\right)$$

Rayleigh-Ritz Principle

Step 3. Obtain the coefficients a_o, a_1, etc by setting

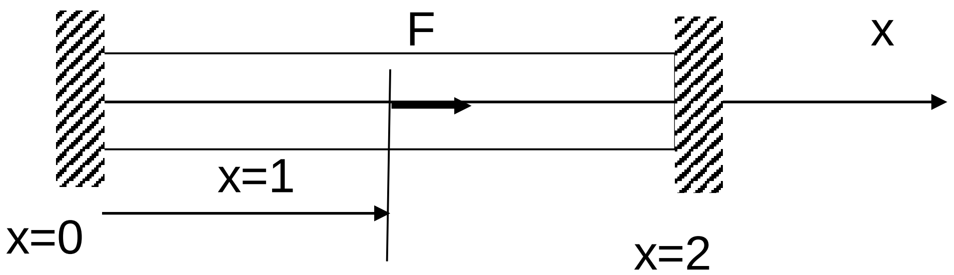
$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,\ldots$$

The approximate solution is

$$u(x) = a_0 \varphi_o(x) + a_1 \varphi_1(x) + a_2 \varphi_2(x) + \ldots$$

Where the coefficients have been obtained from step 3

Example of application of Rayleigh Ritz Principle



E=A=1
F=2

The potential energy of this bar (of length 2) is

$$\Pi(u) = \underbrace{\frac{1}{2}\int_0^2 \left(\frac{du}{dx}\right)^2 dx}_{\text{Strain Energy}} - \underbrace{Fu(x=1)}_{\substack{\text{Potential Energy} \\ \text{of load } F \text{ applied} \\ \text{at } x=1}}$$

Let us assume a polynomial "admissible" displacement field

$$u = a_0 + a_1 x + a_2 x^2$$

Note that this is NOT the analytical solution for this problem.

Example of application of Rayleigh Ritz Principle

For this "admissible" displacement to satisfy the **displacement boundary conditions** the following conditions must be satisfied:

$$\mathrm{u}(\mathrm{x}=0)=a_0=0$$

$$\mathrm{u}(\mathrm{x}=2)=a_0+2a_1+4a_2=0$$

Hence, we obtain

$$a_0=0$$

$$a_1=-2a_2$$

Hence, the "admissible" displacement simplifies to

$$\begin{aligned}\mathrm{u}&=a_0+a_1x+a_2x^2\\&=a_2\left(-2x+x^2\right)\end{aligned}$$

Now we apply **Rayleigh Ritz principle**, which says that if I plug this approximation into the expression for the potential energy Π, I can obtain the unknown (in this case a_2) by minimizing Π

$$\Pi(u) = \frac{1}{2}\int_0^2 \left(\frac{du}{dx}\right)^2 dx - Fu(x=1)$$

$$= \frac{1}{2}\int_0^2 \left(\frac{d}{dx}\left\{a_2\left(-2x+x^2\right)\right\}\right)^2 dx - F\left\{a_2\left(-2x+x^2\right)\right\}_{\substack{evaluated \\ at\ x=1}}$$

$$= \frac{4}{3}a_2^{\ 2} + 2a_2$$

$$\frac{\partial \Pi}{\partial a_2} = 0$$

$$\Rightarrow \frac{8}{3}a_2 + 2 = 0$$

$$\Rightarrow a_2 = -\frac{3}{4}$$

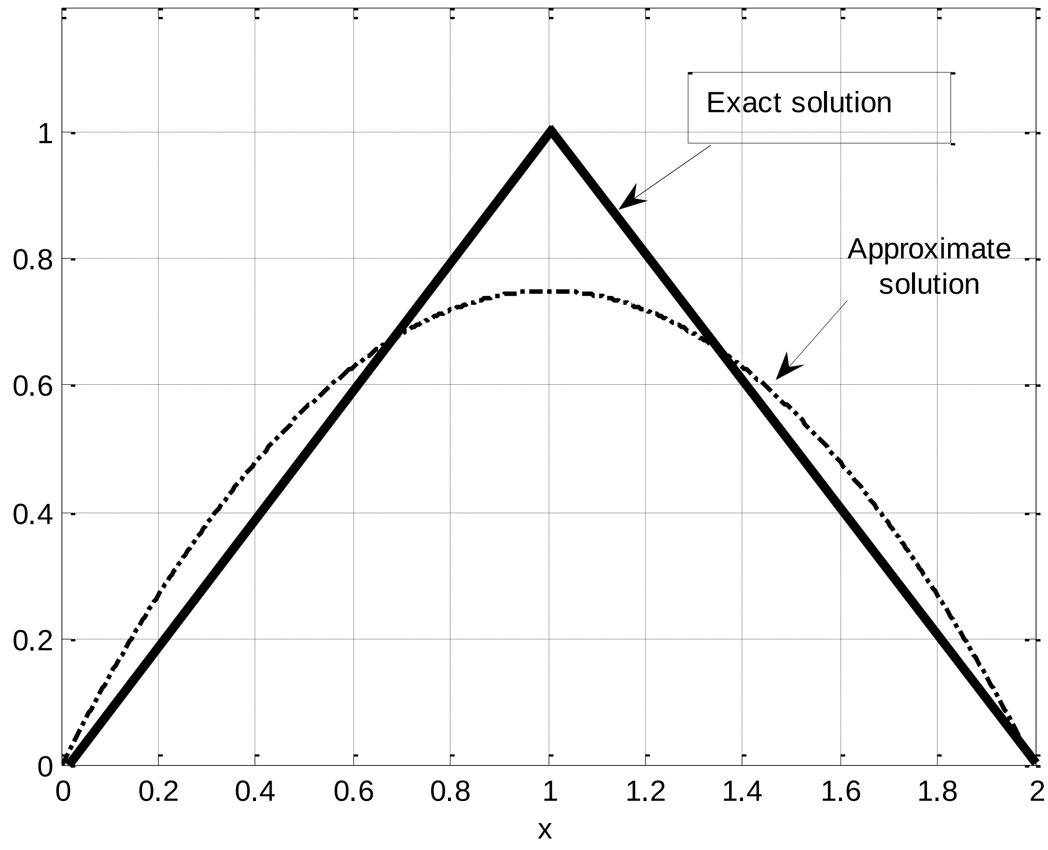
Hence the approximate solution to this problem, using the Rayleigh-Ritz principle is

$$
\begin{aligned}
\mathrm{u} &= a_0 + a_1 x + a_2 x^2 \\
&= a_2\left(-2x + x^2\right) \\
&= -\frac{3}{4}\left(-2x + x^2\right)
\end{aligned}
$$

Notice that the exact answer to this problem (can you prove this?) is

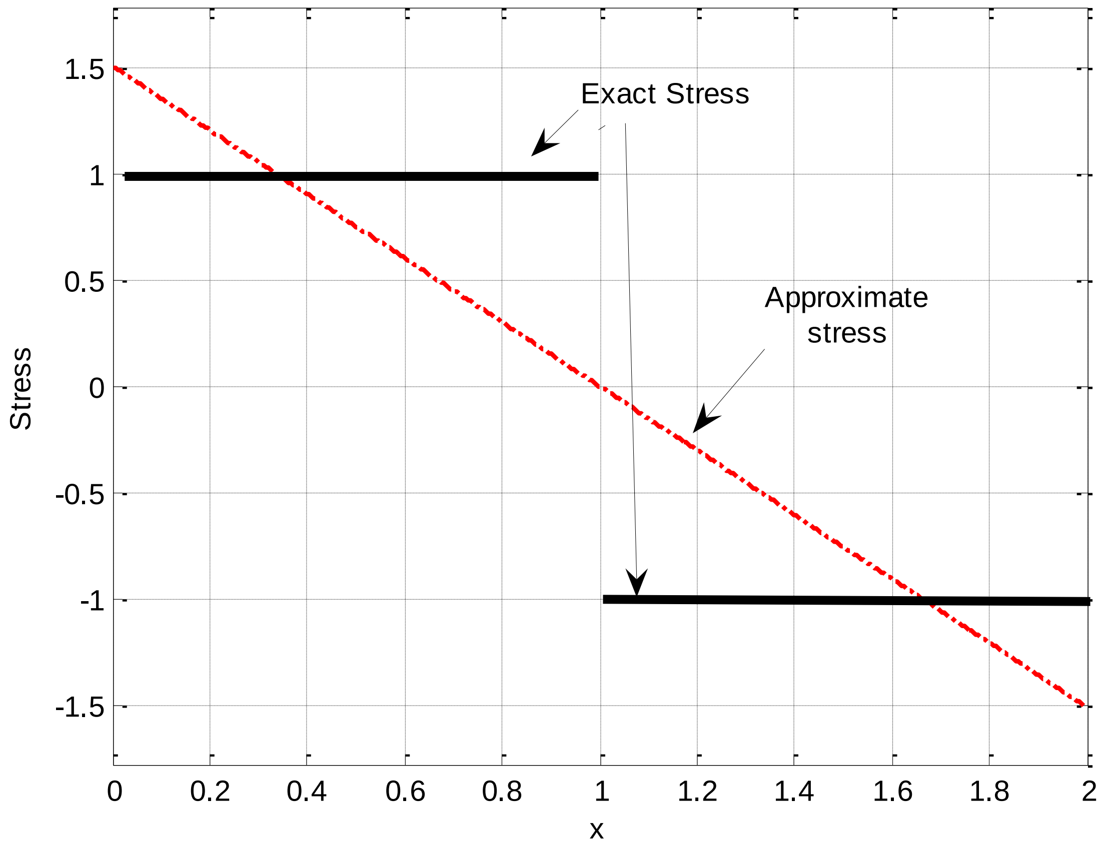
$$
\mathrm{u}_{\text{exact}} = \begin{cases} x & \text{for } 0 \le x < 1 \\ 2 - x & \text{for } 1 \le x \le 2 \end{cases}
$$

The **displacement** solution :



How can you improve the approximation?

The **stress** within the bar:



DEPARTMENT OF MECHANICAL ENGINEERING

INTRODUCTION TO FINITE ELEMENTS

FINITE ELEMENT FORMULATION FOR 1D ELASTICITY USING THE RAYLEIGH-RITZ PRINCIPLE

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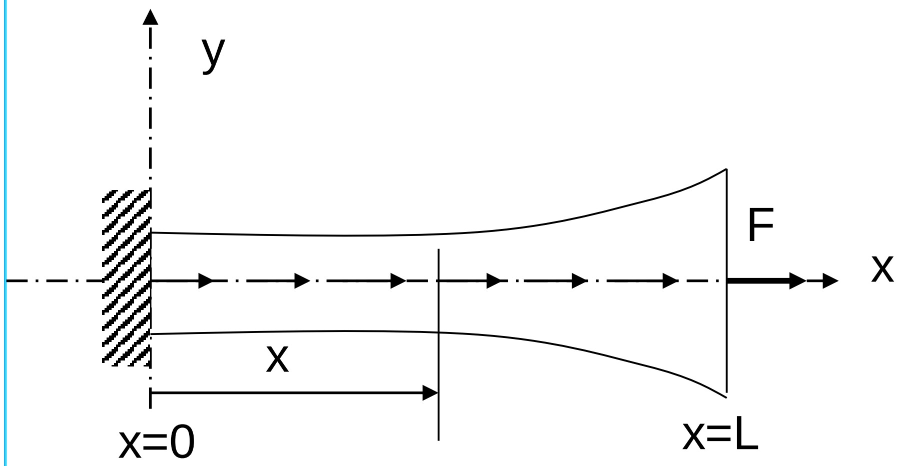
Reading assignment:

Lecture notes

Summary:

- **Stiffness matrix and nodal load vectors for 1D elasticity problem**

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution u(x)

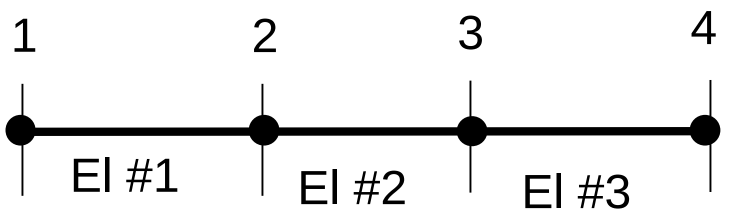
$$\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\ dx - Fu(x = L)$$

Potential energy of the bar corresponding to an **admissible displacement** w(x)

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

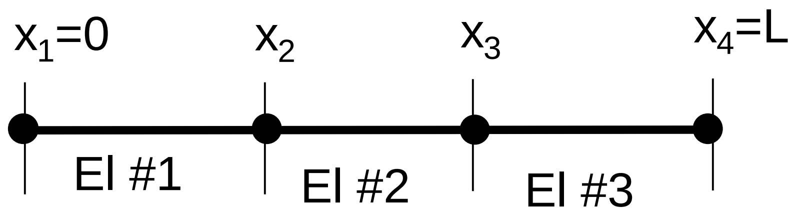
Finite element idea:

Step 1: Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$



Total potential energy

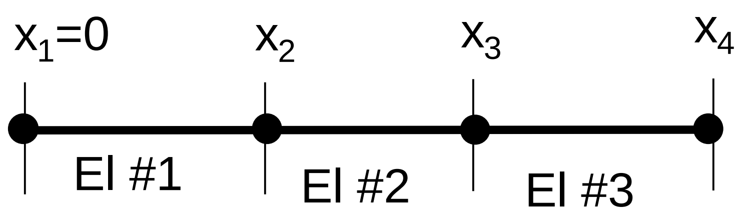
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\ dx$$

Potential energy of element 2:

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\ dx$$



Potential energy of element 3:

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\, dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

Step 2: Describe the behavior of each element

In the "**direct stiffness**" approach, we derived the stiffness matrix of each element directly (See lecture on Springs/Trusses).

Now, we will first approximate the **displacement** inside each element and then show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT
TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT
TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT (this class) USING THE RAYLEIGH-RITZ PRINCIPLE

Summary

Inside an element, the three most important approximations **in terms of the nodal displacements** ($\underline{d}$) are:

Displacement approximation in terms of shape functions

$$w(x) = \underline{N}\,\underline{d} \qquad (1)$$

Strain approximation in terms of strain-displacement matrix

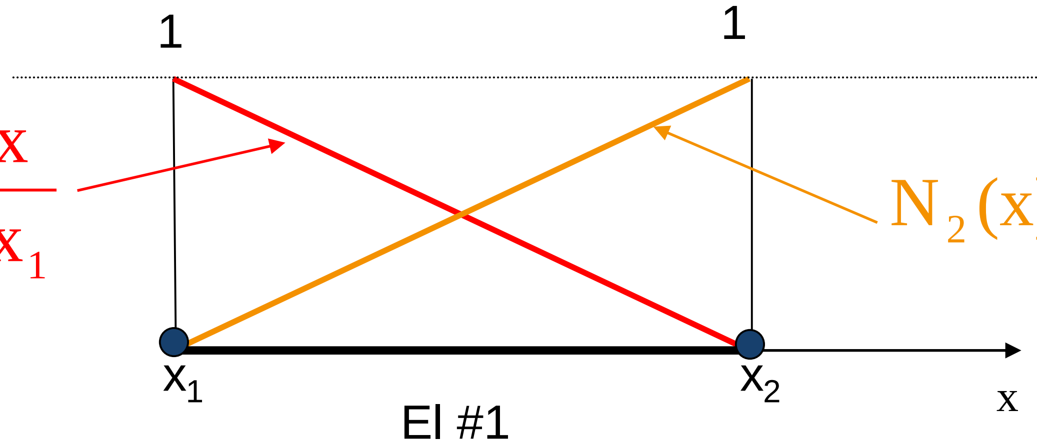
$$\varepsilon(x) = \underline{B}\,\underline{d} \qquad (2)$$

Stress approximation in terms of strain-displacement matrix and Young's modulus

$$\sigma = E\underline{B}\,\underline{d} \qquad (3)$$

The shape functions for a **1D linear element**

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Within the element, the displacement approximation is

$$w(x) = \frac{x_2 - x}{x_2 - x_1} d_{1x} + \frac{x - x_1}{x_2 - x_1} d_{2x}$$

For a **linear element**

Displacement approximation in terms of shape functions

$$w(x) = \begin{bmatrix} \frac{x_2 - x}{x_2 - x_1} & \frac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

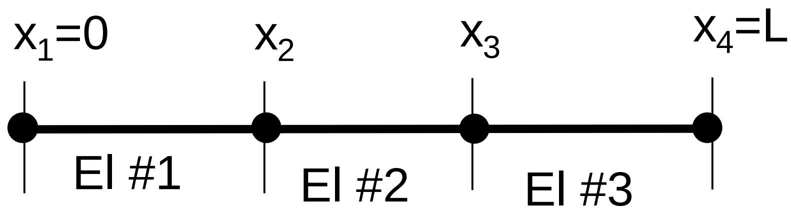
Strain approximation

$$\varepsilon = \frac{dw}{dx} = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Stress approximation

$$\sigma = E\varepsilon = \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Why is the approximation "admissible"?



For the entire bar, the displacement approximation is

$$w(x) = w^{(1)}(x) + w^{(2)}(x) + w^{(3)}(x)$$

Where $w^{(i)}(x)$ is the displacement approximation within element (i).
Let use set d_{1x}=0. Then, can you seen that the above approximation does satisfy the two conditions of being an admissible function on the entire bar, i.e.,

$$(1)\ w(x=0)=0$$

$$(2)\ \frac{dw}{dx}\ \text{exists}$$

TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT USING THE RAYLEIGH-RITZ PRINCIPLE

Potential energy of element 1:

$$\Pi_1(\mathrm{w}) = \frac{1}{2}\int_{x_1}^{x_2} \sigma \;\; \varepsilon \;\; A\mathrm{dx} - \int_{x_1}^{x_2} \mathrm{bw} \;\; \mathrm{dx}$$

Lets plug in the **approximation**

$$\boxed{\mathrm{w(x)} = \underline{\mathrm{N}}\;\underline{\mathrm{d}}} \qquad \boxed{\varepsilon(\mathrm{x}) = \underline{\mathrm{B}}\;\underline{\mathrm{d}}} \qquad \boxed{\sigma = \mathrm{E}\underline{\mathrm{B}}\;\underline{\mathrm{d}}}$$

$$\Pi_1(\underline{\mathrm{d}}) = \frac{1}{2}\underline{\mathrm{d}}^T \left(\int_{x_1}^{x_2} \underline{\mathrm{B}}^{\mathrm{T}}\mathrm{E}\underline{\mathrm{B}} \;\; A\mathrm{dx}\right)\underline{\mathrm{d}} - \underline{\mathrm{d}}^T\left(\int_{x_1}^{x_2} \underline{\mathrm{N}}^T \;\; \mathrm{b} \;\; \mathrm{dx}\right)$$

Lets see what the matrix

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

is for a 1D linear element

Recall that

$$\boxed{\underline{B} = \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}}$$

Hence

$$\underline{B}^T \; E \; \underline{B} = \frac{1}{x_2 - x_1}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} E \frac{1}{x_2 - x_1}\begin{bmatrix} -1 & 1 \end{bmatrix}$$

$$= \frac{E}{(x_2 - x_1)^2}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix} = \frac{E}{(x_2 - x_1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \frac{1}{(x_2 - x_1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_{x_1}^{x_2} AEdx = \left(\int_{x_1}^{x_2} AEdx\right)\frac{1}{(x_2 - x_1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now, if we **assume** E and A are constant

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \left(\int_{x_1}^{x_2} AEdx\right)\frac{1}{(x_2 - x_1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{AE(x_2 - x_1)}{(x_2 - x_1)^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{AE}{(x_2 - x_1)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Remembering that $(x_2 - x_1)$ is the length of the element, this is the **stiffness matrix** we had derived directly before using the **direct stiffness** approach!!

Then why is it necessary to go through this complicated procedure??

1. Easy to handle **nonuniform** E and A
2. Easy to handle **distributed loads**

For nonuniform E and A, i.e. E(x) and A(x), the **stiffness matrix** of the linear element will **NOT** be

$$\frac{EA}{(x_2 - x_1)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

But it will **ALWAYS be**

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

Now lets go back to

$$\Pi_1(\underline{d}) = \frac{1}{2}\underline{d}^T\left(\underbrace{\int_{x_1}^{x_2}\underline{B}^T E \underline{B}\ \ A dx}_{\underline{k}}\right)\underline{d} - \underline{d}^T\left(\underbrace{\int_{x_1}^{x_2}\underline{N}^T\ \ b\ \ dx}_{\underline{f_b}}\right)$$

$$= \frac{1}{2}\underline{d}^T\underline{k}\,\underline{d} - \underline{d}^T\ \underline{f_b}$$

Element **stiffness matrix**

$$\boxed{\underline{k} = \int_{x_1}^{x_2}\underline{B}^T E \underline{B}\ \ A dx}$$

Element **nodal load vector due to distributed body force**

$$\boxed{\underline{f_b} = \int_{x_1}^{x_2}\underline{N}^T\ \ b\, dx}$$

Apply Rayleigh-Ritz principle for the 1D linear element

$$\left.\begin{aligned}\frac{\partial \Pi_1(\underline{d})}{\partial d_{1x}} = 0 \\ \frac{\partial \Pi_1(\underline{d})}{\partial d_{2x}} = 0\end{aligned}\right\} \Rightarrow \frac{\partial \Pi_1(\underline{d})}{\partial \underline{d}} = 0$$

Recall from linear algebra (Lecture notes on Linear Algebra)

$$\Pi_1(\underline{d}) = \frac{1}{2}\underline{d}^T \underline{k}\,\underline{d} - \underline{d}^T \underline{f_b}$$

$$\Rightarrow \frac{\partial \Pi_1(\underline{d})}{\partial \underline{d}} = \underline{k}\,\underline{d} - \underline{f_b}$$

Hence

$$\frac{\partial \Pi_1(\underline{d})}{\partial \underline{d}} = 0$$

$$\boxed{\Rightarrow \underline{k}\underline{d} = \underline{f_b}}$$

Exactly the same equation that we had before, except that the stiffness matrix and nodal force vectors are more general

Recap of the properties of the element stiffness matrix

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^{T} E \underline{B} \; A dx$$

1. The stiffness matrix is **singular** and is therefore non-invertible
2. The stiffness matrix is **symmetric**
3. **Sum of any row** (or **column**) of the stiffness matrix is zero!

k_{11}

Why?

Sum of any row (or column) of the stiffness matrix is zero

Consider a **rigid body motion** of the element

$$\underline{d} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Element strain $\varepsilon = 0 = \underline{B}\,\underline{d}$

$$\Rightarrow \underline{k}\ \underline{d} = \left(\int_{x_1}^{x_2} \underline{B}^T E \underline{B}\ A dx \right) \underline{d}$$

$$= \int_{x_1}^{x_2} \underline{B}^T E \left(\underline{B}\underline{d} \right)\ A dx$$

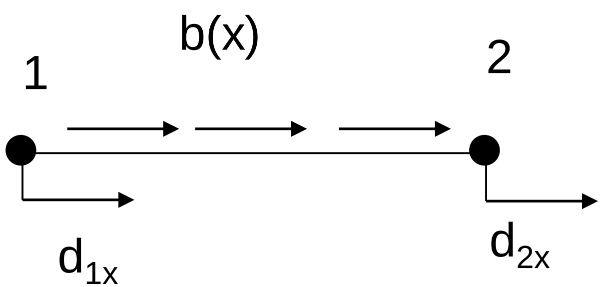
$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\underline{k}\ \underline{d} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow k_{11} + k_{12} = 0 \ \text{ and } \ k_{21} + k_{22} = 0$$

The nodal load vector

$$\underline{f}_b = \int_{x_1}^{x_2} \underline{N}^T \text{ b dx}$$



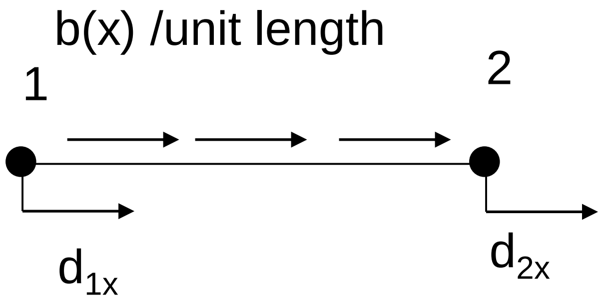
$$\underline{f}_b = \int_{x_1}^{x_2} \underline{N} \text{ b dx} = \int_{x_1}^{x_2} \begin{Bmatrix} N_1(x) \\ N_2(x) \end{Bmatrix} \text{ b dx}$$

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{Bmatrix} \int_{x_1}^{x_2} N_1(x) \text{ b dx} \\ \int_{x_1}^{x_2} N_2(x) \text{ b dx} \end{Bmatrix}$$

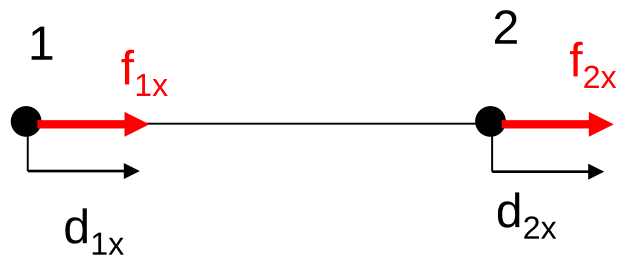
$$f_{1x} = \int_{x_1}^{x_2} N_1(x) \text{ b dx}$$

$$f_{2x} = \int_{x_1}^{x_2} N_2(x) \text{ b dx}$$

"**Consistent**" nodal loads



Replaced by



A distributed load is represented by two nodal loads in a consistent manner

e.g., if b=1

$$f_{1x} = \int_{x_1}^{x_2} N_1(x)\,\mathrm{b}\,\mathrm{dx} = \int_{x_1}^{x_2} N_1(x)\ \ \mathrm{dx} = \frac{\mathrm{x}_2 - x_1}{2}$$

$$f_{2x} = \int_{x_1}^{x_2} N_2(x)\,\mathrm{b}\,\mathrm{dx} = \int_{x_1}^{x_2} N_2(x)\ \ \mathrm{dx} = \frac{\mathrm{x}_2 - x_1}{2}$$

Divide the **total force** into two equal halves and lump them at the nodes

What happens if b(x)=x?

Summary: For each element

Displacement approximation in terms of shape functions

$$w(x) = \underline{N}\,\underline{d}$$

Strain approximation in terms of strain-displacement matrix

$$\varepsilon(x) = \underline{B}\,\underline{d}$$

Stress approximation

$$\sigma = E\underline{B}\,\underline{d}$$

Element stiffness matrix

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

Element nodal load vector

$$\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T \; b \, dx$$

What happens for element #3?

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\ dx - Fw(x = L)$$

For element 3

$$w(x) = \begin{bmatrix} \dfrac{x_4 - x}{x_4 - x_3} & \dfrac{x - x_3}{x_4 - x_3} \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{4x} \end{Bmatrix}$$

$$\Rightarrow w(x = L) = d_{4x}$$

The discretized form of the potential energy

$$\Pi_3(\underline{d}) = \frac{1}{2}\underline{d}^T \left(\int_{x_3}^{x_4} \underline{B}^T E \underline{B}\ \ A dx\right)\underline{d} - \underline{d}^T\left(\int_{x_3}^{x_4} \underline{N}^T\ \ b\ \ dx\right) - Fd_{4x}$$

What happens for element #3?

Now apply Rayleigh-Ritz principle

$$\frac{\partial \Pi_3(\underline{\mathrm{d}})}{\partial \underline{\mathrm{d}}} = 0$$

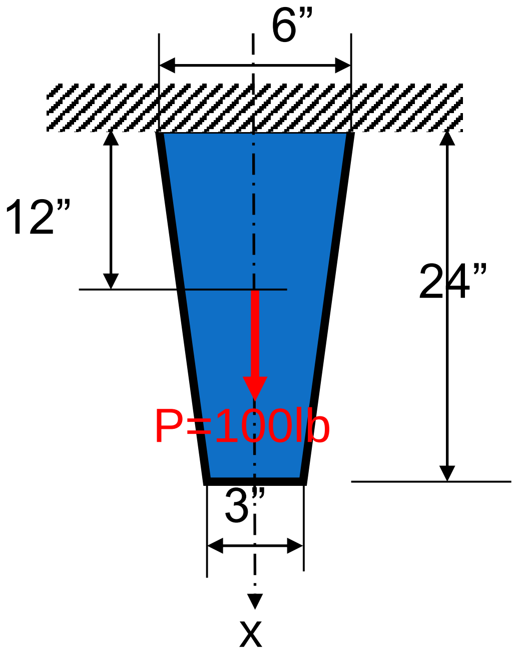
$$\Rightarrow \underline{k}\underline{\mathrm{d}} = \underline{f_b} + \begin{Bmatrix} 0 \\ \mathrm{F} \end{Bmatrix}$$

Hence there is an extra load term on the right hand side due to the concentrated force F applied to the right end of the bar.

NOTE that whenever you have a concentrated load at ANY node, that load should be applied as an extra right hand side term.

Step3:Assembly exactly as you had done before, assemble the global stiffness matrix and global load vector and solve the resulting set of equations by properly taking into account the displacement boundary conditions

Problem:



$E=30x10^6$ psi

ρ=0.2836 lb/in^3

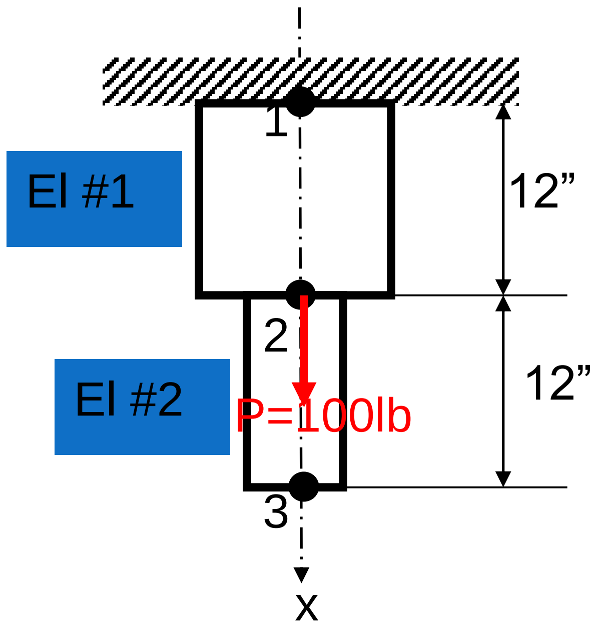
Thickness of plate, t=1”

Model the plate as 2 finite elements and

(1) Write the expression for element stiffness matrix and body force vectors
(2) Assemble the global stiffness matrix and load vector
(3) Solve for the unknown displacements
(4) Evaluate the stress in each element
(5) Evaluate the reaction in each support

Solution (1)

Finite element model



Node-element connectivity chart

Element #	Node 1	Node 2
1	1	2
2	2	3

Stiffness matrix of El #1

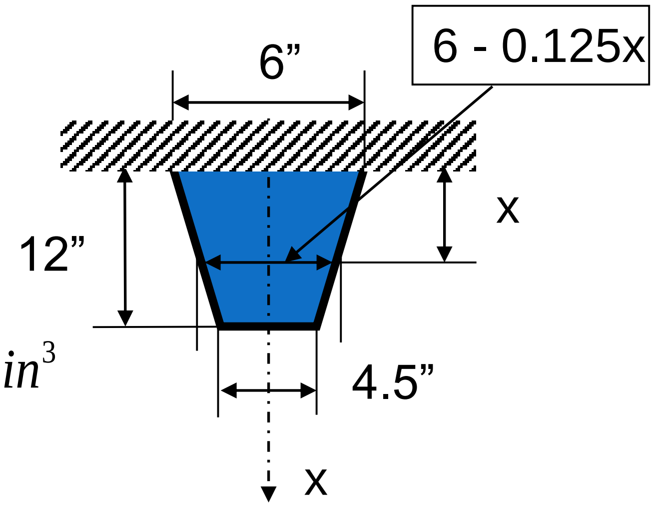
$$\underline{k}^{(1)} = \int_0^{12} \underline{B}^T E \underline{B} \; A dx = \frac{E}{(12)^2}\left(\int_0^{12} A(x)dx\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\int_0^{12} A(x)dx = \int_0^{12} t(6-0.125x)dx = t\int_0^{12}(6-0.125x)dx = 63\, in^3$$

$$\Rightarrow \underline{k}^{(1)} = \frac{E}{(12)^2}(63)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 13.125\times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Stiffness matrix of El #2

$$\underline{k}^{(2)} = \int_{12}^{24} \underline{B}^T E \underline{B} \; A dx = \frac{E}{(12)^2}\left(\int_{12}^{24} A(x)dx\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\int_{12}^{24} A(x)dx = \int_{12}^{24} t(6-0.125x)dx = t\int_{12}^{24}(6-0.125x)dx = 45\ in^3$$

$$\Rightarrow \underline{k}^{(2)} = \frac{E}{(12)^2}(45)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 9.375\times 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Now compute the element load vector due to distributed body force (weight)

$$\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T \text{ b dx}$$

For element #1

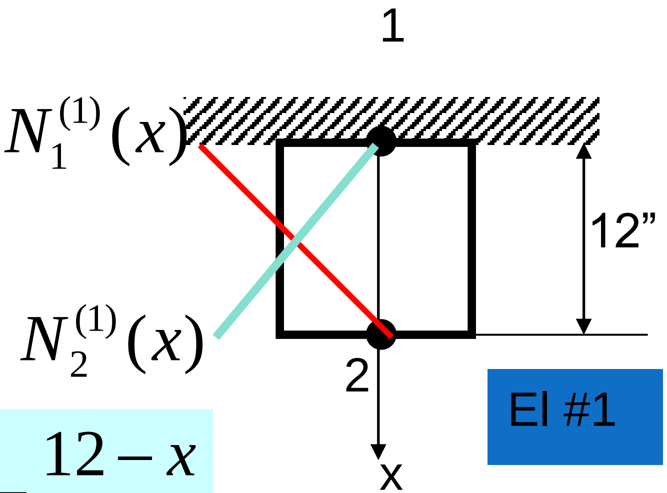
$$\underline{f_b}^{(1)} = \int_0^{12} \underline{N}^T \, b \, dx = \int_0^{12} \underline{N}^T \, (\rho A) \, dx$$

$$= \rho \int_0^{12} \underline{N}^T \, A \, dx$$

$$= \rho \int_0^{12} \begin{Bmatrix} N_1^{(1)}(x) \\ N_2^{(1)}(x) \end{Bmatrix} \underbrace{t(6-0.125x)}_{A(x)} \, dx$$

$$= 0.2836 \begin{Bmatrix} 33 \\ 30 \end{Bmatrix} lb$$

$$= \begin{Bmatrix} 9.3588 \\ 8.508 \end{Bmatrix} lb$$



$$N_1^{(1)}(x) = \frac{12-x}{12}$$

$$N_2^{(1)}(x) = \frac{x}{12}$$

Superscript in parenthesis indicates element number

For element #2

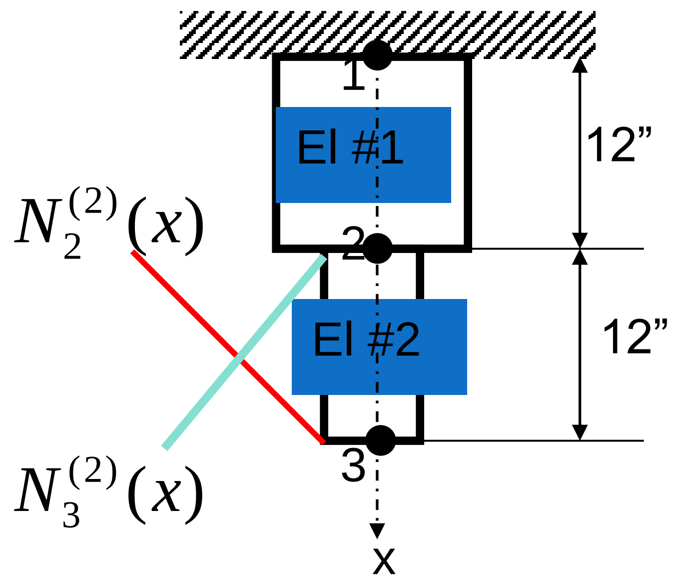
$$\underline{f_b}^{(2)} = \int_{12}^{24} \underline{\mathrm{N}}^T \, \mathrm{b} \, \mathrm{dx} = \int_{12}^{24} \underline{\mathrm{N}}^T \, (\rho \mathrm{A}) \, \mathrm{dx}$$

$$= \rho \int_{12}^{24} \underline{\mathrm{N}}^T \, A \, \mathrm{dx}$$

$$= \rho \int_{12}^{24} \begin{Bmatrix} N_2^{(2)}(x) \\ N_3^{(2)}(x) \end{Bmatrix} \underbrace{t(6-0.125x)}_{A(x)} \, \mathrm{dx}$$

$$= 0.2836 \begin{Bmatrix} 24 \\ 21 \end{Bmatrix} \, lb$$

$$= \begin{Bmatrix} 6.8064 \\ 5.9556 \end{Bmatrix} lb$$



$$N_2^{(2)}(x) = \frac{24-x}{12}$$

$$N_3^{(2)}(x) = \frac{x-12}{12}$$

Solution (2) Assemble the system equations

$$\underline{K} = 10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix}$$

$$\underline{f} = \underline{f_b} + \underline{f}_{concentrated\ load}$$

$$\underline{f_b} = \begin{Bmatrix} 9.3588 \\ 8.508 + 6.8064 \\ 5.9556 \end{Bmatrix} lb$$

$$\underline{f}_{concentrated\ load} = \begin{Bmatrix} 0 \\ 100 \\ 0 \end{Bmatrix} lb$$

$$\Rightarrow \underline{f} = \begin{Bmatrix} 9.3588 \\ 115.3144 \\ 5.9556 \end{Bmatrix} lb$$

Solution (3)

Hence we need to solve

$$10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 9.3588 + R_1 \\ 115.3144 \\ 5.9556 \end{Bmatrix}$$

R_1 is the reaction at node 1.
Notice that since the boundary condition at x=0 (d_{1x}=0) has not been taken into account, the system matrix is not invertible.
Incorporating the boundary condition d_{1x}=0 we need to solve the following set of equations

$$10^6 \times \begin{bmatrix} 22.5 & -9.375 \\ -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 115.3144 \\ 5.9556 \end{Bmatrix}$$

Solve to obtain

$$\begin{Bmatrix} d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 0.92396 \times 10^{-5} \\ 0.98749 \times 10^{-5} \end{Bmatrix} in$$

Solution (4) Stress in elements

Notice that since we are using linear elements, the stress within each element is constant.

In element #1

$$\sigma^{(1)} = E \underline{B}^{(1)} \underline{d}^{(1)}$$

$$= \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} \mathrm{d}_{1x} \\ \mathrm{d}_{2x} \end{Bmatrix}$$

$$= \frac{30 \times 10^6}{12} \mathrm{d}_{2x} \quad \because \mathrm{d}_{1x} = 0$$

$$= 23.099 \ psi$$

In element #2

$$\sigma^{(2)} = E\underline{B}^{(2)}\underline{d}^{(2)}$$

$$= \frac{E}{x_3 - x_2}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}\mathrm{d}_{2x}\\ \mathrm{d}_{3x}\end{Bmatrix}$$

$$= \frac{30\times10^6}{12}\left(\mathrm{d}_{3x}\text{-}\mathrm{d}_{2x}\right)$$

$$=1.5882\ psi$$

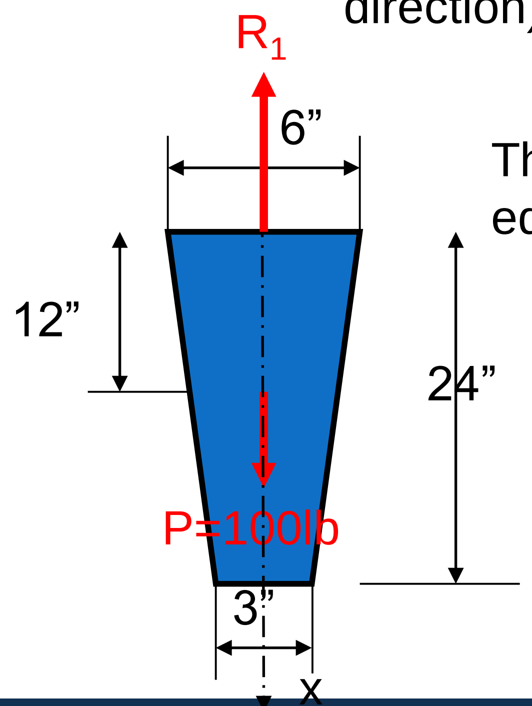
Solution (5) Reaction at support

Go back to the ***first line*** of the global equilibrium equations…

$$10^6 \times \begin{bmatrix} 13.125 & -13.125 & 0 \\ -13.125 & 22.5 & -9.375 \\ 0 & -9.375 & 9.375 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} 9.3588 + R_1 \\ 115.3144 \\ 5.9556 \end{Bmatrix}$$

$\Rightarrow R_1 = -130.6288\, lb$ (The –ve sign indicates that the force is in the –ve x-direction)

Check



The reaction at the wall from force equilibrium in the x-direction

$$R_1 = P + \int_{x=0}^{24} \rho A(x)\, dx$$

$$= 100 + \rho t \int_{x=0}^{24} (6 - 0.125x)\, dx$$

$$= 130.6288\, lb$$

Problem: Can you solve for the displacement and stresses analytically?

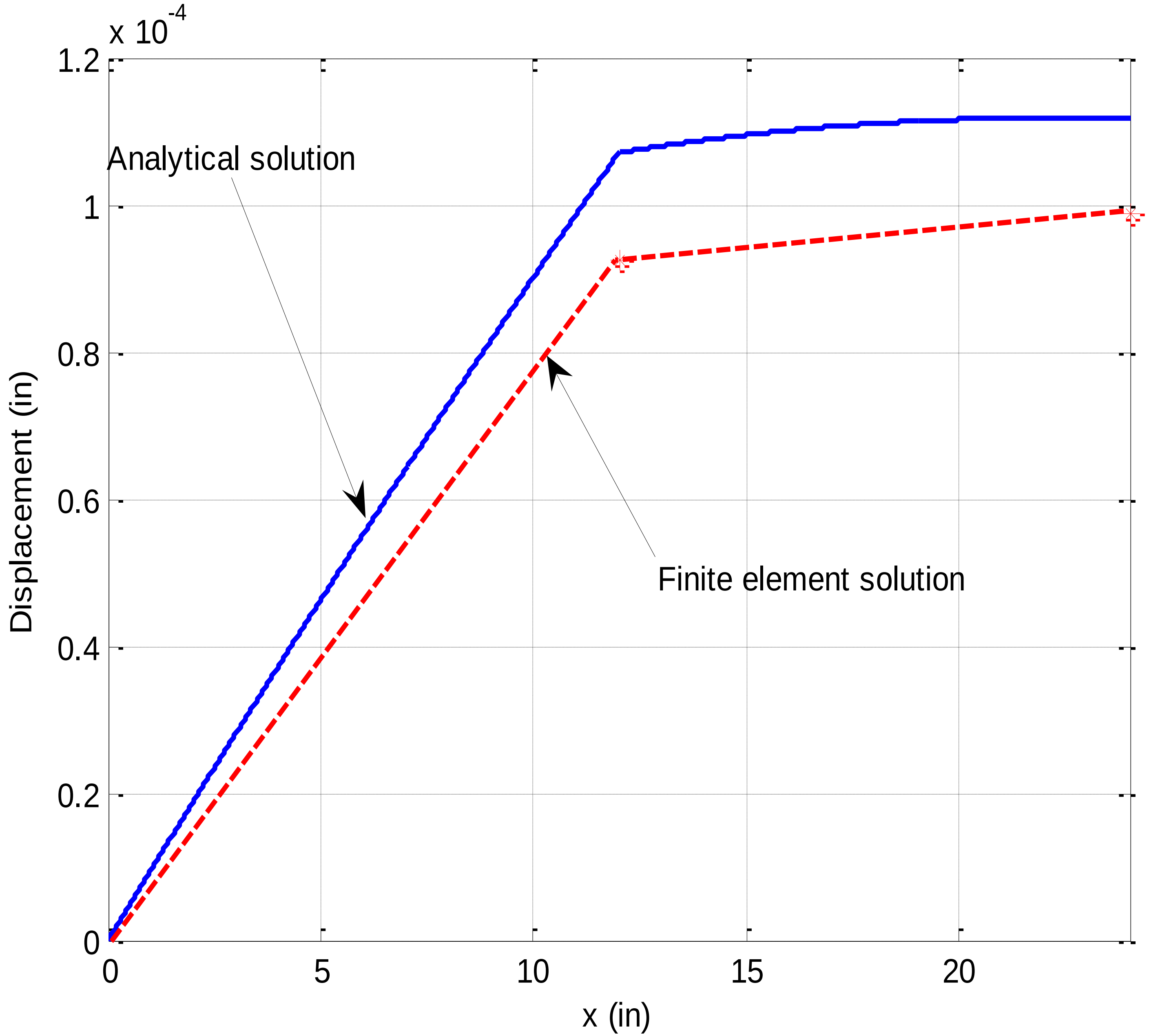
Check out

$$u_{anal} = \begin{cases} -4.727\times10^{-9}x^2 + 9.487\times10^{-7}x & for\ 0 \le x < 12 \\ -4.727\times10^{-9}x^2 + 2.0797\times10^{-7}x + 8.89\times10^{-6} & for\ 12 \le x \le 24 \end{cases}$$

Stress

$$\sigma(x)_{anal} = E\frac{du_{anal}}{dx} = 30\times10^{6}\frac{du_{anal}}{dx}$$

Comparison of displacement solutions



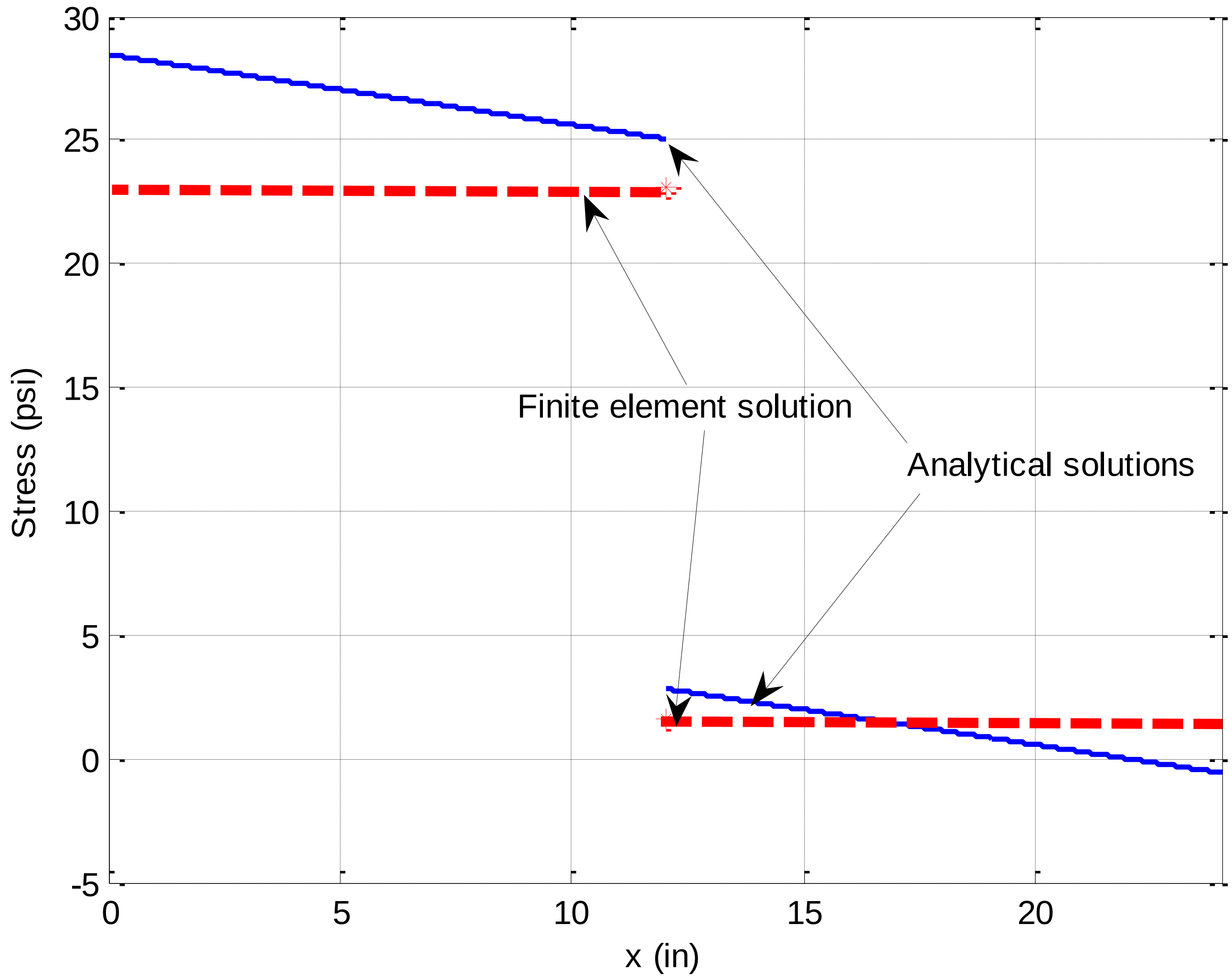
Notice:

1. Slope discontinuity at x=12 (why?)
2. The finite element solution does not produce the exact solution even at the nodes
3. We may improve the solution by

   (1) Increasing the number of elements

   (2) Using higher order elements (e.g., quadratic instead of linear)

Comparison of stress solutions



The analytical as well as the finite element stresses are discontinuous across the elements

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INTRODUCTION TO FINITE ELEMENTS

SHAPE FUNCTIONS IN 1D

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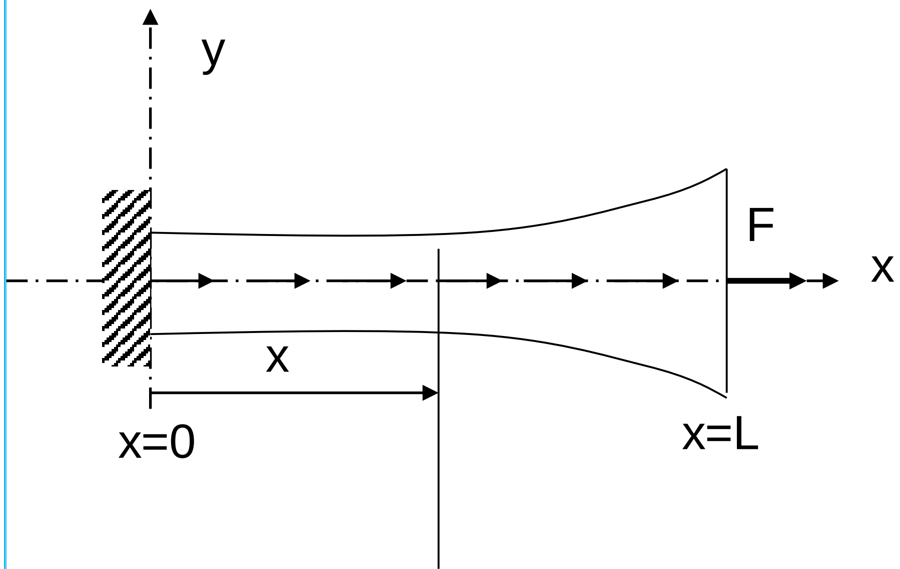
Reading assignment:

Lecture notes, Logan 2.2, 3.1

Summary:

- **Linear shape functions in 1D**
- **Quadratic and higher order shape functions**
- **Approximation of strains and stresses in an element**

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

Potential energy of the axially loaded bar corresponding to the exact solution u(x)

$$\Pi(u) = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - \int_0^L bu\ dx - Fu(x = L)$$

Finite element formulation, takes as its starting point, not the strong formulation, but the **Principle of Minimum Potential Energy**.

Task is to find the function 'w' that minimizes the potential energy of the system

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$

From the Principle of Minimum Potential Energy, that function 'w' is the exact solution.

Rayleigh-Ritz Principle

Step 1. Assume a solution

$$w(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + \ldots$$

Where $\varphi_o(x)$, $\varphi_1(x)$,… are "admissible" functions and a_o, a_1, etc are constants to be determined.

Step 2. Plug the approximate solution into the potential energy

$$\Pi(\mathrm{w}) = \frac{1}{2}\int_0^L \mathrm{EA}\left(\frac{\mathrm{dw}}{\mathrm{dx}}\right)^2 \mathrm{dx} - \int_0^L \mathrm{bw\ dx} - \mathrm{Fw(x = L)}$$

Step 3. Obtain the coefficients a_o, a_1, etc by setting

$$\frac{\partial \Pi(\mathrm{w})}{\partial a_i} = 0, \quad i = 0,1,2,\ldots$$

UGC Autonomous

The approximate solution is

$$u(x) = a_0\varphi_o(x) + a_1\varphi_1(x) + a_2\varphi_2(x) + ...$$

Where the coefficients have been obtained from step 3

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Need to find a systematic way of choosing the approximation functions.

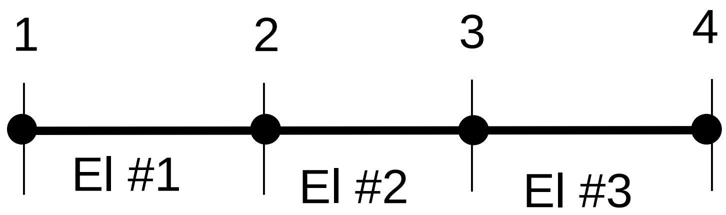
One idea: Choose polynomials!

$w(x) = a_0$ Is this good? (Is '1' an "admissible" function?)

$w(x) = a_1 x$ Is this good? (Is 'x' an "admissible" function?)

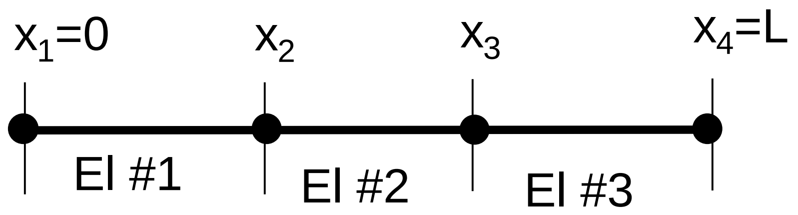
Finite element idea:

Step 1: Divide the truss into **finite elements** connected to each other through special points ("**nodes**")



Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\, dx - Fw(x = L)$$



Total potential energy

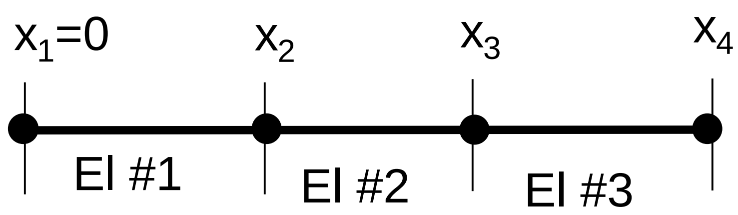
$$\Pi(w) = \frac{1}{2}\int_0^L EA\left(\frac{dw}{dx}\right)^2 dx - \int_0^L bw\ dx - Fw(x = L)$$

Potential energy of element 1:

$$\Pi_1(w) = \frac{1}{2}\int_{x_1}^{x_2} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_1}^{x_2} bw\ dx$$

Potential energy of element 2:

$$\Pi_2(w) = \frac{1}{2}\int_{x_2}^{x_3} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_2}^{x_3} bw\ dx$$



Potential energy of element 3:

$$\Pi_3(w) = \frac{1}{2}\int_{x_3}^{x_4} EA\left(\frac{dw}{dx}\right)^2 dx - \int_{x_3}^{x_4} bw\, dx - Fw(x = L)$$

Total potential energy=sum of potential energies of the elements

$$\Pi(w) = \Pi_1(w) + \Pi_2(w) + \Pi_3(w)$$

Step 2: Describe the behavior of each element

Recall that in the "**direct stiffness**" approach for a bar element, we derived the stiffness matrix of each element directly (See lecture on Trusses) using the following steps:

TASK 1: Approximate the displacement within each bar as a straight line
TASK 2: Approximate the strains and stresses and realize that a bar (with the approximation stated in Task 1) is exactly like a spring with k=EA/L
TASK 3: Use the principle of **force equilibrium** to generate the stiffness matrix

Now, we will show you a systematic way of deriving the stiffness matrix (sections 2.2 and 3.1 of Logan).

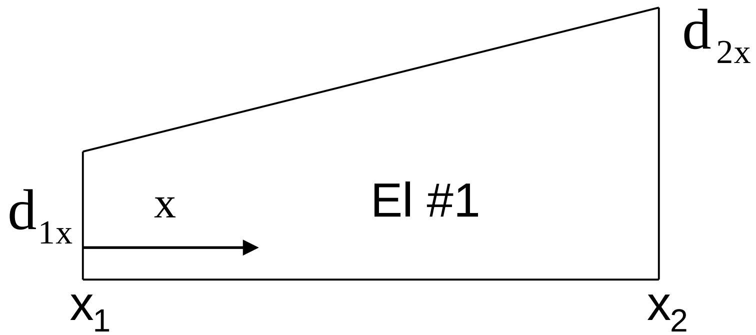
TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT
TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT
TASK 3: DERIVE THE STIFFNESS MATRIX OF EACH ELEMENT (next class) USING THE PRINCIPLE OF MIN. POT ENERGY

Notice that the first two tasks are similar in the two methods. The only difference is that now we are going to use the principle of minimum potential energy, rather than force equilibrium, to derive the stiffness matrix.

TASK 1: APPROXIMATE THE DISPLACEMENT WITHIN EACH ELEMENT

Simplest assumption: displacement varying linearly inside each bar

$$w(x) = a_0 + a_1 x$$



How to obtain a_0 and a_1?

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$
$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

$$w(x_1) = a_0 + a_1 x_1 = d_{1x}$$

$$w(x_2) = a_0 + a_1 x_2 = d_{2x}$$

Solve simultaneously

$$a_0 = \frac{x_2}{x_2 - x_1} d_{1x} - \frac{x_1}{x_2 - x_1} d_{2x}$$

$$a_1 = -\frac{1}{x_2 - x_1} d_{1x} + \frac{1}{x_2 - x_1} d_{2x}$$

Hence

$$w(x) = a_0 + a_1 x = \underbrace{\frac{x_2 - x}{x_2 - x_1}}_{N_1(x)} d_{1x} + \underbrace{\frac{x - x_1}{x_2 - x_1}}_{N_2(x)} d_{2x} = N_1(x) d_{1x} + N_2(x) d_{2x}$$

"**Shape functions**" $N_1(x)$ and $N_2(x)$

In matrix notation, we write

$$\boxed{w(x) = \underline{N}\,\underline{d}} \qquad (1)$$

Vector of nodal shape functions

$$\underline{N} = \left[N_1(x) \quad N_2(x) \right] = \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

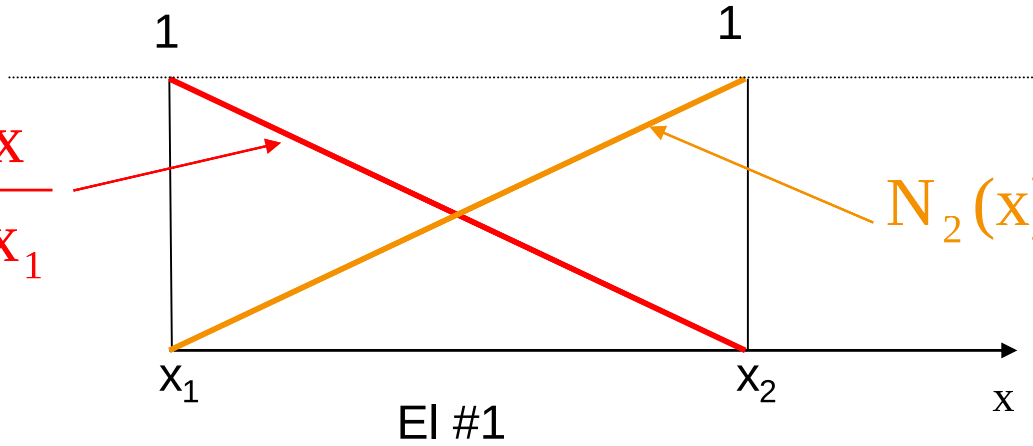
Vector of nodal displacements

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

NOTES: PROPERTIES OF THE SHAPE FUNCTIONS

1. <u>Kronecker delta property</u>: The shape function at any node has a value of 1 at that node and a value of zero at ALL other nodes.

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

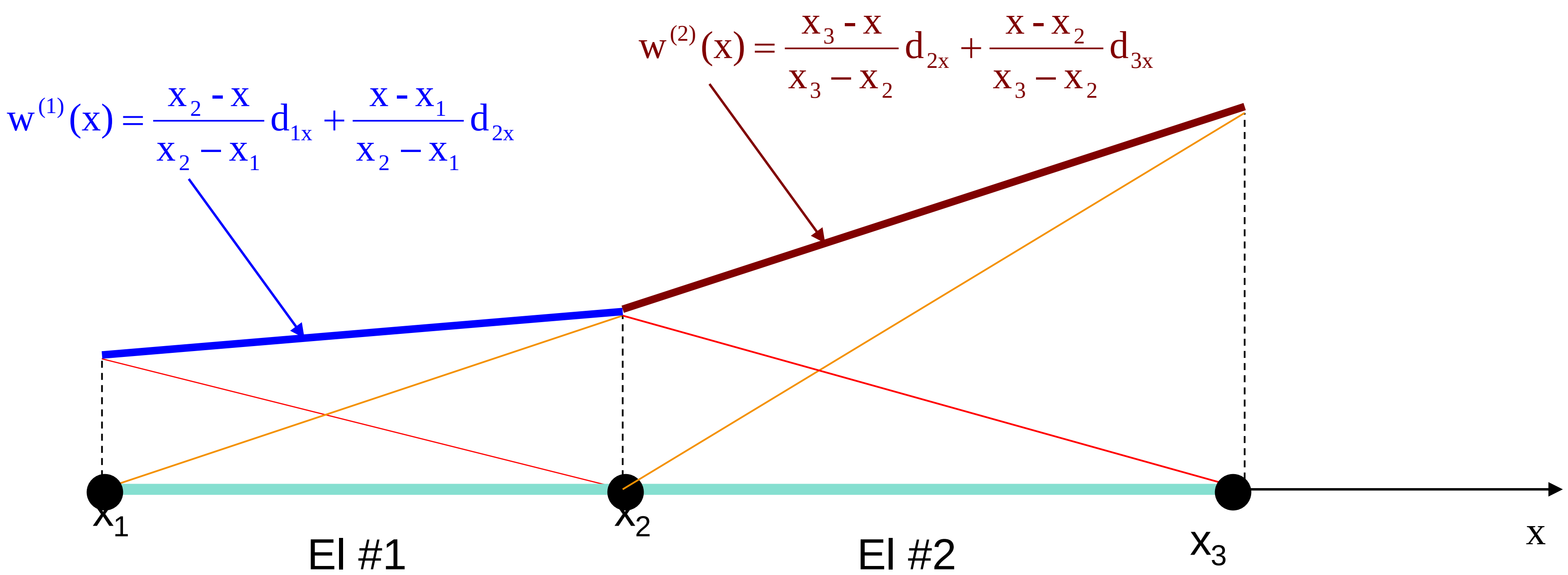
Check

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$

$$\Rightarrow N_1(x = x_1) = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} = 0$$

2. Compatibility: The displacement approximation is **continuous across element boundaries**



At $x=x_2$

$$w^{(1)}(x = x_2) = \frac{x_2 - x_2}{x_2 - x_1} d_{1x} + \frac{x_2 - x_1}{x_2 - x_1} d_{2x} = d_{2x}$$

$$w^{(2)}(x = x_2) = \frac{x_3 - x_2}{x_3 - x_2} d_{2x} + \frac{x_2 - x_2}{x_3 - x_2} d_{3x} = d_{2x}$$

Hence the displacement approximation is continuous across elements

3. Completeness

$$N_1(x) + N_2(x) = 1 \quad \text{for all } x$$
$$N_1(x)x_1 + N_2(x)x_2 = x \quad \text{for all } x$$

Use the expressions

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1};$$

$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

And check

$$N_1(x) + N_2(x) = \frac{x_2 - x}{x_2 - x_1} + \frac{x - x_1}{x_2 - x_1} = 1$$

$$\text{and } N_1(x)\ x_1 + N_2(x)\ x_2 = \frac{x_2 - x}{x_2 - x_1} x_1 + \frac{x - x_1}{x_2 - x_1} x_2 = x$$

Rigid body mode

$$N_1(x) + N_2(x) = 1 \quad \text{for all } x$$

What do we mean by "**rigid body modes**"?

Assume that $d_{1x}=d_{2x}=1$, this means that the element should translate in the positive x direction by 1. Hence ***ANY point*** (x) on the bar should have unit displacement. Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x) + N_2(x) = 1$$

YES!

Constant strain states

$$N_1(x)x_1 + N_2(x)x_2 = x \quad \text{at all } x$$

<u>What do we mean by "**constant strain states**"?</u>

Assume that $d_{1x}=x_1$ and $d_{2x}=x_2$. The strain at ***<u>ANY point</u>*** (x) within the bar is

$$\varepsilon(x) = \frac{d_{2x} - d_{1x}}{x_2 - x_1} = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

Let us see whether the displacement approximation allows this.

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} = N_1(x)x_1 + N_2(x)x_2 = x$$

$$\text{Hence, } \varepsilon(x) = \frac{dw(x)}{dx} = 1$$

YES!

Completeness = Rigid body modes + Constant Strain states

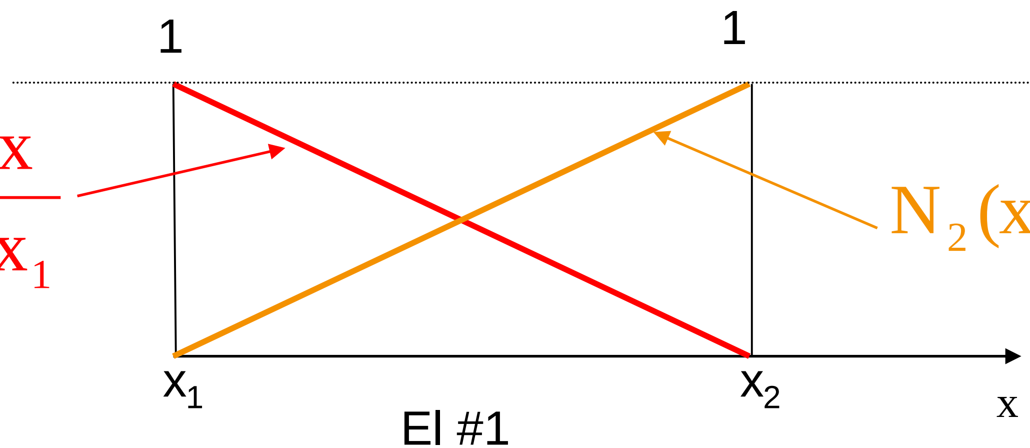
Compatibility + Completeness $\Rightarrow$ Convergence

Ensure that the solution gets better as more elements are introduced and, in the limit, approaches the exact answer.

4. How to write the expressions for the shape functions easily (without having to derive them each time):

Start with the **Kronecker delta property** (the shape function at any node has value of 1 at that node and a value of zero at all other nodes)

$$N_1(x) = \frac{x_2 - x}{x_2 - x_1}$$



$$N_2(x) = \frac{x - x_1}{x_2 - x_1}$$

Node at which N_1 is 0

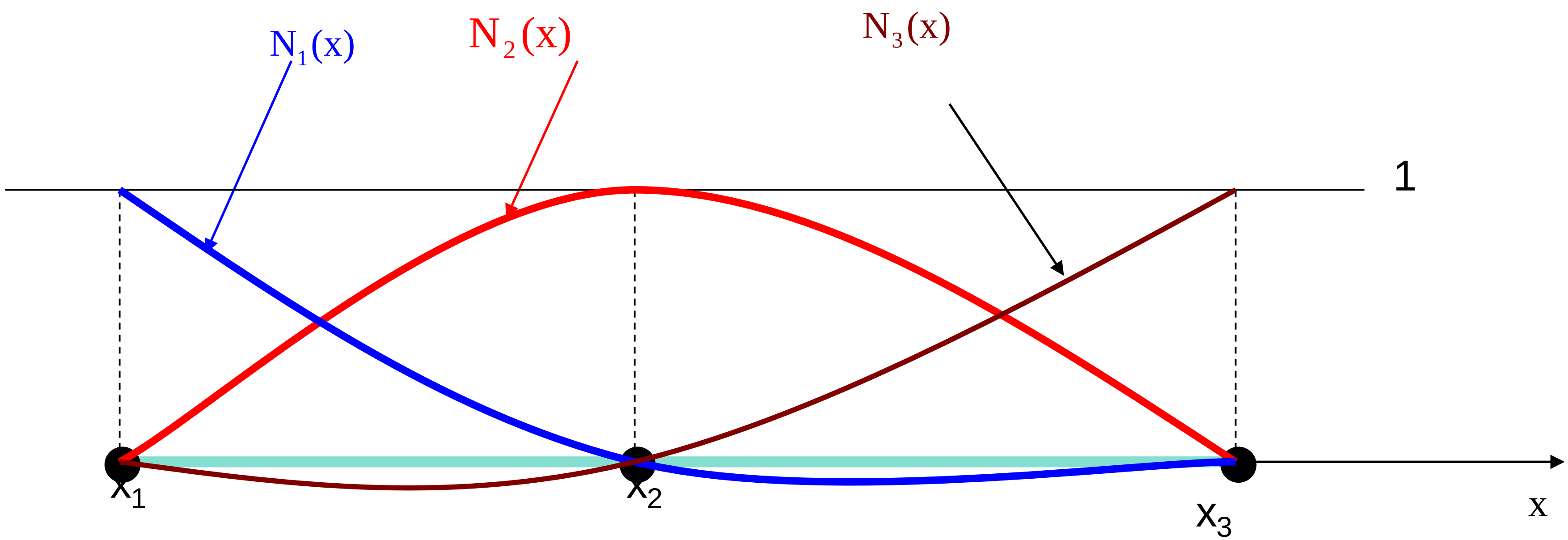
$$N_1(x) = \frac{(x_2 - x)}{(x_2 - x_1)}$$

$$N_2(x) = \frac{(x_1 - x)}{(x_1 - x_2)} = \frac{(x - x_1)}{(x_2 - x_1)}$$

Notice that the length of the element = x_2-x_1

The denominator is the numerator evaluated at the node itself

A slightly fancier assumption:
displacement varying **quadratically** inside each bar



El #1

$$N_1(x) = \frac{(x_2 - x)(x_3 - x)}{(x_2 - x_1)(x_3 - x_1)}$$

$$N_2(x) = \frac{(x_1 - x)(x_3 - x)}{(x_1 - x_2)(x_3 - x_2)}$$

$$N_3(x) = \frac{(x_1 - x)(x_2 - x)}{(x_1 - x_3)(x_2 - x_3)}$$

$$w(x) = N_1(x)d_{1x} + N_2(x)d_{2x} + N_3(x)d_{3x}$$

This is a **quadratic finite element** in 1D and it has three nodes and three associated shape functions per element.

TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

From equation (1), the displacement within each element

$$w(x) = \underline{N}\,\underline{d}$$

Recall that the **strain** in the bar

$$\varepsilon = \frac{dw}{dx}$$

Hence

$$\boxed{\varepsilon = \left[\frac{d\underline{N}}{dx}\right]\underline{d} = \underline{B}\,\underline{d}} \qquad (2)$$

The matrix $\underline{B}$ is known as the "**strain-displacement matrix**"

$$\underline{B} = \left[\frac{d\underline{N}}{dx}\right]$$

For a linear finite element

$$\underline{N} = [N_1(x) \quad N_2(x)] = \left[\frac{x_2 - x}{x_2 - x_1} \quad \frac{x - x_1}{x_2 - x_1} \right]$$

Hence

$$\underline{B} = \left[\frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] = \frac{1}{x_2 - x_1} [-1 \quad 1]$$

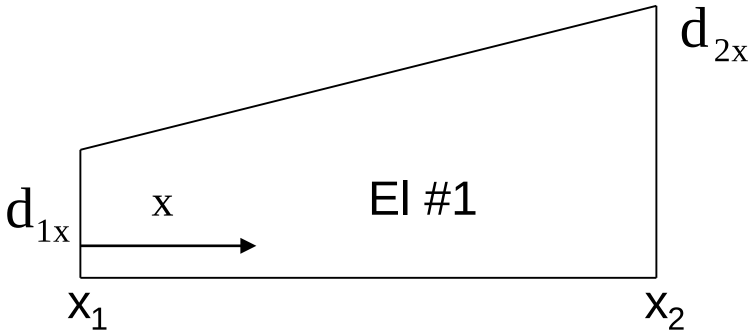
$$\varepsilon = \underline{B}\,\underline{d} = \left[\frac{-1}{x_2 - x_1} \quad \frac{1}{x_2 - x_1} \right] \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

$$= \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$

Hence, strain is a **constant** within each element (only for a linear element)!

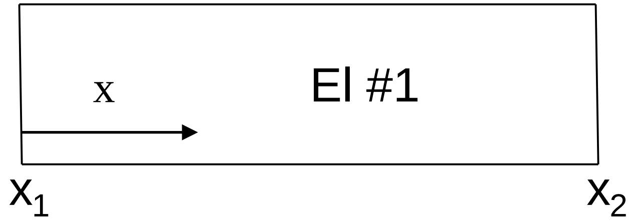
Displacement is linear

$$w(x) = a_0 + a_1 x$$



Strain is constant

$$\varepsilon = \frac{d_{2x} - d_{1x}}{x_2 - x_1}$$



Recall that the **stress** in the bar $$\sigma = E\varepsilon = E\frac{du}{dx}$$

Hence, inside the element, the approximate stress is

$$\boxed{\sigma = E\underline{B}\,\underline{d}} \qquad (3)$$

For a linear element the stress is also constant inside each element. This has the implication that the stress (and strain) is **discontinuous across element boundaries** in general.

Summary

Inside an element, the three most important approximations **in terms of the nodal displacements** ($\underline{d}$) are:

Displacement approximation in terms of shape functions

$$\boxed{u(x) = \underline{N}\,\underline{d}} \qquad (1)$$

Strain approximation in terms of strain-displacement matrix

$$\boxed{\varepsilon(x) = \underline{B}\,\underline{d}} \qquad (2)$$

Stress approximation in terms of strain-displacement matrix and Young's modulus

$$\boxed{\sigma = E\underline{B}\,\underline{d}} \qquad (3)$$

Summary

For a **linear element**

Displacement approximation in terms of shape functions

$$u(x) = \begin{bmatrix} \dfrac{x_2 - x}{x_2 - x_1} & \dfrac{x - x_1}{x_2 - x_1} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Strain approximation

$$\varepsilon = \frac{1}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

Stress approximation

$$\sigma = \frac{E}{x_2 - x_1} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject : FINITE ELEMENT METHODS

<u>UNIT- I</u>

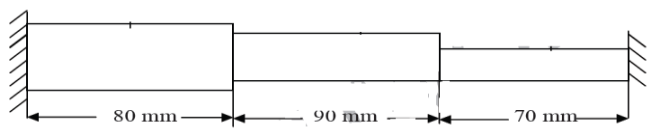
<u>TUTORIAL</u>

1. **a.)** Derive the equilibrium equation for an elastic continuum using potential energy by displacement approach.

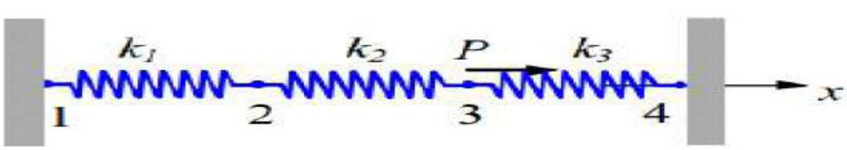
b.) Explain the following methods used for the formulation of element characteristics and load matrices: i) Variational approach ii) Galerkin approach

2. For the three-stepped bar shown in Figure, determine the nodal displacements, nodal forces and stresses in the elements.

Aluminum	Brass	Steel
E = 70 GPa	E = 105 Gpa	E = 200 GPa
A = 900 mm^2	A = 400 mm^2	A = 200 mm^2



3. For the spring system shown in the Fig., find the displacements at the nodes and the reactions. Given K_1=100 N/mm, K_2=200 N/mm, K_3= 100 N/mm, P=500 N.

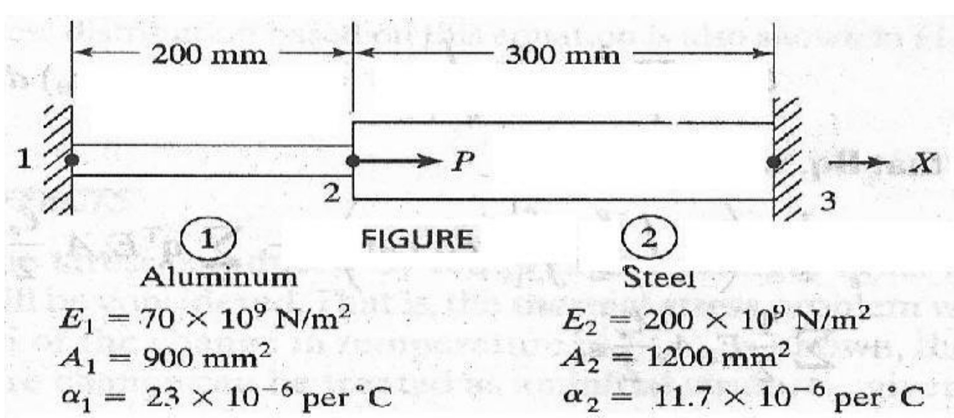


4. Describe the advantages, disadvantages and applications of finite element analysis.

5. An axial load P=300X103N is applied at 200 C to the rod as shown in Figure below. The temperature is the raised to 600 C .

a) Assemble the K and F matrices.

b) Determine the nodal displacements and stresses.



MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

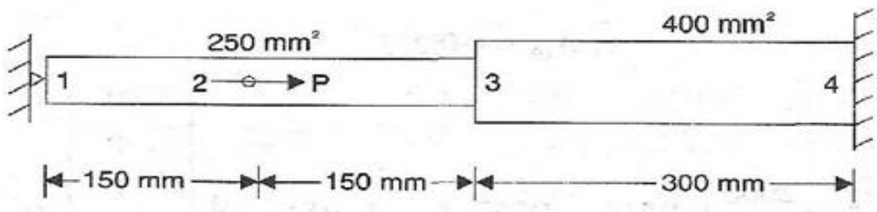
Subject : FINITE ELEMENT METHODS

UNIT- I

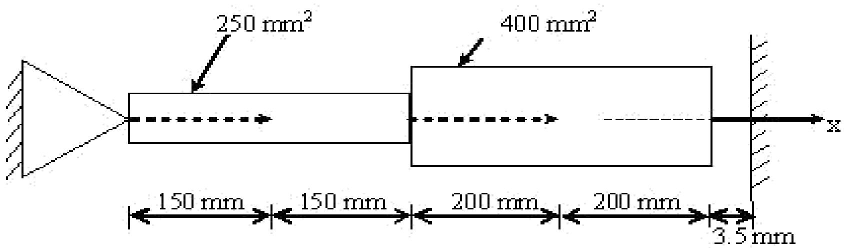
ASSIGNMENT

1. a) Derive the equations of equilibrium for a 3D Solid body
   b.) What are the applications of FEM
2. a.) Explain the Steps involved (Discretization) in FEM
   b.) Explain the Concept of Minimum Potential Energy approach
3. a.) Describe Rayleigh's Ritz method.
   b.) Explain about Natural coordinate System

4. Determine the nodal displacement and Element stresses for axially loaded bar as shown in the Fig. Consider axial load P = 2KN



5. Determine the nodal displacements, element stresses and support reactions for the bar as shown in Figure. Take $E = 200 \times 10^9$ N/m^2



UNIT 2

TRUSS ANALYSIS&2-D ANALYSIS WITH CST ELEMENT

Syllabus:

Analysis of Trusses: Stiffness matrix for plane truss element, Stress calculations and problems. Finite element modeling & two dimensional stress analyses with CST element and treatment of boundary conditions. Convergence requirements

OBJECTIVE:

To learn the principles involved in the discretization of domains with various elements, polynomial and interpolation and assembly of global arrays.

To learn the applications of FEM equations in 2D Plane problems with CST elements.

OUTCOME

Able to formulate Stiffness and load matrices & Solve the problems in Trusses and 2-D plane problems.

UNIT II

Analysis of Trusses

- The links of a truss are two-force members, where the direction of loading is along the axis of the member. Every truss element is in direct tension or compression.
- All loads and reactions are applied only at the joints and all members are connected together at their ends by frictionless pin joints. This makes the truss members very similar to a 1D spar element.

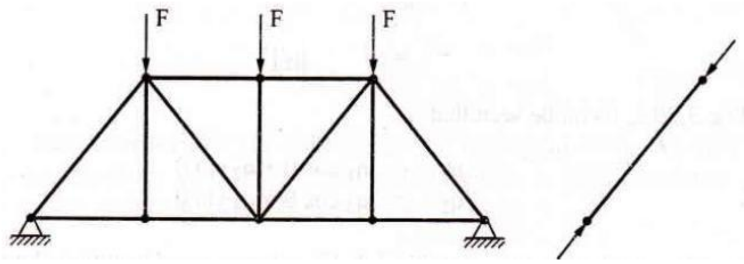


Fig. 1.28 Truss

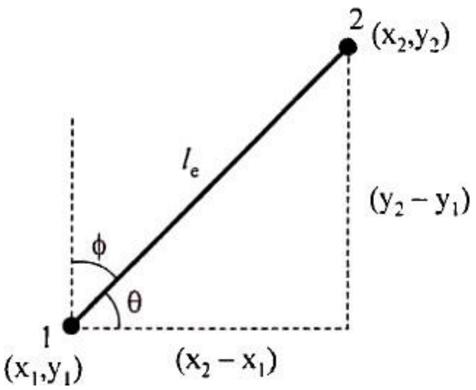


Fig. 1.29 Direction Cosines

The direction cosines l and m can be expressed as:

$$l = \cos\theta = \frac{x_2 - x_1}{l_e}$$

$$m = \cos\phi = \sin\theta = \frac{y_2 - y_1}{l_e}$$

$$\sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2}$$

$q_1' = q_1 l + q_2 m$

$q_2' = q_3 l + q_4 m$

$$[K] = \frac{A_e E_e}{L_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q$$

- **Thermal Effect In Truss Member**

(1) Thermal Load , P = $A_e E_e \varepsilon_e \begin{bmatrix} -l \\ -m \\ l \\ m \end{bmatrix}$

(2) Stress for an element, $\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q - E_e \alpha \Delta t$

(3) Remaining steps will be same as earlier.

Example 1.6: A two member truss is as shown in Fig. 1.30. The cross-sectional area ofeach member is 200 mm^2 and the modulus of elasticity is 200 GPa. Determine the deflections,reactions and stresses in each of the members.

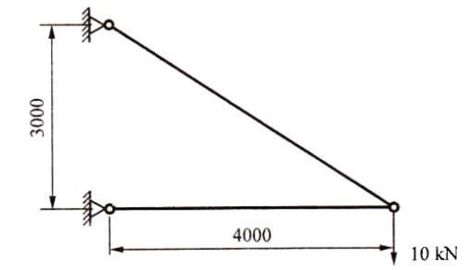


Fig. 1.30

In globalterms, each node would have 2 dof. These dof are marked as shown in Fig.1.31. The position of the nodes, with respect to origin (considered at node l) are as tabulated below:

Node	Xi	Yi
1	0	0
2	4000	0
3	0	3000

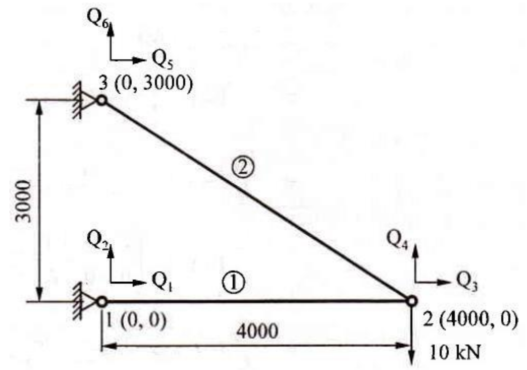


Fig.1.31

For all elements, A=200 mm^2

and E= 200 x 10^3 N/mm^2

The element connectivity table with the relevant terms are:

Element	N_i	N_j	$l_e = \sqrt{(x_j - x_i)^2 - (y_j - y_i)^2}$	$\frac{A_e E_e}{l_e}$	$l = \frac{x_j - x_i}{l_e}$	$m = \frac{y_j - y_i}{l_e}$	l^2	m^2	lm
(1)	1	2	$\sqrt{(4000-0)^2 - (0-0)^2}$ $= 4000$	10000	$\frac{4000-0}{4000}$ $= 1$	$\frac{0-0}{4000} = 0$	1	0	0
(2)	2	3	$\sqrt{(0-4000)^2 + (3000-0)^2}$ $= 5000$	8000	$-\frac{4000}{5000}$ $= -0.8$	$\frac{3000}{5000} = 0.6$	0.64	0.36	-0.48

As each node has two dof in global form, for every element, the element stiffness matrix would be in a 4 x 4 form. For element 1 defined by nodes l-2,the dof are Q_1, Q_2, Q_3 andQ_4 and that for element 2 defined by nodes 2-3, would be Q_3, Q_4, Q_5 andQ_6.

For a truss element

$$\lceil \quad -lm \quad \rceil$$

Element 1: The element stiffness matrix would be :

$$K^1 = 10 \times 10^3 \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \Downarrow \qquad \text{columns: } \overbrace{1 \quad 2}^{\text{Node1}} \; \overbrace{3 \quad 4}^{\text{Node2}} \Leftarrow \text{Global dof}$$

$$= 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 \\ -10 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \Downarrow \qquad \text{columns: } 1 \quad 2 \quad 3 \quad 4 \Leftarrow \text{Global dof}$$

Element 2: The element stiffness matrix would be :

$$K^2 = 8 \times 10^3 \begin{bmatrix} 0.64 & -0.48 & -0.64 & 0.48 \\ -0.48 & 0.36 & 0.48 & -0.36 \\ -0.64 & 0.48 & 0.64 & -0.48 \\ 0.48 & -0.36 & -0.48 & 0.36 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \Downarrow \qquad \text{columns: } \overbrace{3 \quad 4}^{\text{Node2}} \; \overbrace{5 \quad 6}^{\text{Node3}} \Leftarrow \text{Global dof}$$

$$= 10^3 \begin{bmatrix} 5.12 & -3.84 & -5.12 & 3.84 \\ -0.48 & 2.88 & 3.84 & -2.88 \\ -5.12 & 3.84 & 5.12 & -3.84 \\ 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \Downarrow \qquad \text{columns: } 3 \quad 4 \quad 5 \quad 6 \Leftarrow \text{Global dof}$$

The global stiffness matrix would be :

$$K = 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & (10+5.12) & (0-3.84) & -5.12 & 3.84 \\ 0 & 0 & (0-3.84) & (0+2.88) & 3.84 & -2.88 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \Downarrow \qquad \text{columns: } 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \Leftarrow \text{Global dof}$$

$$
= 10^3 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \Leftarrow \text{Global dof} \\ \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 15.12 & -3.84 & -5.12 & 3.84 \\ 0 & 0 & -3.84 & 2.88 & 3.84 & -2.88 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \Downarrow \end{array}
$$

In this case, node 1 and node 3 are completely fixed and hence,

$$Q_1 = Q_2 = Q_5 = Q_6 = 0$$

Hence, rows and columns 1,2,5 and 6 can be eliminated

Also the external nodal forces,

$F_1 = F_2 = F_3 = F_5 = F_6 = 0$

$F4 = -10 \times 10^3$ N

The global force vector would be,

$$
\mathbf{F} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -10 \times 10^3 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \Downarrow \quad \text{Global dof}
$$

In global form, after using the elimination approach

$$\mathbf{KQ} = \mathbf{F}$$

$$10^3 \begin{bmatrix} 15.12 & -3.84 \\ -3.84 & 2.88 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \times 10^3 \end{Bmatrix}$$

$$10^3 (15.12\, Q_3 - 3.84\, Q_4) = 0$$

$$Q_3 = 0.254\, Q_4$$

$10^3 (-3.84\, Q_3 + 2.88\, Q_4) = -10 \times 10^3$

$-3.84\, Q_3 + 2.88\, Q_4 = -10$

$-3.84\,(0.254\, Q_4) + 2.88\, Q_4 = -10$

$Q_4 = -5.25$mm

$Q_3 = -1.334$mm

The reactions can be found by using the equation:

$R = KQ - F$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_5 \\ R_6 \end{Bmatrix} = 10^3 \begin{bmatrix} 10 & 0 & -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5.12 & 3.84 & 5.12 & -3.84 \\ 0 & 0 & 3.84 & -2.88 & -3.84 & 2.88 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1.334 \\ -5.25 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$R_1 = -10 \times 10^3 \times (-1.334) = 13340$ N

$R_2 = 0$ N

$R_5 = -5.12 \times 10^3 \times (-1.334) + 3.84 \times 10^3 \times (-5.25) = -13340$ N

$R_6 = 3.84 \times 10^3 \times (-1.334) - 2.88 \times 10^3 \times (-5.25) = 9997.44$ N

To determine stresses: $\sigma = \frac{E_e}{l_e}\begin{bmatrix} -l & -m & l & m \end{bmatrix} q$

Element 1:

$$\sigma_1 = \frac{200 \times 10^3}{4000} [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 0 \\ 0 \\ -1.334 \\ -5.25 \end{Bmatrix}$$

$$= -66.7 \text{ N/mm}^2$$

Element 2:

$$\sigma_2 = \frac{200 \times 10^3}{5000} [0.8 \quad -0.6 \quad -0.8 \quad 0.6] \begin{Bmatrix} -1.334 \\ -5.25 \\ 0 \\ 0 \end{Bmatrix}$$

$$= 83.312 \text{ N/mm}^2$$

Example 1.7:Consider the four-bar truss shown in Fig. 1.32(a). It is given that $E = 29.5 \times 10^6$ psi andA_e = lin.^{2} for all elements. Complete the following:

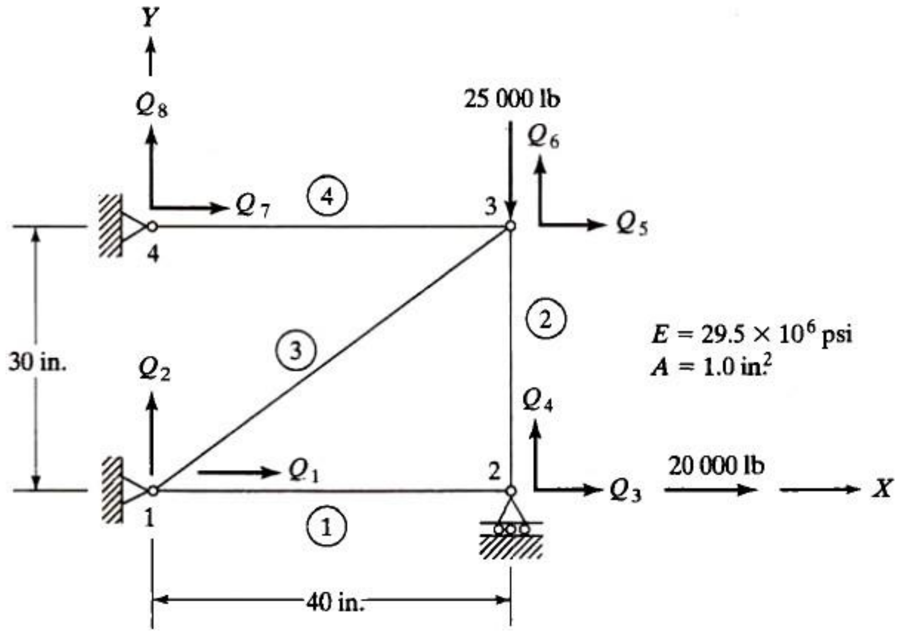
(a) Determine the element stiffness matrix for each element.

(b) Assemble the structural stiffness matrix K for the entire truss

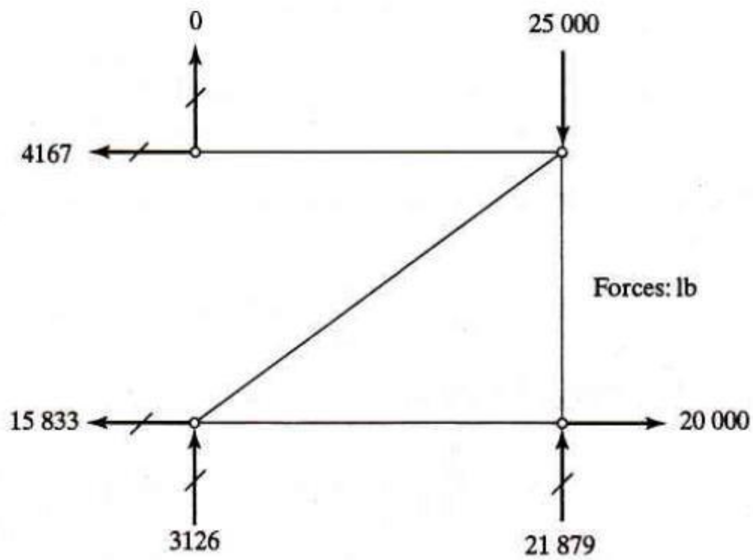
(c) Using the elimination approach, solve for the nodal displacement.

(d) Recover the stresses in each element.

(e) Calculate the reaction forces.



(a)



(b)

Fig. 1.32

(a) It is recommended that a tabular form be used for representing nodal coordinate data and element information. The nodal coordinate data are as follows:

Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

The element connectivity table is

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

Note that the user has a choice in defining element connectivity. For example, the connectivityof element 2 can be defined as 2-3 instead of 3-2 as in the previous table. However,calculations of the direction cosines will be consistent with the adopted connectivityscheme. Using formulas,together with the nodal coordinate data andthe given element connectivity information, we obtain the direction cosines table:

Element	l_e	l	m
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0

For example, the direction cosines of elements 3 are obtained as

$l = (x_3 - x_1)l_e$ = (40 - 0)/50 = 0.8 and $m = (y_3 - y_1)l_e$ = (30 - 0)/50 = 0.6.

Now, the element stiffness matrices for element 1 can be written as

$$\mathbf{k}^1 = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

(Column global dofs: 1 2 3 4 ← Global dof)

The global dofs associated with element 1, which is connected between nodes 1 and2,are indicated in k^1 earlier.These global dofs are shown in Fig. 1.32(a) and assistin assembling the various element stiffness matrices.The element stiffness matrices of elements 2,3and 4 are as follows:

$$\mathbf{k}^2 = \frac{29.5 \times 10^6}{30} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix}$$

(Column global dofs: 5 6 3 4)

$$\mathbf{k}^3 = \frac{29.5 \times 10^6}{50} \begin{bmatrix} .64 & .48 & -.64 & -.48 \\ .48 & .36 & -.48 & -.36 \\ -.64 & -.48 & .64 & .48 \\ -.48 & -.36 & .48 & .36 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 5 \\ 6 \end{matrix}$$

(Column global dofs: 1 2 5 6)

$$\mathbf{k}^4 = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 7 \\ 8 \\ 5 \\ 6 \end{matrix}$$

(columns: 7, 8, 5, 6)

(b) The structural stiffness matrix K is now assembled from the element stiffness matrices.By adding the element stiffness contributions, noting the element connectivity, we get

$$\mathbf{K} = \frac{29.5 \times 10^6}{600} \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\ -5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{matrix}$$

(columns: 1, 2, 3, 4, 5, 6, 7, 8)

(c) The structural stiffness matrix K given above needs to be modified to account for the boundary conditions. The elimination approach will be used here. The rows and columns corresponding to dofs 1, 2, 4, 7, and 8, which correspond to fixed supports, are deleted from the K matrix. The reduced finite element equations are given as

$$\frac{29.5 \times 10^6}{600} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 20\,000 \\ 0 \\ -25\,000 \end{Bmatrix}$$

Solution of these equations yields the displacements

$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 27.12 \times 10^{-3} \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \end{Bmatrix} \text{in.}$$

The nodal displacement vector for the entire structure can therefore be written as

$Q = [0, 0, 27.12 \times 10^{-3}, 0, 5.65 \times 10^{-3}, -22.25 \times 10^{-3}, 0, 0]^T$ in.

(d) The stress in each element can now be determined as shown below.

The connectivity of element 1 is 1 - 2. Consequently, the nodaldisplacementvectorforelementlisgivenby $q = [0, 0, 27.72 \times 10^{-3}, 0]^T$

$$\sigma_1 = \frac{29.5 \times 10^6}{40} [-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= 20\,000.0 \text{ psi}$$

The stress in member 2 is given by

$$\sigma_2 = \frac{29.5 \times 10^6}{30}[0 \quad 1 \quad 0 \quad -1]\begin{Bmatrix} 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ -27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= -21\,880.0 \text{ psi}$$

Following similar steps, we get

ζ3 = 5208.0Psi

ζ4= 4L67.0Psi

(e) The final step is to determine the support reactions. We need to determine the reaction forces along dofs 1, 2, 4, 7and 8, which correspond to fixed supports. These are obtained by substituting for Q into the original finite element equation R = KQ - F. In this substitution, only those rows of K corresponding to the support dofs are needed, and F = 0 for those dofs. Thus, we have

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \frac{29.5 \times 10^6}{600}\begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

Which results in

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 3126.0 \\ 21879.0 \\ -4167.0 \\ 0 \end{Bmatrix} \text{lb}$$

Principle of Minimum Potential Energy

- This principle states that for all kinematically admissible displacementfields corresponding to equilibrium extremize the total potential energy. Ifextreme condition is a minimum, the equilibrium is stable. It means that if thesystem is stable and steady, then total potential energy of the system is zero.
- The application of this principle can be made in a spring or elastic system subjected to the loading conditions.
- Consider a system of four springs as shown in Fig. 1.33subjected to the deflections under the application of force. This system has three nodal points.

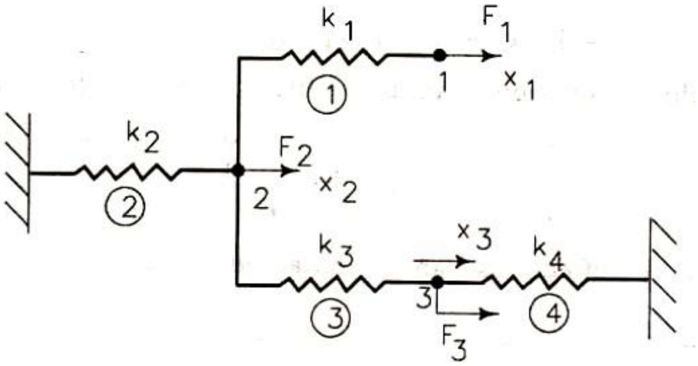


Fig. 1.33

Consider,a system of four spring having the nodes at point 1, 2, 3.

The potential energy P of the system is

$$P = \frac{1}{2}k_1\delta_1^2 + \frac{1}{2}k_2\delta_2^2 + \frac{1}{2}k_3\delta_3^2 + \frac{1}{2}k_4\delta_4^2 - F_1x_1 - F_3x_3$$

Where,

$\delta_1, \delta_2, \delta_3, \delta_4$ = deflections of the four springs

x_1, x_2, x_3 = displacement at nodal points 1, 2, 3

F_1, F_3 = forces at nodal points 1 and 3

k_1, k_2, k_3, k_4 = stiffnesses of four springs

$\delta_1 = x_1 - x_2$

$\delta_2 = x_2$

$\delta_3 = x_3 - x_2$

$\delta_4 = -x_3$

Potential energy of the system (P) is given by

$$P = \frac{1}{2}k_1(x_1 - x_2)^2 + \frac{1}{2}k_2x_2^2 + \frac{1}{2}k_3(x_3 - x_2)^2 + \frac{1}{2}k_4x_3^2 - F_1x_1 - F_3x_3$$

For equilibrium of this three degrees of freedom system, we need to minimize (P) with respect to x_1, x_2 and x_3.

According to principle of minimum potential energy, the potential energy is differentiated with respect to each displacement and equated to zero for minimum condition of potential energy.

$$\frac{dP}{dx_i} = 0 \text{ where } i = 1, 2, 3$$

$$\frac{dP}{dx_1} = k_1(x_1 - x_2) - F_1 = 0$$

$$= k_1x_1 - k_1x_2 = F_1$$

$$\frac{dP}{dx_2} = -k_1(x_1 - x_2) + k_2x_2 - k_3(x_3 - x_2) = 0$$

$$= -k_1x_1 + (k_1 + k_2 + k_3)x_2 - k_3x_3 = 0$$

$$\frac{dP}{dx_3} = k_3(x_3 - x_2) + k_4 x_3 - F_3 = 0$$

$$= -k_3 x_2 + (k_3 + k_4) x_3 = F_3.$$

The above three equations can be written in matrix form as

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ F_3 \end{bmatrix} \quad \text{.......(4)}$$

The x_1, x_2, $x_3$deflectionscanbeobtainedbyusing numerical methods.

On the other hand, we proceed to write the equilibrium equations of the system by considering the equilibrium of each separate node as shown in Fig. 1.34.

$k_1\delta_1 = F_1$

$k_2\delta_2 - k_1\delta_1 - k_3\delta_3 = 0$

$k_3\delta_3 - k_4\delta_4 = F_3$

which is precisely the set of equations represented in Eq. (4).

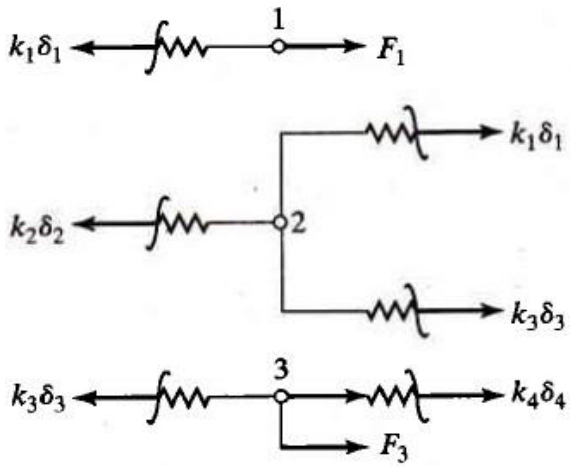


Fig. 1.34.

We see clearly that the set of equations (4) is obtained in a routine manner using thepotential energy approach,without any reference to the free-body diagrams. This makes thepotential energy approach attractive for large and complex problems.

Example 1.8: Calculate the deflections of the points at node 1, 2 and 3 for the spring system shown in Fig. 1.33. The stiffnesses are k_1 = 80 N/mm, k_2= 50 N/mm, k_3=60 N/mm, k_4 = 40 N/mm, the loads are F_1= 40 N, F_3 = 60 N.

Find the deflections x_1, x_2 and x_3.

The model of the above system is used and stiffness, deflection and load matrix can be written as

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ 0 \\ F_3 \end{bmatrix}$$

$$\begin{bmatrix} 80 & -80 & 0 \\ -80 & 80+50+60 & -60 \\ 0 & -60 & 60+40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 40 \\ 0 \\ 60 \end{bmatrix}.$$

Solving the above matrices, the value of x_1= 0.4 mm, x_2= 0.1 mm, x_3= 0.66 mm.

Example 1.9: Fig. 1.35 shows a cluster of four springs. One end of the assembly is fixed and a force of 1000 N is applied at the end. Using the finite element method, determine:

(a) The deflection of each spring.
(b) The reaction forces at support.

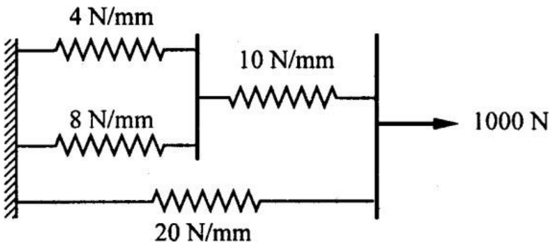


Fig. 1.35

The system of springs can be represented by a finite element model as shown in Fig. 1.36

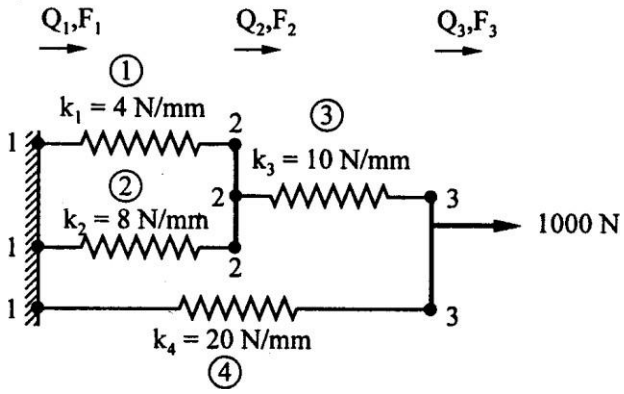


Fig. 1.36

The element connectivity table is as shown:

Element Connectivity

Element	Node Numbers		
ⓔ	1	2	⇐ Local dof
①	1	2	
②	1	2	⇐ Global dof
③	2	3	
④	1	3	

The element stiffness matrices are as under:

$$K^1 = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \quad \Downarrow \qquad (1 \quad 2 \Leftarrow \text{Global dof})$$

$$K^2 = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix} \begin{matrix} 1 \\ 2 \end{matrix} \quad \Downarrow \qquad (1 \quad 2 \Leftarrow \text{Global dof})$$

$$K^3 = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix} \begin{matrix} 2 \\ 3 \end{matrix} \quad \Downarrow \qquad (2 \quad 3 \Leftarrow \text{Global dof})$$

$$K^4 = \begin{bmatrix} 20 & -20 \\ -20 & 20 \end{bmatrix} \begin{matrix} 1 \\ 3 \end{matrix} \quad \Downarrow \qquad (1 \quad 3 \Leftarrow \text{Global dof})$$

The overall stiffness matrix would be:

$$\begin{matrix} 1 & 2 & 3 & \Leftarrow \text{Global dof} \end{matrix}$$

$$K = \begin{bmatrix} 4+8+20 & -4-8 & -20 \\ -4-8 & 4+8+10 & -10 \\ -20 & -10 & 10+20 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad \Downarrow$$

$$\begin{matrix} 1 & 2 & 3 & \Leftarrow \text{Global dof} \end{matrix}$$

$$= \begin{bmatrix} 32 & -12 & -20 \\ -12 & 22 & -10 \\ -20 & -10 & 30 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \quad \Downarrow$$

In global terms,

$$\mathbf{KQ} = \mathbf{F}$$

$$\begin{bmatrix} 32 & -12 & -20 \\ -12 & 22 & -10 \\ -20 & -10 & 30 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

The boundary conditions for this problem are:

$Q_1 = 0$

$F_1 = F_2 = 0$

$F_3 = 1000$ N

$$\begin{bmatrix} 32 & -12 & -20 \\ -12 & 22 & -10 \\ -20 & -10 & 30 \end{bmatrix} \begin{Bmatrix} 0 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 1000 \end{Bmatrix}$$

By elimination approach:

$22\ Q_2 - 10\ Q_3 = 0$

$Q_2 = 0.4545\ Q_3$

$$-10\,Q_2 + 30\,Q_3 = 1000$$

$$-10\,(0.4545\,Q_3) + 30\,Q_3 = 1000$$

$$Q_3 = 39.286 \text{ mm}$$

$$Q_2 = 0.4545 \times 39.286 = 17.857 \text{ mm}$$

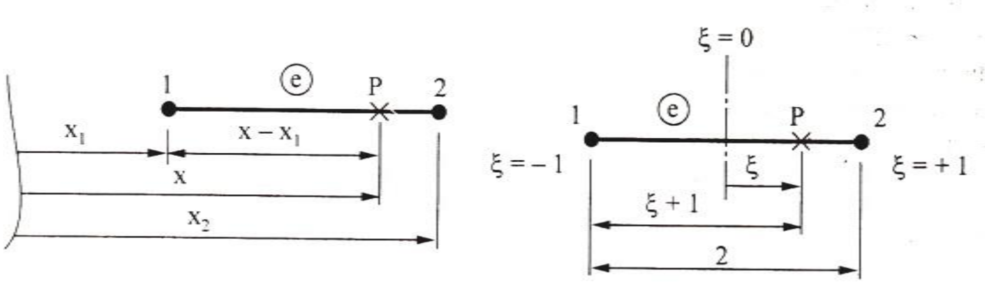
To determine the reaction forces,

$$R = KQ - F$$

$$\{R_1\} = [38 \quad -12 \quad -20] \begin{Bmatrix} 0 \\ 17.857 \\ 39.286 \end{Bmatrix} - \{0\}$$

$$\begin{aligned} R_1 &= -12 \times 17.857 - 20 \times 39.286 \\ &= -1000.004 \text{ N} \end{aligned}$$

Natural or Intrinsic Coordinate System:



(a) Global Coordinate System (b) Natural Coordinate System

Coordinate Systems

Fig.1.37

- Consider a element (e), having node numbers 1 & 2 shown in Fig. 1.37 (a). The first node 1 would be at a distance x_1 and second node would be at a distance x_2 from the reference. A convenient coordinate system called as the natural coordinate system is defined, as it helps in formulating individual element matrix which can than be used to combine and form a global stiffness matrix.
- In natural coordinate system, the centre of the element is considered as 0 and the node 1 and node 2 are placed at a distance - 1 and + 1 respectively Fig. 1.37(b). The variable of measurement of the distance in this case is represented as ξ. Thus node 1 is at coordinate position $\xi = -1$ and node 2 is at $\xi = +1$. Total length of the element would thus be 2 units and this length of the element is covered in the range $\xi = -1$ to $+1$.
- To establish relationship between two coordinate system consider any point P situated at a distance x, in the Global coordinate system Fig.1.37 (a) and correspondingly at a distance ξ from the origin as shown in Fig.1.37 (b).

Now

$$\frac{\text{Length of element in Natural system}}{\text{Length of element in Global system}} = \frac{\text{Dist. of Point P from Node 1 in Natural system}}{\text{Dist. of Point P from Node 1 in Global system}}$$

$$\frac{2}{(x_2 - x_1)} = \frac{\xi + 1}{(x - x_1)}$$

$$\therefore \xi = \frac{2(x - x_1)}{(x_2 - x_1)} - 1 \qquad \_\_\_\_\_\_(a)$$

Confirm the validity of equation (a)

$$\text{At } x = x_1 \rightarrow \xi = \frac{2(x_1 - x_1)}{(x_2 - x_1)} - 1 = 0 - 1 = -1$$

$$\text{At } x = x_2 \rightarrow \xi = \frac{2(x_2 - x_1)}{(x_2 - x_1)} - 1 = 2 - 1 = 1$$

This confirm the relation of the two coordinate system.

Shape function in Natural Coordinate System:

– The natural coordinate can be used to define shape functions. This makes it convenient to isolate the element from the continuum and develop the necessary element stiffness matrix. The shape function as defined earlier is used to interpolate the deflections or degree of freedom within the element. The accuracy of calculations would increase with increase in number of elements. Consider linear distribution as represented by Fig.1.38.

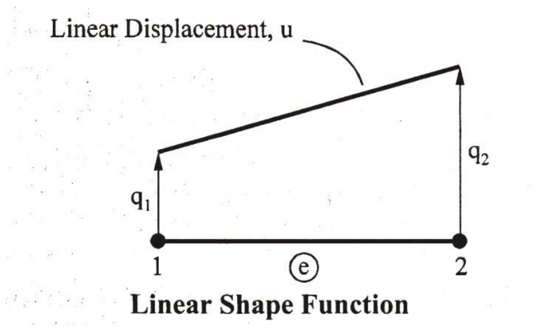


Fig.1.38

– The shape function N_1 and N_2 in natural coordinate term can be developed by considering Fig.1.39 (a) and Fig.1.39 (b).

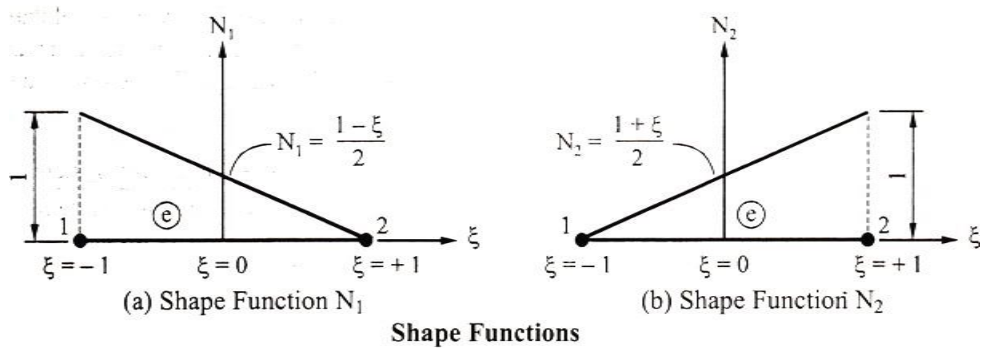


Fig.5.16

– Fig. 1.39 (a) General line equation is

$$y = mx + C \qquad \left\{ \begin{array}{l} m = \text{slope} = -\frac{1}{2} \\ \text{For } N_1 = 1, \xi = -1 \end{array} \right\}$$

$$\therefore N_1 = -\frac{1}{2}\xi + C$$

$$1 = -\frac{1}{2}(-1) + C \qquad \rightarrow C = \frac{1}{2}$$

$$\therefore N_1 = -\frac{1}{2}\xi + \frac{1}{2}$$

$$N_1(\xi) = \frac{1-\xi}{2}$$

– Fig. 1.39(b) line equation is

$$y = mx + C \qquad \left\{ \begin{array}{l} m = \text{slope} = \frac{1}{2} \\ \text{For } N_2 = 0, \xi = -1 \end{array} \right\}$$

$$\therefore N_2 = \frac{1}{2}\xi + C$$

$$0 = \frac{1}{2}(-1) + C$$

$$\therefore C = \frac{1}{2}$$

$$\therefore N_2 = \frac{1}{2}\xi + \frac{1}{2}$$

$$N_2(\xi) = \frac{1+\xi}{2}$$

– Once the shape functions are defined, the linear displacement field within the element can be written in terms of nodal displacements q_1 and q_2 as….

$\therefore u = N_1 q_1 + N_2 q_2$ .... (b)

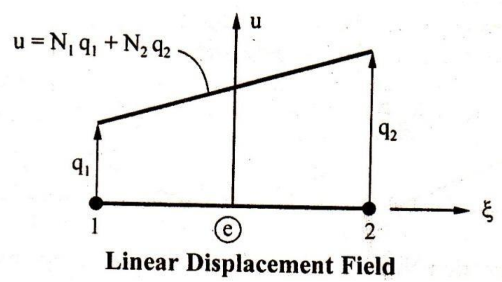


Fig. 1.40

which in Matrix form will be

$u = N\,q$

where $N = [N_1 , N_2]$

$$q = [q_1 , q_2]^T = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

– The term q is referred to as element displacement vector, and the verification of equation (b) can be done by considering the equation of shape functions.

At Node 1: $\xi = -1$

$$N_1 = \frac{1-\xi}{2} = \frac{1-(-1)}{2} = 1$$

$$N_2 = \frac{1+\xi}{2} = \frac{1+(-1)}{2} = 0$$

So displacement at Node 1 will be

$$\begin{aligned} \therefore\ u &= N_1 q_1 + N_2 q_2 \\ &= 1\ (q_1) + 0\ (q_2) \\ &= q_1 \end{aligned}$$

At Node 2: $\xi = +1$

$$N_1 = \frac{1-\xi}{2} = \frac{1-1}{2} = 0$$

$$N_2 = \frac{1+\xi}{2} = \frac{1+1}{2} = 1$$

So displacement at Node 2 will be

$$\begin{aligned} \therefore\ u &= N_2 q_2 + N_1 q_1 \\ &= 1\ (q_2) + 0\ (q_1) \\ &= q_2 \end{aligned}$$

– Thus it is seen that as per equation, the displacement at Node 1 and 2 are q_1 and q_2 which are the expected results.

– We know equation $\xi = \dfrac{2\,(x - x_1)}{(x_2 - x_1)} - 1$

$$\therefore \frac{\xi+1}{2} = \frac{x-x_1}{x_2-x_1}$$

$$\therefore x = \left(\frac{\xi+1}{2}\right)(x_2-x_1) + x_1$$

$$= \left(\frac{\xi+1}{2}\right)x_2 - \left(\frac{\xi+1}{2}\right)x_1 + x_1$$

$$\therefore x = \left(\frac{\xi+1}{2}\right)x_2 + \left(1-\frac{\xi+1}{2}\right)x_1$$

$$= \left(\frac{\xi+1}{2}\right)x_2 + \left(\frac{1-\xi}{2}\right)x_1 \qquad \left\{\because \left(\frac{\xi+1}{2}\right)=N_2, \left(\frac{1-\xi}{2}\right)=N_1\right\}$$

$$\therefore x = N_1 x_1 + N_2 x_2 \qquad \left\{\begin{array}{l} x = N_1 x_1 + N_2 x_2 \\ u = N_1 q_1 + N_2 q_2 \end{array}\right\}$$

– Comparing equation, it is seen that both, the displacement u and the coordinate x can be interpolated within the element using the same shape function N_1 and N_2. This is referred to as the " Isoparametric Formulation".

Example – 1.10: Temp. at Node 1 is 100° C and Node 2 is 40° C. The length of the element is 200 mm. Evaluate the shape function associated with Node 1 and Node 2. Calculate the temp. at point P situated at 150 mm from Node 1. Assume a linear shape function.

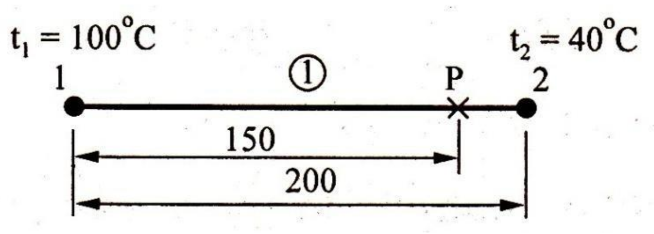


Fig. 1.41

Solution:

At Node 1:

$x_1 = 0$

$x_2 = 200$ mm

$x = 0$

$$\xi = \frac{2(x-x_1)}{(x_2-x_1)} - 1 = -1$$

$$N_1(\xi) = \frac{1-\xi}{2} = 1$$

$$N_2(\xi) = \frac{1+\xi}{2} = 0$$

At Node 2:

$$x_1 = 0$$
$$x_2 = 200 \text{ mm}$$
$$x \ = 200 \text{ mm}$$
$$\xi = \frac{2(x - x_1)}{(x_2 - x_1)} - 1 = \frac{2(200 - 0)}{(200 - 0)} - 1 = 1$$
$$N_1(\xi) = \frac{1-\xi}{2} = 0$$
$$N_2(\xi) = \frac{1+\xi}{2} = 1$$

At Point P:

$$x_1 = 0 \qquad \rightarrow t_1 = 100^\circ C$$
$$x_2 = 200 \text{ mm} \qquad \rightarrow t_2 = 40^\circ C$$
$$x \ = 150 \text{ mm}$$

$$\xi = \frac{2(x - x_1)}{(x_2 - x_1)} - 1 = \frac{2(150 - 0)}{(200 - 0)} - 1 = 0.5$$
$$N_1(\xi) = \frac{1-\xi}{2} = \frac{1-0.5}{2} = 0.25$$
$$N_2(\xi) = \frac{1+\xi}{2} = \frac{1+0.5}{2} = 0.75$$

$$\therefore t = N_1\, t_1 + N_2\, t_2$$
$$= 0.25(100) + 0.75(40)$$
$$t = 55^\circ C$$

Example – 1.11 : A 1D spar element having a linear shape function as shown in fig. If the temp.at Node 1 is 50° C and Node 2 is -20°C. Find the temp. at point P.

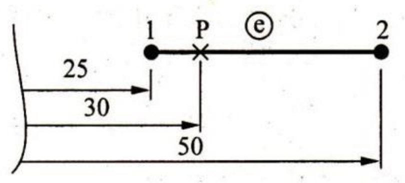


Fig. 1.42

Solution:

At Point P:

$$x_1 = 25 \qquad \rightarrow t_1 = 50^\circ C$$
$$x_2 = 50 \qquad \rightarrow t_2 = -20^\circ C$$
$$x \ = 30$$

Let

$$\xi = \frac{2(x - x_1)}{(x_2 - x_1)} - 1 = \frac{2(30 - 25)}{(50 - 25)} - 1 = -0.6$$

Now

$$N_1(\xi) = \frac{1-\xi}{2} = \frac{1-(-0.6)}{2} = 0.8$$

$$N_2(\xi) = \frac{1+\xi}{2} = \frac{1+(-0.6)}{2} = 0.2$$

$$t = N_1 t_1 + N_2 t_2$$
$$= 0.8(50) + 0.2(-20)$$
$$t = 30^\circ C$$

Example – 1.12: Consider an element having a linear shape function shown in fig. Evaluate the natural coordinate and shape functions for point P. If the displacement at Node 1 and Node 2 are 2 mm and -1 mm respectively, determine the value of displacement at point P. Also determine in global terms the point where the displacement would be zero. Also determine the shape function at zero displacement point.

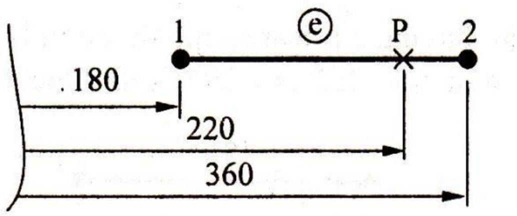


Fig. 1.43

Solution:

At point P:

$x_1 = 180$ mm $\rightarrow q_1 = 2$ mm

$x_2 = 360$ mm $\rightarrow q_2 = -1$ mm

$x = 220$ mm

Let

$$\xi = \frac{2(x - x_1)}{(x_2 - x_1)} - 1 = \frac{2(220 - 180)}{(360 - 180)} - 1 = -0.556$$

$$N_1(\xi) = \frac{1-\xi}{2} = \frac{1-(-0.556)}{2} = 0.778$$

$$N_2(\xi) = \frac{1+\xi}{2} = \frac{1+(-0.556)}{2} = 0.222$$

Let

$$u = N_1 q_1 + N_2 q_2$$
$$= 0.778(2) + 0.222(-1)$$
$$u = 1.334 \text{ mm}$$

To determine the position of point where displacement is zero ($q_1 = 2$ mm , $q_2 = -1$ mm).

Let

$$u = N_1 q_1 + N_2 q_2$$
$$0 = N_1(2) + N_2(-1)$$
$$\therefore 2N_1 = N_2$$
$$\therefore 2\left(\frac{1-\xi}{2}\right) = \left(\frac{1+\xi}{2}\right)$$
$$\therefore 2 - 2\xi = 1 + \xi$$
$$\therefore \xi = \frac{1}{3} = 0.333$$

Let

$$\xi = \frac{2(x - x_1)}{(x_2 - x_1)} - 1$$
$$0.333 = \frac{2(x - 180)}{(360 - 180)} - 1$$
$$x = 300 \text{ mm}$$
$$N_1 = \frac{1-\xi}{2} = \frac{1-0.333}{2} = 0.333$$
$$N_2 = 2N_1 = 2(0.333) = 0.667$$

– x can be find out

$$x = N_1 x_1 + N_2 x_2$$
$$= 0.333(180) + 0.667(360)$$
$$x = 300 \text{ mm}$$

Example – 1.13: Determine the temperature at x = 40 mm if the temperature at nodes $Ø_i$ = 120° C and $Ø_j$ = 80° C and x_i = 10 mm, x_j = 60 mm.

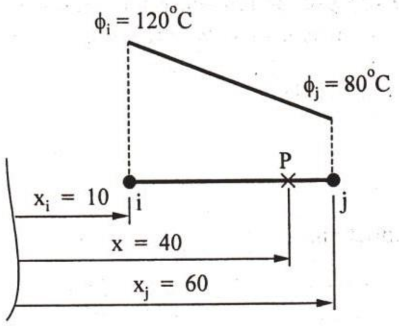


Fig. 1.44

Solution:

$x =$

1
0

x

j

=

6
0

$x \frac{1}{2} = 40$

$\phi_i = 120^\circ C$

$\phi_j = 80^\circ C$

Let

$$\xi = \frac{2\left(x - x_1\right)}{\left(x_2 - x_1\right)} - 1$$

$$= \frac{2\left(40 - 10\right)}{\left(60 - 10\right)} - 1$$

$$= 0.2$$

$$N_1(\xi) = \frac{1-\xi}{2} = \frac{1 - 0.2}{2} = 0.4$$

$$N_2(\xi) = \frac{1+\xi}{2} = \frac{1 + 0.2}{2} = 0.6$$

Let

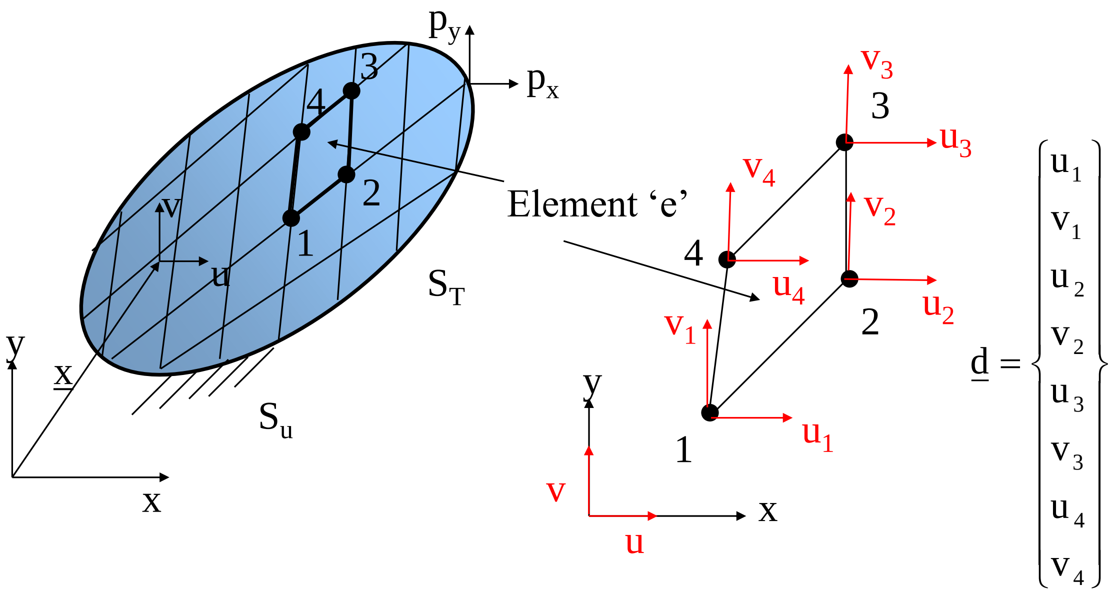
$$\phi = N_1\,\phi_i + N_2\,\phi_j$$

$$= 0.4\left(120\right) + 0.6\left(80\right)$$

$$= 96^\circ\ C$$

Finite element formulation for 2D:

Step 1: Divide the body into **finite elements** connected to each other through special points ("**nodes**")



$$u(x, y) \approx N_1(x, y)\, u_1 + N_2(x, y)\, u_2 + N_3(x, y)\, u_3 + N_4(x, y)\, u_4$$
$$v(x, y) \approx N_1(x, y)\, v_1 + N_2(x, y)\, v_2 + N_3(x, y)\, v_3 + N_4(x, y)\, v_4$$

$$\underline{u} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \left[\begin{array}{cc:cc:cc:cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\boxed{\underline{u} = \underline{N}\, \underline{d}}$$

TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

Approximation of the strain in element 'e'

$$\varepsilon_x = \frac{\partial u(x, y)}{\partial x} \approx \frac{\partial N_1(x, y)}{\partial x} u_1 + \frac{\partial N_2(x, y)}{\partial x} u_2 + \frac{\partial N_3(x, y)}{\partial x} u_3 + \frac{\partial N_4(x, y)}{\partial x} u_4$$
$$\varepsilon_y = \frac{\partial v(x, y)}{\partial y} \approx \frac{\partial N_1(x, y)}{\partial y} v_1 + \frac{\partial N_2(x, y)}{\partial y} v_2 + \frac{\partial N_3(x, y)}{\partial y} v_3 + \frac{\partial N_4(x, y)}{\partial y} v_4$$
$$\gamma_{xy} = \frac{\partial u(x, y)}{\partial y} + \frac{\partial v(x, y)}{\partial x} \approx \frac{\partial N_1(x, y)}{\partial y} u_1 + \frac{\partial N_1(x, y)}{\partial x} v_1 + \ldots\ldots$$

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} & 0 & \frac{\partial N_2(x,y)}{\partial x} & 0 & \frac{\partial N_3(x,y)}{\partial x} & 0 & \frac{\partial N_4(x,y)}{\partial x} & 0 \\ 0 & \frac{\partial N_1(x,y)}{\partial y} & 0 & \frac{\partial N_2(x,y)}{\partial y} & 0 & \frac{\partial N_3(x,y)}{\partial y} & 0 & \frac{\partial N_4(x,y)}{\partial y} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial x} & \frac{\partial N_4(x,y)}{\partial y} & \frac{\partial N_4(x,y)}{\partial x} \end{bmatrix}}_{\underline{B}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\boxed{\underline{\varepsilon} = \underline{B}\,\underline{d}}$$

Summary: For each element

Displacement approximation in terms of shape functions

$$\boxed{\underline{u} = \underline{N}\,\underline{d}}$$

Strain approximation in terms of strain-displacement matrix

$$\boxed{\underline{\varepsilon} = \underline{B}\,\underline{d}}$$

Stress approximation

$$\boxed{\underline{\sigma} = \underline{D}\underline{B}\,\underline{d}}$$

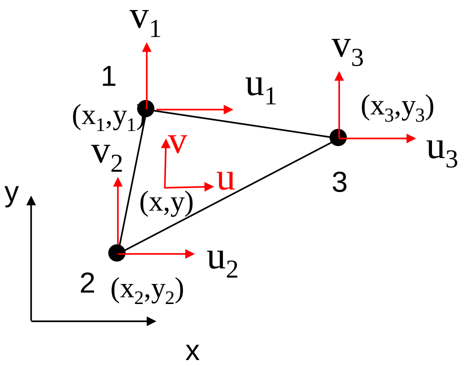
Element stiffness matrix

$$\boxed{\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\,\underline{B}\, dV}$$

Element nodal load vector

$$\boxed{\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\, dS}_{\underline{f}_S}}$$

<u>Constant Strain Triangle (CST) :</u> Simplest 2D finite element



- 3 nodes per element
- 2 dofs per node (each node can move in x- and y- directions)
- Hence 6 dofs per element

The <u>displacement approximation</u> in terms of shape functions is

$$u\ (x,y) \approx N_1 u_1 + N_2 u_2 + N_3 u_3$$

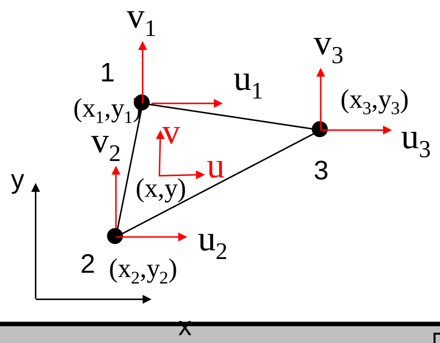
$$v(x,y) \approx N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\underline{u} = \begin{Bmatrix} u\ (x,y) \\ v\ (x,y) \end{Bmatrix} = \left[\begin{array}{cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \hline u_2 \\ v_2 \\ \hline u_3 \\ v_3 \end{Bmatrix}$$

$$\boxed{\underline{u}_{2\times1} = \underline{N}_{2\times6}\ \underline{d}_{6\times1}}$$

$$\underline{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

Formula for the shape functions are



$$N_1 = \frac{a_1 + b_1 x + c_1 y}{2A}$$
$$N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$$
$$N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$$

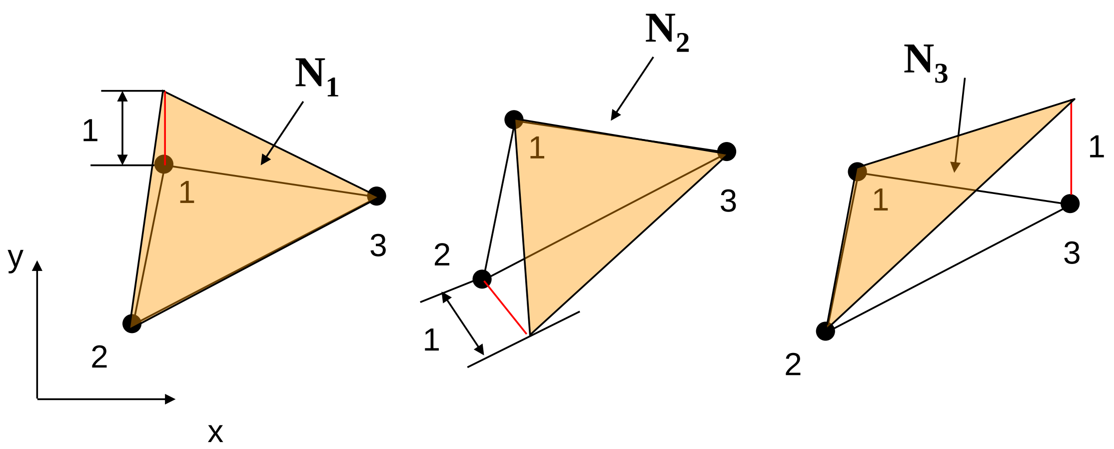
where

$$A = area\ of\ triangle = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

$$a_1 = x_2 y_3 - x_3 y_2 \quad b_1 = y_2 - y_3 \quad c_1 = x_3 - x_2$$
$$a_2 = x_3 y_1 - x_1 y_3 \quad b_2 = y_3 - y_1 \quad c_2 = x_1 - x_3$$
$$a_3 = x_1 y_2 - x_2 y_1 \quad b_3 = y_1 - y_2 \quad c_3 = x_2 - x_1$$

Properties of the shape functions:

1. The shape functions N_1, N_2 and N_3 are linear functions of x and y



$$N_i = \begin{cases} 1 & at\ node\ 'i' \\ 0 & at\ other\ nodes \end{cases}$$

2. At every point in the domain

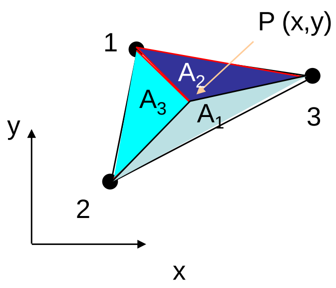
$$\sum_{i=1}^{3} N_i = 1$$

$$\sum_{i=1}^{3} N_i x_i = x$$

$$\sum_{i=1}^{3} N_i y_i = y$$

3. Geometric interpretation of the shape functions

At any point P(x,y) that the shape functions are evaluated,



$$N_1 = \frac{A_1}{A}$$

$$N_2 = \frac{A_2}{A}$$

$$N_3 = \frac{A_3}{A}$$

Approximation of the strains

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \approx \underline{B}\underline{d}$$

$$\underline{B} = \begin{bmatrix} \dfrac{\partial N_1(x,y)}{\partial x} & 0 & \dfrac{\partial N_2(x,y)}{\partial x} & 0 & \dfrac{\partial N_3(x,y)}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1(x,y)}{\partial y} & 0 & \dfrac{\partial N_2(x,y)}{\partial y} & 0 & \dfrac{\partial N_3(x,y)}{\partial y} \\ \dfrac{\partial N_1(x,y)}{\partial y} & \dfrac{\partial N_1(x,y)}{\partial x} & \dfrac{\partial N_2(x,y)}{\partial y} & \dfrac{\partial N_2(x,y)}{\partial x} & \dfrac{\partial N_3(x,y)}{\partial y} & \dfrac{\partial N_3(x,y)}{\partial x} \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Inside each element, all components of strain are constant: hence the name **Constant Strain Triangle** Element stresses (constant inside each element)

$$\boxed{\underline{\boldsymbol{\sigma}} = \underline{\mathbf{D}}\underline{\mathbf{B}}\,\underline{\mathbf{d}}}$$

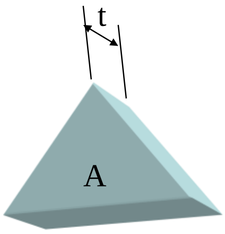
IMPORTANT NOTE:

1. The displacement field is continuous across element boundaries

2. The strains and stresses are NOT continuous across element boundaries

Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\, \underline{B}\, dV$$



Since $\underline{B}$ is constant

$$\underline{k} = \underline{B}^T \underline{D}\, \underline{B} \int_{V^e} dV = \underline{B}^T \underline{D}\, \underline{B}\, At$$

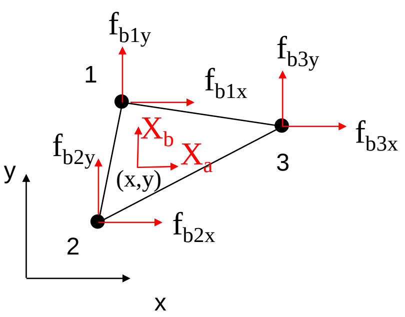
t=thickness of the element
A=surface area of the element

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\, dS}_{\underline{f}_s}$$

Element nodal load vector due to body forces

$$\underline{f}_b = \int_{V^e} \underline{N}^T \underline{X}\, dV = t \int_{A^e} \underline{N}^T \underline{X}\, dA$$



$$\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a\, dA \\ t\int_{A^e} N_1 X_b\, dA \\ t\int_{A^e} N_2 X_a\, dA \\ t\int_{A^e} N_2 X_b\, dA \\ t\int_{A^e} N_3 X_a\, dA \\ t\int_{A^e} N_3 X_b\, dA \end{Bmatrix}$$

EXAMPLE:

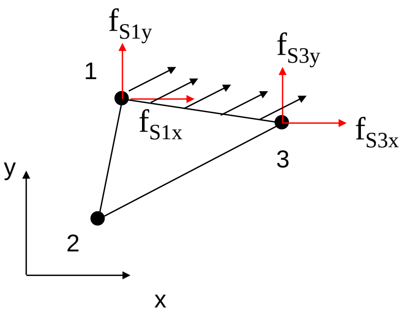
If X_a=1 and X_b=0

$$\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a \, dA \\ t\int_{A^e} N_1 X_b \, dA \\ t\int_{A^e} N_2 X_a \, dA \\ t\int_{A^e} N_2 X_b \, dA \\ t\int_{A^e} N_3 X_a \, dA \\ t\int_{A^e} N_3 X_b \, dA \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 dA \\ 0 \\ t\int_{A^e} N_2 \, dA \\ 0 \\ t\int_{A^e} N_3 dA \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \end{Bmatrix}$$

Element nodal load vector due to traction

$$\boxed{\underline{f}_S = \int_{S_T^e} \underline{\mathbf{N}}^T \underline{T}_S \, dS}$$

EXAMPLE:



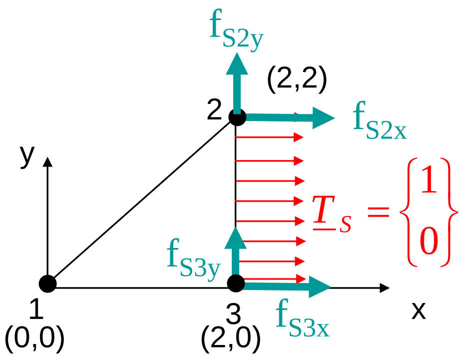
$$\underline{f}_S = t\int_{l_{1-3}^e} \underline{\mathbf{N}}^T \Big|_{along\,1-3} \underline{T}_S \, dS$$

Recommendations for use of CST

1. Use in areas where strain gradients are small
2. Use in mesh transition areas (fine mesh to coarse mesh)
3. Avoid CST in critical areas of structures (e.g., stress concentrations, edges of holes, corners)
4. In general CSTs are not recommended for general analysis purposes as a very large number of these elements are required for reasonable accuracy.

Element nodal load vector due to traction

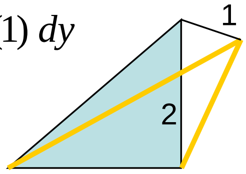
EXAMPLE:



$$\underline{f}_S = t\int_{l_{2-3}^e} \underline{\mathrm{N}}^T\Big|_{along\,2-3} \underline{T}_S \; dS$$

$$f_{S_{2x}} = t\int_{l_{2-3}^e} N_2\Big|_{along\,2-3}(1)\; dy$$

$$= t\left(\frac{1}{2}\right)\times 2\times 1 = t$$



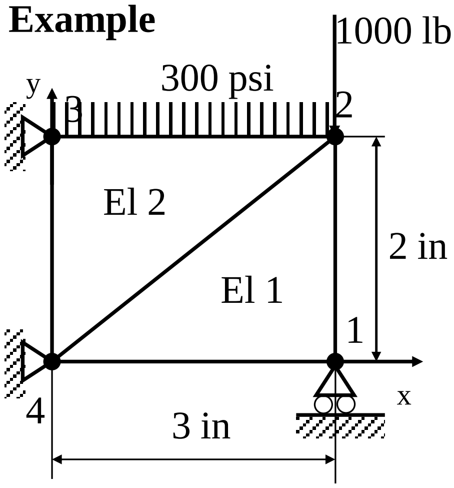
Similarly, compute

$$f_{S_{2y}} = 0$$

$$f_{S_{3x}} = t$$

$$f_{S_{3y}} = 0$$

Example



Thickness (t) = 0.5 in
E= 30×10^6 psi
ν=0.25

(a) Compute the unknown nodal displacements.
(b) Compute the stresses in the two elements.

Realize that this is a plane stress problem and therefore we need to use

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2 & 0.8 & 0 \\ 0.8 & 3.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \times 10^7 \ psi$$

<u>Step 1</u>: Node-element connectivity chart

ELEMENT	Node 1	Node 2	Node 3	Area (sqin)
1	1	2	4	3
2	3	4	2	3

Node	x	y
1	3	0
2	3	2
3	0	2
4	0	0

Nodal coordinates

<u>Step 2</u>: Compute strain-displacement matrices for the elements

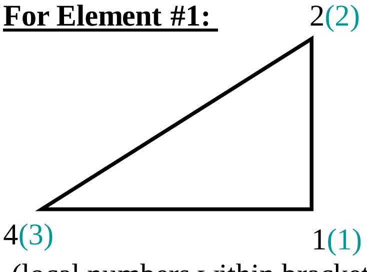
Recall

$$\underline{B} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

with

$$b_1 = y_2 - y_3 \quad b_2 = y_3 - y_1 \quad b_3 = y_1 - y_2$$
$$c_1 = x_3 - x_2 \quad c_2 = x_1 - x_3 \quad c_3 = x_2 - x_1$$

<u>For Element #1:</u>



(local numbers within brackets)

$$y_1 = 0;\ y_2 = 2;\ y_3 = 0$$
$$x_1 = 3;\ x_2 = 3;\ x_3 = 0$$

Hence

$$b_1 = 2 \quad b_2 = 0 \quad b_3 = -2$$
$$c_1 = -3 \quad c_2 = 3 \quad c_3 = 0$$

Therefore

$$\underline{B}^{(1)} = \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

<u>For Element #2:</u>

$$\underline{B}^{(2)} = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

<u>Step 3</u>: Compute element stiffness matrices

$$\underline{k}^{(1)} = At\underline{B}^{(1)^T}\,\underline{D}\,\underline{B}^{(1)} = (3)(0.5)\underline{B}^{(1)^T}\,\underline{D}\,\underline{B}^{(1)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

$u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_4 \quad v_4$

$$\underline{k}^{(2)} = At\underline{B}^{(2)^T}\,\underline{D}\,\underline{B}^{(2)} = (3)(0.5)\underline{B}^{(2)^T}\,\underline{D}\,\underline{B}^{(2)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

$u_3 \quad v_3 \quad u_4 \quad v_4 \quad u_2 \quad v_2$

<u>Step 5</u>: Compute consistent nodal loads

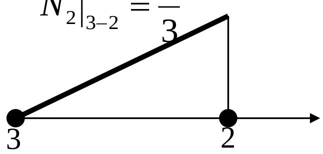
$$\underline{f} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{2y} \end{Bmatrix}$$

$$f_{2y} = -1000 + f_{S_{2y}}$$

The consistent nodal load due to traction on the edge 3-2

$$\begin{aligned} f_{S_{2y}} &= \int_{x=0}^{3} N_3\big|_{3-2}(-300)t\,dx \\ &= (-300)(0.5)\int_{x=0}^{3} N_3\big|_{3-2}\,dx \\ &= -150\int_{x=0}^{3} \frac{x}{3}\,dx \\ &= -50\left[\frac{x^2}{2}\right]_0^3 = -50\left(\frac{9}{2}\right) = -225\ lb \end{aligned}$$

$$N_2\big|_{3-2} = \frac{x}{3}$$



Hence

$$\begin{aligned} f_{2y} &= -1000 + f_{S_{2y}} \\ &= -1225\ \ lb \end{aligned}$$

<u>Step 6</u>: Solve the system equations to obtain the unknown nodal loads

$$\underline{K}\,\underline{d} = \underline{f}$$

$$10^7 \times \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1225 \end{Bmatrix}$$

Solve to get

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0.2337\times10^{-4}\ in \\ 0.1069\times10^{-4}\ in \\ -0.9084\times10^{-4}\ in \end{Bmatrix}$$

<u>Step 7</u>: Compute the stresses in the elements

<u>In Element #1</u>

$$\underline{\sigma}^{(1)} = \underline{D}\,\underline{B}^{(1)}\underline{d}^{(1)}$$

With

$$\begin{aligned}\underline{d}^{(1)T} &= \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{bmatrix} \\ &= \begin{bmatrix} 0.2337\times10^{-4} & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} & 0 & 0 \end{bmatrix}\end{aligned}$$

Calculate

$$\underline{\sigma}^{(1)} = \begin{bmatrix} -114.1 \\ -1391.1 \\ -76.1 \end{bmatrix} \; psi$$

<u>In Element #2</u>

$$\underline{\sigma}^{(2)} = \underline{D}\,\underline{B}^{(2)}\underline{d}^{(2)}$$

With

$$\begin{aligned}\underline{d}^{(2)T} &= \begin{bmatrix} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} \end{bmatrix}\end{aligned}$$

Calculate

$$\underline{\sigma}^{(2)} = \begin{bmatrix} 114.1 \\ 28.52 \\ -363.35 \end{bmatrix} \; psi$$

Notice that the stresses are constant in each element

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INTRODUCTION TO FINITE ELEMENTS

DEVELOPMENT OF TRUSS EQUATIONS

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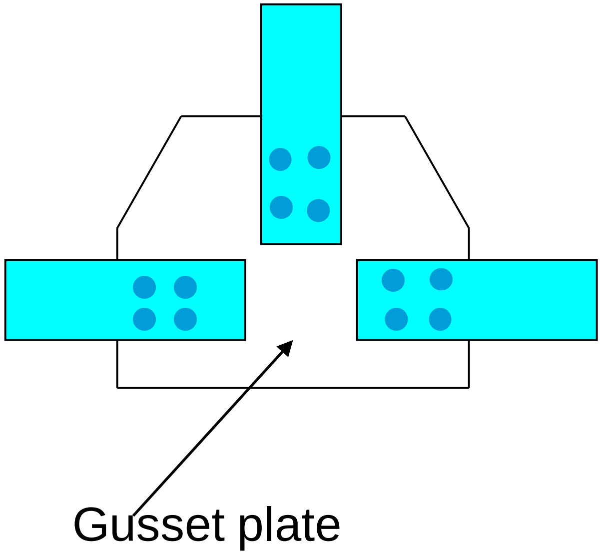
Chapter 2: Lecture notes

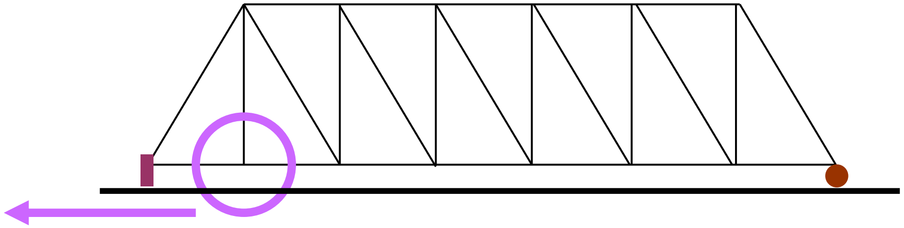
Summary:

- **Stiffness matrix of a bar/truss element**
- **Coordinate transformation**
- **Stiffness matrix of a truss element in 2D space**
- **Problems in 2D truss analysis (including multipoint constraints)**

Trusses: Engineering structures that are composed only of *two-force members. e.g., bridges, roof supports*

Actual trusses: Airy structures composed of slender members (I-beams, channels, angles, bars etc) joined together at their ends by welding, riveted connections or large bolts and pins



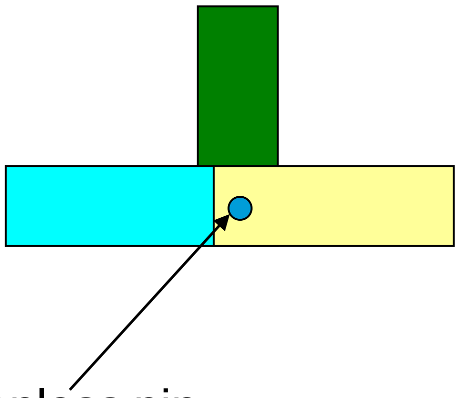


A typical truss structure

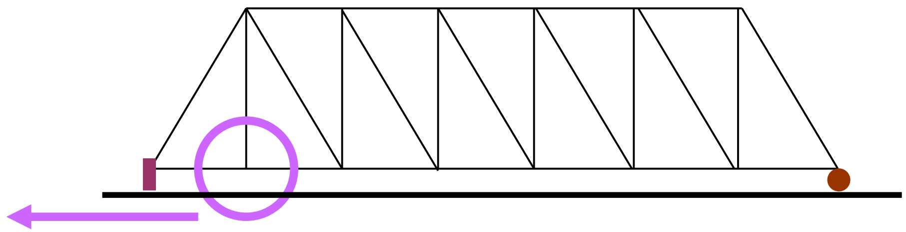
Ideal trusses:

Assumptions

- Ideal truss members are connected only at their ends.
- Ideal truss members are connected by frictionless pins (no moments)
- The truss structure is loaded only at the pins
- Weights of the members are neglected

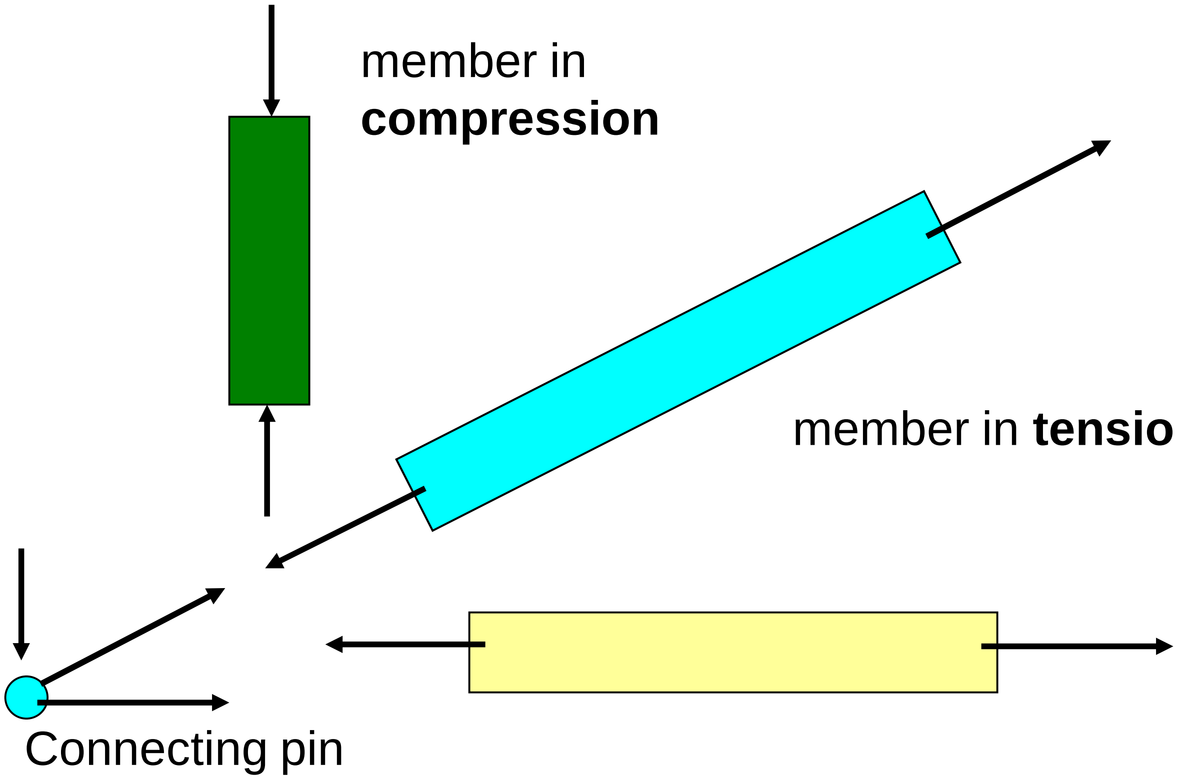


Frictionless pin



A typical truss structure

These assumptions allow us to idealize each truss member as a two-force member (members loaded **only** at their extremities by equal opposite and collinear forces)



FEM analysis scheme

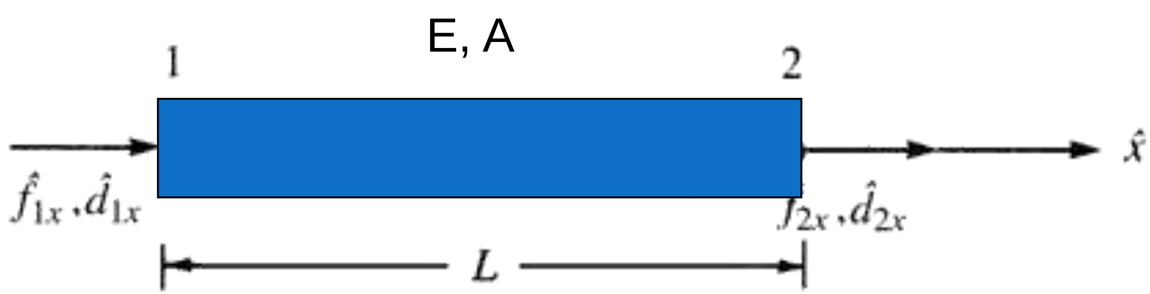
Step 1: Divide the truss into **bar/truss elements** connected to each other through special points ("**nodes**")

Step 2: Describe the behavior of each bar element (i.e. derive its **stiffness matrix** and **load vector** in local AND global coordinate system)

Step 3: Describe the behavior of the entire truss by putting together the behavior of each of the bar elements (by **assembling** their stiffness matrices and load vectors)

Step 4: Apply appropriate boundary conditions and solve

Stiffness matrix of bar element



L: Length of bar

A: Cross sectional area of bar

E: Elastic (Young's) modulus of bar

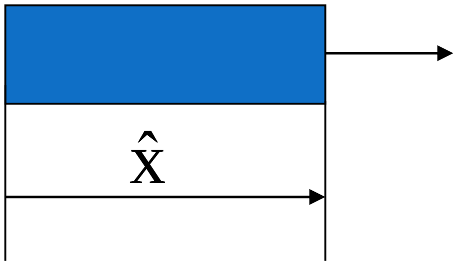
$\hat{u}(\hat{x})$:displacement of bar as a function of local coordinate $\hat{x}$ of bar

The **strain** in the bar at $\hat{x}$

$$\varepsilon(\hat{x}) = \frac{d\hat{u}}{d\hat{x}}$$

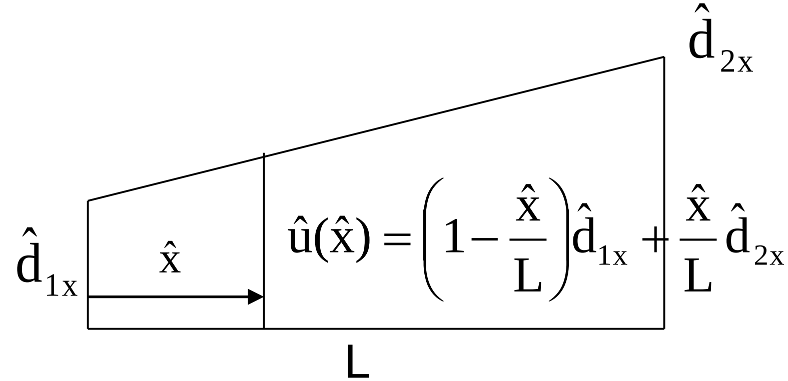
The **stress** in the bar (Hooke's law)

$$\sigma(\hat{x}) = E\,\varepsilon(\hat{x})$$



Tension in the bar

$$T(\hat{x}) = EA\varepsilon$$



Assume that the displacement $\hat{u}(\hat{x})$ is varying **linearly** along the bar

$$\hat{u}(\hat{x}) = \left(1 - \frac{\hat{x}}{L}\right)\hat{d}_{1x} + \frac{\hat{x}}{L}\hat{d}_{2x}$$

Then, strain is **constant** along the bar:

$$\varepsilon = \frac{d\hat{u}}{d\hat{x}} = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L}$$

Stress is also **constant** along the bar:

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

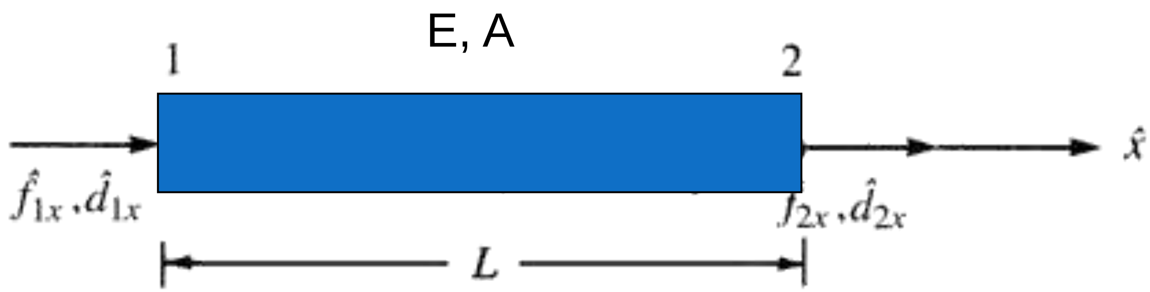
Tension is **constant** along the bar:

$$T = EA\varepsilon = \underbrace{\frac{EA}{L}}_{k}\left(\hat{d}_{2x} - \hat{d}_{1x}\right)$$

The bar is acting like a spring with stiffness

$$k = \frac{EA}{L}$$

Recall the lecture on springs



Two nodes: 1, 2

Nodal displacements: $\hat{d}_{1x} \quad \hat{d}_{2x}$

Nodal forces: $\hat{f}_{1x} \quad \hat{f}_{2x}$

Spring constant: $k = \frac{EA}{L}$

Element stiffness matrix in local coordinates

$$\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$$

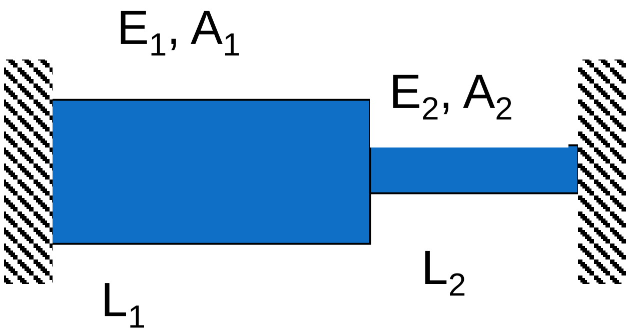
Element force vector

Element **stiffness**

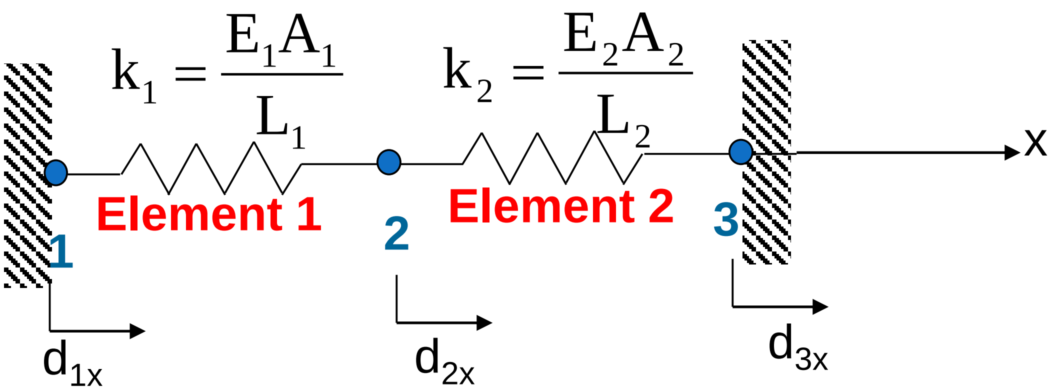
Element nodal displacement vector

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}_{\underline{\hat{k}}} \underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}}_{\underline{\hat{d}}}$$

What if we have 2 bars?

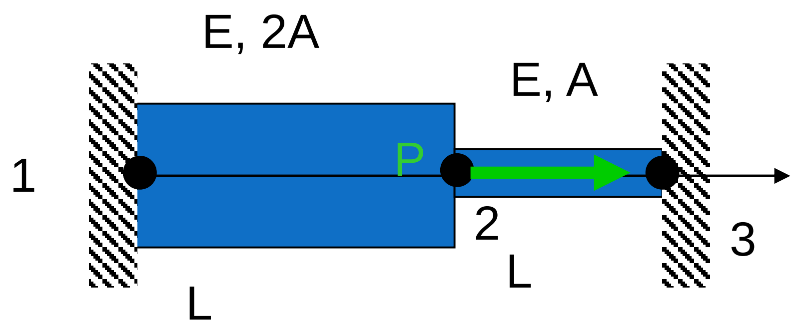


This is equivalent to the following system of springs

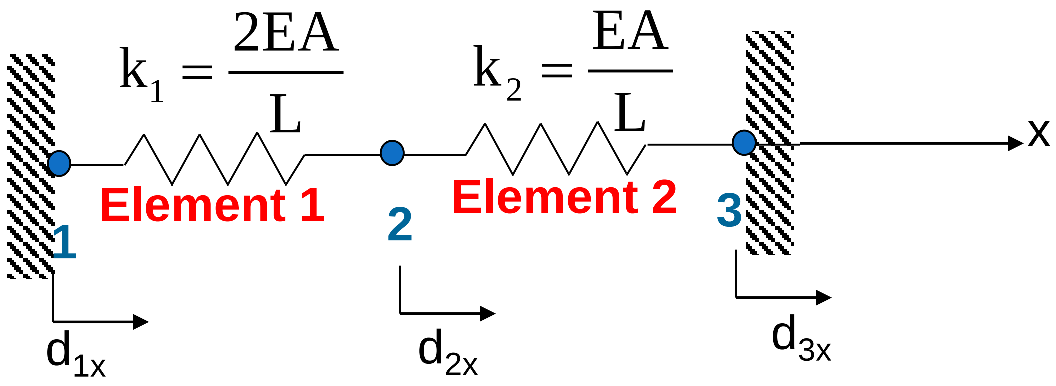


PROBLEM

Problem 1: Find the stresses in the two-bar assembly loaded as shown below



Solution: This is equivalent to the following system of springs



We will first compute the displacement at node 2 and then the stresses within each element

The global set of equations can be generated using the technique developed in the lecture on “springs”

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \end{Bmatrix}$$

here $d_{1x} = d_{3x} = 0 \quad and \quad F_{2x} = P$

Hence, the above set of equations may be explicitly written as

$$-k_1 d_{2x} = F_{1x} \qquad (1)$$

$$(k_1 + k_2) d_{2x} = P \qquad (2)$$

$$-k_2 d_{2x} = F_{3x} \qquad (3)$$

From equation (2) $$d_{2x} = \frac{P}{k_1 + k_2} = \frac{PL}{3EA}$$

To calculate the **stresses**:

For element #1 first compute the element strain

$$\varepsilon^{(1)} = \frac{d_{2x} - d_{1x}}{L} = \frac{d_{2x}}{L} = \frac{P}{3EA}$$

and then the stress as

$$\sigma^{(1)} = E\varepsilon^{(1)} = \frac{P}{3A}$$

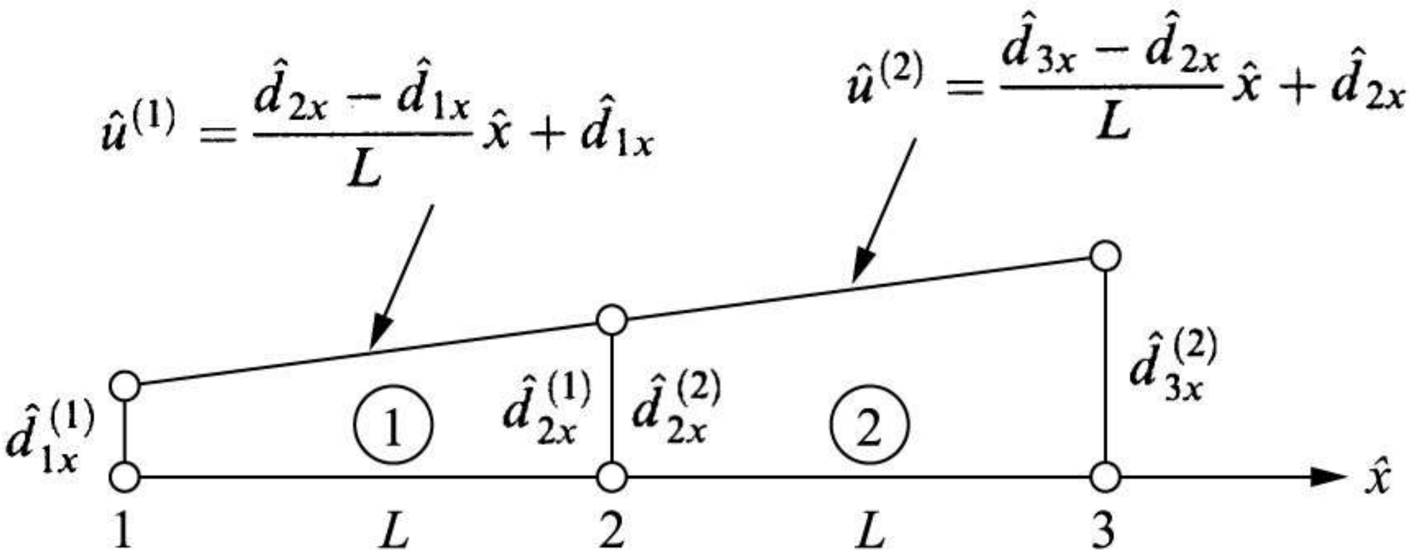
(element in tension)

Similarly, in element # 2

$$\varepsilon^{(2)} = \frac{d_{3x} - d_{2x}}{L} = -\frac{d_{2x}}{L} = -\frac{P}{3EA}$$

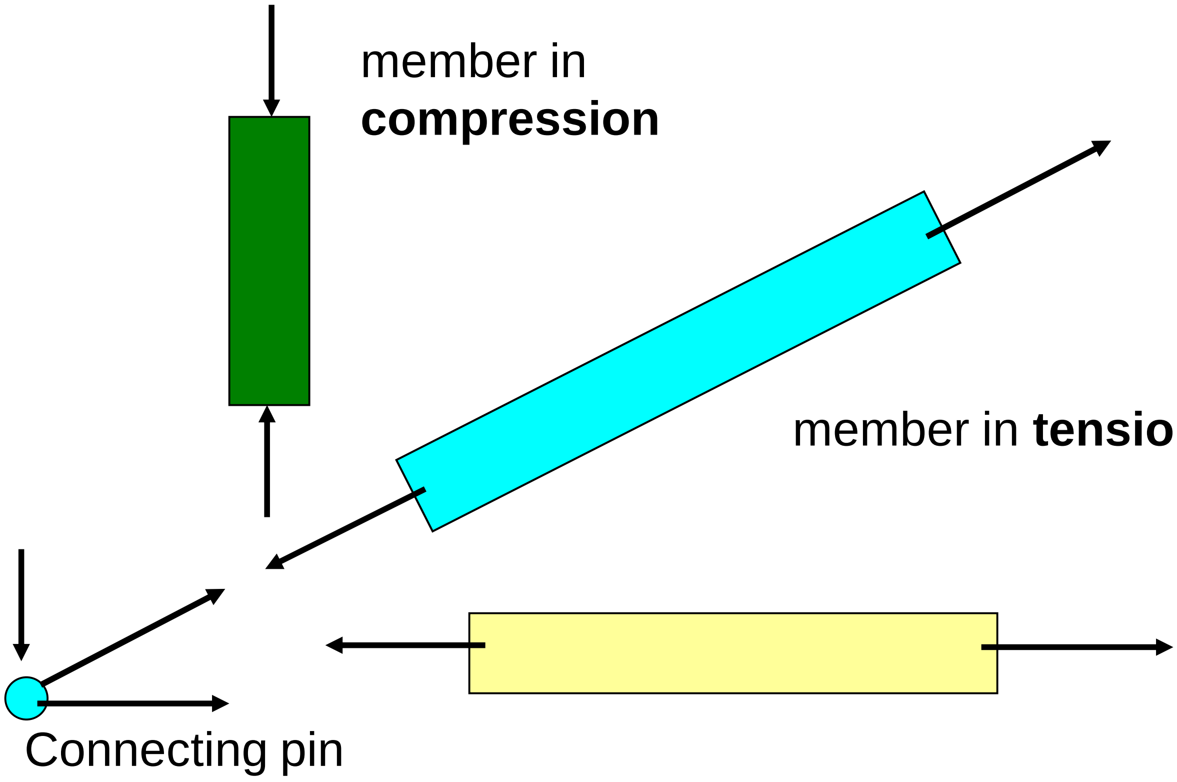
$$\sigma^{(2)} = E\varepsilon^{(2)} = -\frac{P}{3A}$$

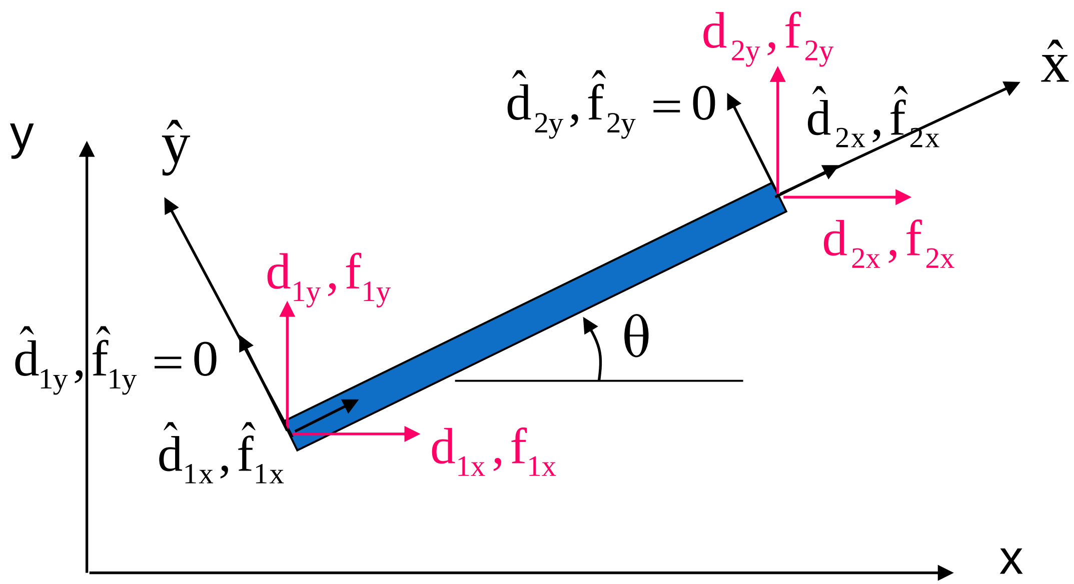
(element in compression)



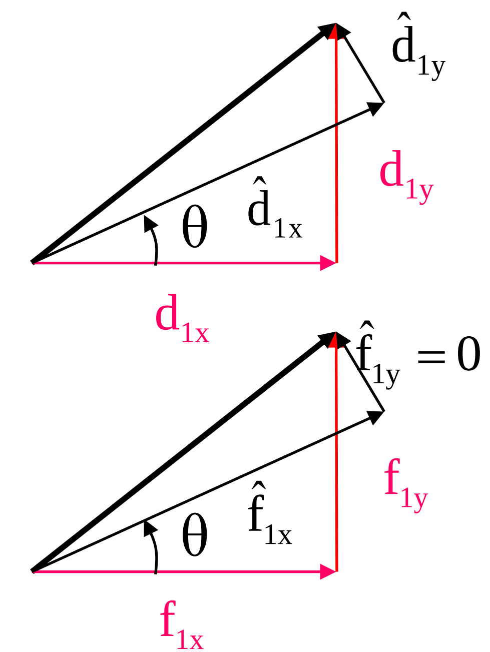
Inter-element continuity of a two-bar structure

Bars in a truss have various orientations

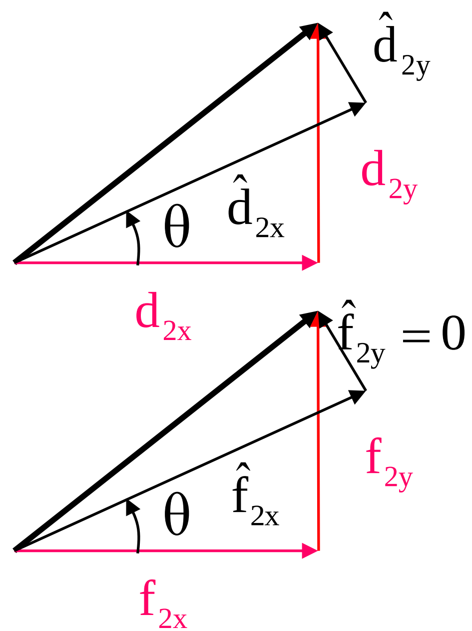




At node 1:



At node 2:



In the **global coordinate system**, the vector of nodal displacements and loads

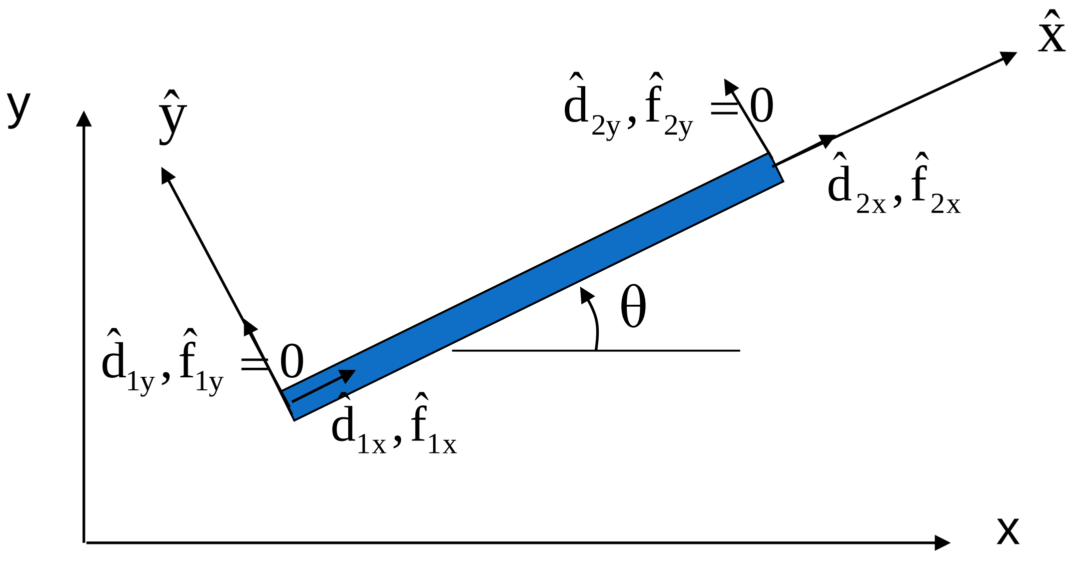
$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}; \qquad \underline{f} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$$

Our objective is to obtain a relation of the form

$$\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}}\ \underset{4\times1}{\underline{d}}$$

Where $\underline{k}$ is the 4x4 element stiffness matrix in global coordinate system

The key is to look at the local coordinates



$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{2x} \end{Bmatrix}$$

$$k = \frac{EA}{L}$$

Rewrite as

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

$$\boxed{\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}}$$

NOTES

1. **Assume** that there is **no stiffness** in the local y direction.
^

2. If you consider the displacement at a point along the local x direction as a vector, then the components of that vector along the global x and y directions are the global x and y displacements.

3. The expanded stiffness matrix in the local coordinates is symmetric and singular.

NOTES

5. In local coordinates we have

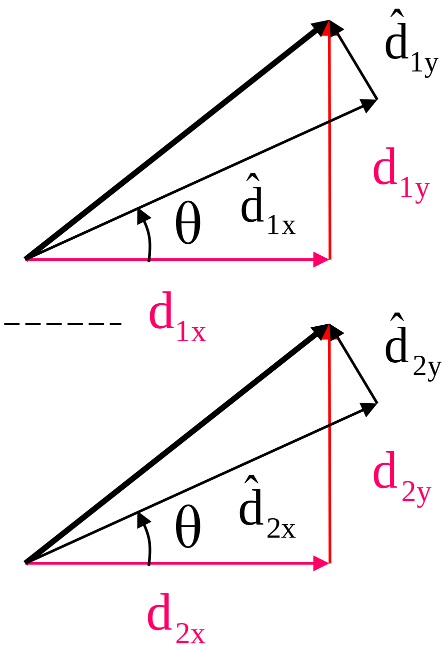
$$\underset{4\times1}{\hat{\underline{f}}} = \underset{4\times4}{\hat{\underline{k}}}\ \underset{4\times1}{\hat{\underline{d}}}$$

But or **goal** is to obtain the following relationship

$$\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}}\ \underset{4\times1}{\underline{d}}$$

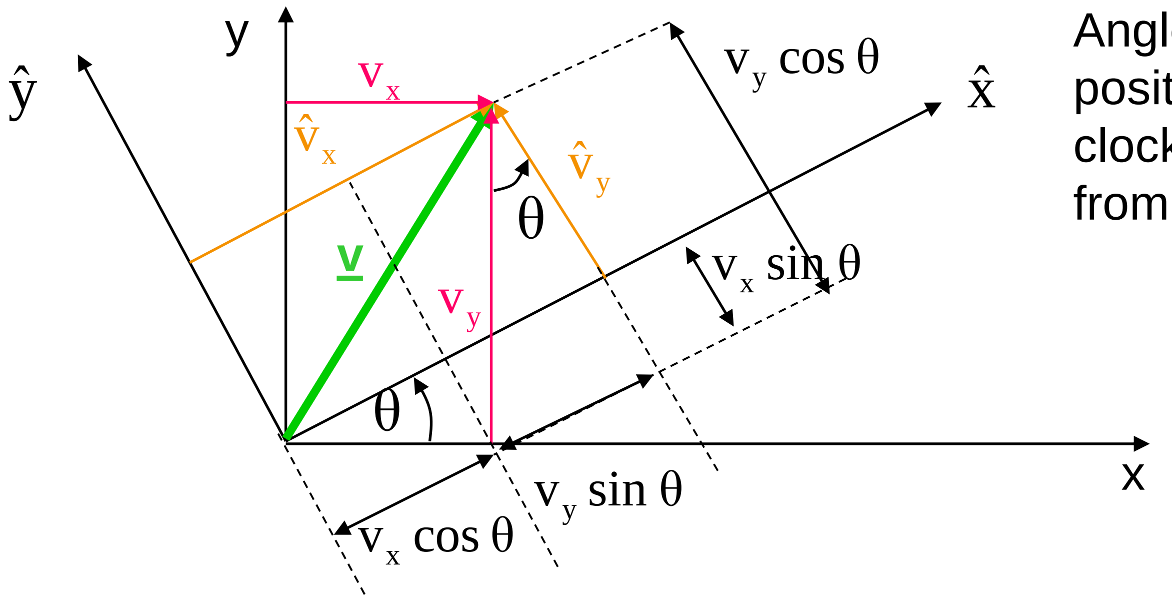
Hence, need a relationship between $\hat{\underline{d}}$ and $\underline{d}$ and between $\hat{\underline{f}}$ and $\underline{f}$

$$\underline{d} = \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \qquad \hat{\underline{d}} = \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$



Need to understand how the components of a vector change with coordinate transformation

Transformation of a vector in two dimensions



Angle θ is measured positive in the counter clockwise direction from the +x axis)

The vector **v** has components (v_x, v_y) in the global coordinate system and $(\hat{v}_x, \hat{v}_y)$ in the local coordinate system. From geometry

$$\hat{v}_x = v_x \cos\theta + v_y \sin\theta$$

$$\hat{v}_y = -v_x \sin\theta + v_y \cos\theta$$

In matrix form

$$\begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$$

Or

$$\begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$$

where

Direction cosines

$$l = \cos\theta$$

$$m = \sin\theta$$

Transformation matrix for a single vector in 2D

$$\underline{T}^* = \begin{bmatrix} l & m \\ -m & l \end{bmatrix}$$

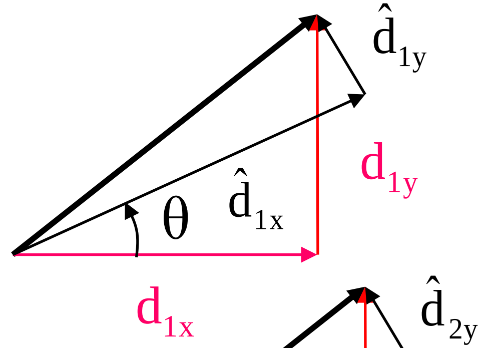
relates $\underline{\hat{v}} = \underline{T}^* \underline{v}$

where $\underline{\hat{v}} = \begin{Bmatrix} \hat{v}_x \\ \hat{v}_y \end{Bmatrix}$ and $\underline{v} = \begin{Bmatrix} v_x \\ v_y \end{Bmatrix}$ are components of the **same vector** in local and global coordinates, respectively.

Relationship between $\hat{\underline{d}}$ and $\underline{d}$ for the truss element

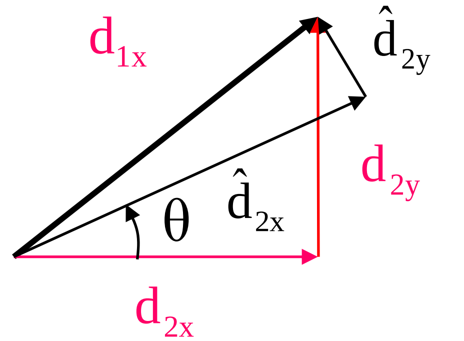
At node 1

$$\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \end{Bmatrix} = \underline{T}^{*} \begin{Bmatrix} d_{1x} \\ d_{1y} \end{Bmatrix}$$



At node 2

$$\begin{Bmatrix} \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix} = \underline{T}^{*} \begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix}$$



Putting these together

$$\boxed{\hat{\underline{d}} = \underline{T}\,\underline{d}}$$

$$\underbrace{\begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}}_{\hat{\underline{d}}} = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}}_{\underline{T}} \underbrace{\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}}_{\underline{d}}$$

$$\underset{4\times 4}{\underline{T}} = \begin{bmatrix} \underline{T}^{*} & \underline{0} \\ \underline{0} & \underline{T}^{*} \end{bmatrix}$$

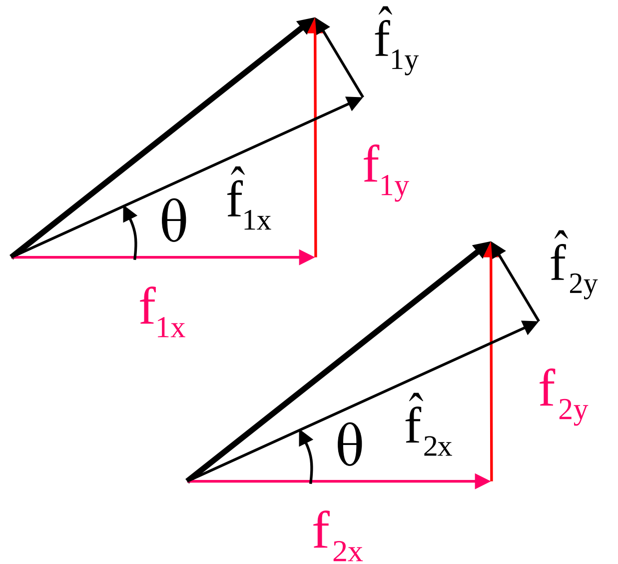
Relationship between $\underline{\hat{f}}$ and $\underline{f}$ for the truss element

At node 1

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} f_{1x} \\ f_{1y} \end{Bmatrix}$$

At node 2

$$\begin{Bmatrix} \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix} = \underline{T}^* \begin{Bmatrix} f_{2x} \\ f_{2y} \end{Bmatrix}$$



Putting these together

$$\boxed{\underline{\hat{f}} = \underline{T}\,\underline{f}}$$

$$\underbrace{\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \end{Bmatrix}}_{\underline{\hat{f}}} = \underbrace{\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}}_{\underline{T}} \underbrace{\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}}_{\underline{f}}$$

$$\underset{4\times4}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

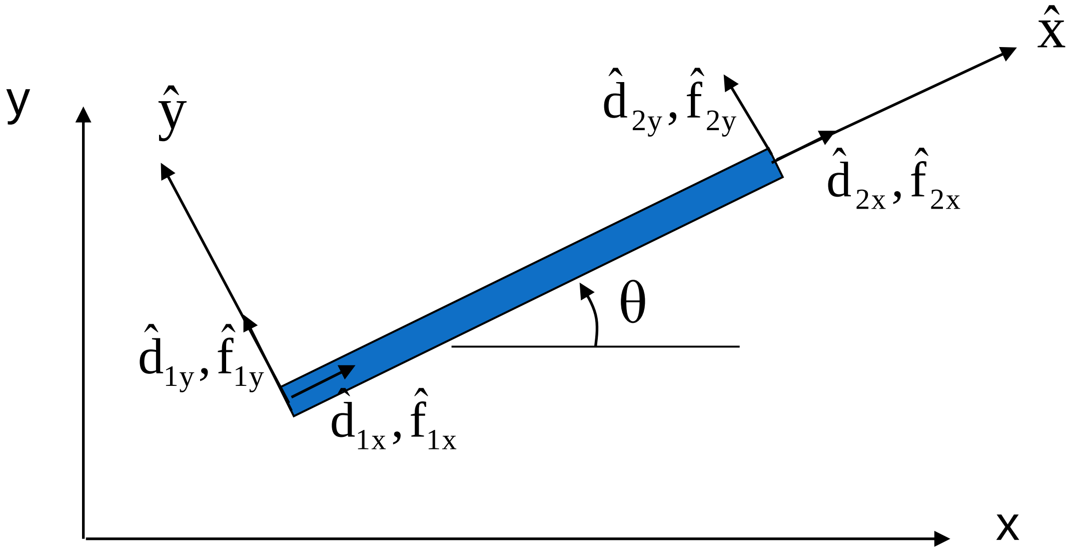
Important property of the transformation matrix $\underline{T}$

The transformation matrix is *orthogonal, i.e.* its inverse is its transpose

$$\underline{T}^{-1} = \underline{T}^{T}$$

Use the property that $l^2+m^2=1$

Putting all the pieces together



$$\underline{\hat{f}} = \underline{T}\,\underline{f}$$

$$\underline{\hat{d}} = \underline{T}\,\underline{d}$$

$$\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$$

$$\Rightarrow \underline{T}\underline{f} = \underline{\hat{k}}\,\underline{T}\underline{d}$$

$$\Rightarrow \underline{f} = \underbrace{\left(\underline{T}^{-1}\underline{\hat{k}}\,\underline{T}\right)}_{\underline{k}}\underline{d}$$

The desired relationship is

$$\underset{4\times1}{\underline{f}} = \underset{4\times4}{\underline{k}}\;\underset{4\times1}{\underline{d}}$$

Where

$$\underset{4\times4}{\underline{k}} = \underset{4\times4}{\underline{T}^{T}}\;\underset{4\times4}{\underline{\hat{k}}}\;\underset{4\times4}{\underline{T}}$$

is the **element stiffness matrix in the global coordinate system**

$$\underline{T} = \begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}$$

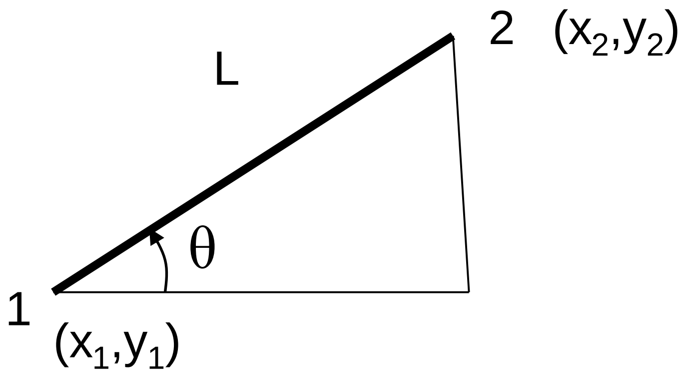
$$\underline{\hat{k}} = \begin{bmatrix} k & 0 & -k & 0 \\ 0 & 0 & 0 & 0 \\ -k & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{k} = \underline{T}^T \underline{\hat{k}} \underline{T} = \frac{EA}{L} \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Computation of the direction cosines

$$l = \cos\theta = \frac{x_2 - x_1}{L}$$

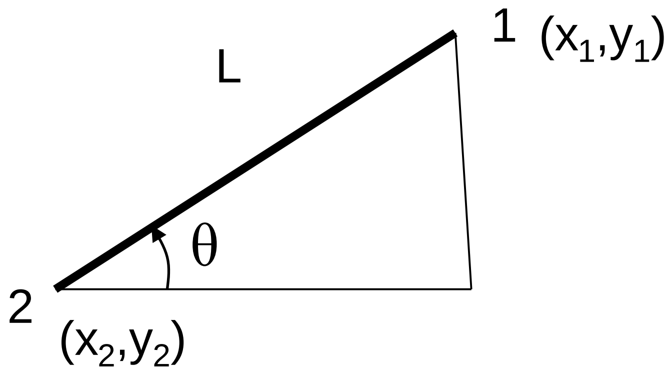
$$m = \sin\theta = \frac{y_2 - y_1}{L}$$



What happens if I reverse the node numbers?

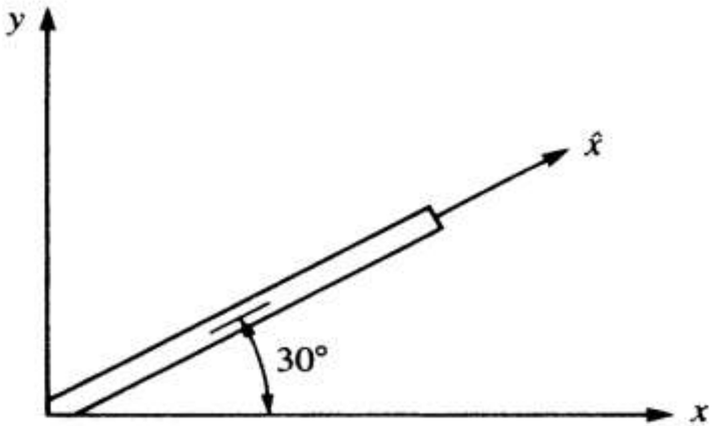
$$l' = \cos\theta = \frac{x_1 - x_2}{L} = -l$$

$$m' = \sin\theta = \frac{y_1 - y_2}{L} = -m$$



Question: Does the stiffness matrix change?

Example Bar element for stiffness matrix evaluation



$$E = 30\times10^{6}\ psi$$

$$A = 2\ in^{2}$$

$$L = 60\ \ in$$

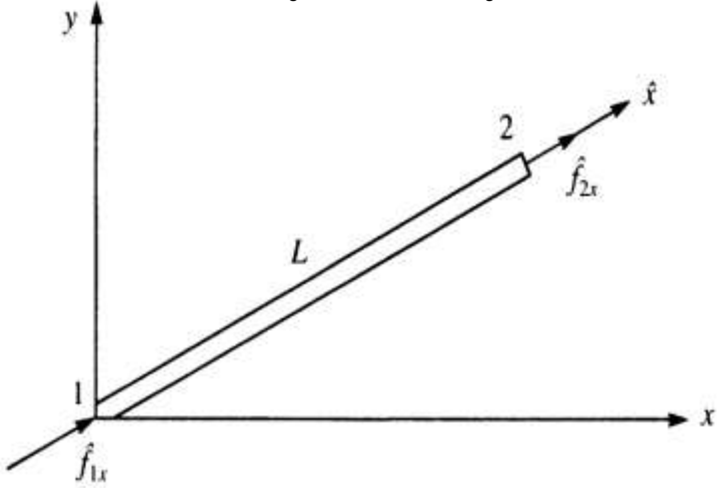
$$\theta = 30^{\circ}$$

$$l = \cos 30 = \frac{\sqrt{3}}{2}$$

$$m = \sin 30 = \frac{1}{2}$$

$$\underline{k} = \frac{(30\times10^{6})(2)}{60}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}\frac{lb}{in}$$

Computation of element strains



Recall that the element strain is

$$\varepsilon = \frac{\hat{d}_{2x} - \hat{d}_{1x}}{L} = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \end{Bmatrix}$$

$$= \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \underline{\hat{d}}$$

$$= \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix} \underline{T}\,\underline{d}$$

$$\varepsilon = \frac{1}{L}\begin{bmatrix} -1 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ -m & l & 0 & 0 \\ 0 & 0 & l & m \\ 0 & 0 & -m & l \end{bmatrix}\underline{d}$$

$$= \frac{1}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{d}$$

$$= \frac{1}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

Computation of element stresses stress and tension

Recall that the element **stress** is

$$\sigma = E\varepsilon = \frac{E}{L}\left(\hat{d}_{2x} - \hat{d}_{1x}\right) = \frac{E}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{d}$$

Recall that the element **tension** is

$$T = EA\varepsilon = \frac{EA}{L}\begin{bmatrix} -l & -m & l & m \end{bmatrix}\underline{d}$$

Steps in solving a problem

Step 1: Write down the **node-element connectivity table** linking local and global nodes; also form the **table of direction cosines** (*l, m*)

Step 2: Write down the **stiffness matrix of each element in global coordinate system with global numbering**

Step 3: **Assemble** the element stiffness matrices to form the global stiffness matrix for the entire structure using the node element connectivity table

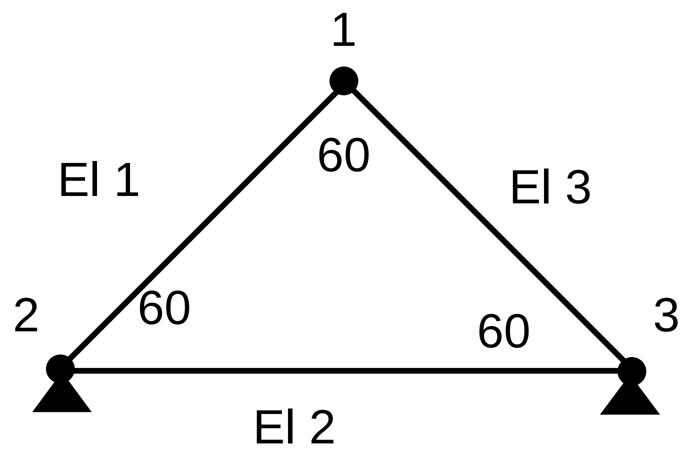
Step 4: Incorporate appropriate **boundary conditions**

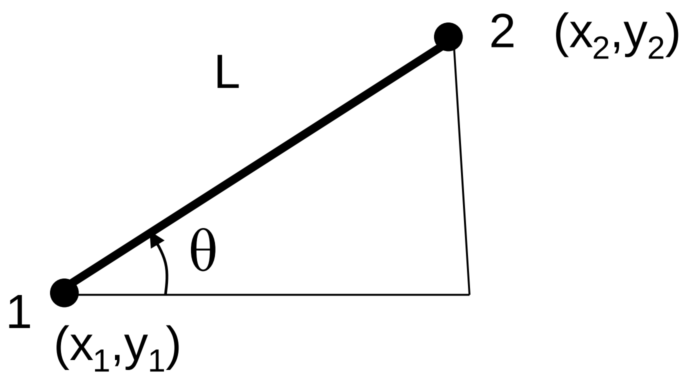
Step 5: Solve resulting set of reduced equations for the unknown displacements

Step 6: Compute the unknown nodal forces

Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3
3	3	1





Stiffness matrix of element 1

$\underline{k}^{(1)} =$

	d_{1x}	d_{1y}	d_{2x}	d_{2y}
d_{1x}				
d_{1y}				
d_{2x}				
d_{2y}				

Stiffness matrix of element 2

$\underline{k}^{(2)} =$

	d_{2x}	d_{2y}	d_{3x}	d_{3y}
d_{2x}				
d_{2y}				
d_{3x}				
d_{3y}				

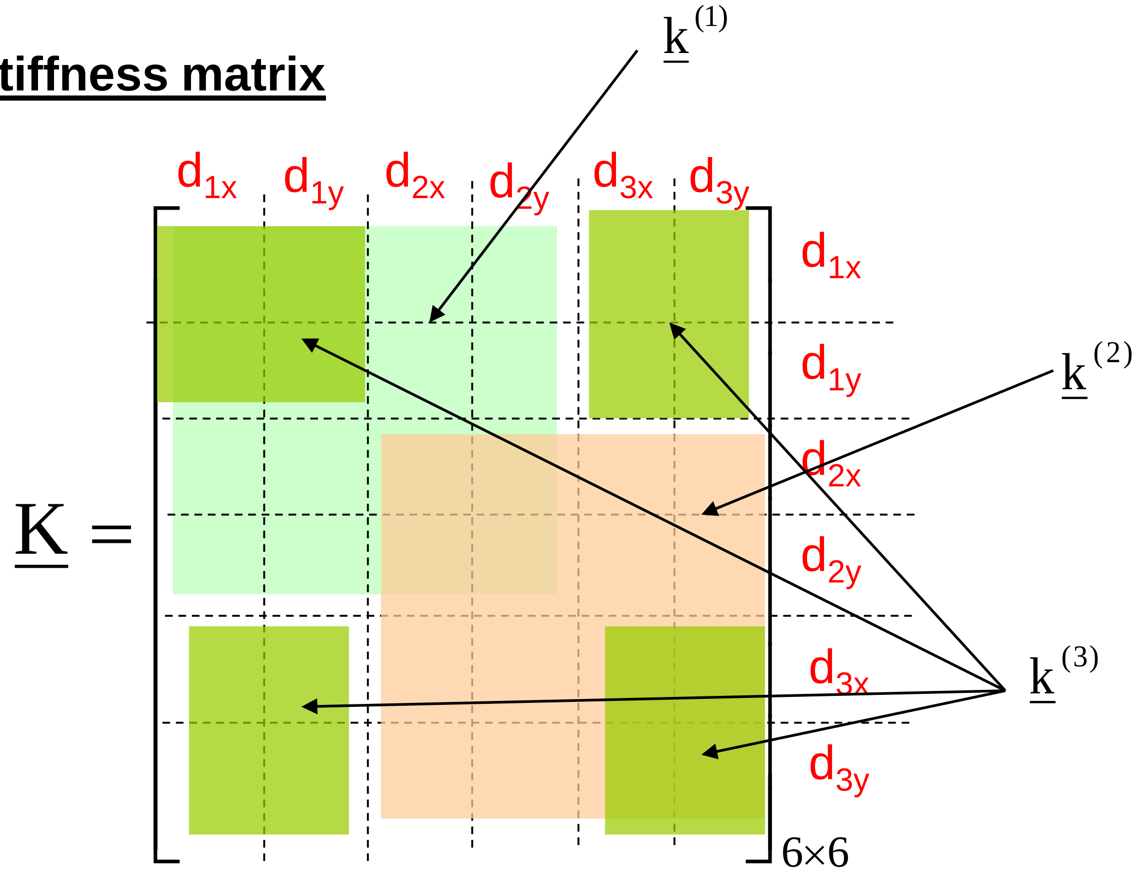
Stiffness matrix of element 3

$\underline{k}^{(3)} =$

	d_{3x}	d_{3y}	d_{1x}	d_{1y}
d_{3x}				
d_{3y}				
d_{1x}				
d_{1y}				

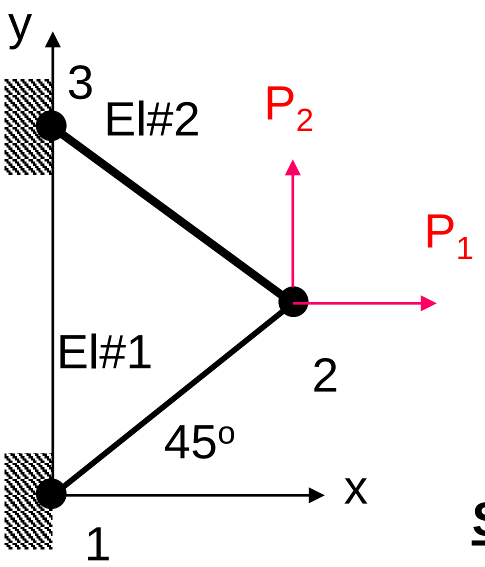
There are 4 **degrees of freedom** (**dof**) per element (2 per node)

Global stiffness matrix



How do you incorporate **boundary conditions**?

Example 2



The length of bars 12 and 23 are equal (L)
E: Young's modulus
A: Cross sectional area of each bar
Solve for
(1) d_{2x} and d_{2y}
(2) Stresses in each bar

Solution

Step 1: Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3

Table of nodal coordinates

Node	x	y
1	0	0
2	Lcos45	Lsin45
3	0	2Lsin45

Table of direction cosines

ELEMENT	Length	$l = \frac{x_2 - x_1}{length}$	$m = \frac{y_2 - y_1}{length}$
1	L	cos45	sin45
2	L	-cos45	sin45

Step 2: Stiffness matrix of each element in global coordinates with global numbering

Stiffness matrix of element 1

$$\underline{k}^{(1)} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$\begin{matrix} & d_{1x} & d_{1y} & d_{2x} & d_{2y} & \end{matrix}$$

$$= \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{matrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{matrix}$$

Stiffness matrix of element 2

$$\underline{k}^{(2)} = \frac{EA}{2L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Columns: d_{2x}, d_{2y}, d_{3x}, d_{3y}; rows: d_{2x}, d_{2y}, d_{3x}, d_{3y}

Step 3: Assemble the global stiffness matrix

$$\underline{K} = \frac{EA}{2L}\begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 2 & 0 & -1 & 1 \\ -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

The final set of equations is $$\underline{K}\,\underline{d} = \underline{F}$$

Step 4: Incorporate boundary conditions

$$\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ d_{2x} \\ d_{2y} \\ 0 \\ 0 \end{Bmatrix}$$

Hence reduced set of equations to solve for unknown displacements at node 2

$$\frac{EA}{2L}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Step 5: Solve for unknown displacements

$$\begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{P_1 L}{EA} \\ \dfrac{P_2 L}{EA} \end{Bmatrix}$$

Step 6: Obtain stresses in the elements

For element #1:

$$\sigma^{(1)} = \frac{\mathrm{E}}{\mathrm{L}} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix}$$

($d_{1x} \rightarrow 0$, $d_{1y} \rightarrow 0$)

$$= \frac{\mathrm{E}}{\sqrt{2}\mathrm{L}}(d_{2x} + d_{2y}) = \frac{P_1 + P_2}{A\sqrt{2}}$$

For element #2:

$$\sigma^{(2)} = \frac{\mathrm{E}}{\mathrm{L}}\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{2y} \\ \cancelto{0}{d_{3x}} \\ \cancelto{0}{d_{3y}} \end{Bmatrix}$$

$$= \frac{\mathrm{E}}{\sqrt{2}\mathrm{L}}(d_{2x} - d_{2y}) = \frac{P_1 - P_2}{A\sqrt{2}}$$

Multi-point constraints

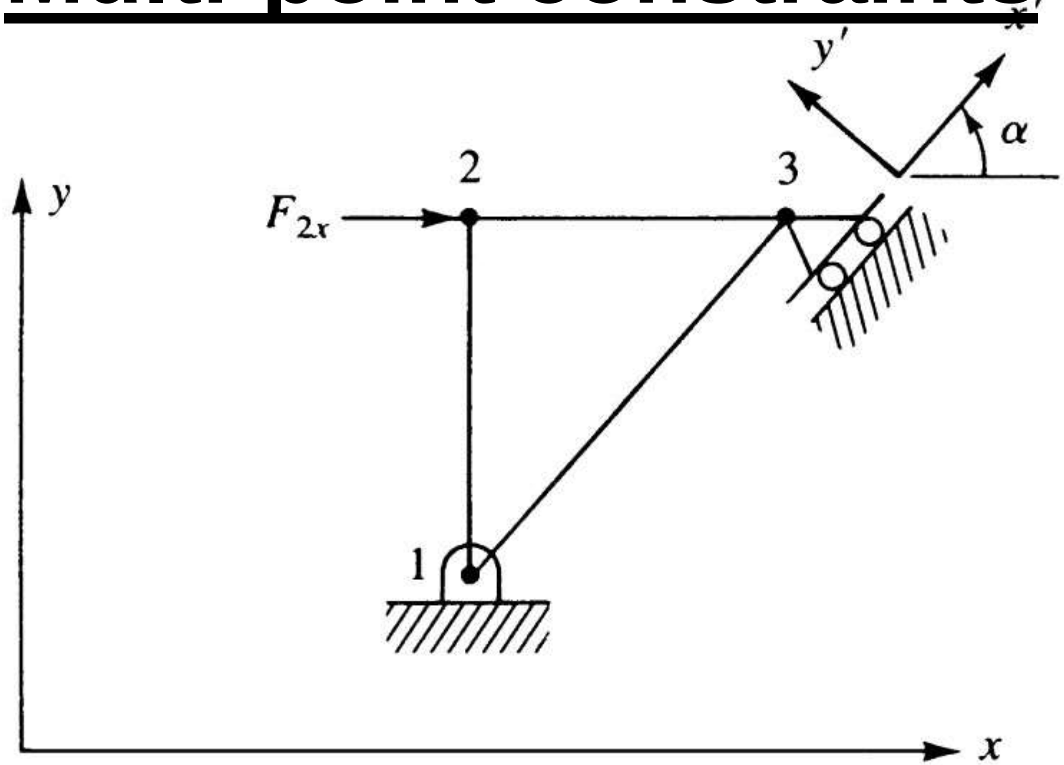
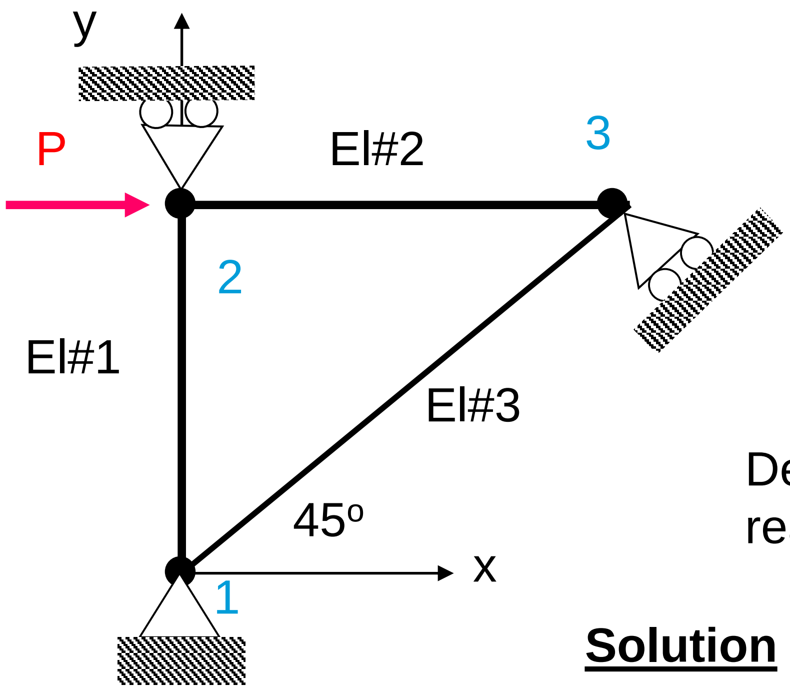


Figure 3-19 Plane truss with inclined boundary conditions at node 3 (see problem worked out in class)

Problem 3: For the plane truss

P=1000 kN,
L=length of elements 1 and 2 = 1m
E=210 GPa
A = 6×10^{-4} m^2 for elements 1 and 2
= 6 $\times10^{-4}$ m2 for element 3

$\sqrt{2}$



Determine the unknown displacements and reaction forces.

Solution

Step 1: Node element connectivity table

ELEMENT	Node 1	Node 2
1	1	2
2	2	3
3	1	3

Table of nodal coordinates

Node	x	y
1	0	0
2	0	L
3	L	L

Table of direction cosines

ELEMENT	Length	$l = \frac{x_2 - x_1}{length}$	$m = \frac{y_2 - y_1}{length}$
1	L	0	1
2	L	1	0
3	$L\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$

Step 2: Stiffness matrix of each element in global coordinates with global numbering

Stiffness matrix of element 1

$$\underline{k}^{(1)} = \frac{EA}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

$$= \frac{(210\times10^9)(6\times10^{-4})}{1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

Columns: d_{1x}, d_{1y}, d_{2x}, d_{2y}; rows: d_{1x}, d_{1y}, d_{2x}, d_{2y}

Stiffness matrix of element 2

$$\underline{k}^{(2)} = \frac{(210\times10^{9})(6\times10^{-4})}{1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns: d_{2x}, d_{2y}, d_{3x}, d_{3y}; rows: d_{2x}, d_{2y}, d_{3x}, d_{3y}

Stiffness matrix of element 3

$$\underline{k}^{(3)} = \frac{(210\times10^{9})(6\sqrt{2}\times10^{-4})}{\sqrt{2}}\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

Columns: d_{1x}, d_{1y}, d_{3x}, d_{3y}; rows: d_{1x}, d_{1y}, d_{3x}, d_{3y}

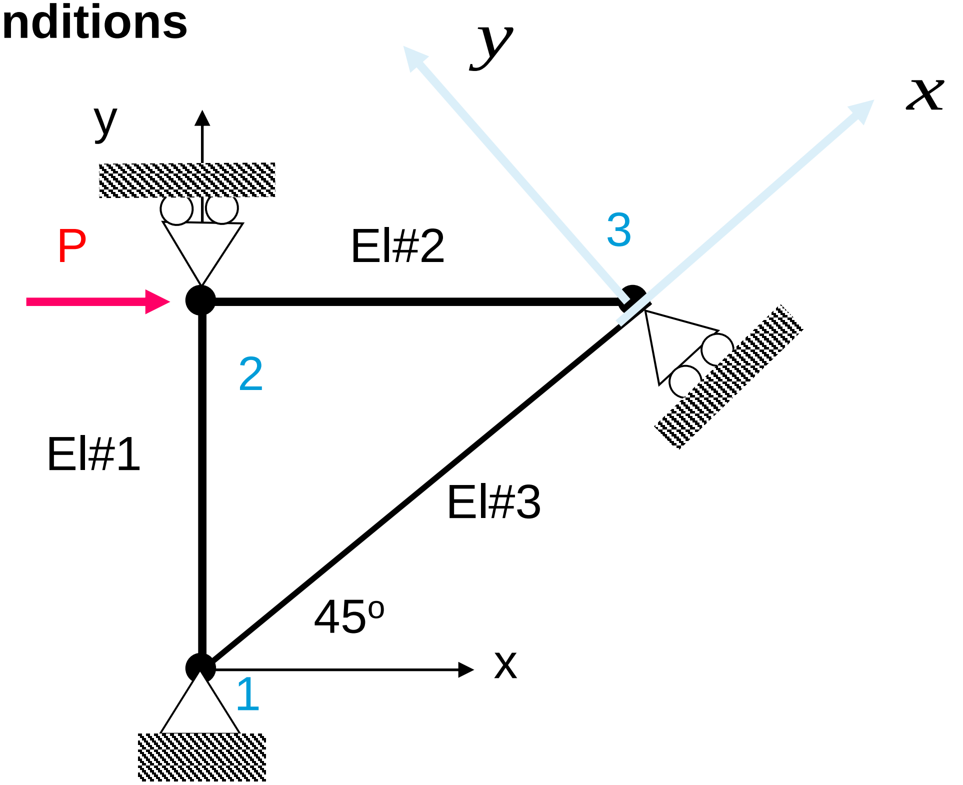
Step 3: Assemble the global stiffness matrix

$$\underline{K} = 1260 \times 10^5 \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ 0.5 & 1.5 & 0 & -1 & -0.5 & -0.5 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -0.5 & -0.5 & -1 & 0 & 1.5 & 0.5 \\ -0.5 & -0.5 & 0 & 0 & 0.5 & 0.5 \end{bmatrix} \text{ N/m}$$

The final set of equations is $$\underline{K}\underline{d} = \underline{F} \qquad \text{Eq(1)}$$

Step 4: Incorporate boundary conditions

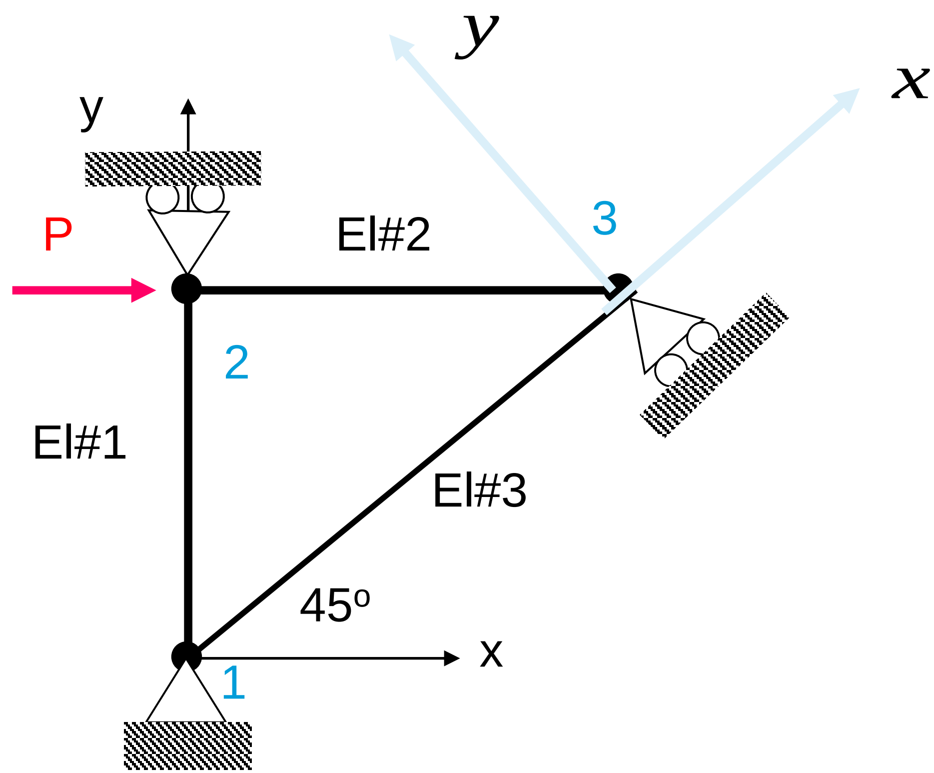
$$\underline{d} = \begin{Bmatrix} 0 \\ 0 \\ d_{2x} \\ 0 \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$



Also, $$d_{3y} = 0$$ in the local coordinate system of element 3

How do I convert this to a boundary condition in the global (x,y) coordinates?

$$\underline{F} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ P \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$



Also, $F_{3x} = 0$ in the local coordinate system of element 3

How do I convert this to a boundary condition in the global (x,y) coordinates?

Using coordinate transformations

$$\begin{Bmatrix} d'_{3x} \\ d'_{3y} \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & l \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix} \quad l = m = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \begin{Bmatrix} d'_{3x} \\ d'_{3y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\sqrt{2}}\left(d_{3x} + d_{3y}\right) \\ \frac{1}{\sqrt{2}}\left(d_{3y} - d_{3x}\right) \end{Bmatrix}$$

$$d'_{3y} = 0$$

(Multi-point constraint)

$$\Rightarrow d'_{3y} = \frac{1}{\sqrt{2}}\left(d_{3y} - d_{3x}\right) = 0$$

$$\Rightarrow d_{3y} - d_{3x} = 0$$

Eq (2)

Similarly for the forces at node 3

$$\begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} l & m \\ -m & n \end{bmatrix} \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} \quad l = m = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{Bmatrix} F_{3x} \\ F_{3y} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\sqrt{2}}\left(F_{3x} + F_{3y}\right) \\ \frac{1}{\sqrt{2}}\left(F_{3y} - F_{3x}\right) \end{Bmatrix}$$

$$\boxed{F_{3x} = 0}$$

$$\Rightarrow F_{3x} = \frac{1}{\sqrt{2}}\left(F_{3y} + F_{3x}\right) = 0$$

$$\Rightarrow F_{3y} + F_{3x} = 0$$

Eq (3)

Therefore we need to solve the following equations simultaneously

$$\underline{K}\,\underline{d} = \underline{F}$$ Eq(1)

$$d_{3y} - d_{3x} = 0$$ Eq(2)

$$F_{3y} + F_{3x} = 0$$ Eq(3)

Incorporate boundary conditions and reduce Eq(1) to

$$1260\times10^{5}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}\begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \end{Bmatrix} = \begin{Bmatrix} P \\ F_{3x} \\ F_{3y} \end{Bmatrix}$$

Write these equations out explicitly

$$1260 \times 10^5 (d_{2x} - d_{3x}) = P$$ Eq(4)

$$1260 \times 10^5 (-d_{2x} + 1.5 d_{3x} + 0.5 d_{3y}) = F_{3x}$$ Eq(5)

$$1260 \times 10^5 (0.5 d_{3x} + 0.5 d_{3y}) = F_{3y}$$ Eq(6)

Add Eq (5) and (6)

$$1260 \times 10^5 (-d_{2x} + 2 d_{3x} + d_{3y}) = F_{3x} + F_{3y} = 0$$ using Eq(3)

$$\Rightarrow 1260 \times 10^5 (-d_{2x} + 3 d_{3x}) = 0$$ using Eq(2)

$$\Rightarrow d_{2x} = 3 d_{3x}$$ Eq(7)

Plug this into Eq(4)

$$\Rightarrow 1260 \times 10^5 (3 d_{3x} - d_{3x}) = P$$

$$\Rightarrow 2520 \times 10^5 d_{3x} = 10^6$$

$$\Rightarrow d_{3x} = 0.003968m$$
$$d_{2x} = 3d_{3x} = 0.0119m$$

Compute the reaction forces

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{Bmatrix} = 1260 \times 10^5 \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{3x} \\ d_{3y} \end{Bmatrix}$$

$$= \begin{Bmatrix} -500 \\ -500 \\ 0 \\ -500 \\ 500 \end{Bmatrix} kN$$

Physical significance of the stiffness matrix

In general, we will have a stiffness matrix of the form

$$\underline{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$

And the finite element force-displacement relation

$$\begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

Physical significance of the stiffness matrix

The first equation is

$$k_{11}d_1 + k_{12}d_2 + k_{13}d_3 = F_1$$

Force equilibrium equation at node 1

Columns of the global stiffness matrix

What if $d_1=1$, $d_2=0$, $d_3=0$?

While **d.o.f** 2 and 3 are held fixed

$F_1 = k_{11}$ Force along **d.o.f** 1 due to unit displacement at **d.o.f** 1

$F_2 = k_{21}$ Force along **d.o.f** 2 due to unit displacement at **d.o.f** 1

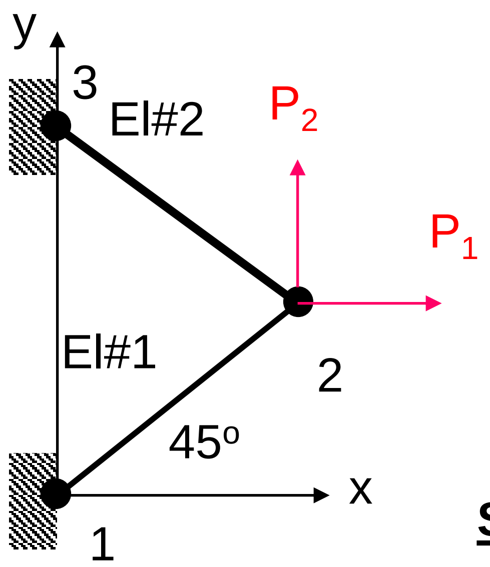
$F_3 = k_{31}$ Force along **d.o.f** 3 due to unit displacement at **d.o.f** 1

Similarly we obtain the physical significance of the other entries of the global stiffness matrix

In general

k_{ij} = Force at d.o.f ‘i’ due to **unit displacement** at d.o.f ‘j’ keeping **all the other d.o.fs fixed**

Example



The length of bars 12 and 23 are equal (L)
E: Young's modulus
A: Cross sectional area of each bar
Solve for d_{2x} and d_{2y} using the "physical interpretation" approach

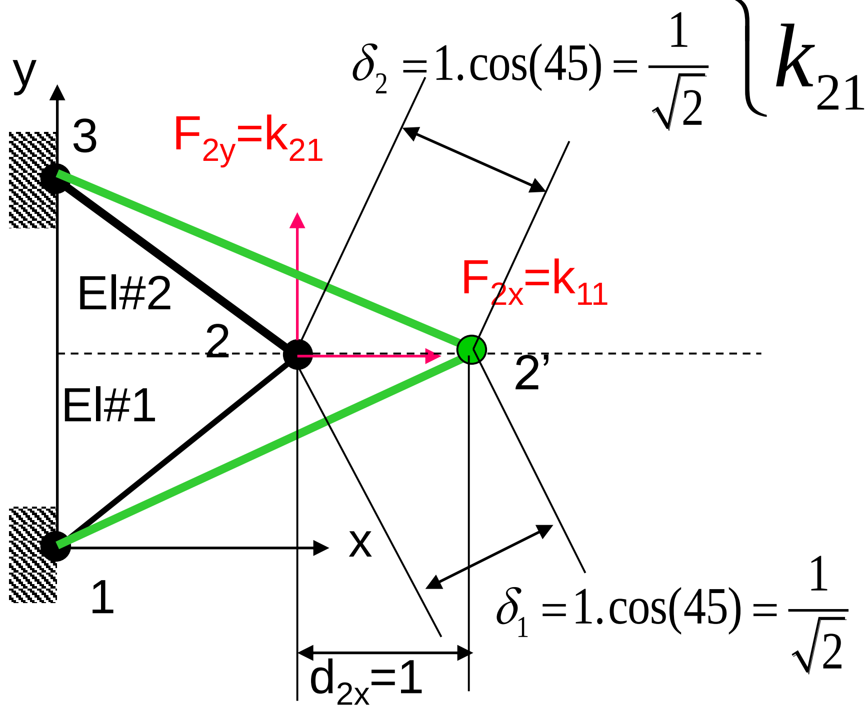
<u>Solution</u>

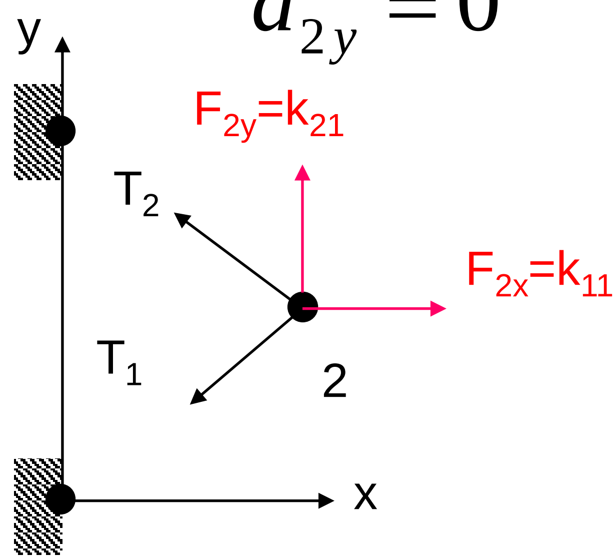
Notice that the final set of equations will be of the form

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} d_{2x} \\ d_{2y} \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

Where k_{11}, k_{12}, k_{21} and k_{22} will be determined using the "physical interpretation" approach

To obtain the first column $\begin{Bmatrix} k_{11} \\ k_{21} \end{Bmatrix}$ apply $d_{2x} = 1$, $d_{2y} = 0$





Force equilibrium

$$\sum F_x = k_{11} - T_1 \cos(45) - T_2 \cos(45) = 0$$

$$\sum F_y = k_{21} - T_1 \sin(45) + T_2 \sin(45) = 0$$

Force-deformation relations

$$T_1 = \frac{EA}{L}\delta_1$$

$$T_2 = \frac{EA}{L}\delta_2$$

Combining force equilibrium and force-deformation relations

$$k_{11} = \frac{(T_1 + T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2)$$

$$k_{21} = \frac{(T_1 - T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2)$$

Now use the geometric (compatibility) conditions (see figure)

$$\delta_1 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

$$\delta_2 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

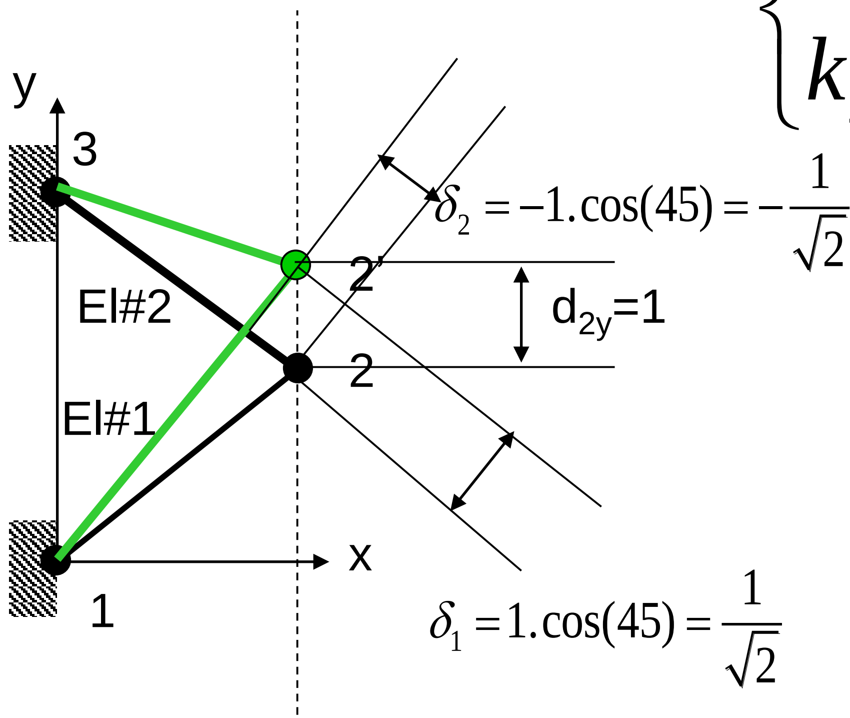
Finally

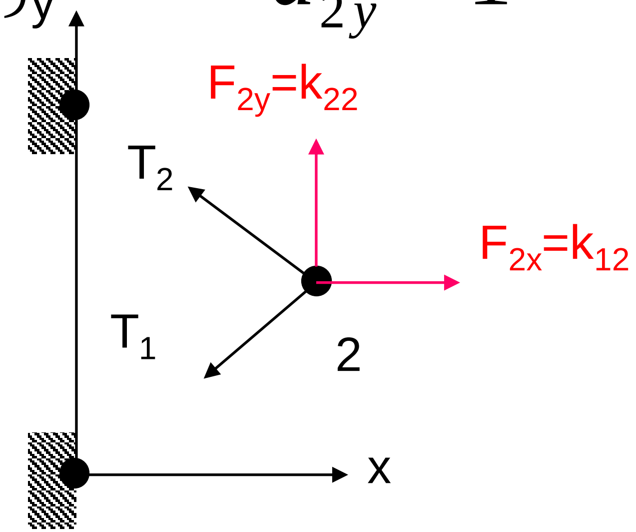
$$k_{11} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2) = \frac{EA}{\sqrt{2}L}(\frac{2}{\sqrt{2}}) = \frac{EA}{L}$$

$$k_{21} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2) = 0$$

To obtain the second column

$$\begin{Bmatrix} k_{12} \\ k_{22} \end{Bmatrix} \text{ apply } \quad d_{2x} = 0, \quad d_{2y} = 1$$





Force equilibrium

$$\sum F_x = k_{12} - T_1 \cos(45) - T_2 \cos(45) = 0$$

$$\sum F_y = k_{22} - T_1 \sin(45) + T_2 \sin(45) = 0$$

Force-deformation relations

$$T_1 = \frac{EA}{L}\delta_1$$

$$T_2 = \frac{EA}{L}\delta_2$$

Combining force equilibrium and force-deformation relations

$$k_{12} = \frac{(T_1 + T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2)$$

$$k_{22} = \frac{(T_1 - T_2)}{\sqrt{2}} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2)$$

Now use the geometric (compatibility) conditions (see figure)

$$\delta_1 = 1.\cos(45) = \frac{1}{\sqrt{2}}$$

$$\delta_2 = -1.\cos(45) = -\frac{1}{\sqrt{2}}$$

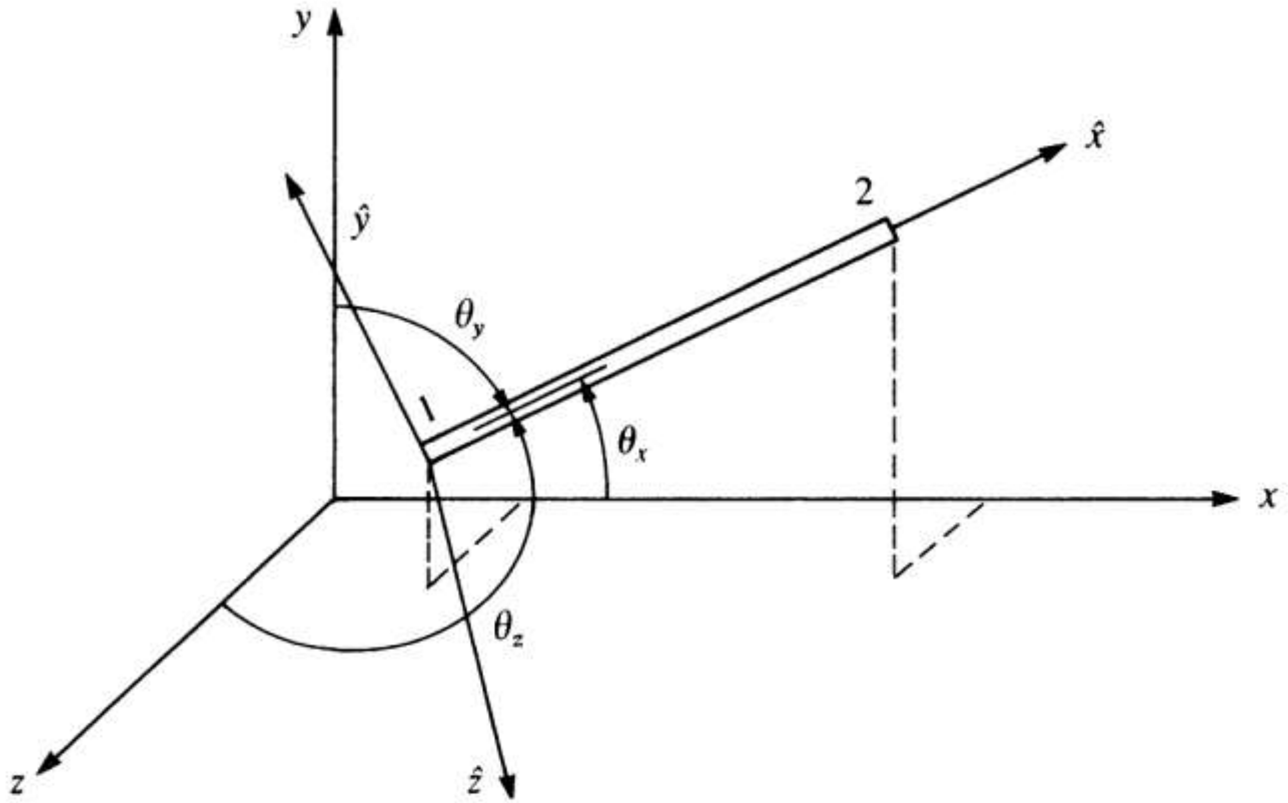
This negative is due to **compression**

Finally

$$k_{12} = \frac{EA}{\sqrt{2}L}(\delta_1 + \delta_2) = 0$$

$$k_{22} = \frac{EA}{\sqrt{2}L}(\delta_1 - \delta_2) = \frac{EA}{\sqrt{2}L}(\frac{2}{\sqrt{2}}) = \frac{EA}{L}$$

3D Truss (space truss)



In local coordinate system

$$\underline{\hat{f}} = \underline{\hat{k}}\,\underline{\hat{d}}$$

$$\begin{Bmatrix} \hat{f}_{1x} \\ \hat{f}_{1y} \\ \hat{f}_{1z} \\ \hat{f}_{2x} \\ \hat{f}_{2y} \\ \hat{f}_{2z} \end{Bmatrix} = \begin{bmatrix} k & 0 & 0 & -k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -k & 0 & 0 & k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \hat{d}_{1x} \\ \hat{d}_{1y} \\ \hat{d}_{1z} \\ \hat{d}_{2x} \\ \hat{d}_{2y} \\ \hat{d}_{2z} \end{Bmatrix}$$

The transformation matrix for a **single vector** in 3D

$$\hat{\underline{d}} = \underline{T}^{*}\underline{d}$$

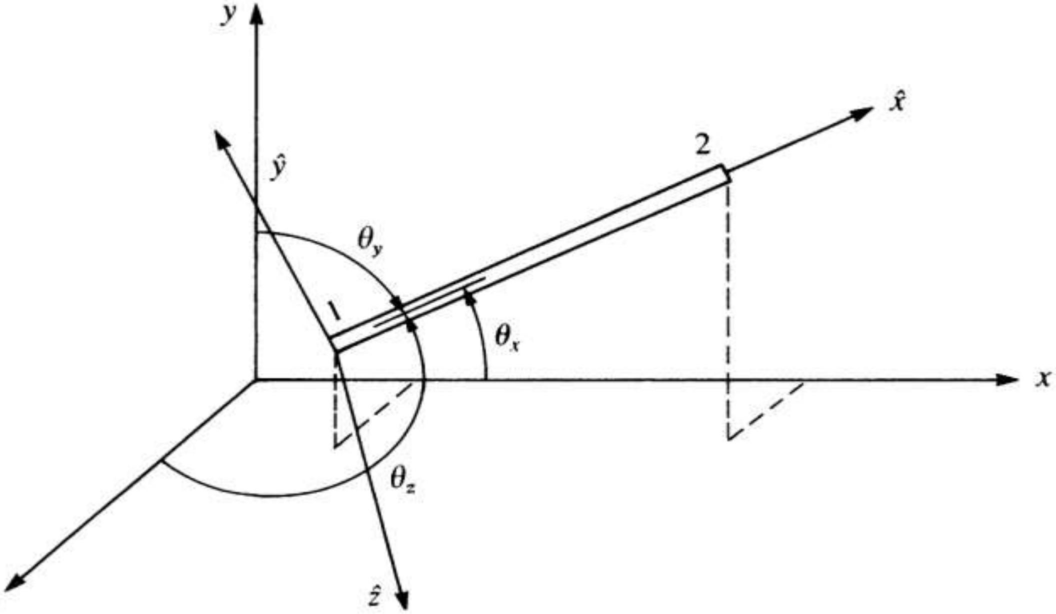
$$\underline{T}^{*} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

l_1, m_1 and n_1 are the direction cosines of $\hat{x}$

$$l_1 = \cos\theta_x$$

$$m_1 = \cos\theta_y$$

$$n_1 = \cos\theta_z$$



Transformation matrix **T** relating the local and global displacement and load vectors of the truss element

$$\hat{\underline{d}} = \underline{T}\,\underline{d}$$

$$\hat{\underline{f}} = \underline{T}\,\underline{f}$$

$$\underset{6\times 6}{\underline{T}} = \begin{bmatrix} \underline{T}^* & \underline{0} \\ \underline{0} & \underline{T}^* \end{bmatrix}$$

Element stiffness matrix in global coordinates

$$\underset{6\times 6}{\underline{k}} = \underset{6\times 6}{\underline{T}^T}\ \underset{6\times 6}{\hat{\underline{k}}}\ \underset{6\times 6}{\underline{T}}$$

$$\underline{k} = \underline{T}^T \underline{\hat{k}}\underline{T} = \frac{EA}{L}\begin{bmatrix} l_1^2 & l_1 m_1 & l_1 n_1 & -l_1^2 & -l_1 m_1 & -l_1 n_1 \\ l_1 m_1 & m_1^2 & m_1 n_1 & -l_1 m_1 & -m_1^2 & -m_1 n_1 \\ l_1 n_1 & m_1 n_1 & n_1^2 & l_1 n_1 & m_1 n_1 & -n_1^2 \\ -l_1^2 & -l_1 m_1 & -l_1 n_1 & l_1^2 & l_1 m_1 & l_1 n_1 \\ -l_1 m_1 & -m_1^2 & -m_1 n_1 & l_1 m_1 & m_1^2 & m_1 n_1 \\ -l_1 n_1 & -m_1 n_1 & -n_1^2 & l_1 n_1 & m_1 n_1 & n_1^2 \end{bmatrix}$$

Notice that the direction cosines of **only** the local x axis enter the $\underline{k}$ matrix

MRCET CAMPUS

UGC Autonomous

DEPARTMENT OF MECHANICAL ENGINEERING

INTRODUCTION TO FINITE ELEMENTS

CONSTANT STRAIN TRIANGLE (CST)

DEPARTMENT OF MECHANICAL ENGINEERING

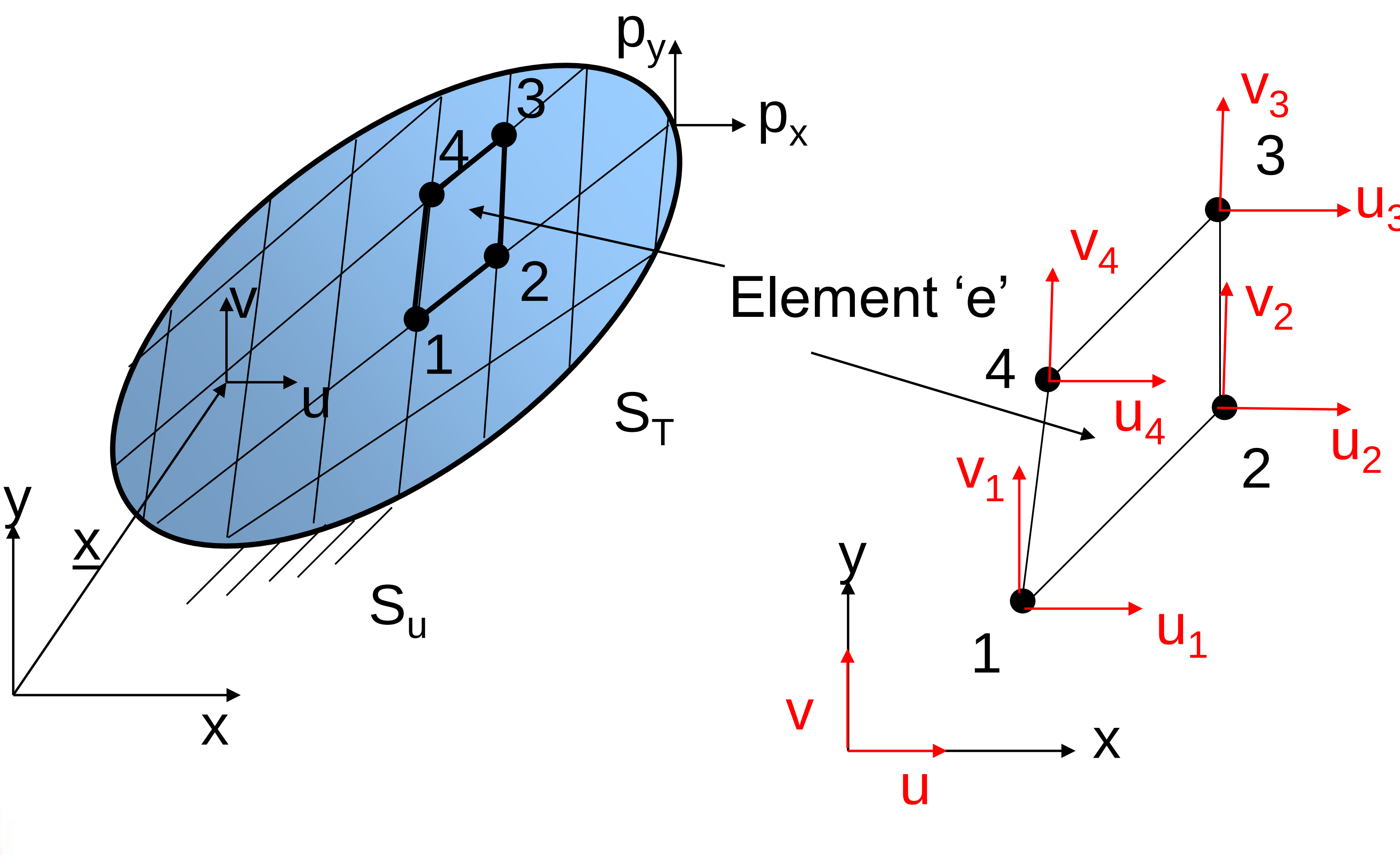
Lecture notes

Summary:

- Computation of shape functions for constant strain triangle
- Properties of the shape functions
- Computation of strain-displacement matrix
- Computation of element stiffness matrix
- Computation of nodal loads due to body forces
- Computation of nodal loads due to traction
- Recommendations for use
- Example problems

Finite element formulation for 2D:

Step 1: Divide the body into **finite elements** connected to each other through special points ("**nodes**")



$$\underline{d} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$u(x,y) \approx N_1(x,y)\,u_1 + N_2(x,y)\,u_2 + N_3(x,y)\,u_3 + N_4(x,y)\,u_4$$

$$v(x,y) \approx N_1(x,y)\,v_1 + N_2(x,y)\,v_2 + N_3(x,y)\,v_3 + N_4(x,y)\,v_4$$

$$\underline{u} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \left[\begin{array}{cc|cc|cc|cc} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \hline u_2 \\ v_2 \\ \hline u_3 \\ v_3 \\ \hline u_4 \\ v_4 \end{Bmatrix}$$

$$\boxed{\underline{u} = \underline{N}\,\underline{d}}$$

TASK 2: APPROXIMATE THE STRAIN and STRESS WITHIN EACH ELEMENT

Approximation of the strain in element 'e'

$$\varepsilon_x = \frac{\partial u\,(x, y)}{\partial x} \approx \frac{\partial N_1(x, y)}{\partial x} u_1 + \frac{\partial N_2(x, y)}{\partial x} u_2 + \frac{\partial N_3(x, y)}{\partial x} u_3 + \frac{\partial N_4(x, y)}{\partial x} u_4$$

$$\varepsilon_y = \frac{\partial v\,(x, y)}{\partial y} \approx \frac{\partial N_1(x, y)}{\partial y} v_1 + \frac{\partial N_2(x, y)}{\partial y} v_2 + \frac{\partial N_3(x, y)}{\partial y} v_3 + \frac{\partial N_4(x, y)}{\partial y} v_4$$

$$\gamma_{xy} = \frac{\partial u\,(x, y)}{\partial y} + \frac{\partial v\,(x, y)}{\partial x} \approx \frac{\partial N_1(x, y)}{\partial y} u_1 + \frac{\partial N_1(x, y)}{\partial x} v_1 + \ldots\ldots$$

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial N_1(x,y)}{\partial x} & 0 & \frac{\partial N_2(x,y)}{\partial x} & 0 & \frac{\partial N_3(x,y)}{\partial x} & 0 & \frac{\partial N_4(x,y)}{\partial x} & 0 \\ 0 & \frac{\partial N_1(x,y)}{\partial y} & 0 & \frac{\partial N_2(x,y)}{\partial y} & 0 & \frac{\partial N_3(x,y)}{\partial y} & 0 & \frac{\partial N_4(x,y)}{\partial y} \\ \frac{\partial N_1(x,y)}{\partial y} & \frac{\partial N_1(x,y)}{\partial x} & \frac{\partial N_2(x,y)}{\partial y} & \frac{\partial N_2(x,y)}{\partial x} & \frac{\partial N_3(x,y)}{\partial y} & \frac{\partial N_3(x,y)}{\partial x} & \frac{\partial N_4(x,y)}{\partial y} & \frac{\partial N_4(x,y)}{\partial x} \end{bmatrix}}_{\underline{B}} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$\boxed{\underline{\varepsilon} = \underline{B}\,\underline{d}}$$

Summary: For each element

Displacement approximation in terms of shape functions

$$\underline{u} = \underline{N}\,\underline{d}$$

Strain approximation in terms of strain-displacement matrix

$$\underline{\varepsilon} = \underline{B}\,\underline{d}$$

Stress approximation

$$\underline{\sigma} = \underline{D}\,\underline{B}\,\underline{d}$$

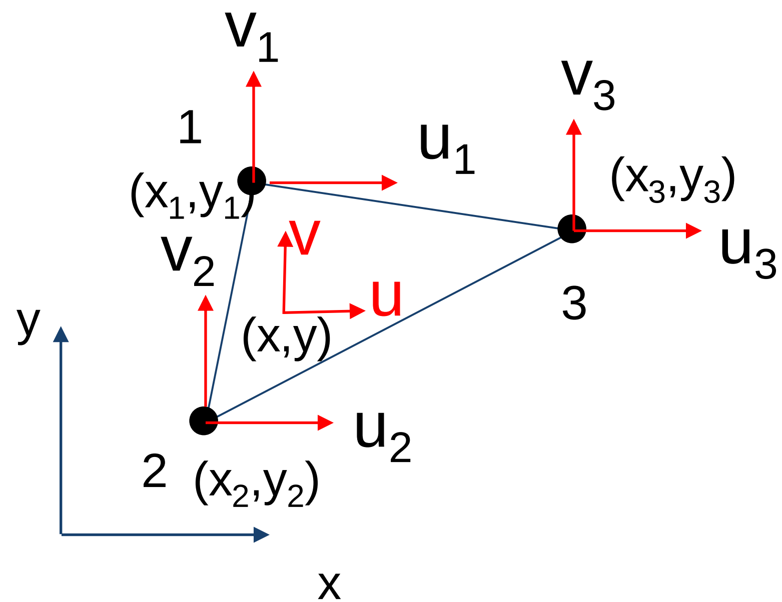
Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\,\underline{B}\,dV$$

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\,dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\,dS}_{\underline{f}_S}$$

Constant Strain Triangle (CST) : Simplest 2D finite element



- 3 nodes per element
- 2 dofs per node (each node can move in x- and y- directions)
- Hence 6 dofs per element

The displacement approximation in terms of shape functions is

$$u(x,y) \approx N_1 u_1 + N_2 u_2 + N_3 u_3$$

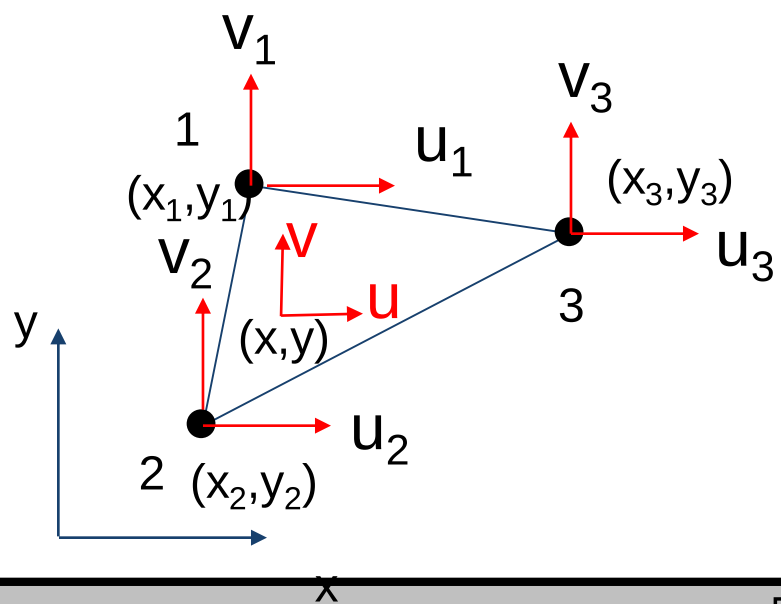
$$v(x,y) \approx N_1 v_1 + N_2 v_2 + N_3 v_3$$

$$\underline{u} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = \left[\begin{array}{cc:cc:cc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array}\right] \begin{Bmatrix} u_1 \\ v_1 \\ \hdashline u_2 \\ v_2 \\ \hdashline u_3 \\ v_3 \end{Bmatrix}$$

$$\boxed{\underline{u}_{2\times1} = \underline{N}_{2\times6}\ \underline{d}_{6\times1}}$$

$$\underline{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}$$

Formula for the shape functions are



$$N_1 = \frac{a_1 + b_1 x + c_1 y}{2A}$$

$$N_2 = \frac{a_2 + b_2 x + c_2 y}{2A}$$

$$N_3 = \frac{a_3 + b_3 x + c_3 y}{2A}$$

where

$$A = area\ of\ triangle = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

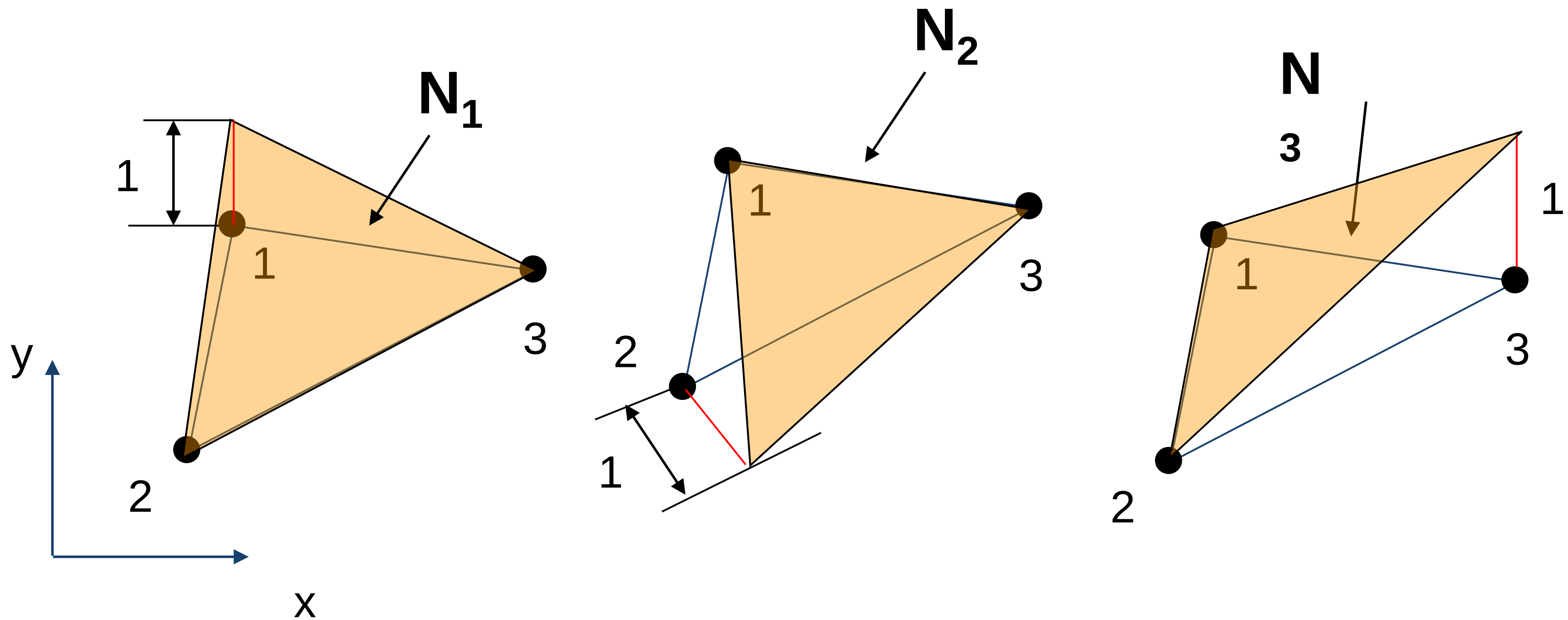
$$a_1 = x_2 y_3 - x_3 y_2 \quad b_1 = y_2 - y_3 \quad c_1 = x_3 - x_2$$

$$a_2 = x_3 y_1 - x_1 y_3 \quad b_2 = y_3 - y_1 \quad c_2 = x_1 - x_3$$

$$a_3 = x_1 y_2 - x_2 y_1 \quad b_3 = y_1 - y_2 \quad c_3 = x_2 - x_1$$

Properties of the shape functions:

1. The shape functions N_1, N_2 and N_3 are linear functions of x and y



$$N_i = \begin{cases} 1 & \textit{at node 'i'} \\ 0 & \textit{at other nodes} \end{cases}$$

2. At every point in the domain

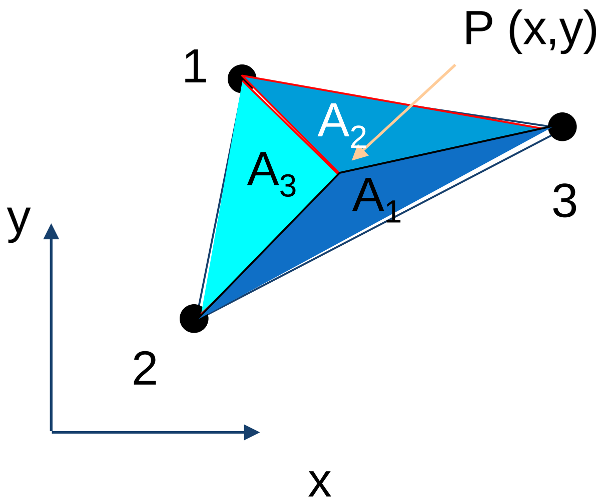
$$\sum_{i=1}^{3} N_i = 1$$

$$\sum_{i=1}^{3} N_i x_i = x$$

$$\sum_{i=1}^{3} N_i y_i = y$$

3. Geometric interpretation of the shape functions

At any point P(x,y) that the shape functions are evaluated,



$$N_1 = \frac{A_1}{A}$$

$$N_2 = \frac{A_2}{A}$$

$$N_3 = \frac{A_3}{A}$$

Approximation of the strains

$$\underline{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \approx \underline{B}\underline{d}$$

$$\underline{B} = \begin{bmatrix} \dfrac{\partial N_1(x,y)}{\partial x} & 0 & \dfrac{\partial N_2(x,y)}{\partial x} & 0 & \dfrac{\partial N_3(x,y)}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1(x,y)}{\partial y} & 0 & \dfrac{\partial N_2(x,y)}{\partial y} & 0 & \dfrac{\partial N_3(x,y)}{\partial y} \\ \dfrac{\partial N_1(x,y)}{\partial y} & \dfrac{\partial N_1(x,y)}{\partial x} & \dfrac{\partial N_2(x,y)}{\partial y} & \dfrac{\partial N_2(x,y)}{\partial x} & \dfrac{\partial N_3(x,y)}{\partial y} & \dfrac{\partial N_3(x,y)}{\partial x} \end{bmatrix}$$

$$= \frac{1}{2A} \begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Inside each element, all components of strain are constant: hence the name **Constant Strain Triangle**

Element stresses (constant inside each element)

$$\underline{\sigma} = \underline{D}\underline{B}\,\underline{d}$$

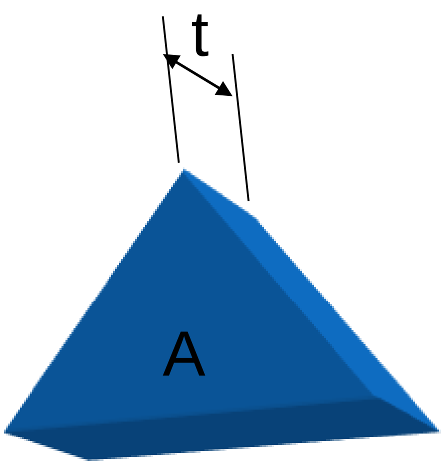
IMPORTANT NOTE:

1. The displacement field is continuous across element boundaries

2. The strains and stresses are NOT continuous across element boundaries

Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D} \, \underline{B} \, dV$$



Since $\underline{B}$ is constant

$$\underline{k} = \underline{B}^T \underline{D} \, \underline{B} \int_{V^e} dV = \underline{B}^T \underline{D} \, \underline{B} \, At$$

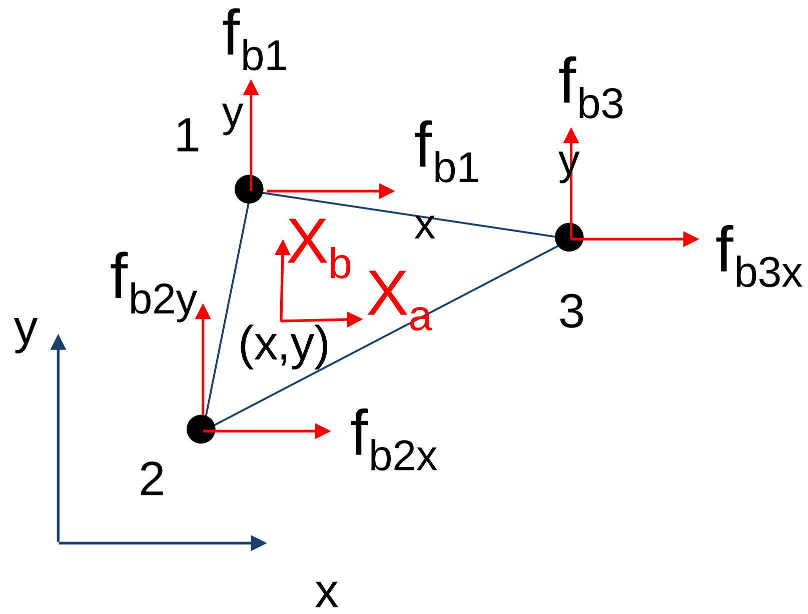
t=thickness of the element
A=surface area of the element

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{\mathrm{N}}^T \underline{X}\, dV}_{\underline{f}_b} + \underbrace{\int_{S_T{}^e} \underline{\mathrm{N}}^T \underline{T}_S \, dS}_{\underline{f}_S}$$

Element nodal load vector due to body forces

$$\underline{f}_b = \int_{V^e} \underline{\mathrm{N}}^T \underline{X}\, dV = t\int_{A^e} \underline{\mathrm{N}}^T \underline{X}\, dA$$



$$\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a\, dA \\ t\int_{A^e} N_1 X_b\, dA \\ t\int_{A^e} N_2 X_a\, dA \\ t\int_{A^e} N_2 X_b\, dA \\ t\int_{A^e} N_3 X_a\, dA \\ t\int_{A^e} N_3 X_b\, dA \end{Bmatrix}$$

EXAMPLE:

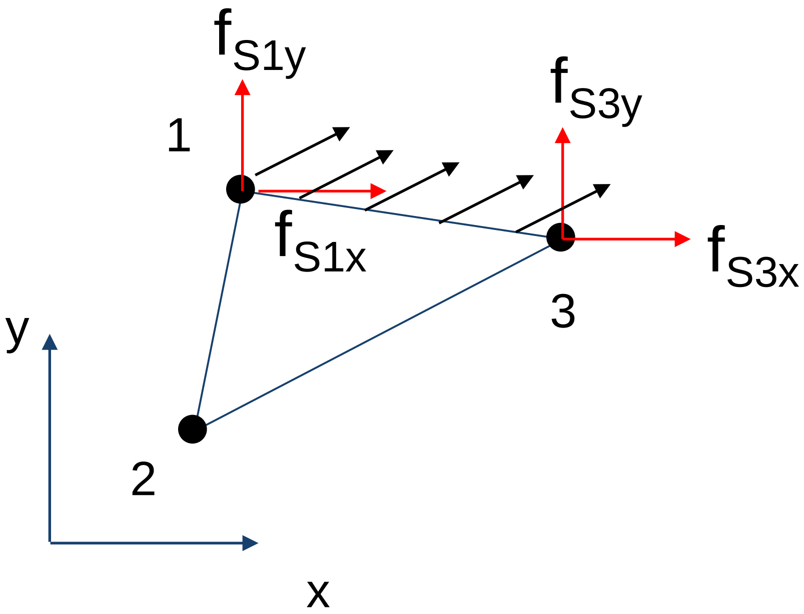
If X_a=1 and X_b=0

$$\underline{f}_b = \begin{Bmatrix} f_{b1x} \\ f_{b1y} \\ f_{b2x} \\ f_{b2y} \\ f_{b3x} \\ f_{b3y} \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 X_a \ dA \\ t\int_{A^e} N_1 X_b \ dA \\ t\int_{A^e} N_2 X_a \ dA \\ t\int_{A^e} N_2 X_b \ dA \\ t\int_{A^e} N_3 X_a \ dA \\ t\int_{A^e} N_3 X_b \ dA \end{Bmatrix} = \begin{Bmatrix} t\int_{A^e} N_1 dA \\ 0 \\ t\int_{A^e} N_2 \ dA \\ 0 \\ t\int_{A^e} N_3 dA \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \\ \frac{tA}{3} \\ 0 \end{Bmatrix}$$

Element nodal load vector due to traction

$$\underline{f}_S = \int_{S_T^e} \underline{\mathrm{N}}^T \underline{T}_S \; dS$$

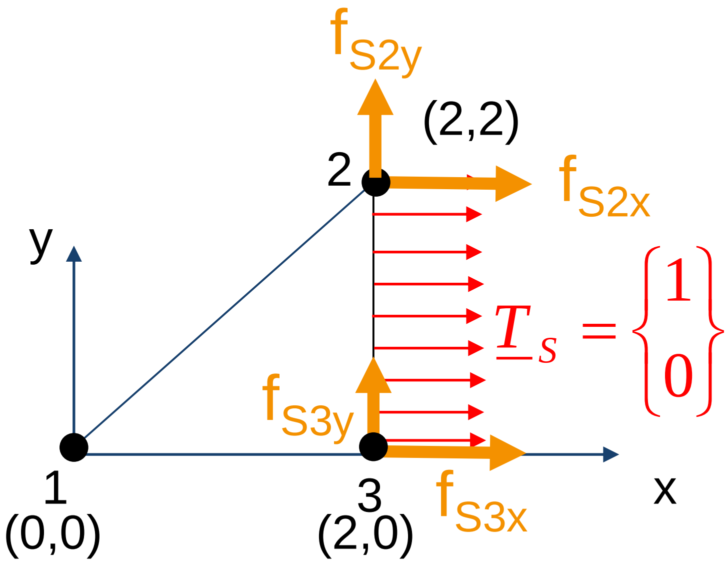
EXAMPLE:



$$\underline{f}_S = t\int_{l_{1-3}^e} \underline{\mathrm{N}}^T \Big|_{along\,1-3} \underline{T}_S \; dS$$

Element nodal load vector due to traction

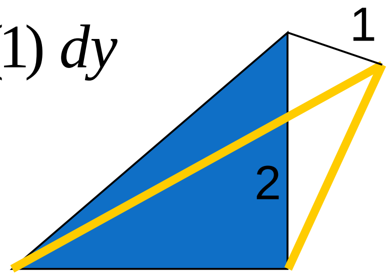
EXAMPLE:



$$\underline{f}_S = t\int_{l_{2-3}^e} \underline{\mathrm{N}}^T\Big|_{along\,2-3} \underline{T}_S \; dS$$

$$f_{S_{2x}} = t\int_{l_{2-3}^e} N_2\Big|_{along\,2-3} (1)\; dy$$

$$= t\left(\frac{1}{2}\right)\times 2\times 1 = t$$



Similarly, compute

$$f_{S_{2y}} = 0$$

$$f_{S_{3x}} = t$$

$$f_{S_{3y}} = 0$$

Recommendations for use of CST

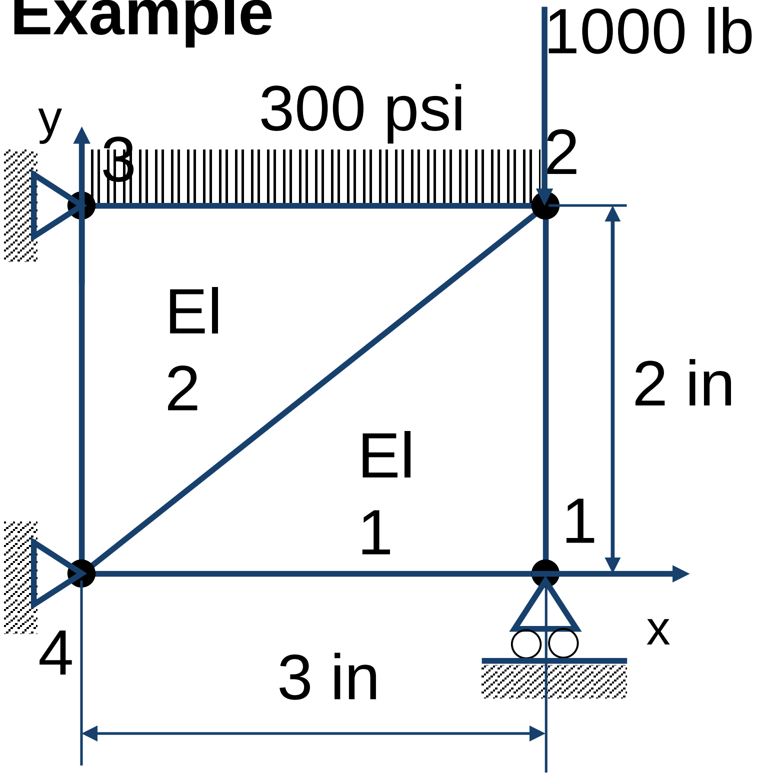
1. Use in areas where strain gradients are small

2. Use in mesh transition areas (fine mesh to coarse mesh)

3. Avoid CST in critical areas of structures (e.g., stress concentrations, edges of holes, corners)

4. In general CSTs are not recommended for general analysis purposes as a very large number of these elements are required for reasonable accuracy.

Example



Thickness (t) = 0.5 in

E= 30×10^6 psi

ν=0.25

(a) Compute the unknown nodal displacements.

(b) Compute the stresses in the two elements.

Realize that this is a plane stress problem and therefore we need to use

$$\underline{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 3.2 & 0.8 & 0 \\ 0.8 & 3.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix} \times 10^7 \ psi$$

<u>Step 1</u>: Node-element connectivity chart

ELEMENT	Node 1	Node 2	Node 3	Area (sqin)
1	1	2	4	3
2	3	4	2	3

Node	x	y
1	3	0
2	3	2
3	0	2
4	0	0

Nodal coordinates

Step 2: Compute strain-displacement matrices for the elements

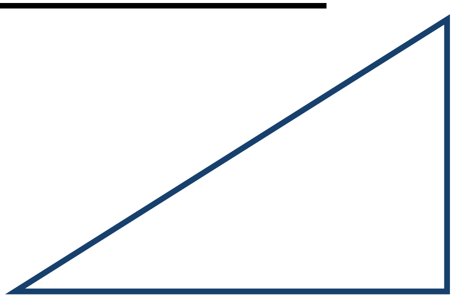
Recall

$$\underline{B} = \frac{1}{2A}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

with

$$b_1 = y_2 - y_3 \quad b_2 = y_3 - y_1 \quad b_3 = y_1 - y_2$$
$$c_1 = x_3 - x_2 \quad c_2 = x_1 - x_3 \quad c_3 = x_2 - x_1$$

For Element #1:



(local numbers within brackets)

$$y_1 = 0;\ y_2 = 2;\ y_3 = 0$$
$$x_1 = 3;\ x_2 = 3;\ x_3 = 0$$

Hence

$$b_1 = 2 \quad b_2 = 0 \quad b_3 = -2$$
$$c_1 = -3 \quad c_2 = 3 \quad c_3 = 0$$

Therefore

$$\underline{B}^{(1)} = \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

For Element #2:

$$\underline{B}^{(2)} = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix}$$

Step 3: Compute element stiffness matrices

$$\underline{k}^{(1)} = At\underline{B}^{(1)^T}\,\underline{D}\,\underline{B}^{(1)} = (3)(0.5)\underline{B}^{(1)^T}\,\underline{D}\,\underline{B}^{(1)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

$u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_4 \quad v_4$

$$\underline{k}^{(2)} = At\underline{B}^{(2)^T}\underline{D}\,\underline{B}^{(2)} = (3)(0.5)\underline{B}^{(2)^T}\underline{D}\,\underline{B}^{(2)}$$

$$= \begin{bmatrix} 0.9833 & -0.5 & -0.45 & 0.2 & -0.5333 & 0.3 \\ & 1.4 & 0.3 & -1.2 & 0.2 & -0.2 \\ & & 0.45 & 0 & 0 & -0.3 \\ & & & 1.2 & -0.2 & 0 \\ & & & & 0.5333 & 0 \\ & & & & & 0.2 \end{bmatrix} \times 10^7$$

u_3 | v_3 | u_4 | v_4 | u_2 | v_2

<u>Step 4</u>: Assemble the global stiffness matrix corresponding to the *<u>nonzero</u>* degrees of freedom

Notice that

$$u_3 = v_3 = u_4 = v_4 = v_1 = 0$$

Hence we need to calculate only a small (3x3) stiffness matrix

$$\underline{K} = \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \times 10^7 \quad \begin{matrix} u_1 \\ u_2 \\ v_2 \end{matrix}$$

$$\begin{matrix} u_1 & u_2 & v_2 \end{matrix}$$

Step 5: Compute consistent nodal loads

$$\underline{f} = \begin{Bmatrix} f_{1x} \\ f_{2x} \\ f_{2y} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{2y} \end{Bmatrix}$$

$$f_{2y} = -1000 + f_{S_{2y}}$$

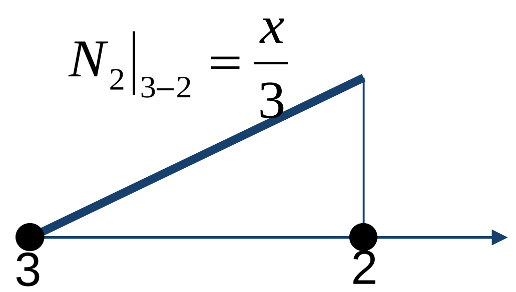
The consistent nodal load due to traction on the edge 3-2

$$f_{S_{2y}} = \int_{x=0}^{3} N_3\big|_{3-2} (-300) t\, dx$$

$$= (-300)(0.5)\int_{x=0}^{3} N_3\big|_{3-2}\, dx$$

$$= -150\int_{x=0}^{3} \frac{x}{3}\, dx$$

$$= -50\left[\frac{x^2}{2}\right]_0^3 = -50\left(\frac{9}{2}\right) = -225\, lb$$



Hence

$$f_{2y} = -1000 + f_{S_{2y}}$$
$$= -1225 \ \ lb$$

<u>Step 6</u>: Solve the system equations to obtain the unknown nodal loads

$$\underline{K}\,\underline{d} = \underline{f}$$

$$10^7 \times \begin{bmatrix} 0.983 & -0.45 & 0.2 \\ -0.45 & 0.983 & 0 \\ 0.2 & 0 & 1.4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -1225 \end{Bmatrix}$$

Solve to get

$$\begin{Bmatrix} u_1 \\ u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0.2337 \times 10^{-4} \ in \\ 0.1069 \times 10^{-4} \ in \\ -0.9084 \times 10^{-4} \ in \end{Bmatrix}$$

Step 7: **Compute the stresses in the elements**

In Element #1

$$\underline{\sigma}^{(1)} = \underline{D}\,\underline{B}^{(1)}\underline{d}^{(1)}$$

With

$$\underline{d}^{(1)^T} = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_4 & v_4 \end{bmatrix}$$
$$= \begin{bmatrix} 0.2337\times10^{-4} & 0 & 0.1069\times10^{-4} & -0.9084\times10^{-4} & 0 & 0 \end{bmatrix}$$

Calculate

$$\underline{\sigma}^{(1)} = \begin{bmatrix} -114.1 \\ -1391.1 \\ -76.1 \end{bmatrix} psi$$

In Element #2

$$\underline{\sigma}^{(2)} = \underline{D}\,\underline{B}^{(2)}\underline{d}^{(2)}$$

With

$$\underline{d}^{(2)^T} = \begin{bmatrix} u_3 & v_3 & u_4 & v_4 & u_2 & v_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1069\times 10^{-4} & -0.9084\times 10^{-4} \end{bmatrix}$$

Calculate

$$\underline{\sigma}^{(2)} = \begin{bmatrix} 114.1 \\ 28.52 \\ -363.35 \end{bmatrix} psi$$

Notice that the stresses are constant in each element

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INTRODUCTION TO FINITE ELEMENTS

PRACTICAL CONSIDERATIONS IN FEM MODELING

LIVE TO LEARN & LEARN TO SHARE

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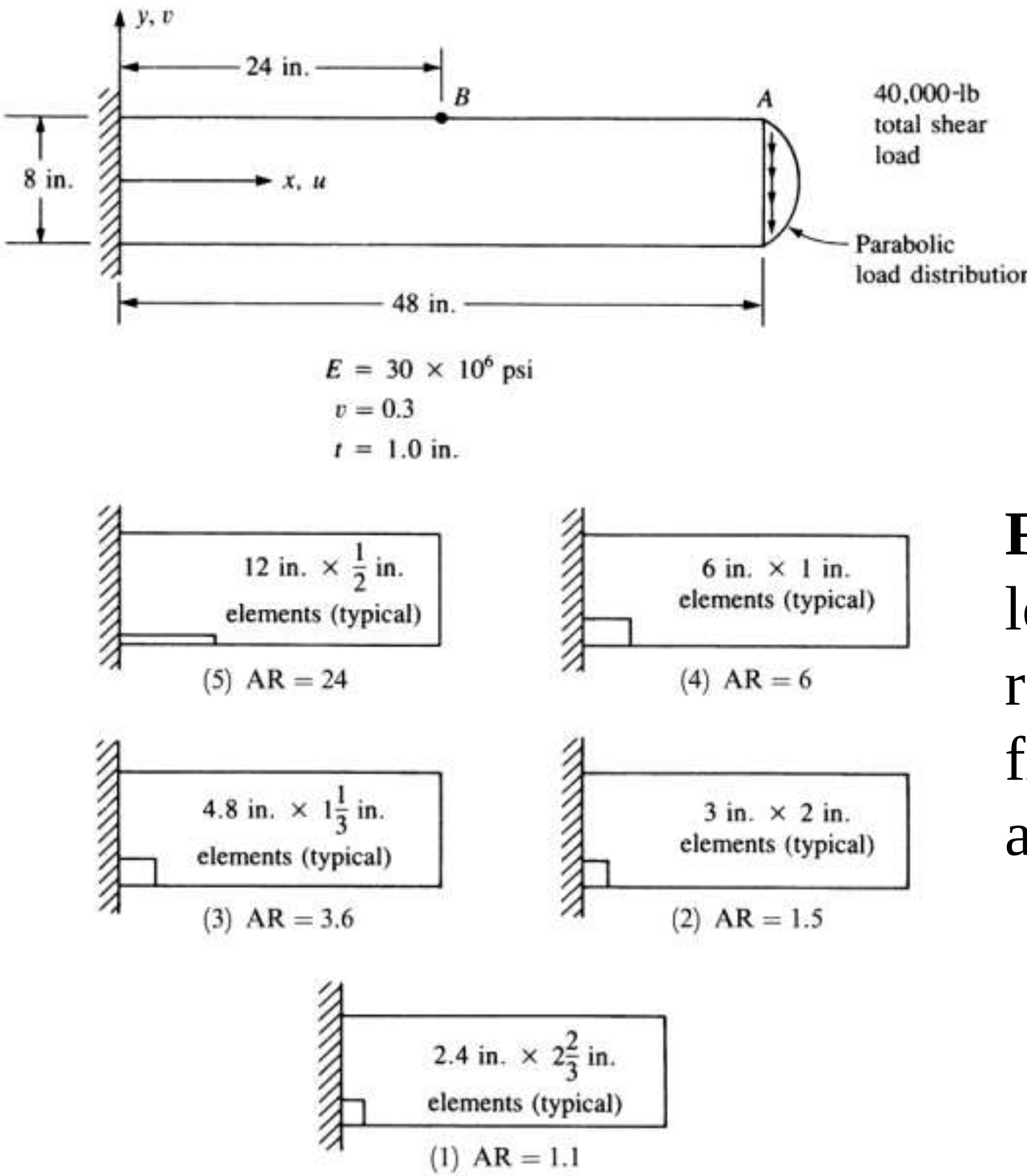
Reading assignment:

Logan Chap 7 + Lecture notes

Summary:

- Aspect ratio and element shapes
- Use of symmetry
- Natural subdivisions at discontinuities
- Stress equilibrium in FEM solutions

Aspect ratio and element shapes



Aspect ratio = longest dimension/ shortest dimension

Figure 7-1a (a) Beam with loading: effects of the aspect ratio (AR) illustrated by the five cases with different aspect ratios

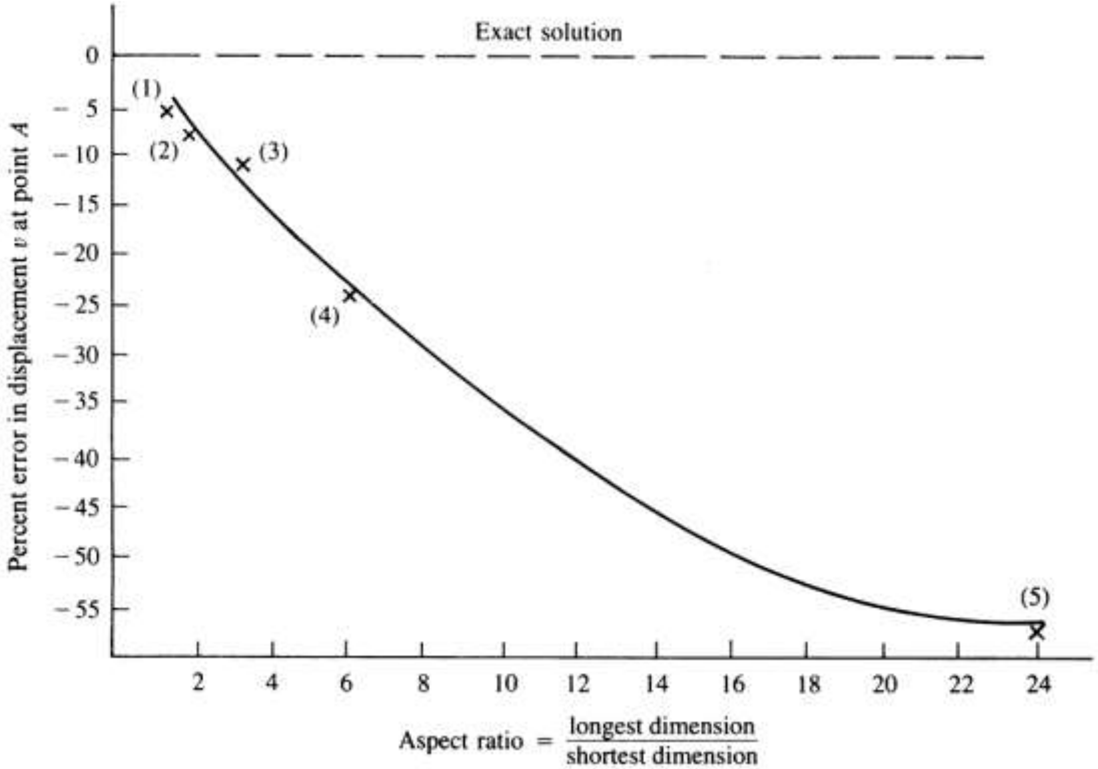
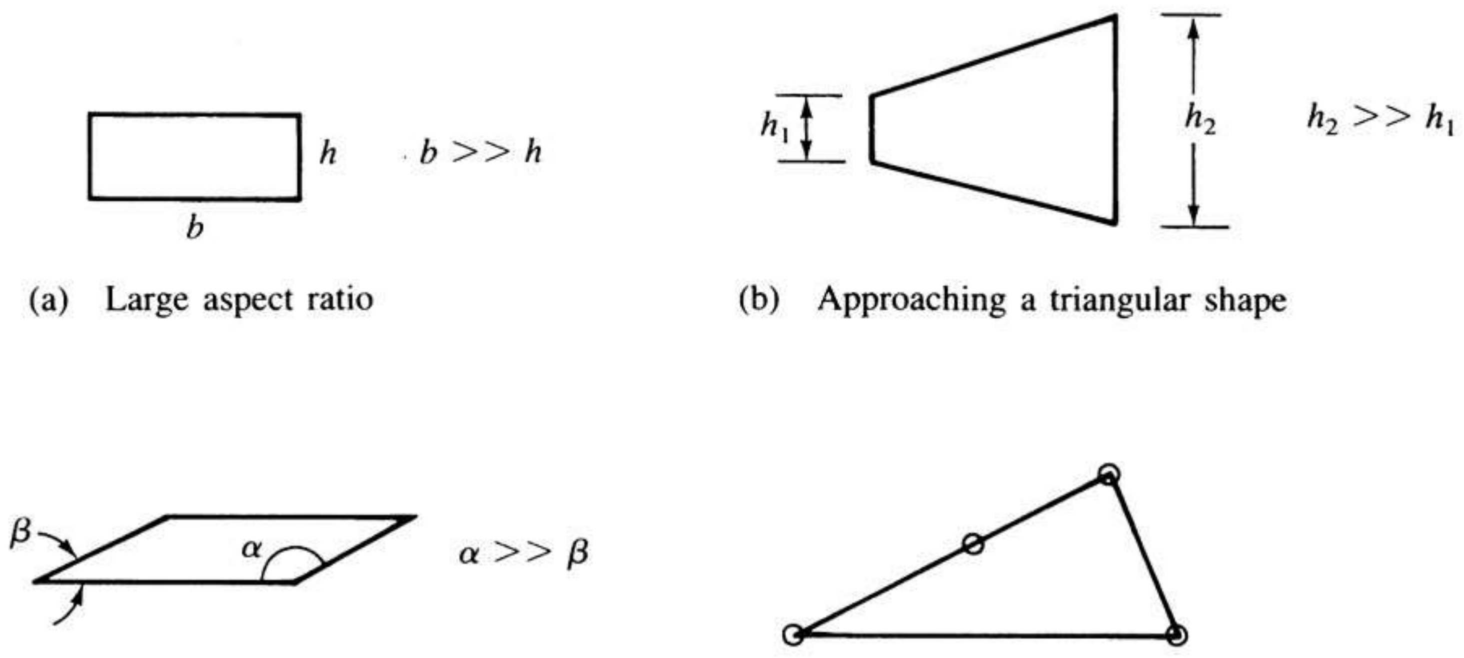


Figure 7-1b (b) Inaccuracy of solution as a function of the aspect ratio (numbers in parentheses correspond to the cases listed in Table 7-1)



(a) Large aspect ratio

(b) Approaching a triangular shape

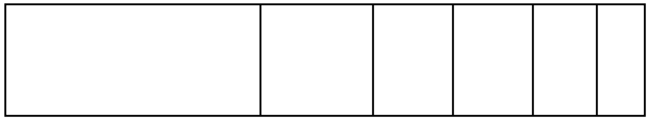
(c) Very large and very small corner angles

(d) Triangular quadrilateral

Figure 7-2 Elements with poor shapes

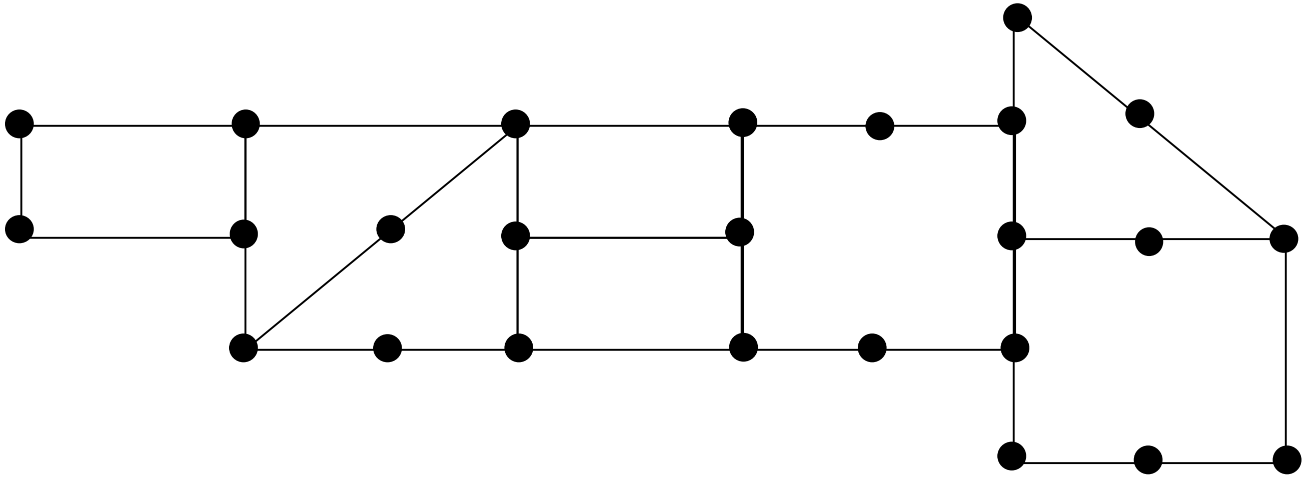
Avoid abrupt changes in element sizes

Abrupt change in element size



Gradual change in element size

Examples of how NOT to connect elements



Use of symmetry in modeling

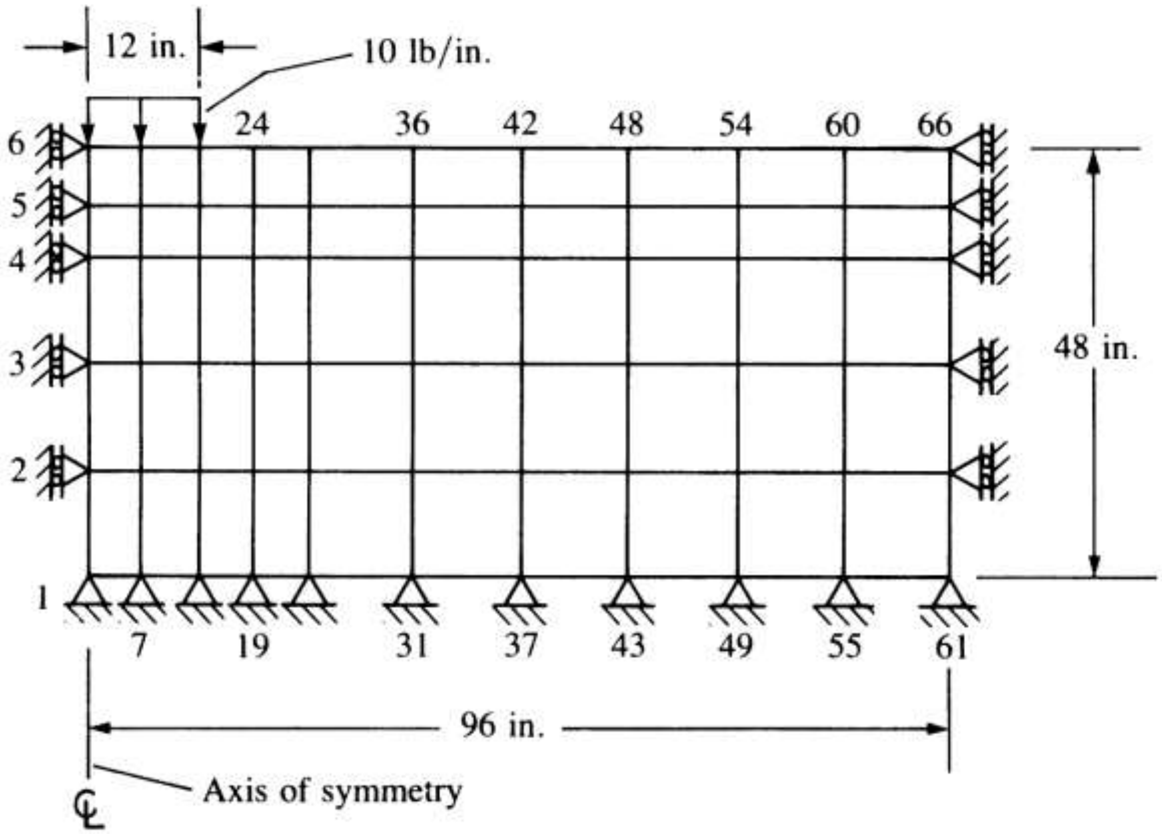
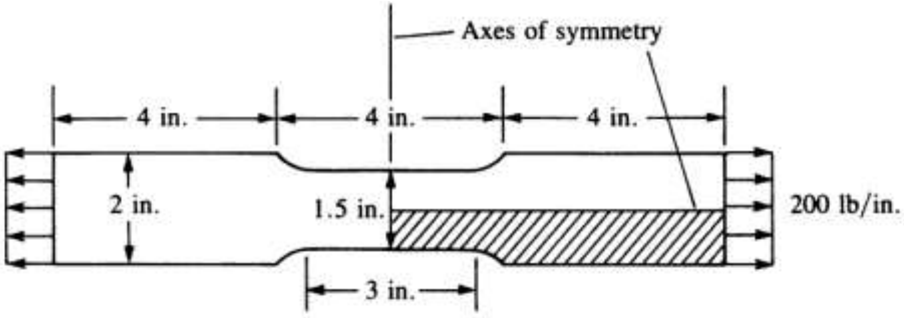
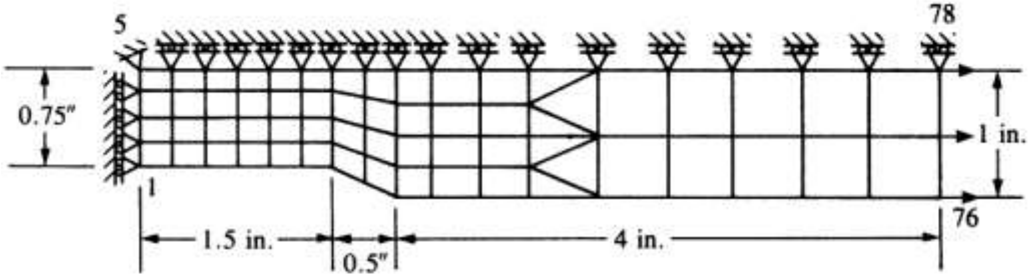


Figure 7-3 Use of symmetry applied to a soil mass subjected to foundation loading (number of nodes = 66, number of elements = 50) (2.54 cm = 1 in., 4.445 N = 1 lb)

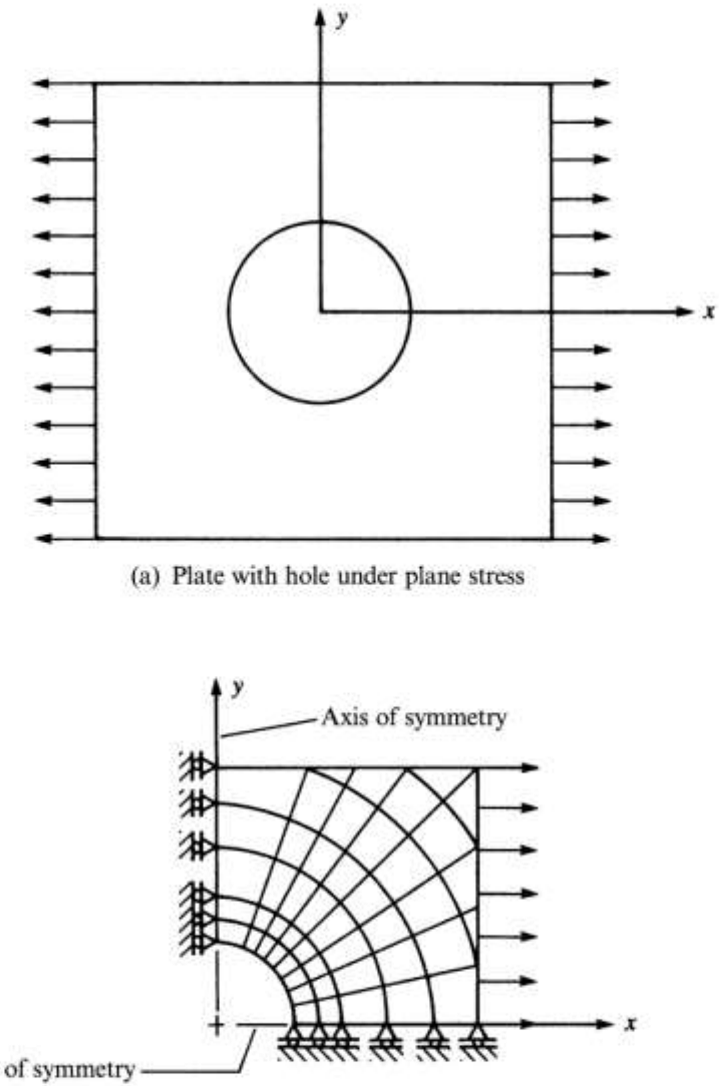


(a) Plane stress uniaxially loaded member with fillet



(b) Enlarged finite element model of the cross-hatched quarter of the member (number of nodes = 78, number of elements = 60) (2.54 cm = 1 in.)

Figure 7-4 Use of symmetry applied to a uniaxially loaded member with a fillet

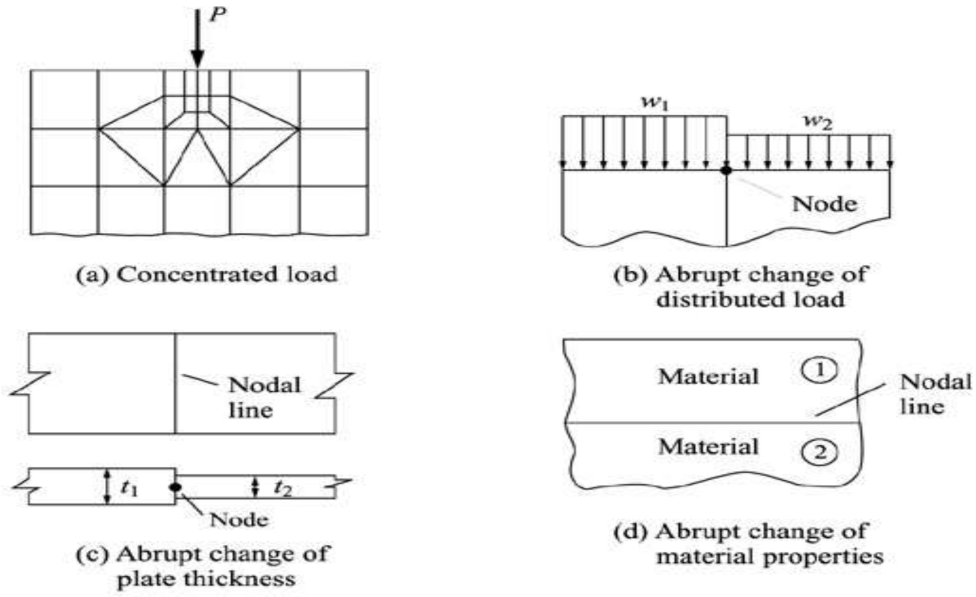


(a) Plate with hole under plane stress

(b) Finite element model of one-quarter of the plate

Figure 7-5 Problem reduction using axes of symmetry applied to a plate with a hole subjected to tensile force

Natural subdivisions at discontinuities



Look before you leap!

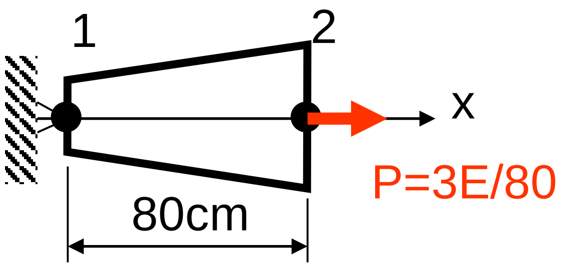
1. Check the **model** that you have developed:
   - Boundary conditions
   - Loadings
   - Symmetry?
   - Element aspect ratios/shapes
   - Mesh gradation

2. Check the **results**
   - Eyeball
   - Anything funny (nonzero displacements where they should be zero?)
   - Are stress concentrations in places that you expect?
   - Comparison with known analytical solution/literature

3. If you remesh the same problem and analyze, do the solutions converge? (specifically check for convergence in **strain energy**)

Stress equilibrium in FEM analysis

Example: Consider a linear elastic bar with varying cross section



$$A(x) = \left(1 + \frac{x}{40}\right)^2 \quad sqcm$$

The governing differential (equilibrium) equation

$$E\frac{d}{dx}\left(A(x)\frac{du}{dx}\right) = 0 \quad for \quad x \in (0,80) \qquad \text{Eq(1)}$$

E: Young's modulus

Boundary conditions

$$u(x=0) = 0$$

$$EA\frac{du}{dx}\bigg|_{x=80cm} = P = \frac{3E}{80}$$

Analytical solution

$$u^{exact}(x) = \frac{3}{2}\left(1 - \frac{1}{1+\frac{x}{40}}\right)$$

Lets us discretize the bar using a 2-noded (linear) bar element. The finite element approximation within the bar is

$$u^{FEM}(x) = N_1(x)u_{1x} + N_2(x)u_{2x}$$

where the **shape functions**

$$N_1(x) = 1 - \frac{x}{80}$$

$$N_2(x) = \frac{x}{80}$$

If we incorporate the boundary condition at x=0

$$u^{FEM}(x) = N_2(x)u_{2x} = \frac{x}{80}u_{2x}$$

Does this solution satisfy the equilibrium equation (Eq 1)?

$$E\frac{d}{dx}\left(A(x)\frac{du^{FEM}}{dx}\right) = E\frac{d}{dx}\left(\frac{\left(1+\frac{x}{40}\right)^2}{80}u_{2x}\right) = \frac{u_{2x}}{80}\frac{d}{dx}\left(1+\frac{x}{40}\right)^2 = ??$$

<u>Conclusion:</u> The FEM displacement field does NOT satisfy the equilibrium equations at every point inside the elements.

However, the solution gets better as the mesh is refined.

<u>Stress equilibrium in FEM analysis</u>

To obtain exact solution of the **mathematical model** in solid mechanics we need to satisfy

1. Compatibility
2. Stress-strain law
3. Stress-equilibrium

at **every point** in the computational domain.

In a FE model one satisfies the first 2 conditions exactly.
But **stress-equilibrium** is **NOT** satisfied point wise.
Question: Then what *is* satisfied?

Let us compute the FEM solution using a bar element
The stiffness matrix is

$$\underline{K} = \frac{E\int_{x=0}^{80} A(x)dx}{80^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{13E}{240}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The system equations to solve are

$$\frac{13E}{240}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{1x} \\ u_{2x} \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ P \end{Bmatrix}$$

With u_{1x}=0; we solve for

$$u_{2x} = \frac{240P}{13E} = \left(\frac{240}{13E}\right)\left(\frac{3E}{80}\right) = \frac{9}{13}\,cm$$

(Note that the **exact** solution for the displacement at node 2 is 1cm!!)

Let us now compute the nodal forces due to element stresses using the formula

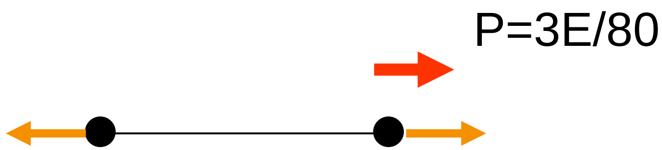
$$\boxed{\underline{f} = \underline{K}\underline{d}}$$

$$\underline{K} = \frac{13E}{240}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\underline{d} = \begin{Bmatrix} 0 \\ \dfrac{9}{13} \end{Bmatrix}$$

$$\Rightarrow \underline{f} = \underline{K}\underline{d} = \frac{13E}{240}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ \dfrac{9}{13} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{3E}{80} \\ \dfrac{3E}{80} \end{Bmatrix}$$

Two observations

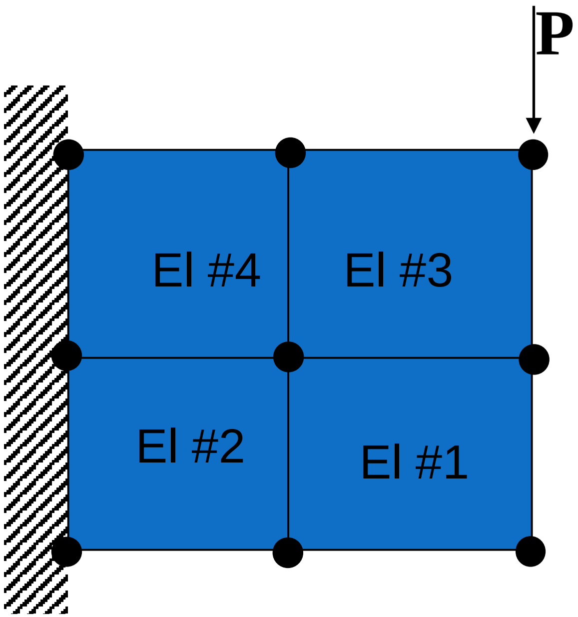


$$f_{1x} = \frac{3E}{80} \qquad f_{2x} = \frac{3E}{80}$$

1. Element equilibrium
2. Nodal equilibrium

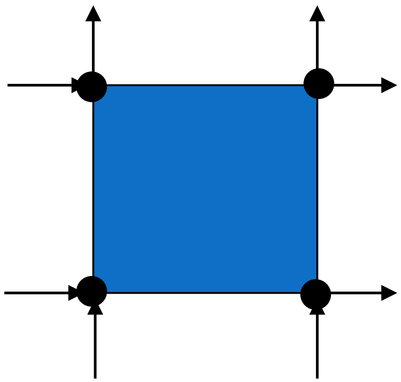
The following two properties are ALWAYS satisfied by the FEM solution using a coarse or a fine mesh

Property 1: Nodal point equilibrium
Property 2. Element equilibrium



PROPERTY 1: (Nodal point equilibrium) At any node the sum of the element nodal point forces is in equilibrium with the externally applied loads (including all effects due to body forces, surface tractions, initial stresses, concentrated loads, inertia, damping and reaction)

How to compute the **nodal reaction forces** for a given finite element?



$$\underline{f} = \underline{k}\,\underline{d} = \left(\int_{V^e} \underline{B}^T \underline{D}\,\underline{B}\, dV\right) \underline{d}$$

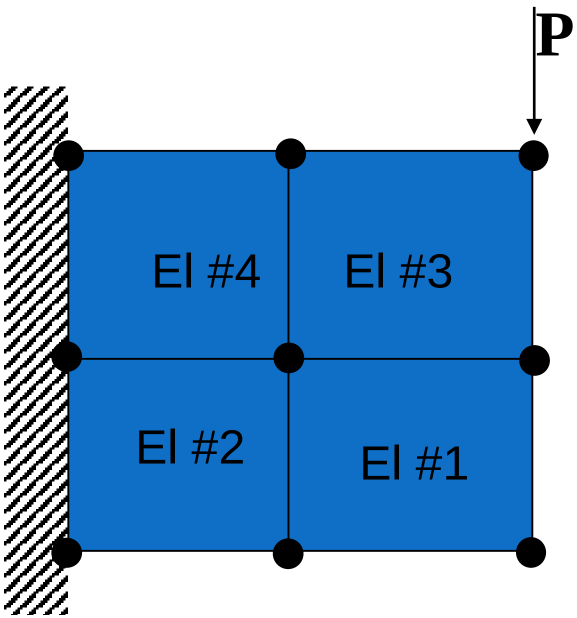
$$= \int_{V^e} \underline{B}^T \left(\underline{D}\,\underline{B}\,\underline{d}\right) dV$$

$$= \int_{V^e} \underline{B}^T \underline{\tau}\, dV$$

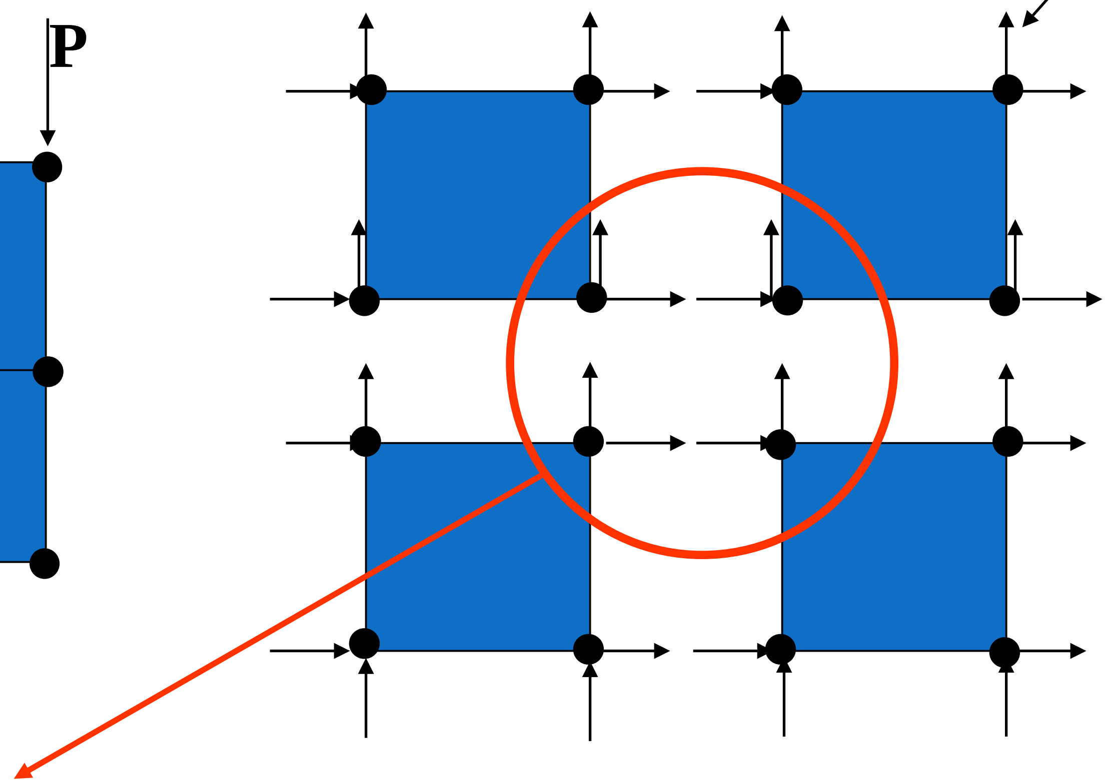
Once we have computed the element stress, we may obtain the nodal reaction forces as

$$\boxed{\underline{f} = \int_{V^e} \underline{B}^T \underline{\tau}\, dV}$$

Nodal point equilibrium implies:



This is equal in magnitude and in the same direction as P

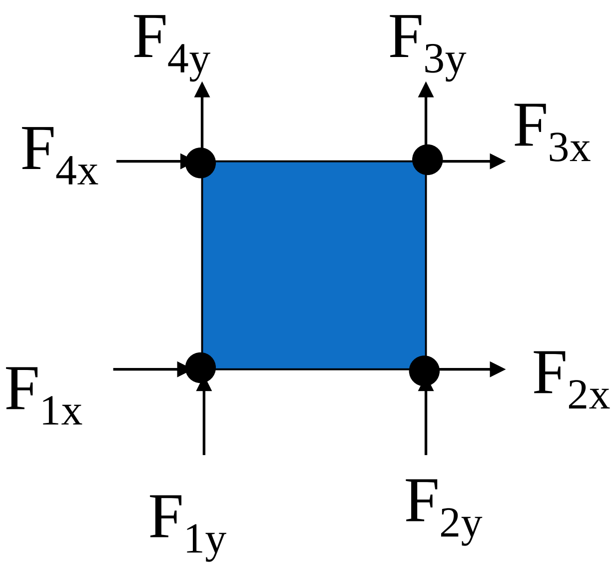


Sum of forces equal externally applied load (=0 at this node)

PROPERTY 2: (Element equilibrium) Each element is in equilibrium under its forces $\underline{f}$

i.e., each element is under **force and moment equilibrium**

e.g.,



Define $\underline{d} = [1\,0\,1\,0\,1\,0\,1\,0]^T$

as a rigid body displacement in x-direction

$$\therefore \underline{d}^T \underline{f} = F_{1x} + F_{2x} + F_{3x} + F_{4x}$$

But

$$\underline{d}^T \underline{f} = \underline{d}^T \left(\int_{V^e} \underline{B}^T \underline{\tau}\, dV \right)$$

$$= \int_{V^e} \left(\underline{d}^T \underline{B}^T \right) \underline{\tau}\, dV$$

$$= 0 \qquad \because \underline{\varepsilon} = \underline{B}\underline{d} = \underline{0}$$

since this is a rigid body displacement, the strains are zero

Hence

$$F_{1x} + F_{2x} + F_{3x} + F_{4x} = 0$$

Example (Finite Element Procedures, Bathe 1996)

EXAMPLE 4.9: The finite element solution to the problem in Fig. E4.6, with $P = 100$, $E = 2.7 \times 10^6$, $\nu = 0.30$, $t = 0.1$, is given in Fig. E4.9. Clearly, the stresses are not continuous between elements, and equilibrium on the differential level is not satisfied. However,

1. Show that $\sum_m \mathbf{F}^{(m)} = \mathbf{R}$ and calculate the reactions.
2. Show that the element forces $\mathbf{F}^{(4)}$ for element 4 are in equilibrium.

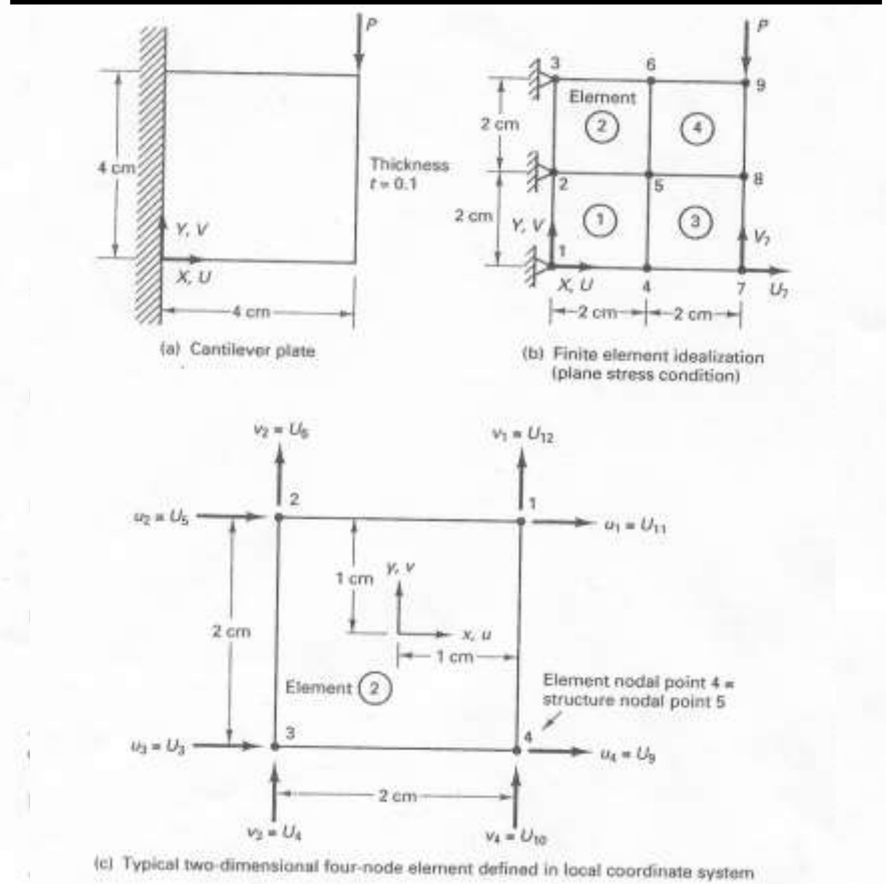


Figure E4.6 Finite element plane stress analysis

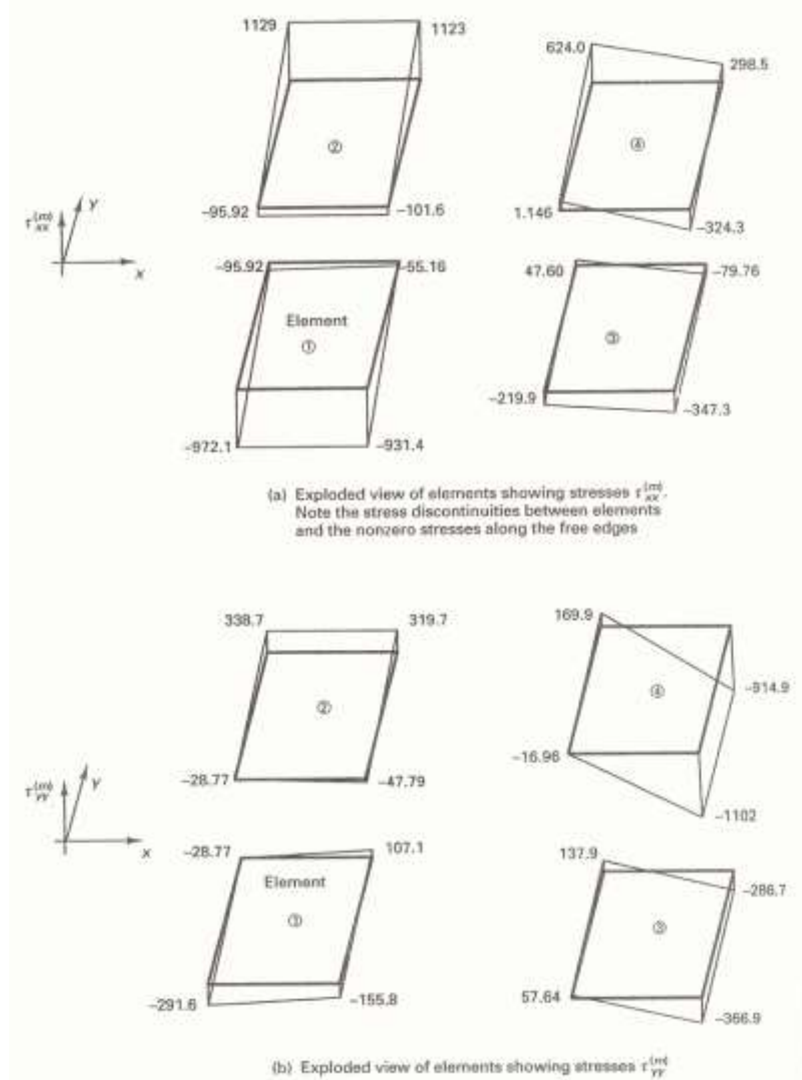
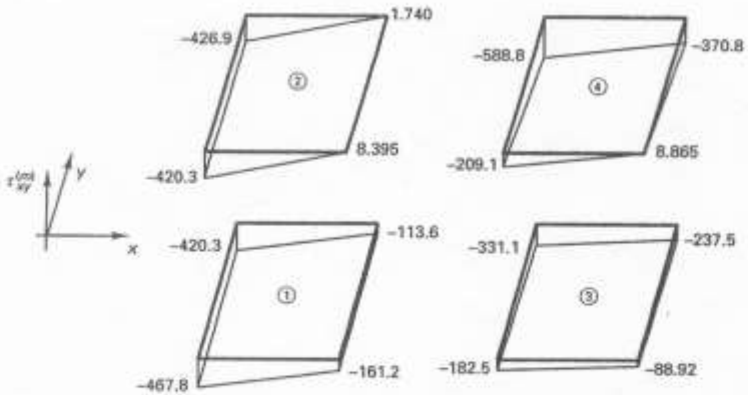
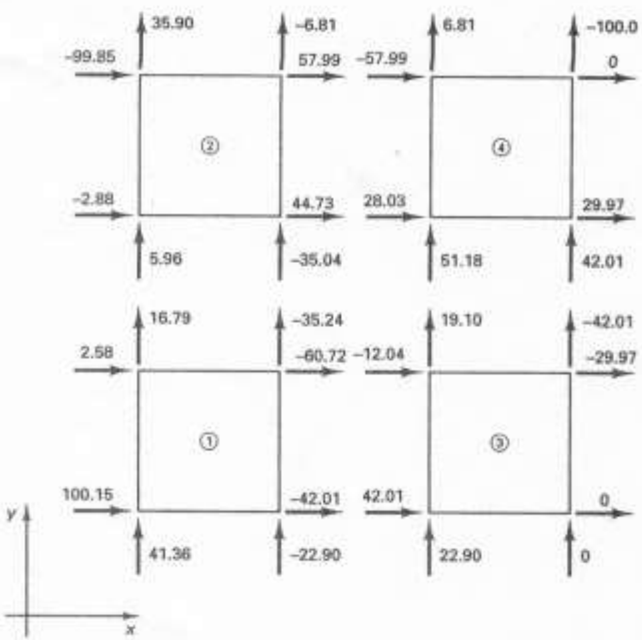


Figure E4.9 Solution results for problem considered in Example 4.6 (rounded to digits shown)



(c) Exploded view of elements showing stresses $\tau_{xy}^{(m)}$



(d) Exploded view of elements showing element nodal point forces equivalent (in the virtual work sense) to the element stresses. The nodal point forces are at each node in equilibrium with the applied forces (including the reactions)

Figure E4.9 *(continued)*

EXAMPLE 4.9: The finite element solution to the problem in Fig. E4.6, with $P = 100$, $E = 2.7 \times 10^6$, $\nu = 0.30$, $t = 0.1$, is given in Fig. E4.9. Clearly, the stresses are not continuous between elements, and equilibrium on the differential level is not satisfied. However,

1. Show that $\Sigma_m \mathbf{F}^{(m)} = \mathbf{R}$ and calculate the reactions.
2. Show that the element forces $\mathbf{F}^{(4)}$ for element 4 are in equilibrium.

The fact that $\Sigma_m \mathbf{F}^{(m)} = \mathbf{R}$ follows from the solution of (4.17), and $\mathbf{R}$ consists of the sum of all nodal point forces. Hence, this relation can also be used to evaluate the reactions.

Referring to the nodal point numbering in Fig. E4.6(b), we find

for node 1:

$$\text{reactions } R_x = 100.15$$
$$R_y = 41.36$$

for node 2:

$$\text{reactions } R_x = 2.58 - 2.88 = -0.30$$
$$R_y = 16.79 + 5.96 = 22.74 \text{ (because of rounding)}$$

for node 3:

$$\text{reactions } R_x = -99.85$$
$$R_y = 35.90$$

for node 4:

$$\text{horizontal force equilibrium: } -42.01 + 42.01 = 0$$
$$\text{vertical force equilibrium: } -22.90 + 22.90 = 0$$

for node 5:

$$\text{horizontal force equilibrium: } -60.72 - 12.04 + 44.73 + 28.03 = 0$$
$$\text{vertical force equilibrium: } -35.24 - 35.04 + 19.10 + 51.18 = 0$$

for node 6:

$$\text{horizontal force equilibrium: } 57.99 - 57.99 = 0$$
$$\text{vertical force equilibrium: } -6.81 + 6.81 = 0$$

And for nodes 7, 8, and 9, force equilibrium is obviously also satisfied, where at node 9 the element nodal force balances the applied load $P = 100$.

Finally, let us check the overall force equilibrium of the model:

horizontal equilibrium:

$$100.15 - 0.30 - 99.85 = 0$$

vertical equilibrium:

$$41.36 + 22.74 + 35.90 - 100 = 0$$

moment equilibrium (about node 2):

$$-100 \times 4 + 100.15 \times 2 + 99.85 \times 2 = 0$$

It is important to realize that this force equilibrium will hold for *any* finite element mesh, howeve coarse the mesh may be, provided properly formulated elements are used (see Section 4.3).

Now consider element 4:

horizontal equilibrium:

$$0 - 57.99 + 28.03 + 29.97 = 0 \text{ (because of rounding)}$$

vertical equilibrium:

$$-100 + 6.81 + 51.18 + 42.01 = 0$$

moment equilibrium (about its local node 3):

$$-100 \times 2 + 57.99 \times 2 + 42.01 \times 2 = 0$$

Hence the element nodal forces are in equilibrium.

NOTE: In a finite element analysis

1. Stress equilibrium violated inside each element
2. Stresses are discontinuous across elements
3. Stresses are not in equilibrium with the applied traction

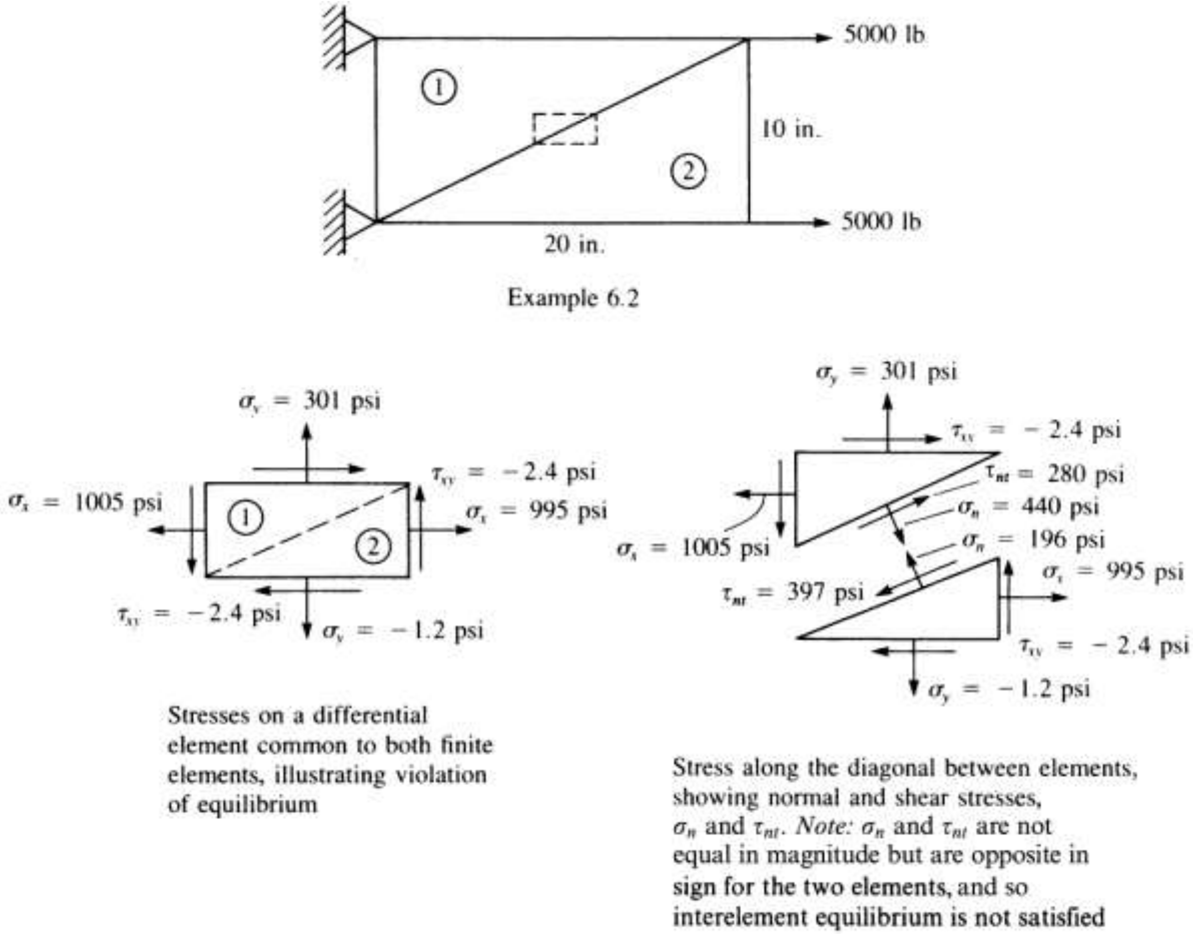


Figure7-10 Example 6.2, illustrating violation of equilibrium of a differential element and along the diagonal edge between two elements (the coarseness of the mesh amplifies the violation of equilibrium)

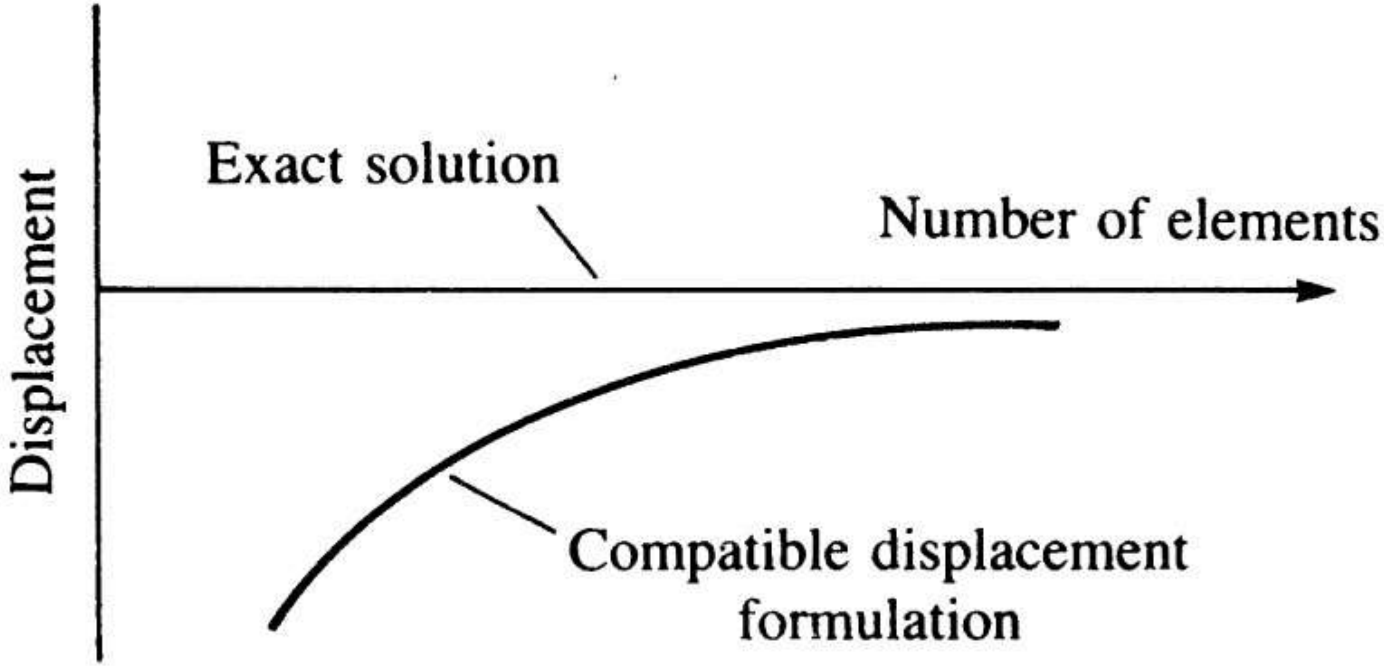


Figure 7-11 Convergence of a finite element solution based on the compatible displacement formulation

Hence a finite element analysis can be interpreted as a process in which

1. The structure or continuum is idealized as an assemblage of elements connected at nodes pertaining to the elements.

2. The externally applied forces are lumped to these nodes to obtain the equivalent nodal load vectors

3. The equivalent nodal loads are equilibriated by the nodal point forces that are equivalent to the element internal stresses.

4. Compatibility and stress-strain relationships are exactly satisfied, but instead of force equilibrium at the differential level, only **global equilibrium** for the complete structure, of the **nodal points** and of **each element** under its nodal point forces is satisfied.

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject : FINITE ELEMENT METHODS

UNIT-II

TUTORIAL - II

1. a.) Determine the shape functions for 4 – nodded quadrilateral element.

   b.) For a beam and loading shown in fig.5, determine the slopes at 2 and 3 and the vertical deflection at the midpoint of the distributed load.

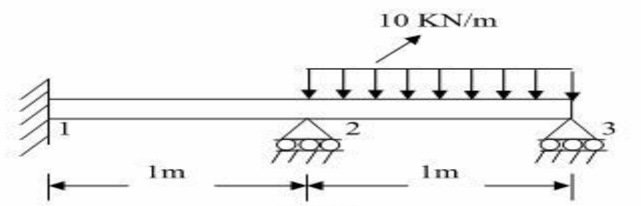
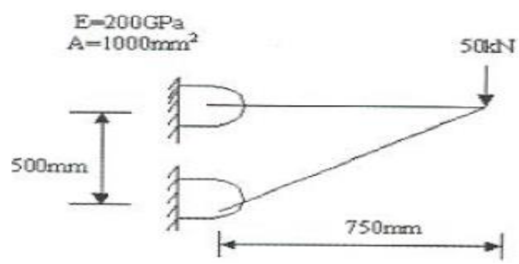


Fig.5

2. Calculate the stiffness matrix, stresses and reactions in the truss structure shown in Figure



3. Derive the stiffness matrix for truss and beam as shown in figure 2

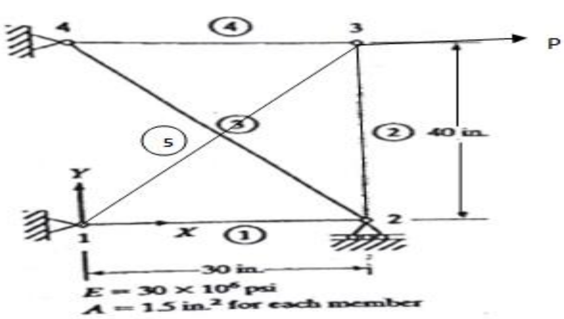
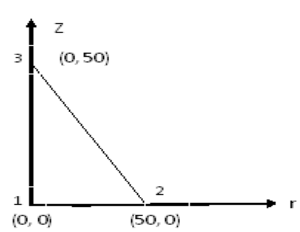


Figure 2

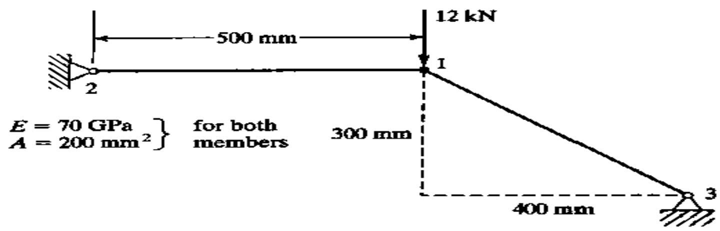
4. For axi-symmetric element shown in figure, determine the strain-displacement matrix. Let E = 2.1×10^5 N/mm^2 and ν= 0.25. The co-ordinates shown in figure are in millimetres.



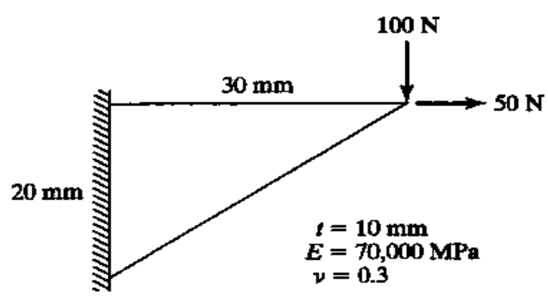
5.Write about different boundary considerations in beams

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY
Subject : FINITE ELEMENT METHODS
UNIT –II
ASSIGNMENT -II

1. Consider the truss shown in Fig.. Determine the displacement and stress in each truss member.



2. For the triangular plane structure given in Fig, determine the deflection at the point of load application using a one-element model. If a mesh of several triangular elements is used, comment on the stress values in the elements close to the tip.



3. a.) Write about different boundary considerations in beams.
   b.) Determine the support reactions and maximum vertical deflection for the Continuous beam shown in Figure.

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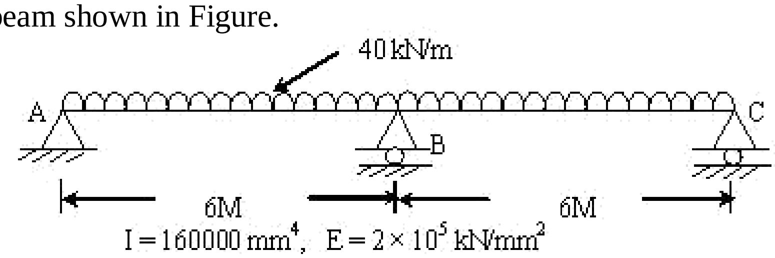


4. Explain the prosperities for hermits shape functions

5. (a) Derive the B Matrix (relating strains and nodal displacements) for an iso parametric triangular element with linear interpolation for the geometry as well as field variables.

b) Explain why the above element is popularly known as CST. Discuss about the advantages and disadvantages of the element

UNIT 3

AXISYMMETRIC ANALYSIS&NUMERICAL INTEGRATION

Syllabus:

Finite element modeling of axi-symmetric solids subjected to axisymmetric loading with triangular elements. Two dimensional four node isoparametric elements and numerical integration.

OBJECTIVE:

To learn the principles involved in the discretization of domains with various elements, polynomial and interpolation and assembly of global arrays.

To learn the applications of FEM for axisymmetric problems with axisymmetric Boundary conditions.

To Understand the need of numerical integration and Iso parametric formulation.

OUTCOME:

Able to solve the problems in Axisymmetric structures.

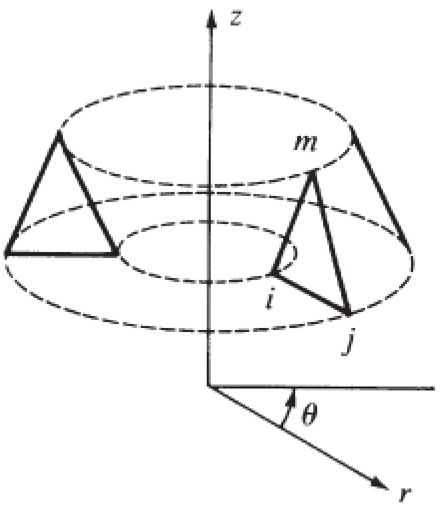
Apply the numerical integration techniques to get element matrices.

UNIT-III

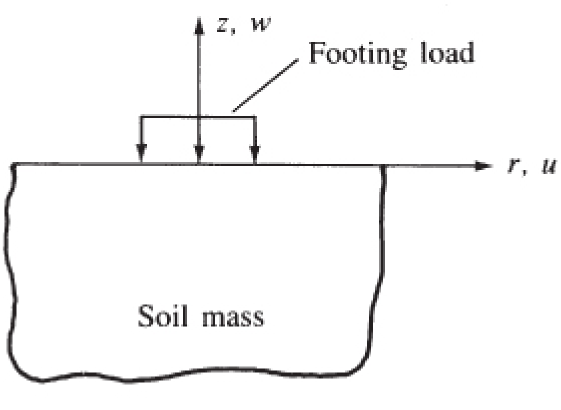
Axisymmetric Elements

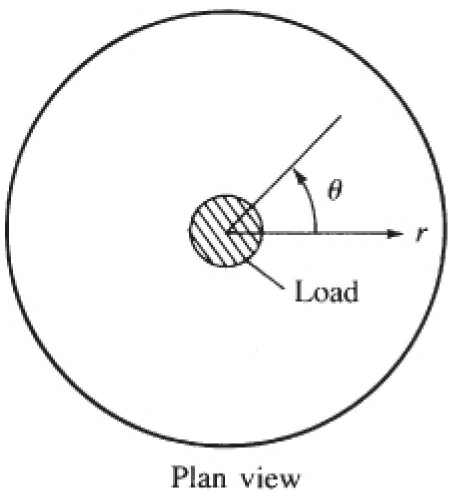
In this chapter, we consider a special two-dimensional element called the ***axisymmetric element***.
This element is quite useful when symmetry with respect to geometry and loading exists about an axis of the body being analyzed.
Problems that involve soil masses subjected to circular footing loads or thick-walled pressure vessels can often be analyzed using the element developed in this chapter.

Derivation of the Stiffness Matrix
Axisymmetric elements are triangular tori such that each element is symmetric with respect to geometry and loading about an axis such as the *z* axis.
Hence, the *z* axis is called the *axis of symmetry* or the *axis of revolution.*
Each vertical cross section of the element is a plane triangle.
The nodal points of an axisymmetric triangular element describe circumferential lines.

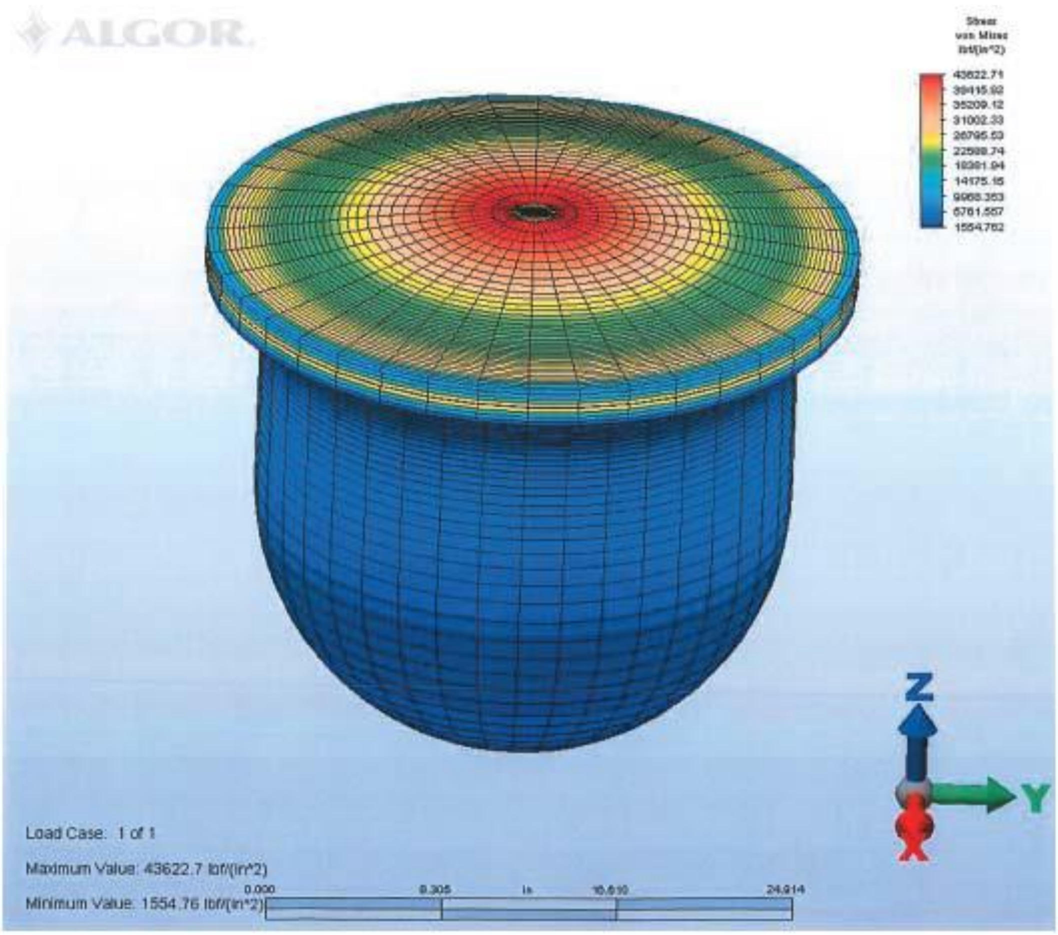


For instance, the axisymmetric problem of a semi-infinite halfspace loaded by a circular area (circular footing) can be solved using the axisymmetric element developed.

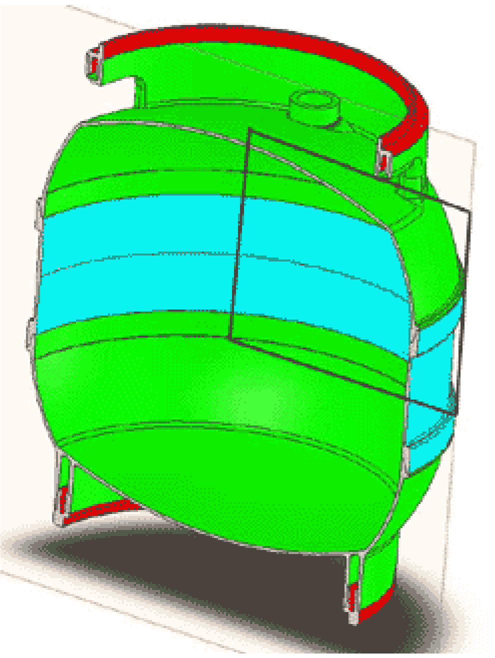


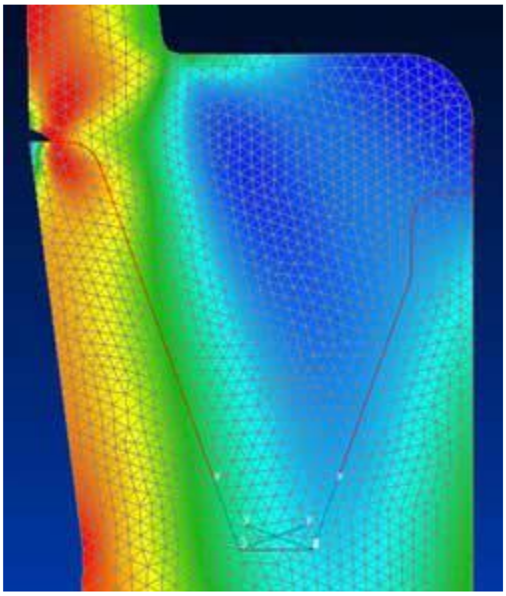


For instance, the axisymmetric problem of a domed pressure vessel can be solved using the axisymmetric element.

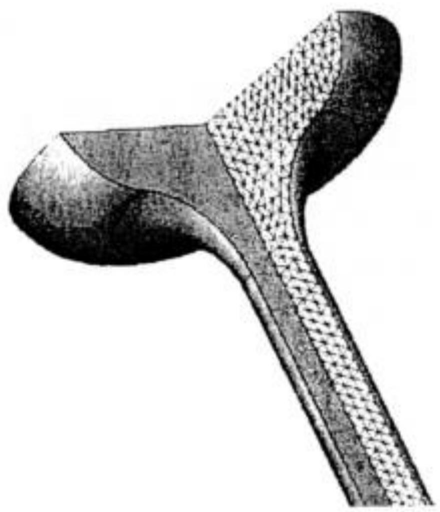


For instance, the axisymmetric problem of stresses acting on the barrel under an internal pressure loading.

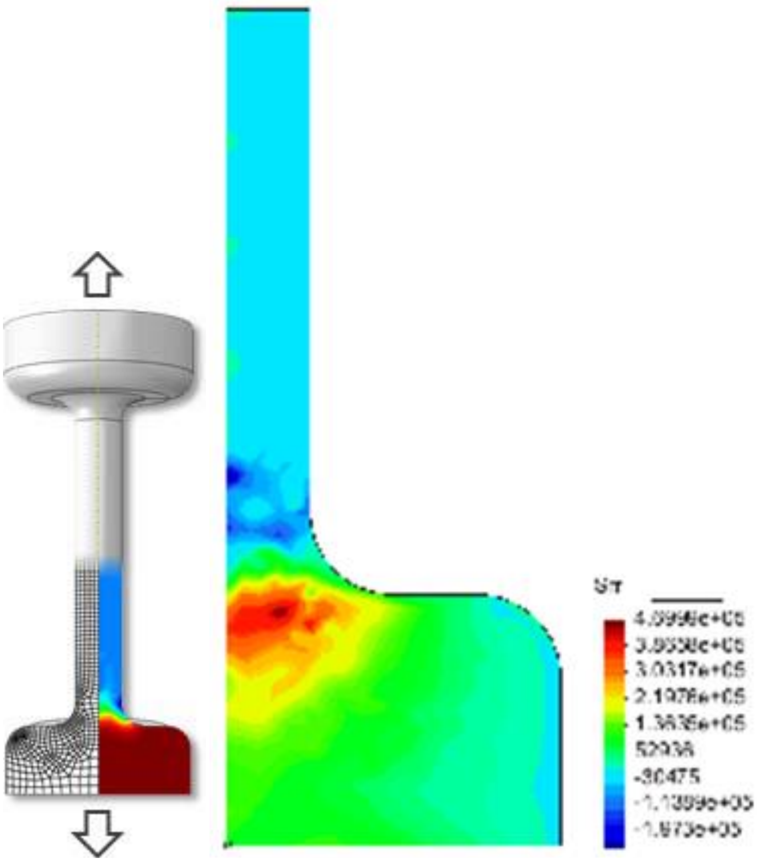




For instance, the axisymmetric problem of an engine valve stem can be solved using the axisymmetric element.

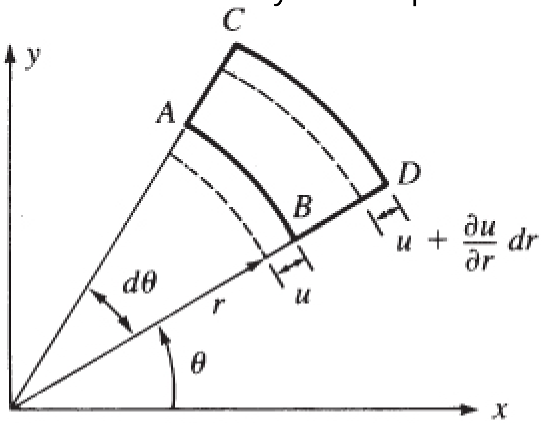


For instance, an axisymmetric specimen loaded under tension compression.

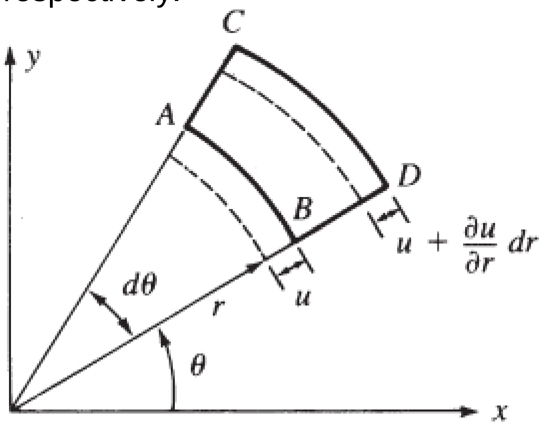


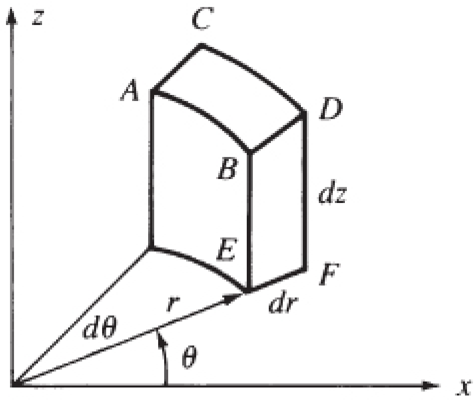
Because of symmetry about the *z* axis, the stresses are independent of the angular coordinate. Therefore, all derivatives with respect to θ vanish, and the displacement component *v* (tangent to the θ direction), the shear strains and the shear stresses with angular θ planes are all zero.

Consider an axisymmetric ring element and its cross section to represent the general state of strain for an axisymmetric problem.



The displacements can be expressed for element *ABCD* in the plane of a cross-section in cylindrical coordinates.
We then let *u* and *w* denote the displacements in the radial and longitudinal directions, respectively.





The side *AB* of the element is displaced an amount *u*, and side *CD* is then displaced an amount $u + (\partial u / \partial r)$ in the radial direction.

The normal strain in the radial direction is then given by: $\varepsilon_r = \dfrac{\partial u}{\partial r}$

The tangential strain is due only to the radial displacement.

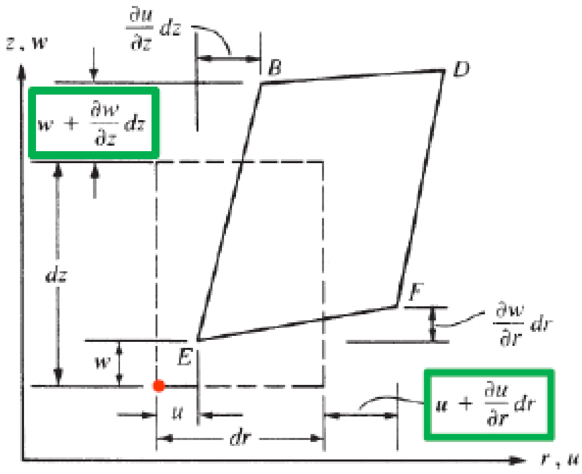
Having only radial displacement *u*, the new length of the arc *AB* is $(r + u)d\theta$, and the tangential strain is then given by:

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

Consider the longitudinal element *BDEF to* obtain the longitudinal strain and the shear strain.

The element displaces by amounts *u* and *w* in the radial and longitudinal directions at point *E*.

The element displaces additional amounts: $(\partial w/\partial z)dz$ along line *BE* and $(\partial u/\partial r)dr$ along line *EF*.



Summarizing the strain-displacement relationships gives:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

The isotropic stress-strain relationship, obtained by simplifying the general stress-strain relationships, is:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

The function u evaluated at node i is: $u(r_i, z_i) = a_1 + a_2 r_i + a_3 z_i$

The general displacement function is then expressed in matrix form as:

$$\{\Psi_i\} = \begin{Bmatrix} a_1 + a_2 r + a_3 z \\ a_4 + a_5 r + a_6 z \end{Bmatrix} = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

By substituting the coordinates of the nodal points into the equation we can solve for the ***a***'s:

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad \Rightarrow \quad \{a\} = [x]^{-1}\{u\}$$

$$\begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix} \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} \qquad \Rightarrow \quad \{a\} = [x]^{-1}\{w\}$$

Performing the inversion operations we have:

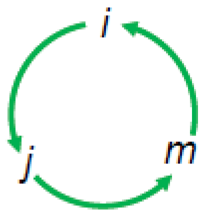
$$[x]^{-1} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \qquad 2A = \begin{vmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{vmatrix}$$

$$2A = r_i\left(z_j - z_m\right) + r_j\left(z_m - z_i\right) + r_m\left(z_i - z_j\right)$$

where A is the area of the triangle

Step 2 - Select Displacement Functions

$$[x]^{-1} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}$$



$$\alpha_i = r_j z_m - z_j r_m \qquad \beta_i = z_j - z_m \qquad \gamma_i = r_m - r_j$$

$$\alpha_j = r_m z_i - z_m r_i \qquad \beta_j = z_m - z_i \qquad \gamma_j = r_i - r_m$$

$$\alpha_m = r_i z_j - z_i r_j \qquad \beta_m = z_i - z_j \qquad \gamma_m = r_j - r_i$$

The values of ***a*** may be written matrix form as:

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

Expanding the above equations:

$$\{u\} = \{1 \quad r \quad z\}\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Substituting the values for ***a*** into the above equation gives:

$$\{u\} = \frac{1}{2A}[1 \quad r \quad z]\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

We will now derive the *u* displacement function in terms of the coordinates *r* and *z*.

$$\{u\} = \frac{1}{2A}\begin{bmatrix}1 & r & z\end{bmatrix}\begin{bmatrix}\alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m\end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$u(r,z) = \frac{1}{2A}\{(\alpha_i + \beta_i r + \gamma_i z)u_i + (\alpha_j + \beta_j r + \gamma_j z)u_j + (\alpha_m + \beta_m r + \gamma_m z)u_m\}$$

We will now derive the *w* displacement function in terms of the coordinates *r* and *z*.

$$\{w\} = \frac{1}{2A}\begin{bmatrix}1 & r & z\end{bmatrix}\begin{bmatrix}\alpha_i w_i + \alpha_j w_j + \alpha_m w_m \\ \beta_i w_i + \beta_j w_j + \beta_m w_m \\ \gamma_i w_i + \gamma_j w_j + \gamma_m w_m\end{bmatrix}$$

Multiplying the matrices in the above equations gives:

$$w(r,z) = \frac{1}{2A}\{(\alpha_i + \beta_i r + \gamma_i z)w_i + (\alpha_j + \beta_j r + \gamma_j z)w_j + (\alpha_m + \beta_m r + \gamma_m z)w_m\}$$

The displacements can be written in a more convenience form as:

$$u(r,z) = N_i u_i + N_j u_j + N_m u_m$$

$$w(r,z) = N_i w_i + N_j w_j + N_m w_m$$

where:

$$N_i = \frac{1}{2A}(\alpha_i + \beta_i r + \gamma_i z)$$

$$N_j = \frac{1}{2A}(\alpha_j + \beta_j r + \gamma_j z)$$

$$N_m = \frac{1}{2A}(\alpha_m + \beta_m r + \gamma_m z)$$

The elemental displacements can be summarized as:

$$\{\Psi_i\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i w_i + N_j w_j + N_m w_m \end{Bmatrix}$$

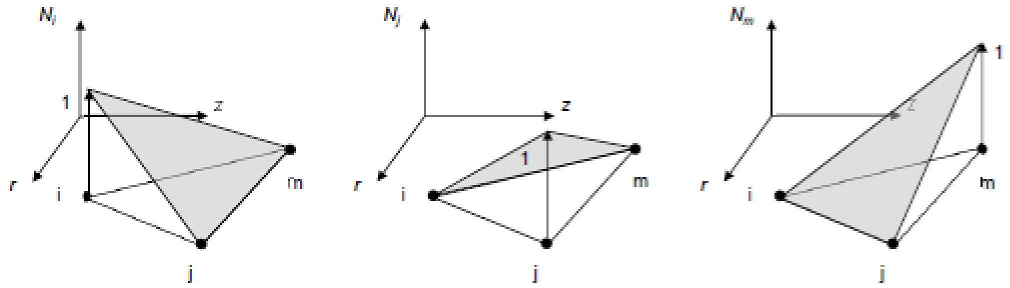
$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

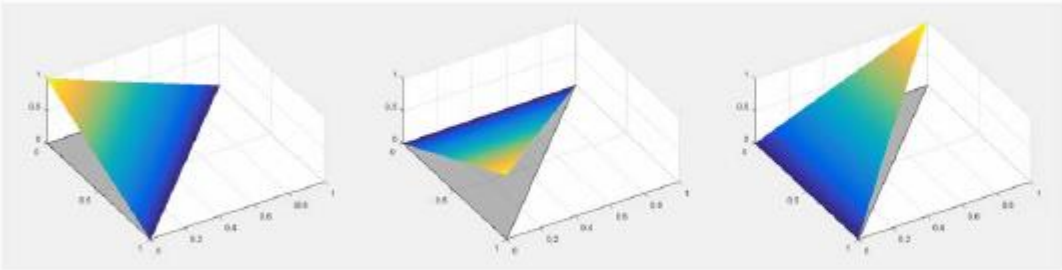
$$\{\Psi\} = [N]\{d\}$$

In another form the equations are:

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

The linear triangular shape functions are illustrated below:





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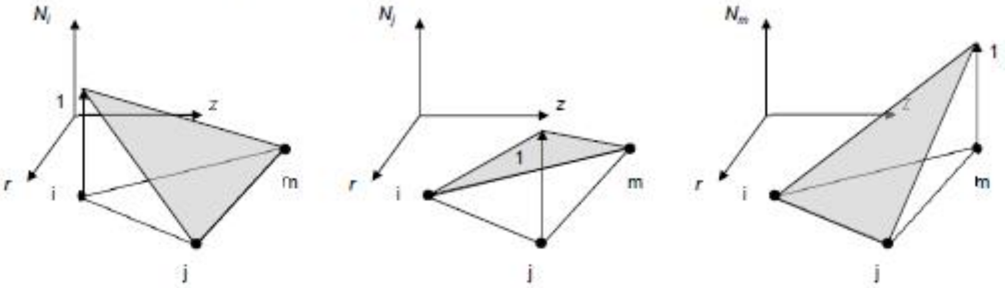
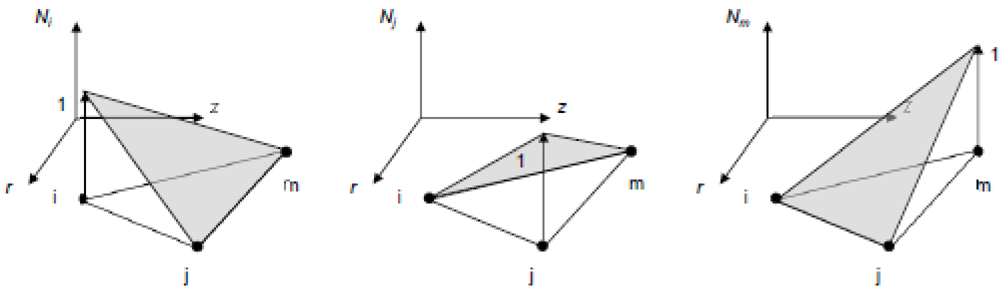


So that u and w will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all r and z locations on the element.

The linear triangular shape functions are illustrated below:



So that u and w will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all r and z locations on the element.

For example, assume all the triangle displaces as a rigid body in the x direction: $u = u_0$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_0 \\ 0 \\ u_0 \\ 0 \\ u_0 \\ 0 \end{Bmatrix} \qquad u_0 = u_0\left(N_i + N_j + N_m\right) \quad \Rightarrow N_i + N_j + N_m = 1$$

So that u and w will yield a constant value for rigid-body displacement, $N_i + N_j + N_m = 1$ for all r and z locations on the element.

For example, assume all the triangle displaces as a rigid body in the z direction: $w = w_0$

$$\{\Psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} 0 \\ w_0 \\ 0 \\ w_0 \\ 0 \\ w_0 \end{Bmatrix} \qquad w_0 = w_0\left(N_i + N_j + N_m\right) \quad \Rightarrow N_i + N_j + N_m = 1$$

Elemental Strains: The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \begin{Bmatrix} a_2 \\ a_6 \\ \dfrac{a_1}{r} + a_2 + \dfrac{a_3 z}{r} \\ a_3 + a_5 \end{Bmatrix}$$

Elemental Strains: The strains over a two-dimensional element are:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \dfrac{1}{r} & 1 & \dfrac{z}{r} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

Substituting our approximation for the displacement gives:

$$\frac{\partial u}{\partial r} = u_{,r} = \frac{\partial}{\partial r}\left(N_i u_i + N_j u_j + N_m u_m\right)$$

$$u_{,r} = N_{i,r} u_i + N_{j,r} u_j + N_{m,r} u_m$$

where the comma indicates differentiation with respect to that variable.

The derivatives of the interpolation functions are:

$$N_{i,r} = \frac{1}{2A}\frac{\partial}{\partial r}\left(\alpha_i + \beta_i r + \gamma_i z\right) = \frac{\beta_i}{2A}$$

$$N_{j,r} = \frac{\beta_j}{2A} \qquad N_{m,r} = \frac{\beta_m}{2A}$$

Therefore:

$$\frac{\partial u}{\partial r} = \frac{1}{2A}\left(\beta_i u_i + \beta_j u_j + \beta_m u_m\right)$$

In a similar manner, the remaining strain terms are approximated as:

$$\frac{\partial w}{\partial z} = \frac{1}{2A}\left(\gamma_i w_i + \gamma_j w_j + \gamma_m w_m \right)$$

$$\frac{u}{r} = \frac{1}{2A}\left[\left(\frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r}\right)u_i + \left(\frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r}\right)u_j + \left(\frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r}\right)u_m\right]$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} = \frac{1}{2A}\left(\beta_i u_i + \gamma_i w_i + \beta_j u_j + \gamma_j w_j + \beta_m u_m + \gamma_m w_m\right)$$

We can write the strains in matrix form as:

$$\begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix}\begin{Bmatrix} d_i \\ d_j \\ d_m \end{Bmatrix}$$

$$\{\varepsilon\} = [B]\{d\}$$

We can write the strains in matrix form as:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} [B_i] & [B_j] & [B_m] \end{bmatrix}\begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

Stress-Strain Relationship: The in-plane stress-strain relationship is:

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} \qquad \{\sigma\} = [D][B]\{d\}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

The stiffness matrix can be defined as:

$$[k] = \int_V [B]^T [D][B] dV$$

For a circumferential differential element the integral becomes:

$$[k] = 2\pi \int_A [B]^T [D][B]\, r\, dr\, dz$$

After integrating along the circumferential boundary, the $[B]$ matrix is a function of r and z.

Therefore, $[k]$ is a function of r and z and is of order 6 x 6.

We can evaluate $[k]$ by one of three methods:

1. Numerical integration (Gaussian quadrature) as discussed in Chapter 10.

2. Explicit multiplication and term-by-term integration.

Therefore, $[k]$ is a function of r and z and is of order 6 x 6.

We can evaluate $[k]$ by one of three methods:

3. Evaluate $[B]$ for a centroidal point $(\bar{r}, \bar{z})$ of the element

$$\left[B(\bar{r}, \bar{z})\right] = \left[\bar{B}\right]$$

$$r = \bar{r} = \frac{r_i + r_j + r_m}{3} \qquad z = \bar{z} = \frac{z_i + z_j + z_m}{3}$$

Therefore, $[k]$ is a function of r and z and is of order 6 x 6.

We can evaluate $[k]$ by one of three methods:

3. Evaluate $[B]$ for a centroidal point $(\bar{r}, \bar{z})$ of the element

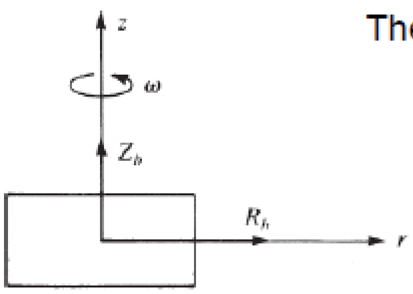
As a first approximation: $[k] = 2\pi\bar{r}A\left[\bar{B}\right]^T[D]\left[\bar{B}\right]$

If the triangular subdivisions are consistent with the final stress distribution (that is, small elements in regions of high stress gradients), then acceptable results can be obtained by Method 3.

Derivation of Force Vectors:

Distributed Body Forces

Loads such as gravity (in the direction of the z axis) or centrifugal forces in rotating machine parts (in the direction of the r axis) are considered to be body forces.

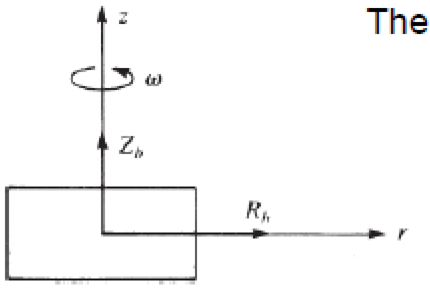


The body forces can be found by:

$$\{f_b\} = 2\pi \int_A [N]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r\,dr\,dz$$

Distributed Body Forces

Where $R_b = \omega^2 \rho r$ for a machine part moving with a constant angular velocity ω about the z axis, with material mass density ρ and radial coordinate r, and Z_b is the body force per unit volume due to the force of gravity.



The body forces can be found by:

$$\{f_b\} = 2\pi \int_A [N]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r\, dr\, dz$$

Considering the body force at node i, we have

$$\{f_{bi}\} = 2\pi \int_A [N_i]^T \begin{Bmatrix} R_b \\ Z_b \end{Bmatrix} r\, dr\, dz \qquad [N_i]^T = \begin{bmatrix} N_i & 0 \\ 0 & N_i \end{bmatrix}$$

Multiplying and integrating yields

$$\{f_{bi}\} = \frac{2\pi}{3} \begin{Bmatrix} \bar{R}_b \\ Z_b \end{Bmatrix} A\bar{r}$$

The origin of the coordinates is the centroid of the element, and R_b is the radially directed body force per unit volume evaluated at the centroid of the element.

The body forces at nodes j and m are identical to those given for node i. Hence, for an element, we have

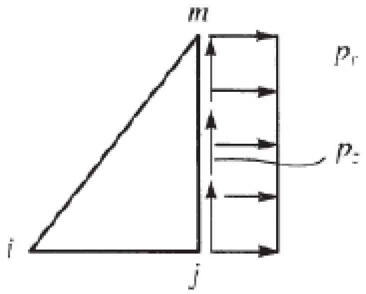
$$\{f_b\} = \frac{2\pi A\bar{r}}{3} \begin{Bmatrix} \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \end{Bmatrix} \qquad \bar{R}_b = \omega^2 \rho \bar{r}$$

Surface Forces

Surface forces can be found by $\{f_s\} = \int_S [N_s]^T \{T\} dS$

Where again $[N_s]$ denotes the shape function matrix evaluated along the surface where the surface traction acts.

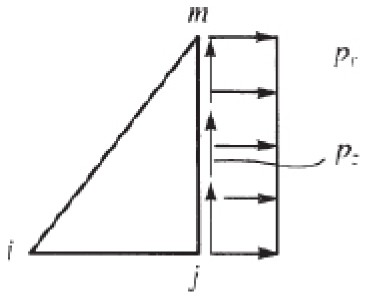
For example, along the vertical face *jm* of an element, let uniform loads p_r, and p_z, be applied along surface $r = r_j$.



For instance, for node *j*, substituting N_j gives

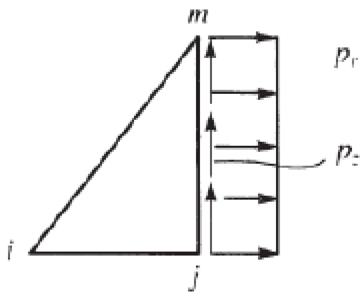
$$\{f_{sj}\} = \int_{z_j}^{z_m} \frac{1}{2A}\begin{bmatrix} \alpha_j + \beta_j r + \gamma_j z & 0 \\ 0 & \alpha_j + \beta_j r + \gamma_j z \end{bmatrix}\begin{Bmatrix} p_r \\ p_z \end{Bmatrix} 2\pi r_j \, dz$$

Evaluated at $r = r_j$ and z



Integrating the equations explicitly along with similar evaluations for f_{si} and f_{sm} the total distribution of surface force to nodes *i*, *j*, and *m* is

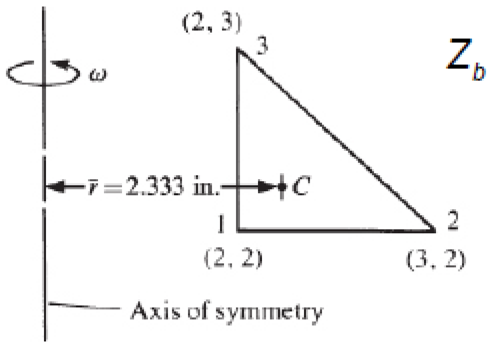
$$\{f_s\} = \frac{2\pi r_j (z_m - z_j)}{2}\begin{Bmatrix} 0 \\ 0 \\ p_r \\ p_z \\ p_r \\ p_z \end{Bmatrix}$$



Example 1

For the element of an axisymmetric body rotating with a constant angular velocity ω = 100 rev/min, evaluate the approximate body force matrix.

Include the weight of the material, where the weight density ρ_w = 0.283 lb./in.^{3}. Dimensions are inches.



$$Z_b = 0.283 \text{ lb./in.}^3$$

Let evaluate the approximate body force matrix.

The body forces per unit volume evaluated at the centroid of the element are:

$$\bar{R}_b = \omega^2 \rho \bar{r}$$

$$= \left[(100\,\text{rpm})\left(2\pi \tfrac{\text{rad}}{\text{rev}}\right)\left(\frac{1\,\text{min}}{60\,\text{sec}}\right)\right]^2 \frac{0.283\text{ lb./in.}^3}{\left(32.2\tfrac{\text{ft.}}{\text{s}^2} \times \dfrac{12\,\text{in.}}{\text{ft.}}\right)}(2.333\,\text{in.})$$

$$= 0.187\,\text{lb./in.}^3$$

$$\frac{2\pi A\bar{r}}{3} = \frac{2\pi\left(0.5\,\text{in.}^2\right)(2.333\,\text{in.})}{3} = 2.44\,\text{in.}^2$$

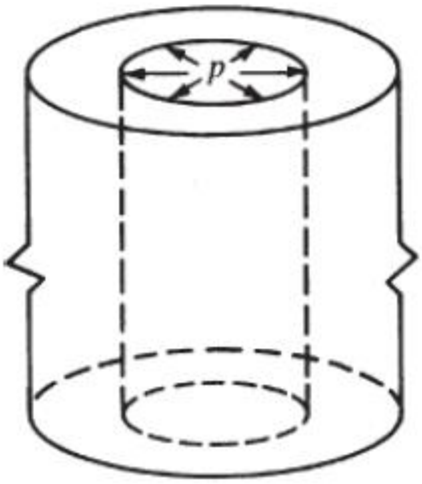
Let evaluate the approximate body force matrix.

The body forces per unit volume evaluated at the centroid of the element are:

$$\{f_b\} = \frac{2\pi A\bar{r}}{3}\begin{Bmatrix} \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \\ \bar{R}_b \\ Z_b \end{Bmatrix} = 2.44\,\text{in.}^3 \begin{Bmatrix} 0.187 \\ -0.283 \\ 0.187 \\ -0.283 \\ 0.187 \\ -0.283 \end{Bmatrix} \frac{\text{lb.}}{\text{in.}^3} = \begin{Bmatrix} 0.457 \\ -0.691 \\ 0.457 \\ -0.691 \\ 0.457 \\ -0.691 \end{Bmatrix} \text{lb.}$$

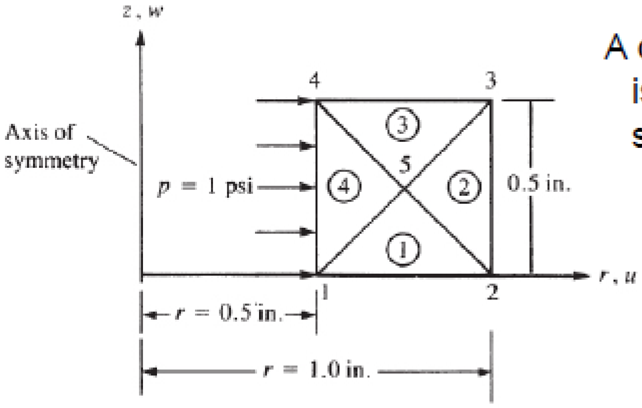
Example 2

For the long, thick-walled cylinder under internal pressure p equal to 1 psi, determine the displacements and stresses.



First discretize the cylinder into four triangular elements.

A horizontal slice of the cylinder represents the total cylinder behavior.



A coarse mesh of elements is used for simplicity's sake

The governing global matrix equation is:

$$\begin{Bmatrix} F_{1r} \\ F_{1z} \\ F_{2r} \\ F_{2z} \\ F_{3r} \\ F_{3z} \\ F_{4r} \\ F_{4z} \\ F_{5r} \\ F_{5z} \end{Bmatrix} = [K] \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \\ u_5 \\ w_5 \end{Bmatrix}$$

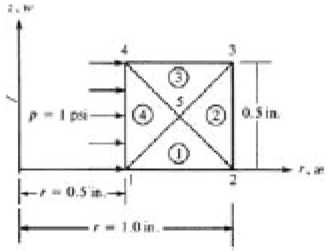
$[K]$ is a matrix of order 10 x 10

The $[K]$ matrix is assembled in the usual manner by superposition of the individual element stiffness matrices.

For simplicity's sake, we will evaluate $[B]$ for a centroidal point $(\bar{r}, \bar{z})$ of the element.

$$[k] = 2\pi \bar{r} A \left[\bar{B}\right]^T [D] \left[\bar{B}\right]$$

Assemblage of the Stiffness Matrix: Element 1

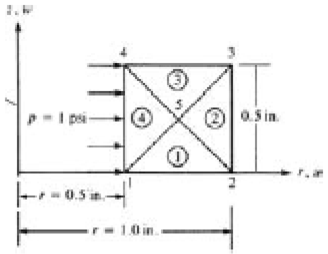


$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[\bar{B}] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{\bar{r}} + \beta_i + \frac{\gamma_i \bar{z}}{\bar{r}} & 0 & \frac{\alpha_j}{\bar{r}} + \beta_j + \frac{\gamma_j \bar{z}}{\bar{r}} & 0 & \frac{\alpha_m}{\bar{r}} + \beta_m + \frac{\gamma_m \bar{z}}{\bar{r}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

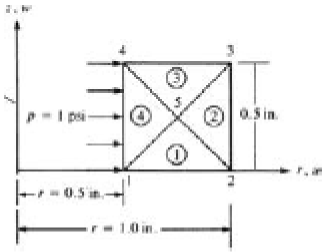
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\alpha_i = r_j z_m - z_j r_m = (1.0)(0.25) - (0.0)(0.75) = 0.25 \text{ in.}^2$$

$$\alpha_j = r_m z_i - z_m r_i = (0.75)(0.0) - (0.25)(0.5) = -0.125 \text{ in.}^2$$

$$\alpha_m = r_i z_j - z_i r_j = (0.5)(0.0) - (0.0)(1.0) = 0$$

Assemblage of the Stiffness Matrix: Element 1



$$r = \bar{r} \qquad z = \bar{z}$$

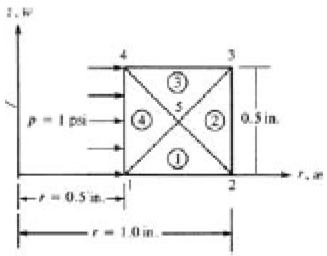
$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\beta_i = z_j - z_m = -0.25\text{in}^2 \qquad \gamma_i = r_m - r_j = -0.25\text{in.}^2$$

$$\beta_j = z_m - z_i = 0.25\text{in}^2 \qquad \gamma_j = r_i - r_m = -0.25\text{in.}^2$$

$$\beta_m = z_i - z_j = 0 \qquad \gamma_m = r_j - r_i = 0.5\text{in.}^2$$

Assemblage of the Stiffness Matrix: Element 1



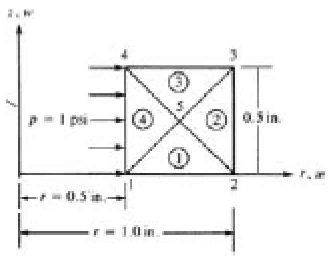
$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.75\text{in.} \qquad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.0833\text{in.}$$

$$A = \frac{1}{2}(0.5)(0.25) = 0.0625\text{in.}^2$$

Assemblage of the Stiffness Matrix: Element 1

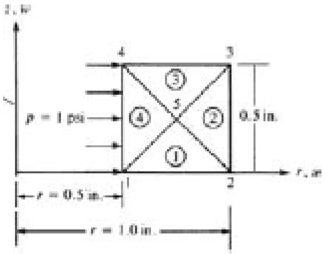


$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[\bar{B}\right] = \frac{1}{0.125} \begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ 0.0556 & 0 & 0.0556 & 0 & 0.0556 & 0 \\ -.025 & -0.25 & -0.25 & 0.25 & 0.5 & 0 \end{bmatrix} {}^{1}\!/_{\text{in.}}$$

Assemblage of the Stiffness Matrix: Element 1



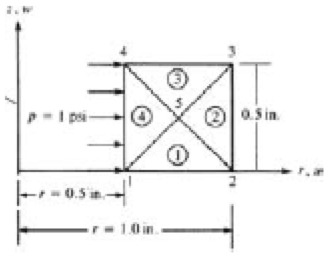
$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & 0 \\ \nu & 1-\nu & 0 & 0 \\ 0 & 0 & 1-\nu & 0 \\ 0 & 0 & 0 & 0.5-\nu \end{bmatrix}$$

Assume that $E = 30 \times 10^6$ psi and $\nu = 0.3$

Assemblage of the Stiffness Matrix: Element 1

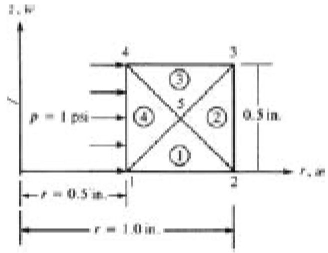


$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[D] = 57.7(10^6) \begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} psi$$

Assemblage of the Stiffness Matrix: Element 1

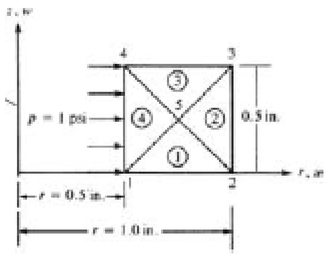


$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[\bar{B}]^T_{6x4}[D]_{4x4} = \frac{57.7(10^6)}{0.125}\begin{bmatrix} -0.158 & -0.0583 & -0.0361 & -0.05 \\ -0.075 & -0.175 & -0.075 & -0.05 \\ 0.192 & 0.0917 & 0.114 & -0.05 \\ -0.075 & -0.175 & -0.075 & 0.05 \\ 0.0167 & 0.0166 & 0.0388 & 0.1 \\ 0.15 & 0.35 & 0.15 & 0 \end{bmatrix}$$

Assemblage of the Stiffness Matrix: Element 1



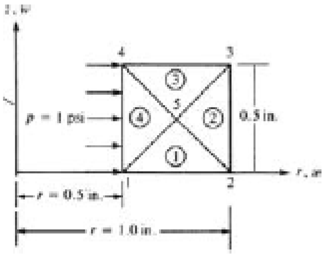
$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.00 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[k^{(1)}] = (10^6)\begin{bmatrix} 54.46 & 29.45 & -31.63 & 2.26 & -29.37 & -31.71 \\ 29.45 & 61.17 & -11.33 & 33.98 & -31.72 & -95.15 \\ -31.63 & -11.33 & 72.59 & -38.52 & -20.31 & 49.84 \\ 2.26 & 33.98 & -38.52 & 61.17 & 22.66 & -95.15 \\ -29.37 & -31.72 & -20.31 & 22.66 & 56.72 & 9.06 \\ -31.71 & -95.15 & 49.84 & -95.15 & 9.06 & 190.31 \end{bmatrix}\frac{\text{lb.}}{\text{in.}}$$

(column groups: $i = 1$, $j = 2$, $m = 5$)

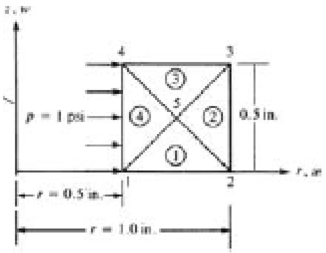
Assemblage of the Stiffness Matrix: Element 2



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$[\bar{B}] = \frac{1}{2A} \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{\bar{r}} + \beta_i + \frac{\gamma_i \bar{z}}{\bar{r}} & 0 & \frac{\alpha_j}{\bar{r}} + \beta_j + \frac{\gamma_j \bar{z}}{\bar{r}} & 0 & \frac{\alpha_m}{\bar{r}} + \beta_m + \frac{\gamma_m \bar{z}}{\bar{r}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$



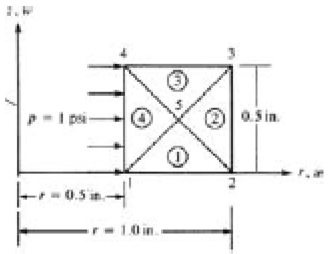
$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\alpha_i = r_j z_m - z_j r_m = (1.0)(0.25) - (0.5)(0.75) = -0.125 \text{in.}^2$$

$$\alpha_j = r_m z_i - z_m r_i = (0.75)(0.0) - (0.25)(1.0) = -0.25 \text{in.}^2$$

$$\alpha_m = r_i z_j - z_i r_j = (1.0)(0.5) - (0.0)(1.0) = 0.5 \text{in.}^2$$



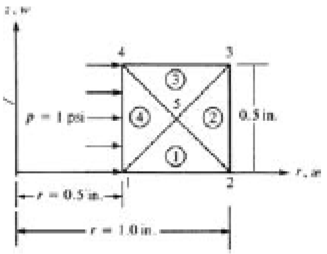
$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\beta_i = z_j - z_m = 0.25 \text{in}^2 \qquad \gamma_i = r_m - r_j = -0.25 \text{in.}^2$$

$$\beta_j = z_m - z_i = 0.25 \text{in}^2 \qquad \gamma_j = r_i - r_m = 0.25 \text{in.}^2$$

$$\beta_m = z_i - z_j = -0.5 \text{in}^2 \qquad \gamma_m = r_j - r_i = 0$$

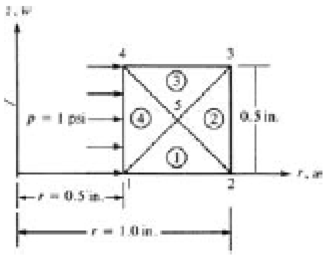


$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.9167 \text{in.} \qquad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.25 \text{in.}$$

$$A = \frac{1}{2}(0.5)(0.25) = 0.0625 \text{in.}^2$$

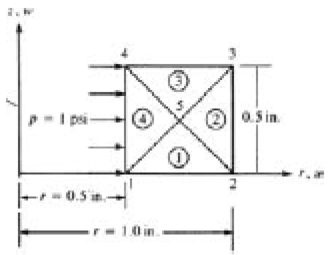


$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 1.00 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[k^{(2)}\right] = (10^6) \overset{\qquad i=2 \qquad\qquad\qquad j=3 \qquad\qquad\qquad m=5}{\left[\begin{array}{cc|cc|cc} 85.75 & -46.07 & 52.52 & 12.84 & -118.92 & 33.23 \\ -46.07 & 74.77 & -12.84 & -41.54 & 45.32 & -33.23 \\ \hline 52.52 & -12.84 & 85.74 & 46.07 & -118.92 & -33.23 \\ 12.84 & -41.54 & 46.07 & 74.77 & -45.21 & -33.23 \\ \hline -118.92 & 45.32 & -118.92 & -45.21 & 216.41 & 0 \\ 33.23 & -33.23 & -33.23 & -33.23 & 0 & 66.46 \end{array}\right]} \frac{\text{lb.}}{\text{in.}}$$

Assemblage of the Stiffness Matrix: Element 3



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ 0.50 \\ 0.75 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.50 \\ 0.25 \end{Bmatrix} \text{in.}$$

$$\left[\bar{B}\right] = \frac{1}{2A} \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{\bar{r}} + \beta_i + \frac{\gamma_i \bar{z}}{\bar{r}} & 0 & \frac{\alpha_j}{\bar{r}} + \beta_j + \frac{\gamma_j \bar{z}}{\bar{r}} & 0 & \frac{\alpha_m}{\bar{r}} + \beta_m + \frac{\gamma_m \bar{z}}{\bar{r}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

$$\alpha_i = r_j z_m - z_j r_m \quad = (0.5)(0.25) - (0.5)(0.75) = -0.25 \text{in.}^2$$

$$\alpha_j = r_m z_i - z_m r_i \quad = (0.75)(0.5) - (0.25)(1.0) = 0.125 \text{in.}^2$$

$$\alpha_m = r_i z_j - z_i r_j \quad = (1.0)(0.5) - (0.5)(0.5) = 0.25 \text{in.}^2$$

$$\beta_i = z_j - z_m = 0.25 \text{in}^2 \qquad \gamma_i = r_m - r_j = 0.25 \text{in.}^2$$

$$\beta_j = z_m - z_i = -0.25 \text{in}^2 \qquad \gamma_j = r_i - r_m = 0.25 \text{in.}^2$$

$$\beta_m = z_i - z_j = 0 \qquad \gamma_m = r_j - r_i = -0.5 \text{in.}^2$$

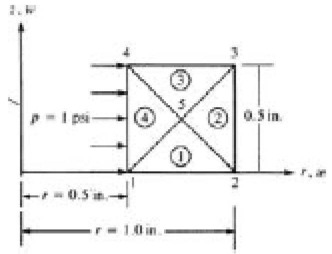
$$\bar{r} = \sum_{i=1}^{3} \frac{r_i}{3} = 0.75 \text{in.} \qquad \bar{z} = \sum_{i=1}^{3} \frac{z_i}{3} = 0.417 \text{in.}$$

$$A = \frac{1}{2}(0.5)(0.25) = 0.0625 \text{in.}^2$$

$$\begin{array}{c} \quad\quad\quad\quad i=3 \quad\quad\quad\quad\quad j=4 \quad\quad\quad\quad\quad m=5 \end{array}$$

$$[k^{(3)}] = (10^6)\left[\begin{array}{cc|cc|cc} 72.58 & 38.52 & -31.63 & 11.33 & -20.31 & -49.84 \\ 38.52 & 61.17 & -2.26 & 33.98 & -22.66 & -95.15 \\ \hline -31.63 & -2.26 & 54.46 & -29.45 & -29.37 & 31.72 \\ 11.33 & 33.98 & -29.45 & 61.17 & 31.72 & -95.15 \\ \hline -20.31 & -22.66 & -29.37 & 31.72 & 56.72 & -9.06 \\ -49.84 & -95.15 & 31.72 & -95.15 & -9.06 & 190.31 \end{array}\right] \frac{\text{lb.}}{\text{in.}}$$

Assemblage of the Stiffness Matrix: Element 4



$$r = \bar{r} \qquad z = \bar{z}$$

$$\begin{Bmatrix} r_i \\ r_j \\ r_m \end{Bmatrix} = \begin{Bmatrix} 0.50 \\ 0.75 \\ 0.50 \end{Bmatrix} \text{in.} \qquad \begin{Bmatrix} z_i \\ z_j \\ z_m \end{Bmatrix} = \begin{Bmatrix} 0.00 \\ 0.25 \\ 0.50 \end{Bmatrix} \text{in.}$$

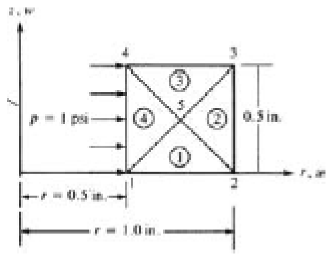
$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \dfrac{\alpha_i}{\bar{r}} + \beta_i + \dfrac{\gamma_i \bar{z}}{\bar{r}} & 0 & \dfrac{\alpha_j}{\bar{r}} + \beta_j + \dfrac{\gamma_j \bar{z}}{\bar{r}} & 0 & \dfrac{\alpha_m}{\bar{r}} + \beta_m + \dfrac{\gamma_m \bar{z}}{\bar{r}} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

$$
\left[k^{(4)}\right] = \left(10^6\right)
\begin{array}{c}
\begin{array}{cccccc} \quad i=1 & & \quad j=5 & & \quad m=4 & \end{array} \\
\left[\begin{array}{cc|cc|cc}
41.53 & 21.90 & -66.45 & -21.14 & 20.39 & -0.75 \\
21.92 & 47.57 & -36.24 & -21.14 & 0.75 & -26.43 \\
\hline
-66.45 & -36.24 & 169.14 & 0 & -66.45 & 36.24 \\
-21.14 & -21.14 & 0 & 42.28 & 21.14 & -21.14 \\
\hline
20.39 & 0.75 & -66.45 & 21.14 & 41.53 & -21.90 \\
-0.75 & -26.43 & 36.24 & -21.14 & -21.90 & 47.57
\end{array}\right]
\end{array}
\frac{\text{lb.}}{\text{in.}}
$$

Using superposition of the element stiffness matrices, where we rearrange the elements of each stiffness matrix in order of increasing nodal degrees of freedom, we can obtain the global stiffness matrix.

$$
[K] = \left(10^6\right)
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\left[\begin{array}{cc|cc|cc|cc|cc}
95.99 & 51.35 & -36.63 & 2.26 & 0 & 0 & 20.39 & -0.75 & -95.82 & -52.86 \\
51.35 & 108.74 & -11.33 & 33.98 & 0 & 0 & 33.98 & -26.43 & -67.96 & -116.3 \\
\hline
-36.63 & -11.33 & 158.34 & -84.59 & 52.52 & 12.84 & 0 & 0 & -139.2 & 83.07 \\
2.26 & 33.98 & -84.59 & 135.94 & -12.84 & -41.54 & 0 & 0 & 67.98 & -128.4 \\
\hline
0 & 0 & 52.52 & -12.84 & 158.33 & 84.59 & -31.63 & 11.33 & -139.2 & -83.07 \\
0 & 0 & 12.84 & -41.54 & 84.59 & 135.94 & -2.26 & 33.98 & -67.98 & -128.4 \\
\hline
20.39 & 33.98 & 0 & 0 & -31.63 & -2.26 & 95.99 & -51.35 & -95.82 & 52.86 \\
-0.75 & -26.43 & 0 & 0 & 11.33 & 33.98 & -51.35 & 108.74 & 67.96 & -116.3 \\
\hline
-95.82 & -67.96 & -139.2 & 67.98 & -139.2 & -67.98 & -95.82 & 67.96 & 498.99 & 0 \\
-52.86 & -116.3 & 83.07 & -128.4 & -83.07 & -128.4 & 52.86 & -116.3 & 0 & 489.36
\end{array}\right]
\end{array}
\frac{lb.}{in.}
$$

The applied nodal forces are given as:



$$
\{f_s\} = \frac{2\pi r_i\left(z_j - z_m\right)}{2}\begin{Bmatrix} p_r \\ p_z \\ 0 \\ 0 \\ p_r \\ p_z \end{Bmatrix} = \begin{Bmatrix} 0.785 \\ 0 \\ 0 \\ 0 \\ 0.785 \\ 0 \end{Bmatrix} \text{lb.}
$$

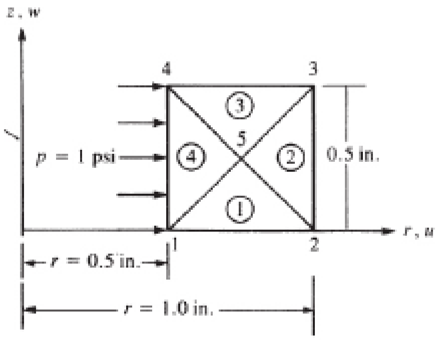
$$
F_{1r} = F_{4r} = \frac{2\pi(0.5\text{in.})(0.5\text{in.})}{2}\left(1 psi\right) = 0.785\, lb.
$$

The resulting equations are:

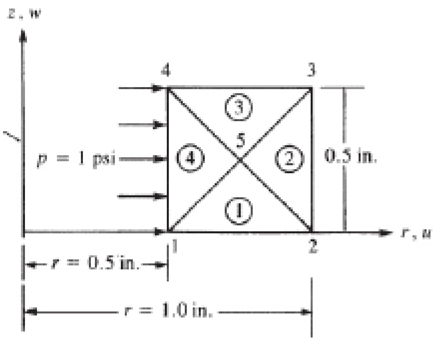
$$(10^6)\frac{lb.}{in.}\begin{bmatrix} 95.99 & 51.35 & -36.63 & 2.26 & 0 & 0 & 20.39 & -0.75 & -95.82 & -52.86 \\ 51.35 & 108.74 & -11.33 & 33.98 & 0 & 0 & 33.98 & -26.43 & -67.96 & -116.3 \\ -36.63 & -11.33 & 158.34 & -84.59 & 52.52 & 12.84 & 0 & 0 & -139.2 & 83.07 \\ 2.26 & 33.98 & -84.59 & 135.94 & -12.84 & -41.54 & 0 & 0 & 67.98 & -128.4 \\ 0 & 0 & 52.52 & -12.84 & 158.33 & 84.59 & -31.63 & 11.33 & -139.2 & -83.07 \\ 0 & 0 & 12.84 & -41.54 & 84.59 & 135.94 & -2.26 & 33.98 & -67.98 & -128.4 \\ 20.39 & 33.98 & 0 & 0 & -31.63 & -2.26 & 95.99 & -51.35 & -95.82 & 52.86 \\ -0.75 & -26.43 & 0 & 0 & 11.33 & 33.98 & -51.35 & 108.74 & 67.96 & -116.3 \\ -95.82 & -67.96 & -139.2 & 67.98 & -139.2 & -67.98 & -95.82 & 67.96 & 498.99 & 0 \\ -52.86 & -116.3 & 83.07 & -128.4 & -83.07 & -128.4 & 52.86 & -116.3 & 0 & 489.36 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \\ u_5 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} 0.785 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.785 \\ 0 \\ 0 \\ 0 \end{Bmatrix} lb.$$

The nodal displacements are:

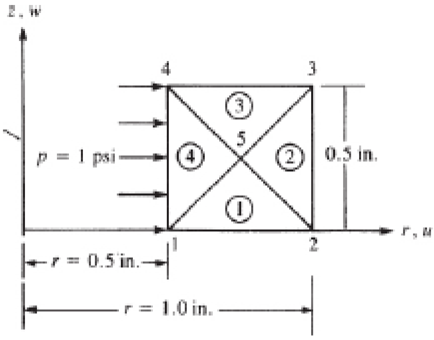
$$\begin{Bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \\ u_5 \\ w_5 \end{Bmatrix} = \begin{Bmatrix} 0.0322 \\ 0.00115 \\ 0.0219 \\ 0.00206 \\ 0.0219 \\ -0.00206 \\ 0.0322 \\ -0.00115 \\ 0.0244 \\ 0.0 \end{Bmatrix} \times (10^{-6}) \text{in.}$$



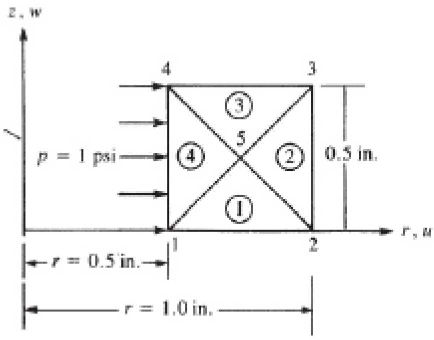
The results for nodal displacements are as expected because radial displacements at the inner edge are equal ($u_1 = u_4$) and those at the outer edge are equal ($u_2 = u_3$).



In addition, the axial displacements at the outer nodes and inner nodes are equal but opposite in sign ($w_1 = -w_4$ and $w_2 = -w_3$) as a result of the Poisson effect and symmetry.



Finally, the axial displacement at the center node is zero ($w_5 = 0$), as it should be because of symmetry.



Determine the stresses in each element as: $\{\sigma\} = [D][\bar{B}]\{d\}$

For Element 1:

$$[D] = 57.7\left(10^6\right)\begin{bmatrix} 0.7 & 0.3 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} psi$$

$$[\bar{B}] = \frac{1}{0.125}\begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ 0.0556 & 0 & 0.0556 & 0 & 0.0556 & 0 \\ -.025 & -0.25 & -0.25 & 0.25 & 0.5 & 0 \end{bmatrix} \frac{1}{in.}$$

For Element 1:

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{xy}\end{Bmatrix}=[D]\left[\bar{B}\right]\begin{Bmatrix}u_1\\ w_1\\ u_2\\ w_2\\ u_5\\ w_5\end{Bmatrix}=\begin{Bmatrix}-0.338\\ -0.0126\\ 0.942\\ -0.1037\end{Bmatrix}psi$$

For Element 2:

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{xy}\end{Bmatrix}=[D]\left[\bar{B}\right]\begin{Bmatrix}u_2\\ w_2\\ u_3\\ w_3\\ u_5\\ w_5\end{Bmatrix}=\begin{Bmatrix}-0.105\\ -0.0747\\ 0.690\\ 0.0\end{Bmatrix}psi$$

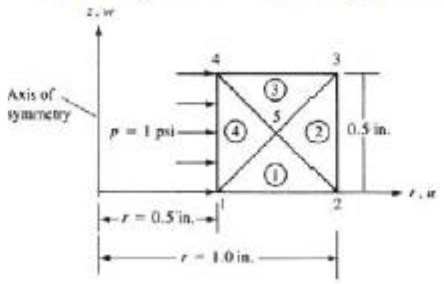
For Element 3:

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{xy}\end{Bmatrix}=[D]\left[\bar{B}\right]\begin{Bmatrix}u_3\\ w_3\\ u_4\\ w_4\\ u_5\\ w_5\end{Bmatrix}=\begin{Bmatrix}-0.337\\ -0.0125\\ 0.942\\ 0.1037\end{Bmatrix}psi$$

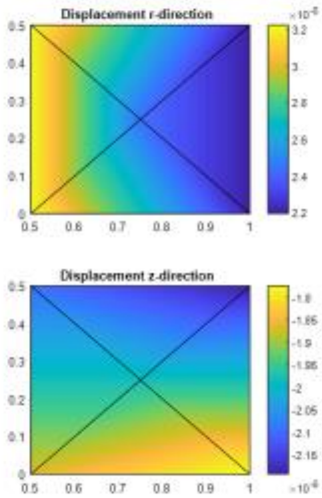
For Element 4:

$$\begin{Bmatrix}\sigma_r\\ \sigma_z\\ \sigma_\theta\\ \tau_{xy}\end{Bmatrix}=[D]\left[\bar{B}\right]\begin{Bmatrix}u_1\\ w_1\\ u_5\\ w_5\\ u_4\\ w_4\end{Bmatrix}=\begin{Bmatrix}-0.470\\ 0.1493\\ 1.426\\ 0.0\end{Bmatrix}psi$$

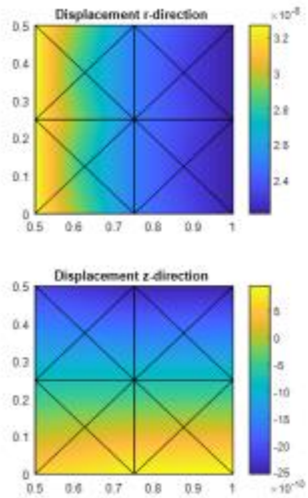
Example 2 – Using Matlab code



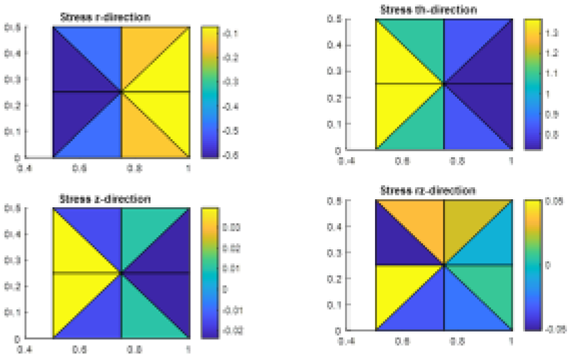
A coarse mesh of 4 CST elements



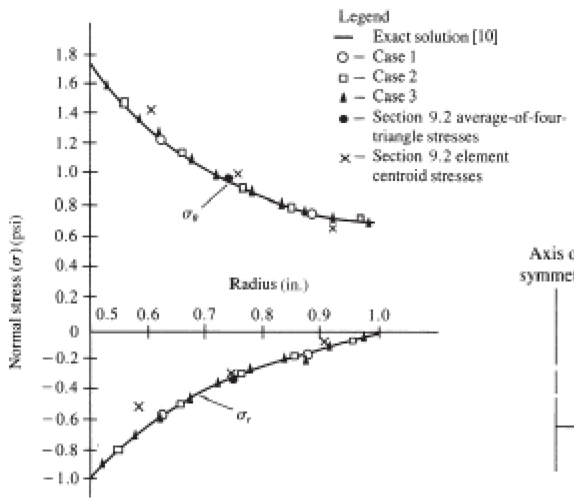
A coarse mesh of 16 CST elements

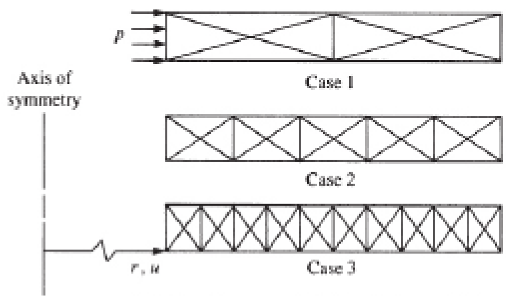


A coarse mesh of 8 CST elements

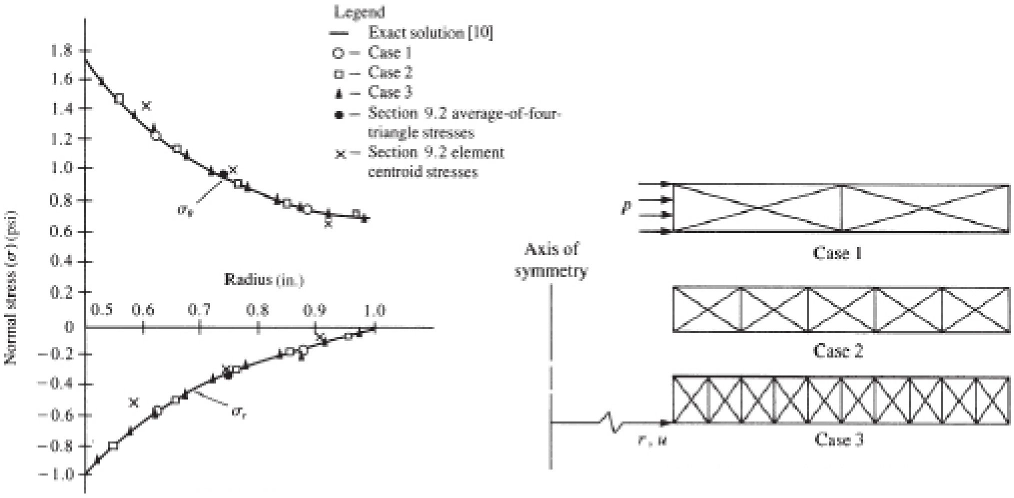


The figure below shows the exact solution along with the results determined here and the other results.

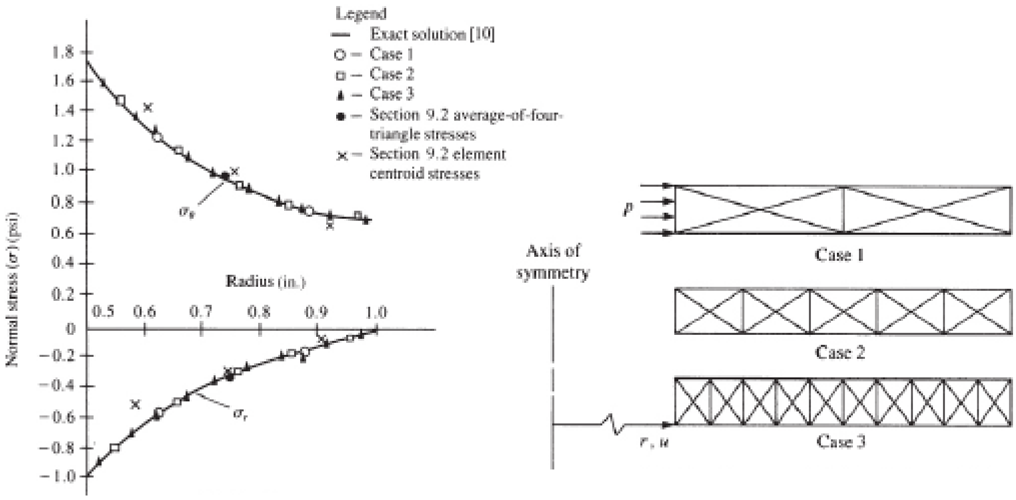




Observe that agreement with the exact solution is quite good except for the limited results due to the very coarse mesh used in the longhand example.

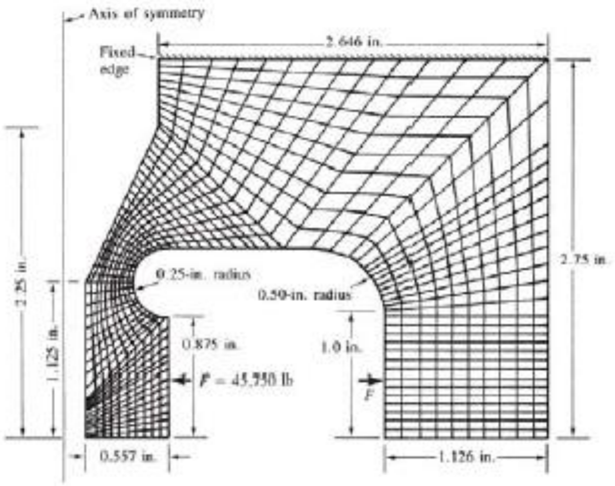


Stresses have been plotted at the center of the quadrilaterals and were obtained by averaging the stresses in the four connecting triangles.

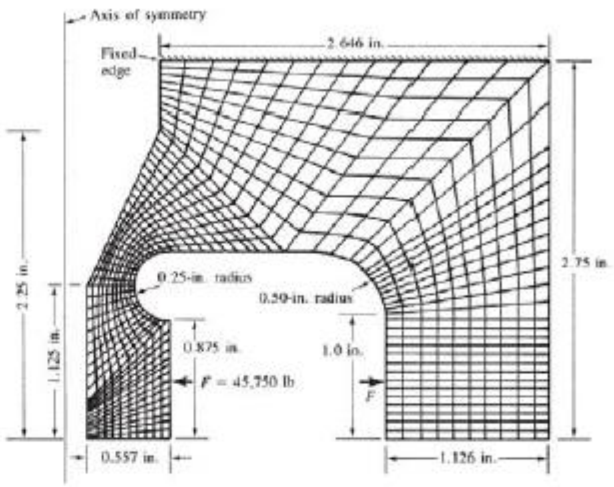


Applications

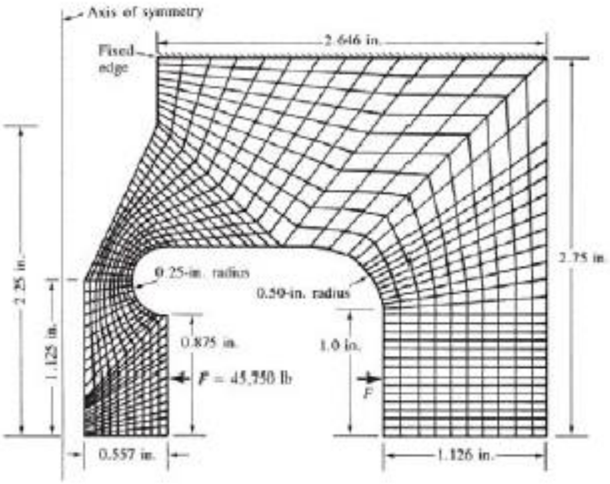
The figure below shows a finite element model of a high-strength steel die used in a thin-plastic-film-making process



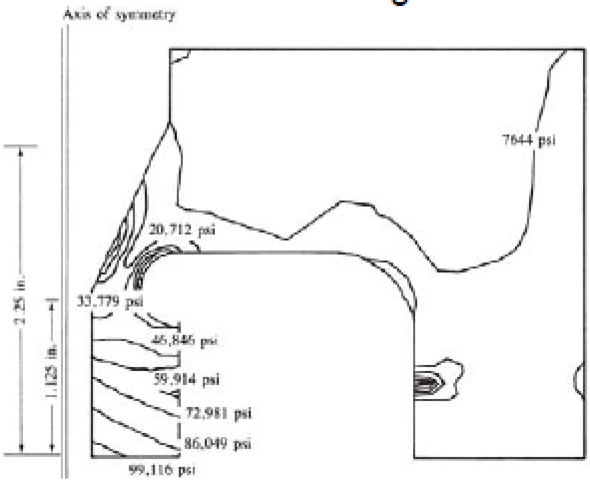
The die is an irregularly shaped disk. An axis of symmetry with respect to geometry and loading exists as shown.



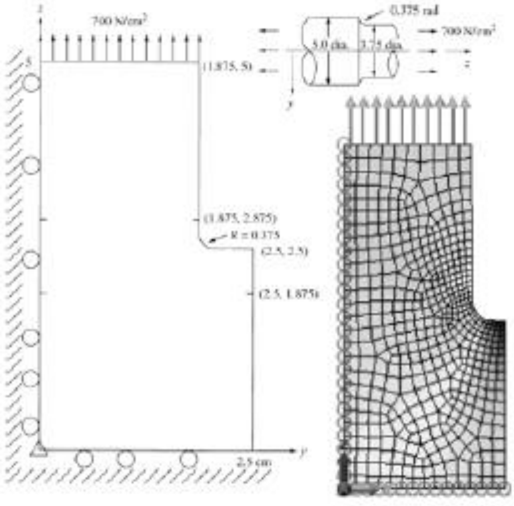
The die was modeled by using simple quadrilateral axisymmetric elements. The locations of high stress were of primary concern.



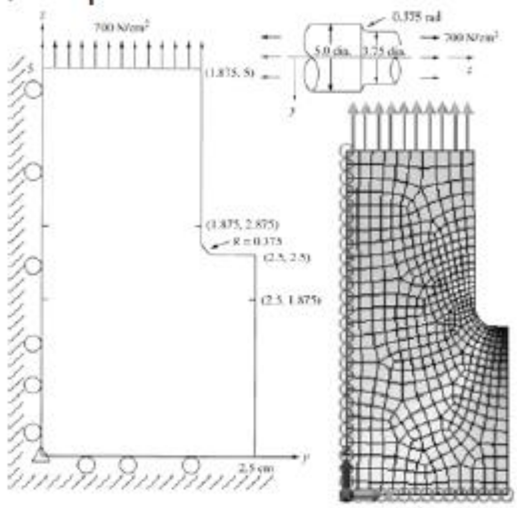
The figure shows a plot of the von Mises stress contours for the die. The von Mises (or equivalent, or effective) stress is often used as a failure criterion in design.



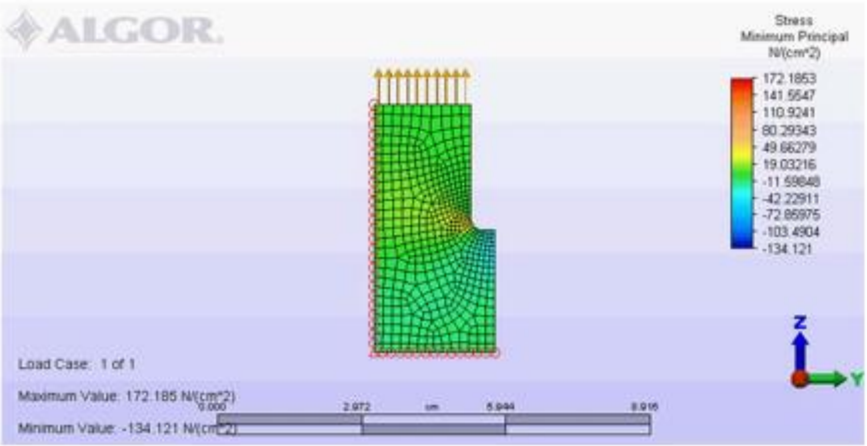
The figure shows a stepped 4130 steel shaft with a fillet radius subjected to an axial pressure of 1,000 psi in tension.

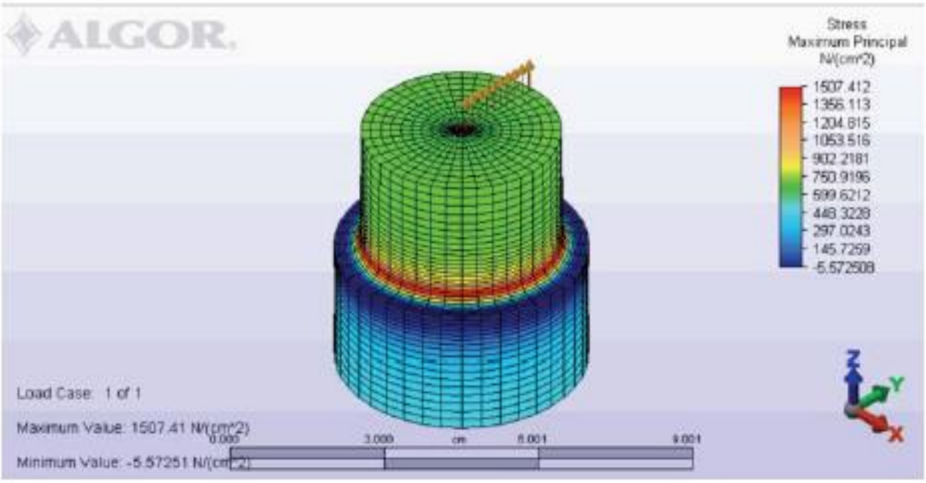


Fatigue analysis for reversed axial loading required an accurate stress concentration factor to be applied to the average axial stress of 1,000 psi.



Fatigue analysis for reversed axial loading required an accurate stress concentration factor to be applied to the average axial stress of 1,000 psi.





Numerical Integration in 2D

Summary:

a. Gauss integration on a 2D square domain
b. Integration on a triangular domain
c. Recommended order of integration
d. “Reduced” vs “Full” integration; concept of “spurious” zero energy modes/ “hour-glass” modes

1D quardrature rule recap

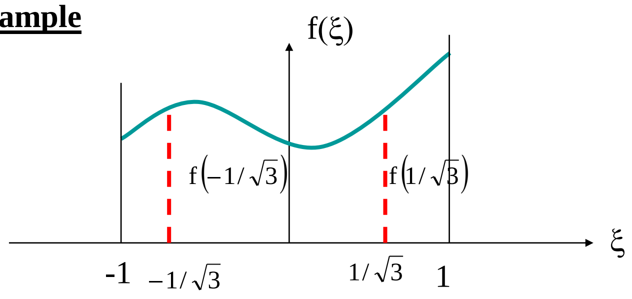
$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{M} W_i \ f(\xi_i)$$

Weight (W_i)   Integration point (ξ_i)

Choose the integration points **and** weights to maximize accuracy

Newton-Cotes	Gauss quadrature
1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘M-1’ 2. More expensive	1. ‘M’ integration points are necessary to exactly integrate a polynomial of degree ‘2M-1’ 2. Less expensive 3. Exponential convergence, error proportional to $\left(\frac{1}{2M}\right)^{2M}$

Example

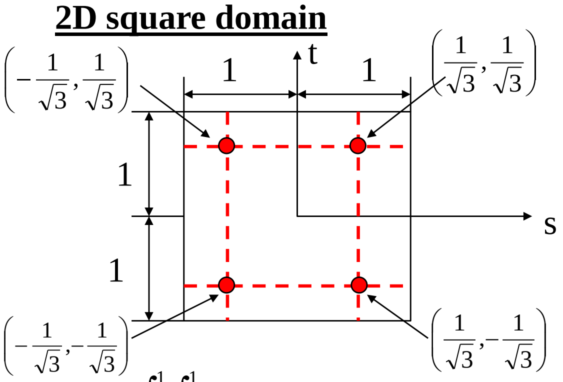


A 2-point Gauss quadrature rule

$$\int_{-1}^{1} f(\xi)\,d\xi \approx f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

is **exact** for a polynomial of degree 3 or less

2D square domain



$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dsdt$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\,dsdt$$

$$\approx \int_{-1}^{1}\left(\sum_{j=1}^{M} W_j f(s,t_j)\right) ds$$ Using 1D Gauss rule to integrate along 't'

$$\approx \sum_{i=1}^{M}\sum_{j=1}^{M} W_i W_j f(s_i,t_j)$$ Using 1D Gauss rule to integrate along 's'

$$= \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij} f(s_i,t_j)$$ Where $W_{ij} = W_i\, W_j$

The rule

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt \approx \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij}\, f(s_i, t_j)$$

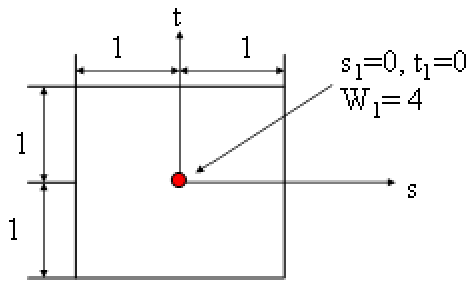
Uses **M^2 integration points** on a nonuniform grid inside the parent element and is **exact for a polynomial of degree (2M-1)** i.e.,

$$\int_{-1}^{1}\int_{-1}^{1} s^{\alpha} t^{\beta} dsdt \overset{exact}{=} \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij} s_i^{\alpha} t_j^{\beta} \quad for \;\; \alpha + \beta \le 2M - 1$$

A **M^2 –point rule is exact for a complete polynomial of degree (2M-1)**

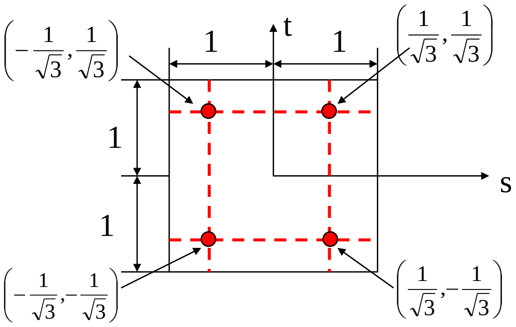
CASE I: M=1 (One-point GQ rule)

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt \approx 4 f(0,0)$$



is exact for a product of two linear polynomials

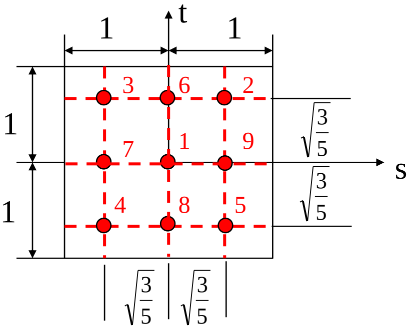
CASE II: M=2 (2x2 GQ rule)



$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij} f(s_i,t_j)$$
$$= f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

is exact for a product of two
cubic polynomials

CASE III: M=3 (3x3 GQ rule)



$$W_1 = \frac{64}{81},$$
$$W_2 = W_3 = W_4 = W_5 = \frac{25}{81}$$
$$W_6 = W_7 = W_8 = W_9 = \frac{40}{81}$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, ds dt \approx \sum_{i=1}^{3}\sum_{j=1}^{3} W_{ij} f(s_i,t_j)$$

is exact for a product of two 1D polynomials of degree 5

Examples

If *f(s,t)=1*

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = 4$$

A **1-point GQ scheme** is sufficient

If *f(s,t)=s*

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = 0$$

A **1-point GQ scheme** is sufficient

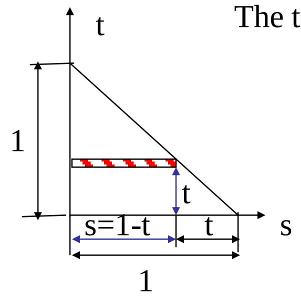
If $f(s,t)=s^2t^2$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = \frac{4}{9}$$

A **3x3 GQ scheme** is sufficient

2D Gauss quadrature for triangular domains

Remember that the **parent element** is a right angled triangle with unit sides



The type of integral encountered

$$I = \int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \mathrm{dsdt}$$

$$I = \int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \mathrm{dsdt}$$

$$\approx \sum_{IP=1}^{M} W_{IP}\, f_{IP}$$

Constraints on the weights

if f(s,t)=1

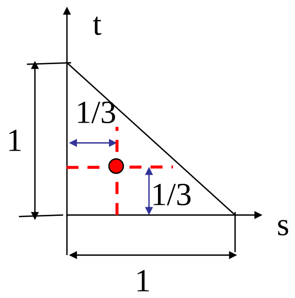
$$I = \int_{t=0}^{1}\int_{s=0}^{1-t} f(s,t)\ \text{dsdt} = \frac{1}{2}$$

$$= \sum_{IP=1}^{M} W_{IP}$$

$$\therefore \sum_{IP=1}^{M} W_{IP} = \frac{1}{2}$$

Example 1. A M=1 point rule is exact for a polynomial

$$f(s,t) \sim \quad 1$$
$$s \quad t$$



$$I \approx \frac{1}{2} f\left(\frac{1}{3},\frac{1}{3}\right)$$

Why?

Assume

$$f(s,t) = \alpha_1 + \alpha_2 s + \alpha_3 t$$

Then

$$\int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \ \mathrm{dsdt} = \frac{1}{2}\alpha_1 + \frac{1}{3!}\alpha_2 + \frac{1}{3!}\alpha_3$$

But

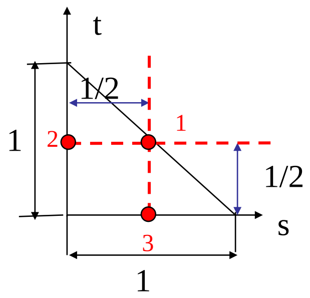
$$\int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \ \mathrm{dsdt} = W_1 f(s_1,t_1)$$

$$\therefore \frac{1}{2}\alpha_1 + \frac{1}{3!}\alpha_2 + \frac{1}{3!}\alpha_3 = W_1(\alpha_1 + \alpha_2 s_1 + \alpha_3 t_1)$$

Hence $W_1 = \frac{1}{2};\ W_1 s_1 = \frac{1}{3!};\ W_1 t_1 = \frac{1}{3!}$

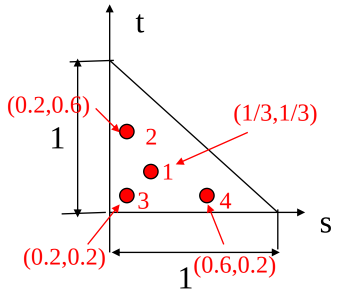
<u>Example 2.</u> A M=3 point rule is exact for a complete polynomial of degree 2 $f(s,t) \sim$

$$\begin{matrix} & 1 & \\ s & & t \\ s^2 & st & t^2 \end{matrix}$$



$$I \approx \frac{1}{6} f\left(\frac{1}{2},\frac{1}{2}\right) + \frac{1}{6} f\left(\frac{1}{2},0\right) + \frac{1}{6} f\left(0,\frac{1}{2}\right)$$

Example 4. A M=4 point rule is exact for a complete polynomial of degree 3



$$
\begin{array}{ccccccc}
f(s,t) \sim & & & 1 & & & \\
 & & s & & t & & \\
 & s^2 & & st & & t^2 & \\
s^3 & & s^2t & & st^2 & & t^3
\end{array}
$$

$$
\boxed{I \approx -\frac{27}{96} f\left(\frac{1}{3},\frac{1}{3}\right) + \frac{25}{96} f(0.2,0.6) + \frac{25}{96} f(0.2,0.2) + \frac{25}{96} f(0.6,0.2)}
$$

Recommended order of integration
"Finite Element Procedures"
by K. –J. Bathe

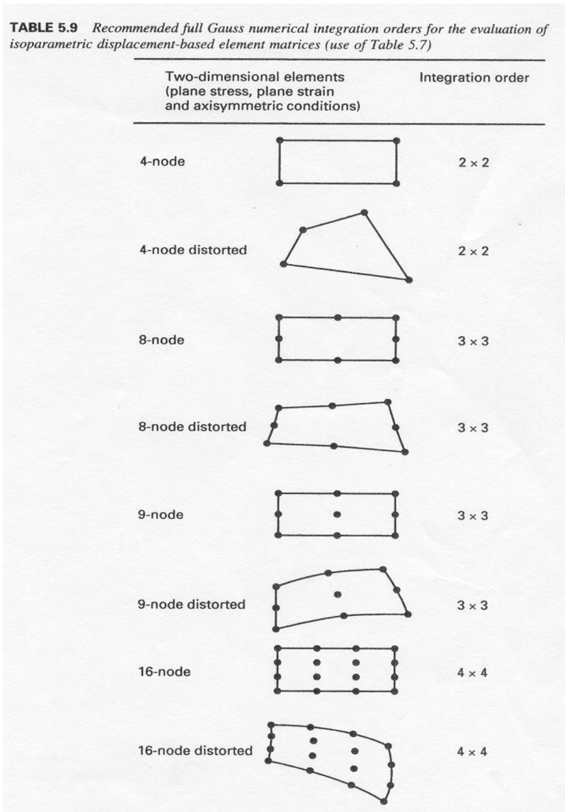


TABLE 5.9 *Recommended full Gauss numerical integration orders for the evaluation of isoparametric displacement-based element matrices (use of Table 5.7)*

Two-dimensional elements (plane stress, plane strain and axisymmetric conditions)	Integration order
4-node	2 × 2
4-node distorted	2 × 2
8-node	3 × 3
8-node distorted	3 × 3
9-node	3 × 3
9-node distorted	3 × 3
16-node	4 × 4
16-node distorted	4 × 4

"Reduced" vs "Full" integration

Full integration: Quadrature scheme sufficient to provide exact integrals of all terms of the stiffness matrix if the element is geometrically undistorted.

Reduced integration: An integration scheme of lower order than required by "full" integration.

Recommendation: Reduced integration is NOT recommended.

MRCET CAMPUS
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DEPARTMENT OF MECHANICAL ENGINEERING

INTRODUCTION TO FINITE ELEMENTS

MAPPED ELEMENT GEOMETRIES AND SHAPE FUNCTIONS: THE ISOPARAMETRIC FORMULATION

LIVE TO LEARN & LEARN TO SHARE

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DEPARTMENT OF MECHANICAL ENGINEERING

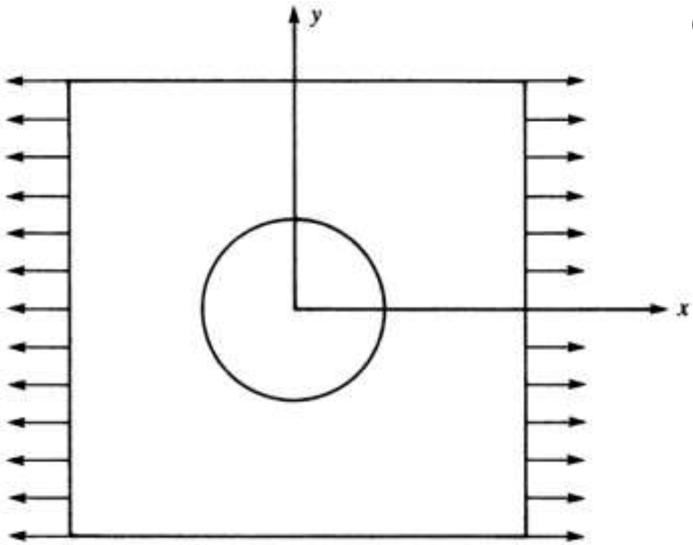
Reading assignment:

Chapter 10.1-10.3, 10.6 + Lecture notes

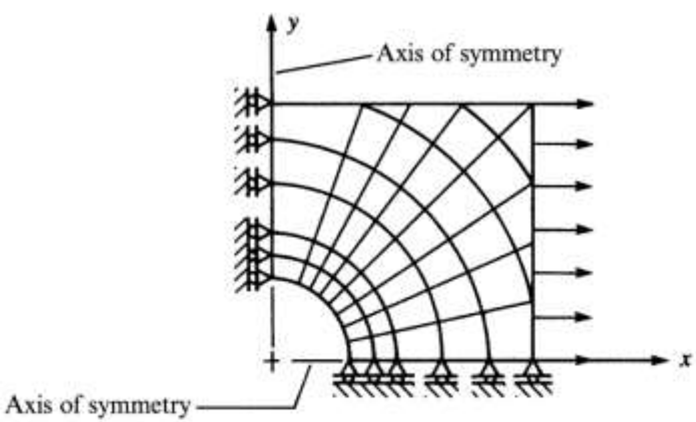
Summary:

- Concept of isoparametric mapping
- 1D isoparametric mapping
- Element matrices and vectors in 1D
- 2D isoparametric mapping : rectangular parent elements
- 2D isoparametric mapping : triangular parent elements
- Element matrices and vectors in 2D

For complex geometries

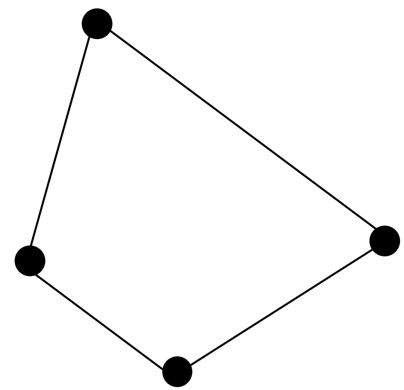


(a) Plate with hole under plane stress

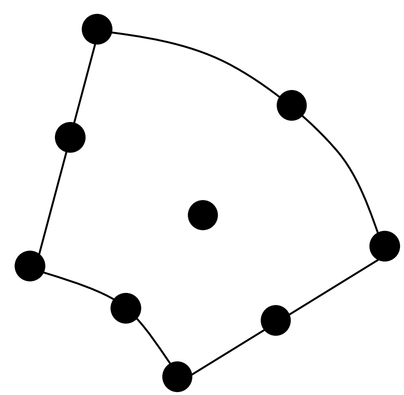


(b) Finite element model of one-quarter of the plate

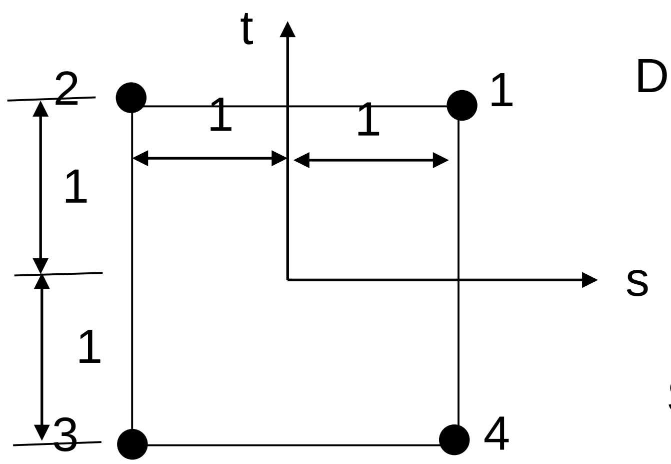
General quadrilateral elements



Elements with curved sides



Consider a special 4-noded rectangle in its local coordinate system (s,t)



Displacement interpolation

$$u \approx N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$
$$v \approx N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

Shape functions in local coord system

$$N_1(s,t) = \frac{1}{4}(1+s)(1+t)$$

$$N_2(s,t) = \frac{1}{4}(1-s)(1+t)$$

$$N_3(s,t) = \frac{1}{4}(1-s)(1-t)$$

$$N_4(s,t) = \frac{1}{4}(1+s)(1-t)$$

Recall that

$$N_1 + N_2 + N_3 + N_4 = 1$$

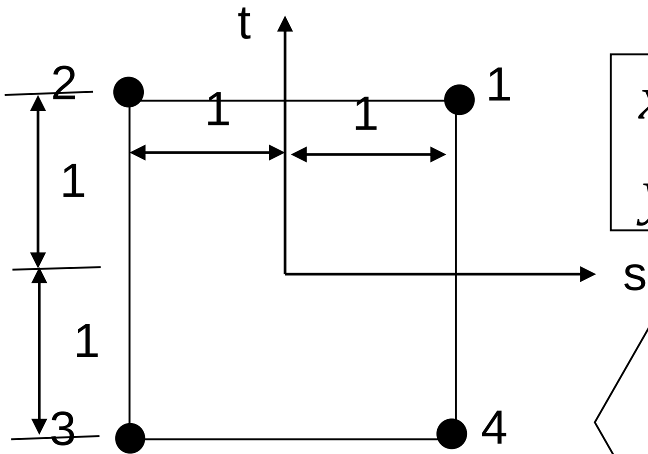
Rigid body modes

$$N_1 s_1 + N_2 s_2 + N_3 s_3 + N_4 s_4 = s$$

$$N_1 t_1 + N_2 t_2 + N_3 t_3 + N_4 t_4 = t$$

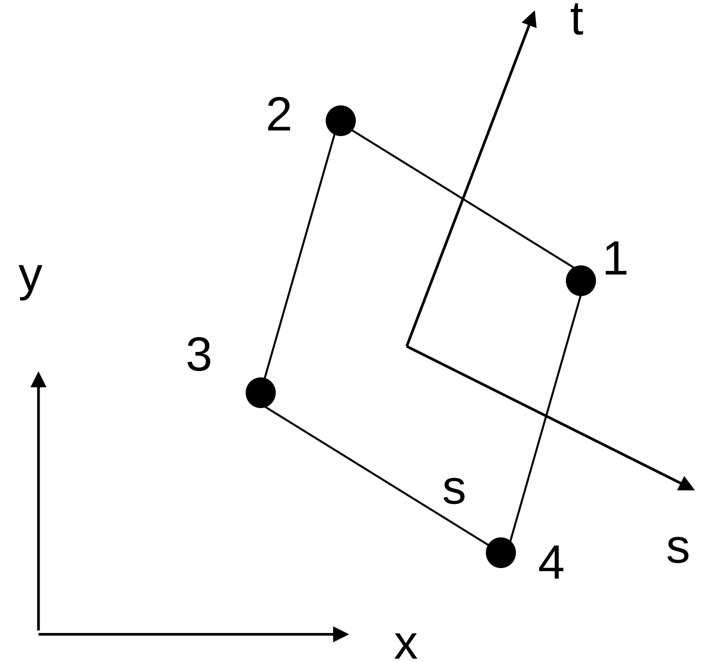
Constant strain states

Goal is to map this element from local coords to a general quadrilateral element in global coord system



$$x = x(s,t)$$
$$y = y(s,t)$$

$$s = s(x,y)$$
$$t = t(x,y)$$



Local coordinate system

Global coordinate system

$$N_1(s,t) \qquad\qquad N_1(x,y)$$

$$N_1(s,t) = N_1(s(x,y),t(x,y)) \equiv N_1(x,y)$$

In the mapped coordinates, the shape functions need to satisfy

1. Kronecker delta property

Then

$$N_i = \begin{cases} 1 \; at \; node \; i \\ 0 \; at \; all \; other \; nodes \end{cases}$$

2. Polynomial completeness

$$\sum_i N_i = 1$$
$$\sum_i N_i x_i = x$$
$$\sum_i N_i y_i = y$$

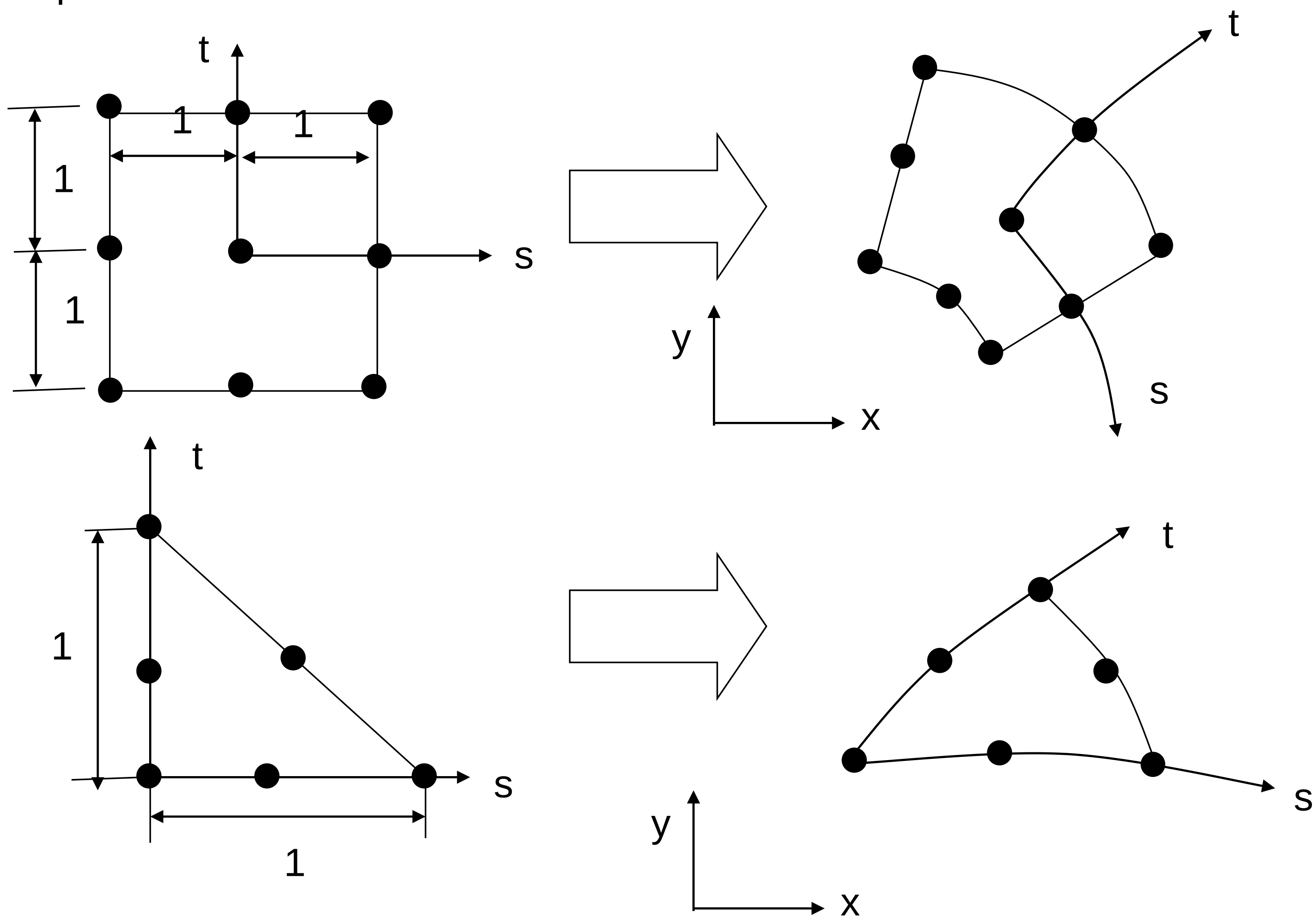
The relationship

$$x = \sum_i N_i(s,t)x_i$$
$$y = \sum_i N_i(s,t)y_i$$

Provides the required mapping from the local coordinate system
To the global coordinate system and is known as **isoparametric mapping**

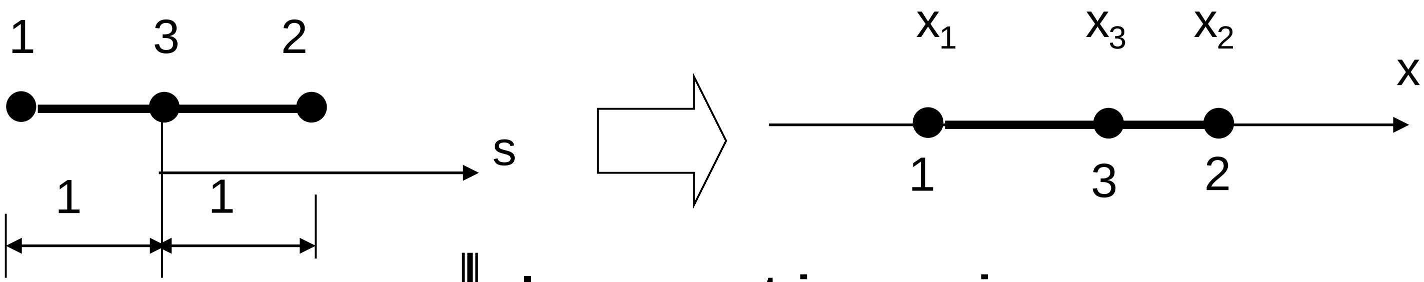
(s,t): isoparametric coordinates
(x,y): global coordinates

Examples



1D isoparametric mapping

3 noded (quadratic) element



Isoparametric mapping

$$x = \sum_{i=1}^{3} N_i(s)x_i$$

$$\Rightarrow x = -\frac{s(1-s)}{2}x_1 + \frac{s(1+s)}{2}x_2 + \left(1-s^2\right)x_3$$

Local (isoparametric) coordinates

$$N_1(s) = -\frac{s(1-s)}{2}$$

$$N_2(s) = \frac{s(1+s)}{2}$$

$$N_3(s) = 1-s^2$$

NOTES

1. Given a point in the isoparametric coordinates, I can obtain the corresponding mapped point in the global coordinates using the isoparametric mapping equation

$$x = -\frac{s(1-s)}{2}x_1 + \frac{s(1+s)}{2}x_2 + (1-s^2)x_3$$

$$At\ s = -1;\quad x = x_1$$

$$At\ s = 0;\quad x = x_3$$

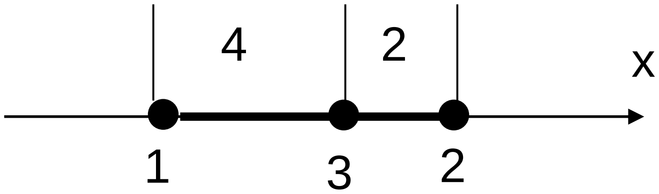
$$At\ s = 1;\quad x = x_2$$

Question

x=? at s=0.5?

2. The shape functions themselves get mapped
In the isoparametric coordinates (s) they are polynomials.
In the global coordinates (x) they are in general nonpolynomials
Lets consider the following numerical example

$$x_1 = 0;\ x_2 = 6;\ x_3 = 4$$



Isoparametric mapping x(s)

$$x = -\frac{s(1-s)}{2}x_1 + \frac{s(1+s)}{2}x_2 + (1-s^2)x_3$$
$$= -\frac{s(1-s)}{2}0 + \frac{s(1+s)}{2}6 + (1-s^2)4$$
$$= 4 + 3s - s^2$$

Simple polynomial

Inverse mapping s(x)

$$s = \frac{3-\sqrt{25-4x}}{2}$$

Complicated function

Now lets compute the **shape functions** in the global coordinates

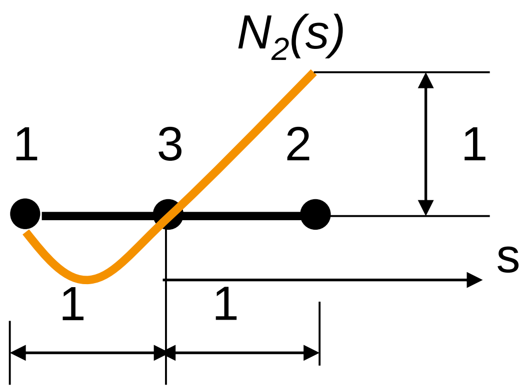
$$N_2(s) = \frac{s(1+s)}{2}$$

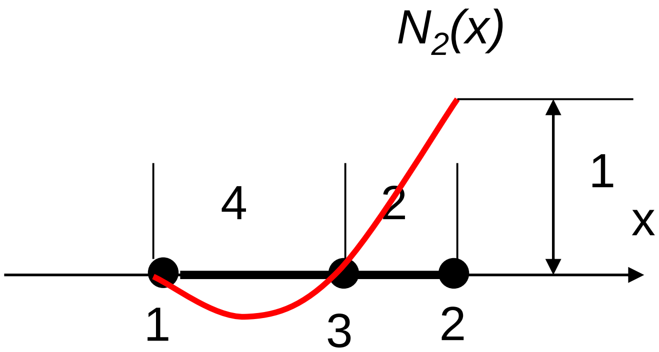
$$= \frac{1}{2}\left(\frac{3-\sqrt{25-4x}}{2}\right)\left(1+\frac{3-\sqrt{25-4x}}{2}\right)$$

$$= \frac{1}{2}\left(10 - x - 2\sqrt{25-4x}\right)$$

$$= N_2(x)$$

Now lets compute the **shape functions** in the global coordinates





$N_2(s)$ is a simple polynomial

$$N_2(s) = \frac{s(1+s)}{2}$$

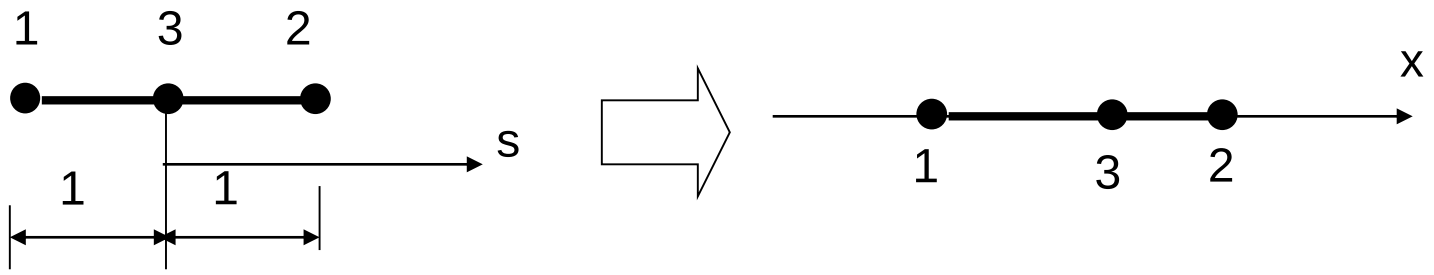
$N_2(x)$ is a complicated function

$$N_2(x) = \frac{1}{2}\left(10 - x - 2\sqrt{25-4x}\right)$$

However, thanks to isoparametric mapping, we always ensure

1. Knonecker delta property
2. Rigid body and constant strain states

Element matrices and vectors for a mapped 1D bar element



Displacement interpolation $$u \approx N_1 u_1 + N_2 u_2 + N_3 u_3 = \underline{N}\,\underline{d}$$

Strain-displacement relation $$\varepsilon = \frac{du}{dx} \approx \frac{dN_1}{dx} u_1 + \frac{dN_2}{dx} u_2 + \frac{dN_3}{dx} u_3 = \underline{B}\,\underline{d}$$

Stress $$\sigma = E\varepsilon = E\underline{B}\,\underline{d}$$

The strain-displacement matrix $$\underline{B} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix}$$

The only difference from before is that the shape functions are in the isoparametric coordinates

$$N_1(s) = -\frac{s(1-s)}{2}$$

$$N_2(s) = \frac{s(1+s)}{2}$$

$$N_3(s) = 1 - s^2$$

We know the isoparametric mapping

$$\boxed{x = \sum_{i=1}^{3} N_i(s) x_i}$$

And we will **not** try to obtain explicitly the inverse map.

How to compute the B matrix?

Using chain rule

$$\frac{dN_i(s)}{dx} = \frac{dN_i(s)}{ds}\frac{ds}{dx} \qquad (*)$$

Do I know $\dfrac{dN_i(s)}{ds}$?

Do I know $\dfrac{ds}{dx}$?

I know $x = \sum_{i=1}^{3} N_i(s) x_i$

Hence $\dfrac{dx}{ds} = \sum_{i=1}^{3} \dfrac{dN_i(s)}{ds} x_i \equiv J \quad (Jacobian\ of\ mapping)$

From (*)

$$\boxed{\frac{dN_i(s)}{dx} = \frac{1}{J}\frac{dN_i(s)}{ds}}$$

What does the Jacobian do?

$$dx = J\ ds$$

Maps a differential element from the isoparametric coordinates to the global coordinates

The strain-displacement matrix

$$\underline{B} = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix}$$

$$= \frac{1}{J}\begin{bmatrix} \frac{dN_1}{ds} & \frac{dN_2}{ds} & \frac{dN_3}{ds} \end{bmatrix}$$

For the 3-noded element

$$J = \sum_{i=1}^{3} \frac{dN_i(s)}{ds} x_i = \frac{2s-1}{2} x_1 + \frac{2s+1}{2} x_2 - 2s x_3$$

$$\underline{B} = \frac{1}{J}\begin{bmatrix} \frac{2s-1}{2} & \frac{2s+1}{2} & -2s \end{bmatrix}$$

The element stiffness matrix

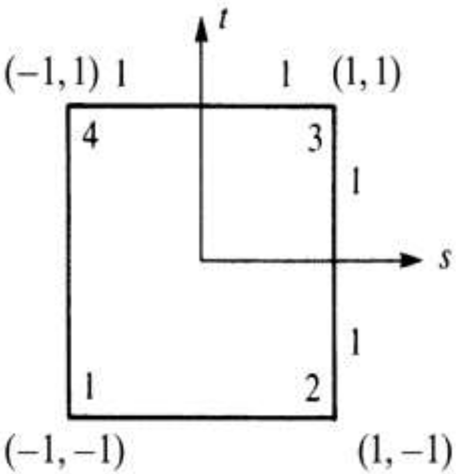
$$\underline{k} = \int_{x_1}^{x_2} EA\, \underline{B}^T\, \underline{B}\, dx$$

$$= \int_{-1}^{1} EA\, \underline{B}^T\, \underline{B}\; Jds \quad \because dx = Jds$$

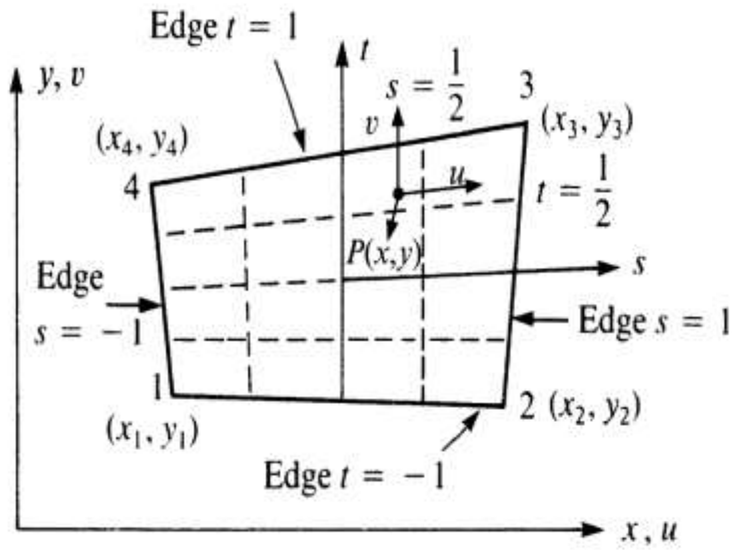
NOTES

1. The integral on ANY element in the global coordinates in now an integral from -1 to 1 in the local coodinates
2. The jacobian is a function of ‘s’ in general and enters the integral. The specific form of ‘J’ is determined by the values of x_1, x_2 and x_3. Gaussian quadrature is used to evaluate the stiffness matrix
3. In general B is a vector of rational functions in ‘s’

Isoparametric mapping in 2D: Rectangular parent elements



(a)



(b)

Parent element

Mapped element in global coordinates

Isoparametric mapping

$$x = \sum_i N_i(s,t)x_i$$

$$y = \sum_i N_i(s,t)y_i$$

Shape functions of parent element in isoparametric coordinates

$$N_1(s,t) = \frac{1}{4}(1-s)(1-t)$$

$$N_2(s,t) = \frac{1}{4}(1+s)(1-t)$$

$$N_3(s,t) = \frac{1}{4}(1+s)(1+t)$$

$$N_4(s,t) = \frac{1}{4}(1-s)(1+t)$$

Isoparametric mapping

$$x = \sum_i N_i(s,t)x_i$$

$$y = \sum_i N_i(s,t)y_i$$

NOTES:

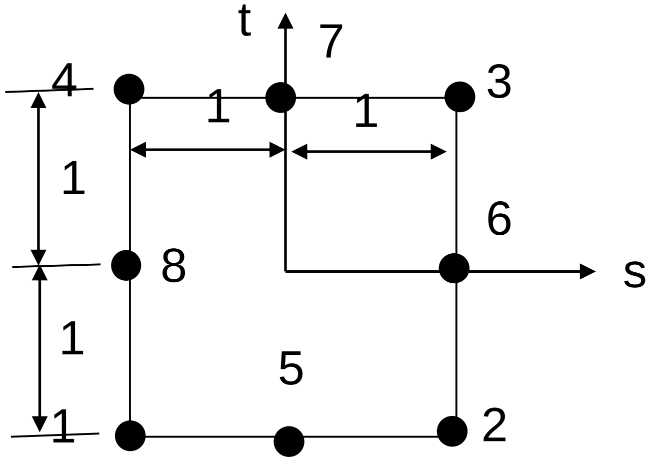
1. The isoparametric mapping provides the map (s,t) to (x,y) , i.e., if you are given a point (s,t) in isoparametric coordinates, then you can compute the coordinates of the point in the (x,y) coordinate system using the equations

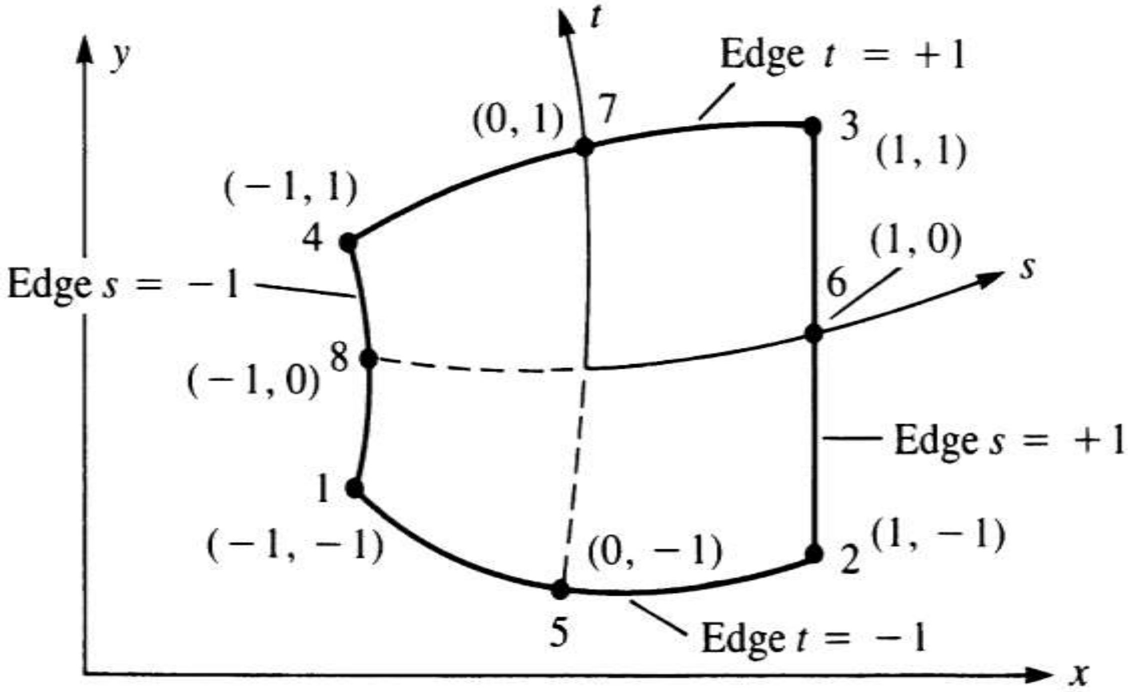
$$x = \sum_i N_i(s,t)\, x_i$$

$$y = \sum_i N_i(s,t)\, y_i$$

2. The inverse map will never be explicitly computed.

8-noded Serendipity element





8-noded Serendipity element: element shape functions in isoparametric coordinates

$$N_1(s,t) = \frac{1}{4}(1-s)(1-t)(-s-t-1) \qquad N_5(s,t) = \frac{1}{2}(1-t)(1+s)(1-s)$$

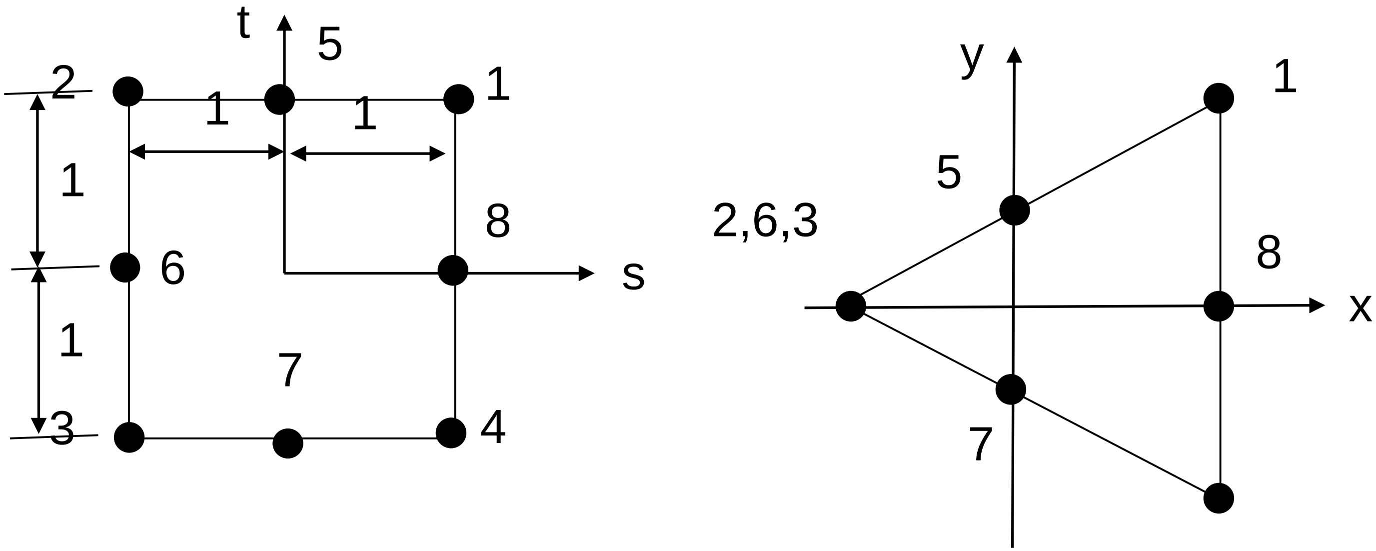
$$N_2(s,t) = \frac{1}{4}(1+s)(1-t)(s-t-1) \qquad N_6(s,t) = \frac{1}{2}(1-t)(1+t)(1+s)$$

$$N_3(s,t) = \frac{1}{4}(1+s)(1+t)(s+t-1) \qquad N_7(s,t) = \frac{1}{2}(1+t)(1+s)(1-s)$$

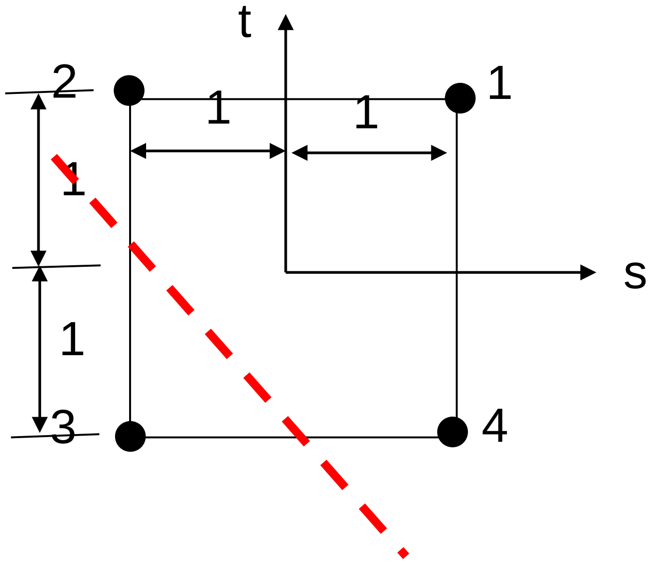
$$N_4(s,t) = \frac{1}{4}(1-s)(1+t)(-s+t-1) \qquad N_8(s,t) = \frac{1}{2}(1-s)(1+t)(1-t)$$

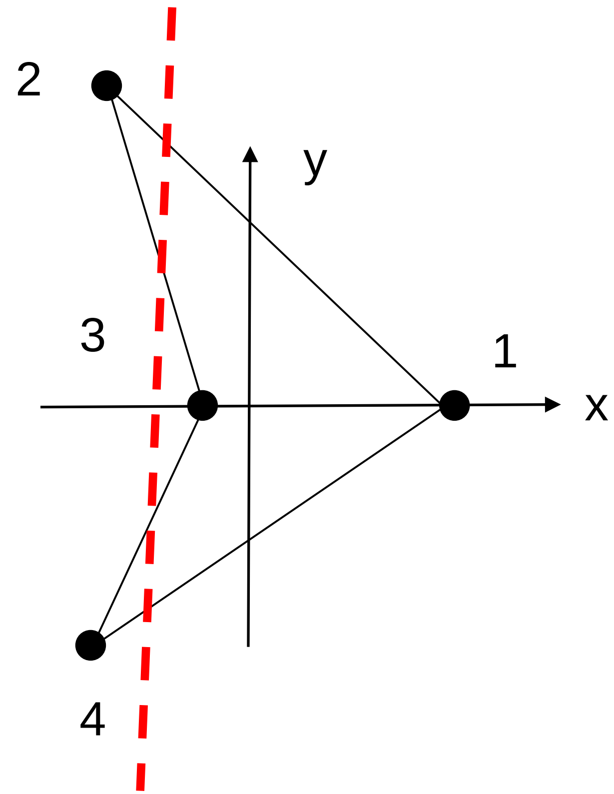
NOTES

1. $N_i(s,t)$ is a simple polynomial in s and t. But $N_i(x,y)$ is a complex function of x and y.
2. The element edges can be curved in the mapped coordinates
3. A "midside" node in the parent element may not remain as a midside node in the mapped element. An extreme example

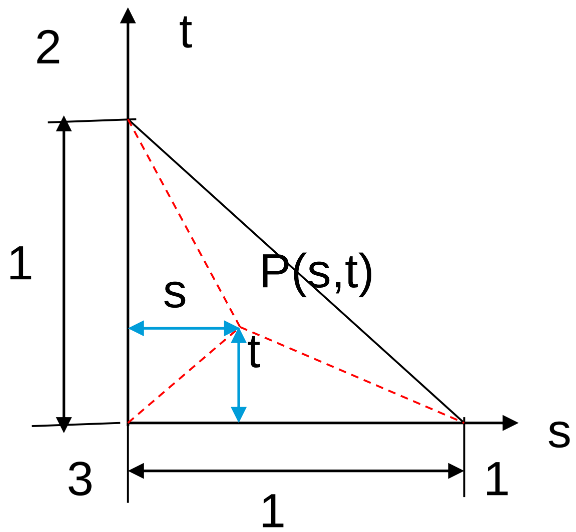


4. Care must be taken to ensure **UNIQUENESS** of mapping





Isoparametric mapping in 2D: Triangular parent elements



Parent element: a right angled triangles with arms of unit length

Key is to link the isoparametric coordinates with the area coordinates

$$\Delta_{123} = \frac{1}{2}$$

$$\Delta_{P31} = \frac{t}{2}$$

$$\Delta_{P23} = \frac{s}{2}$$

$$\Delta_{P12} = \frac{1}{2}(1-s-t)$$

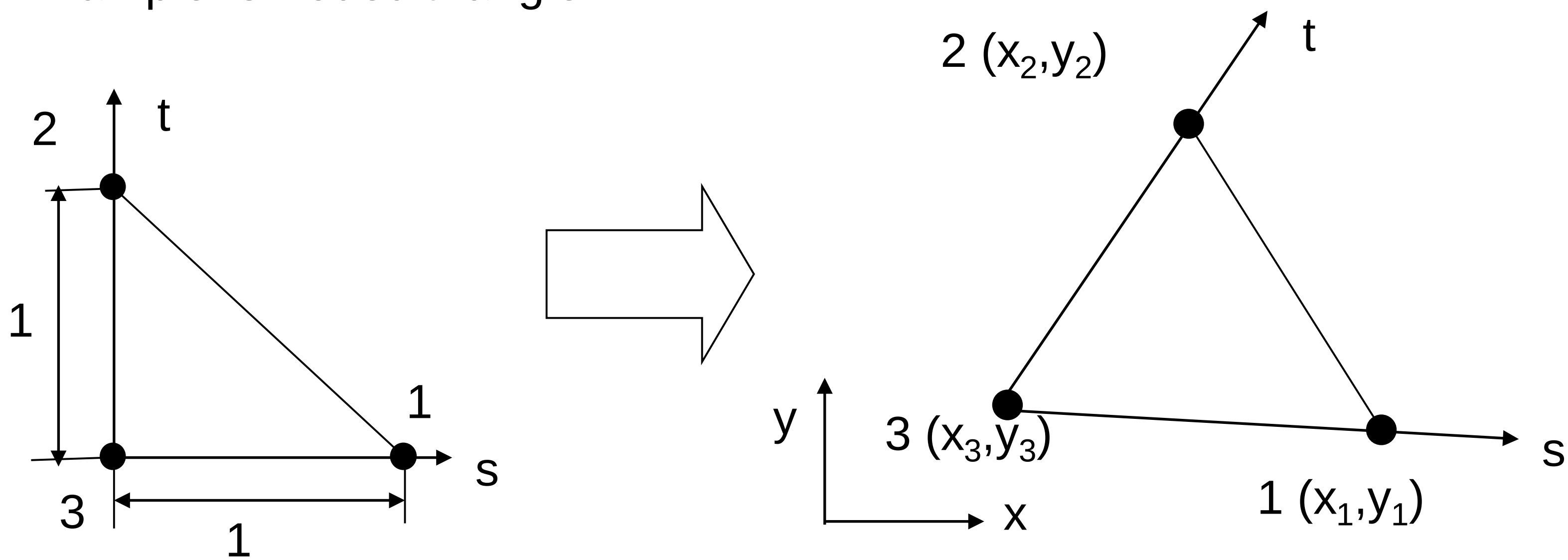
$$L_1 = \frac{\Delta_{P23}}{\Delta_{123}} = s$$

$$L_2 = \frac{\Delta_{P31}}{\Delta_{123}} = t$$

$$L_3 = \frac{\Delta_{P12}}{\Delta_{123}} = 1-s-t$$

Now replace L_1, L_2, L_3 in the formulas for the shape functions of triangular elements to obtain the shape functions in terms of (s,t)

Example: 3-noded triangle



Parent shape functions

Isoparametric mapping

$$N_1 = s$$
$$N_2 = t$$
$$N_3 = 1 - s - t$$

$$x = N_1(s,t)x_1 + N_2(s,t)x_2 + N_3(s,t)x_3$$
$$y = N_1(s,t)y_1 + N_2(s,t)y_2 + N_3(s,t)y_3$$

Element matrices and vectors for a mapped 2D element

Recall: For each element

Displacement approximation

$$\underline{u} = \underline{N}\,\underline{d}$$

Strain approximation

$$\underline{\varepsilon} = \underline{B}\,\underline{d}$$

Stress approximation

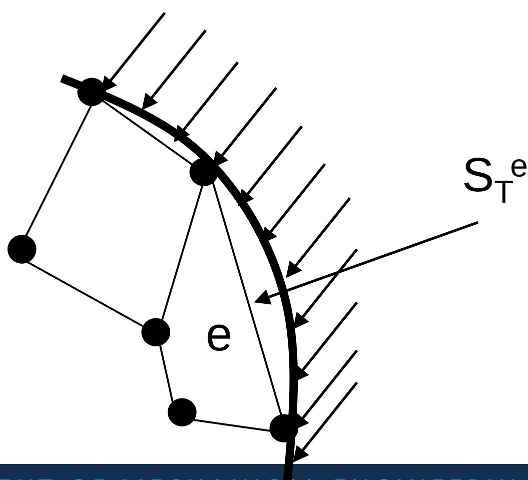
$$\underline{\sigma} = \underline{D}\,\underline{B}\,\underline{d}$$

Element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\,\underline{B}\,dV$$

Element nodal load vector

$$\underline{f} = \underbrace{\int_{V^e} \underline{N}^T \underline{X}\,dV}_{\underline{f}_b} + \underbrace{\int_{S_T^e} \underline{N}^T \underline{T}_S\,dS}_{\underline{f}_S}$$



In isoparametric formulation

1. Shape functions first expressed in (s,t) coordinate system

i.e., $N_i(s,t)$

2. The **isoparamtric mapping** relates the (s,t) coordinates with the global coordinates (x,y)

$$x = \sum_i N_i(s,t)x_i$$

$$y = \sum_i N_i(s,t)y_i$$

3. It is laborious to find the inverse map s(x,y) and t(x,y) and we do not do that. Instead we compute the integrals in the domain of the parent element.

NOTE

1. $N_i(s,t)$ s are already available as simple polynomial functions
2. The first task is to find $\frac{\partial N_i}{\partial x}$ and $\frac{\partial N_i}{\partial y}$

Use chain rule

$$\frac{\partial N_i(x,y)}{\partial s} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial N_i(x,y)}{\partial t} = \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial t}$$

In matrix form

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial s} \\ \dfrac{\partial N_i}{\partial t} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial s} & \dfrac{\partial y}{\partial s} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial y}{\partial t} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix}$$

Can be computed

This is known as the **Jacobian** matrix (J) for the mapping (s,t) → (x,y)

We want to compute these for the B matrix

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial s} \\ \dfrac{\partial N_i}{\partial t} \end{Bmatrix} = \underline{J} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = \underline{J}^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial s} \\ \dfrac{\partial N_i}{\partial t} \end{Bmatrix}$$

How to compute the Jacobian matrix?
Start from

$$x = \sum_i N_i(s,t) x_i$$

$$y = \sum_i N_i(s,t) y_i$$

$$\frac{\partial x}{\partial s} = \sum_i \frac{\partial N_i(s,t)}{\partial s} x_i \quad ; \quad \frac{\partial x}{\partial t} = \sum_i \frac{\partial N_i(s,t)}{\partial t} x_i$$

$$\frac{\partial y}{\partial s} = \sum_i \frac{\partial N_i(s,t)}{\partial s} y_i \; ; \; \frac{\partial y}{\partial t} = \sum_i \frac{\partial N_i(s,t)}{\partial t} y_i$$

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_i \frac{\partial N_i(s,t)}{\partial s} x_i & \sum_i \frac{\partial N_i(s,t)}{\partial s} y_i \\ \sum_i \frac{\partial N_i(s,t)}{\partial t} x_i & \sum_i \frac{\partial N_i(s,t)}{\partial t} y_i \end{bmatrix}$$

Need to ensure that **det($\underline{J}$) > 0** for one-to-one mapping

3. Now we need to transform the integrals from (x,y) to (s,t)

Case 1. Volume integrals

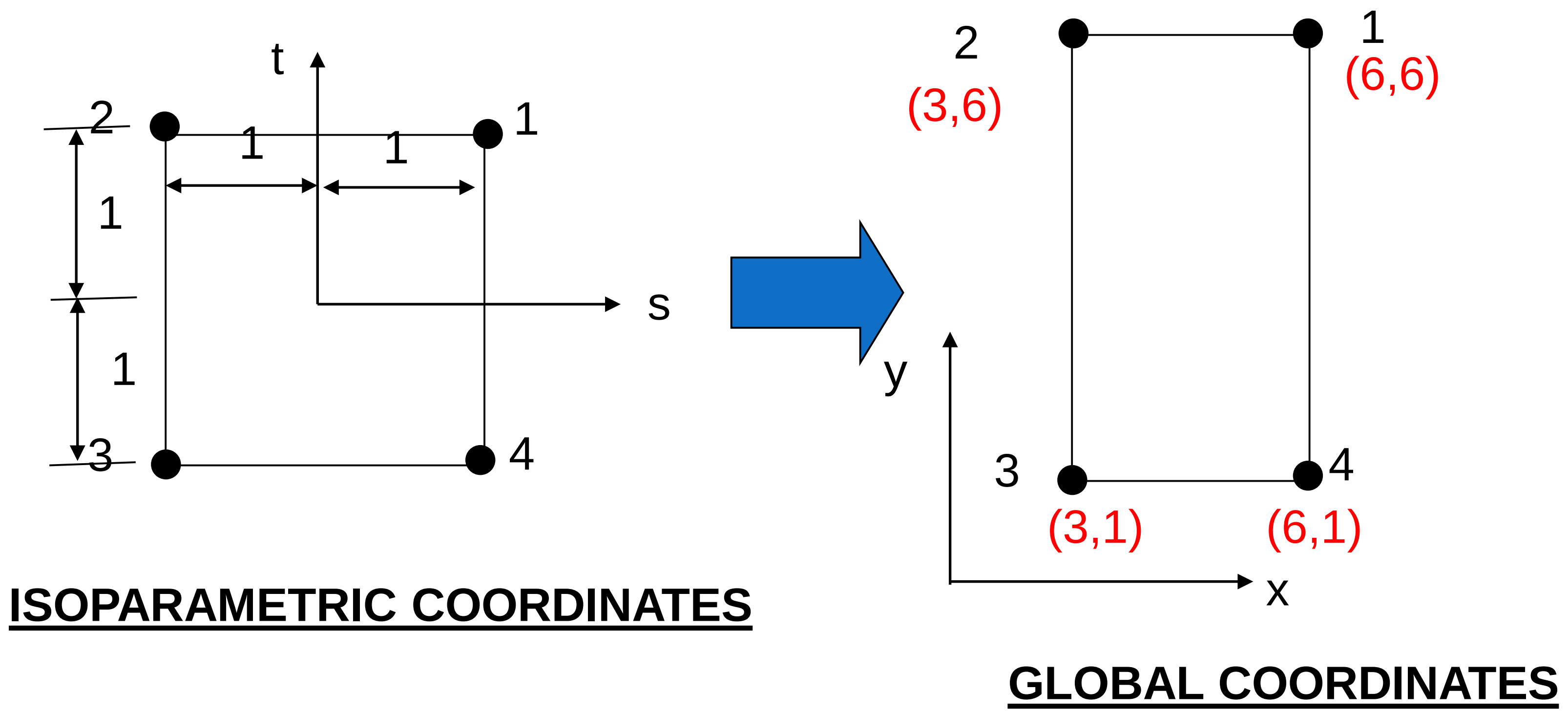
$$\int_{V^e} f(s,t)\, dV = \int_{A^e} f(s,t)\, h\, dA = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, h \det(\underline{J})\, ds dt$$

h=thickness of element

This depends on the key result

$$dA = dx dy = \det(\underline{J})\, ds dt$$

Problem: Consider the following isoparamteric map



Displacement interpolation

$$u \approx N_1 u_1 + N_2 u_2 + N_3 u_3 + N_4 u_4$$

$$v \approx N_1 v_1 + N_2 v_2 + N_3 v_3 + N_4 v_4$$

Shape functions in **isoparametric coord** system

$$N_1(s,t) = \frac{1}{4}(1+s)(1+t)$$

$$N_2(s,t) = \frac{1}{4}(1-s)(1+t)$$

$$N_3(s,t) = \frac{1}{4}(1-s)(1-t)$$

$$N_4(s,t) = \frac{1}{4}(1+s)(1-t)$$

The **isoparamtric map**

$$x = N_1(s,t)x_1 + N_2(s,t)x_2 + N_3(s,t)x_3 + N_4(s,t)x_4$$
$$y = N_1(s,t)y_1 + N_2(s,t)y_2 + N_3(s,t)y_3 + N_4(s,t)y_4$$

$$\Rightarrow x = \frac{(1+s)(1+t)}{4}6 + \frac{(1-s)(1+t)}{4}3 + \frac{(1-s)(1-t)}{4}3 + \frac{(1+s)(1-t)}{4}6$$
$$= \frac{3(1+s)}{2}$$
$$y = \frac{(1+s)(1+t)}{4}6 + \frac{(1-s)(1+t)}{4}6 + \frac{(1-s)(1-t)}{4}1 + \frac{(1+s)(1-t)}{4}1$$
$$= \frac{7+5t}{2}$$

In this case, we may compute the inverse map, but we will **NOT** do that!

The **Jacobian matrix**

$$\underline{J} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{5}{2} \end{bmatrix} \quad \text{since} \quad x = \frac{3(1+s)}{2}, \quad y = \frac{7+5t}{2}$$

NOTE: The diagonal terms are due to stretching of the sides along the x-and y-directions. The off-diagonal terms are zero because the element does not shear.

$$\underline{J}^{-1} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/5 \end{bmatrix} \quad \text{and} \quad \det(\underline{J}) = \frac{15}{4}$$

Hence, if I were to compute the first column of the **B** matrix along the positive x-direction

$$\mathbf{B}_1 = \begin{Bmatrix} \dfrac{\partial N_1}{\partial x} \\ 0 \\ \dfrac{\partial N_1}{\partial y} \end{Bmatrix}$$

I would use

$$\begin{Bmatrix} \dfrac{\partial N_1}{\partial x} \\ \dfrac{\partial N_1}{\partial y} \end{Bmatrix} = \underline{J}^{-1} \begin{Bmatrix} \dfrac{\partial N_1}{\partial s} \\ \dfrac{\partial N_1}{\partial t} \end{Bmatrix} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/5 \end{bmatrix} \begin{Bmatrix} \dfrac{1+t}{4} \\ \dfrac{1+s}{4} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1+t}{6} \\ \dfrac{1+s}{10} \end{Bmatrix}$$

Hence

$$\mathbf{B}_1 = \begin{Bmatrix} \dfrac{\partial N_1}{\partial x} \\ 0 \\ \dfrac{\partial N_1}{\partial y} \end{Bmatrix} = \begin{Bmatrix} \dfrac{1+t}{6} \\ 0 \\ \dfrac{1+s}{10} \end{Bmatrix}$$

The element stiffness matrix

$$\underline{k} = \int_{V^e} \underline{B}^T \underline{D}\, \underline{B}\, dV = \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T \underline{D}\, \underline{B}\, \det(\underline{J}) h\, ds\, dt$$

DEPARTMENT OF MECHANICAL ENGINEERING

INTRODUCTION TO FINITE ELEMENTS

NUMERICAL INTEGRATION IN 1D

DEPARTMENT OF MECHANICAL ENGINEERING

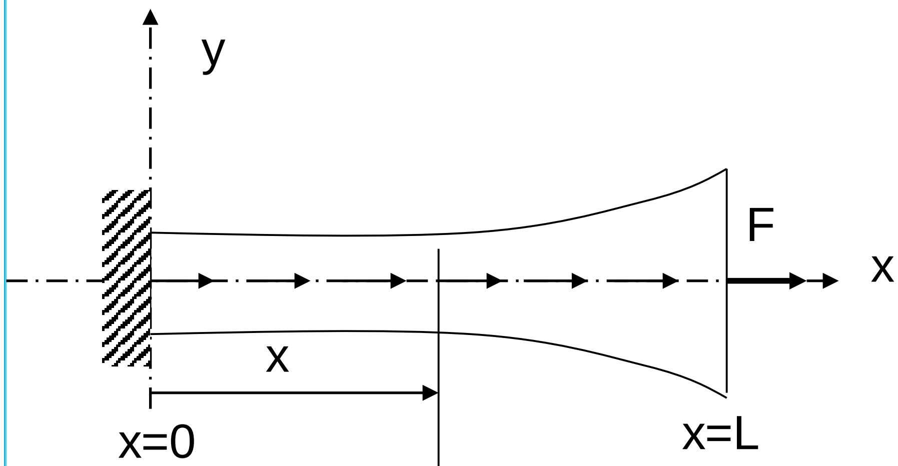
Reading assignment:

Lecture notes, Logan 10.4

Summary:

- **Newton-Cotes Integration Schemes**
- **Gaussian quadrature**

Axially loaded elastic bar



A(x) = cross section at x
b(x) = body force distribution (force per unit length)
E(x) = Young's modulus

For each element

Element stiffness matrix

$$\underline{k} = \int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx$$

$$k_{ij} = \int_{x_1}^{x_2} B_i E B_j \; A dx$$

where $$B_i = \frac{dN_i(x)}{dx}$$

Only for a linear finite element

$$\int_{x_1}^{x_2} \underline{B}^T E \underline{B} \; A dx = \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{x_1}^{x_2} AEdx = \left(\int_{x_1}^{x_2} AEdx \right) \frac{1}{(x_2 - x_1)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Element nodal load vector

$$\underline{f_b} = \int_{x_1}^{x_2} \underline{N}^T \; b \, dx$$

$$f_{bi} = \int_{x_1}^{x_2} N_i \; b \, dx$$

Question: How do we compute these integrals using a computer?

Any integral from x_1 to x_2 can be transformed to the following integral on (-1, 1)

$$I = \int_{-1}^{1} f(\xi)\, d\xi$$

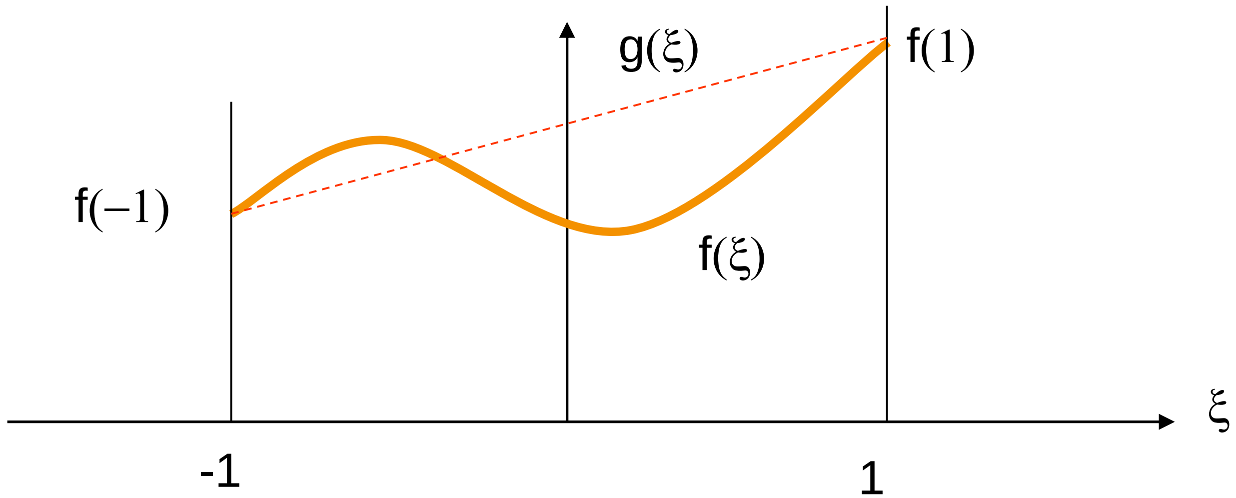
Use the following change of variables

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$

Goal: Obtain a good approximate value of this integral
1. Newton-Cotes Schemes (trapezoidal rule, Simpson's rule, etc)
2. Gauss Integration Schemes

NOTE: Integration schemes in 1D are referred to as "quadrature rules"

Trapezoidal rule: Approximate the function f(ξ) by a straight line g(ξ) that passes through the end points and integrate the straight line



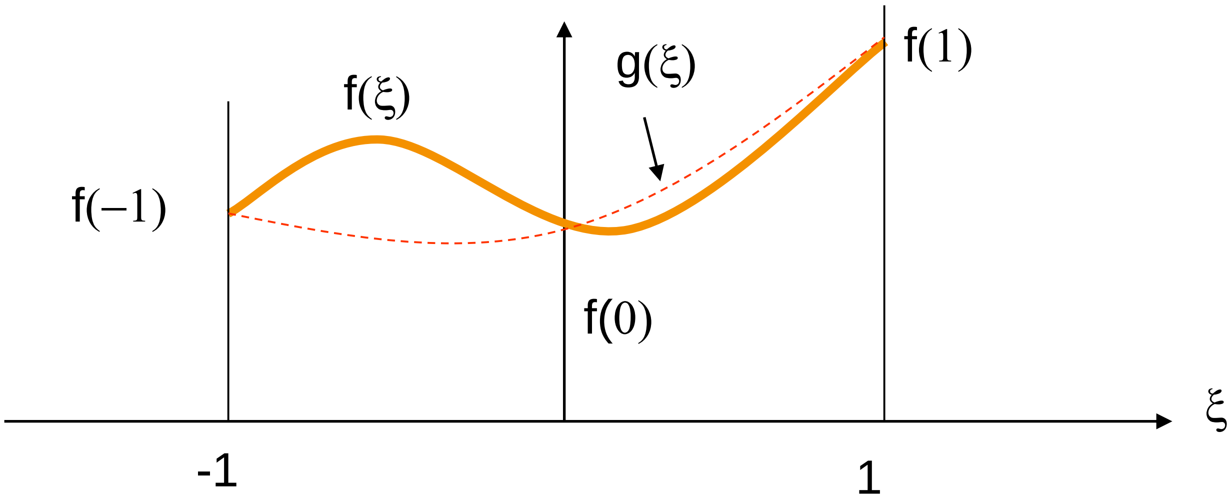
$$g(\xi) = \frac{1-\xi}{2} f(-1) + \frac{1+\xi}{2} f(1)$$

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \int_{-1}^{1} g(\xi)\, d\xi = f(1) + f(-1)$$

- Requires the function f(x) to be evaluated at 2 points (-1, 1)
- Constants and linear functions are exactly integrated
- Not good for quadratic and higher order polynomials

How can I make this better?

Simpson's rule: Approximate the function f(ξ) by a parabola g(ξ) that passes through the end points and through f(0) and integrate the parabola



$$g(\xi) = \frac{\xi(\xi-1)}{2} f(-1) + (1-\xi)(1+\xi) f(0) + \frac{\xi(1+\xi)}{2} f(1)$$

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \int_{-1}^{1} g(\xi)\, d\xi = \frac{1}{3} f(1) + \frac{4}{3} f(0) + \frac{1}{3} f(-1)$$

- Requires the function f(x) to be evaluated at 3 points (-1,0, 1)
- Constants, linear functions and parabolas are exactly integrated
- Not good for cubic and higher order polynomials

How to generalize this formula?

Notice that both the integration formulas had the general form

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{M} W_i \; f(\xi_i)$$

Weight   Integration point

Trapezoidal rule:
M=2

$$W_1 = 1 \qquad \xi_1 = -1$$
$$W_2 = 1 \qquad \xi_2 = 1$$

Accurate for polynomial of degree at most 1 (=M-1)

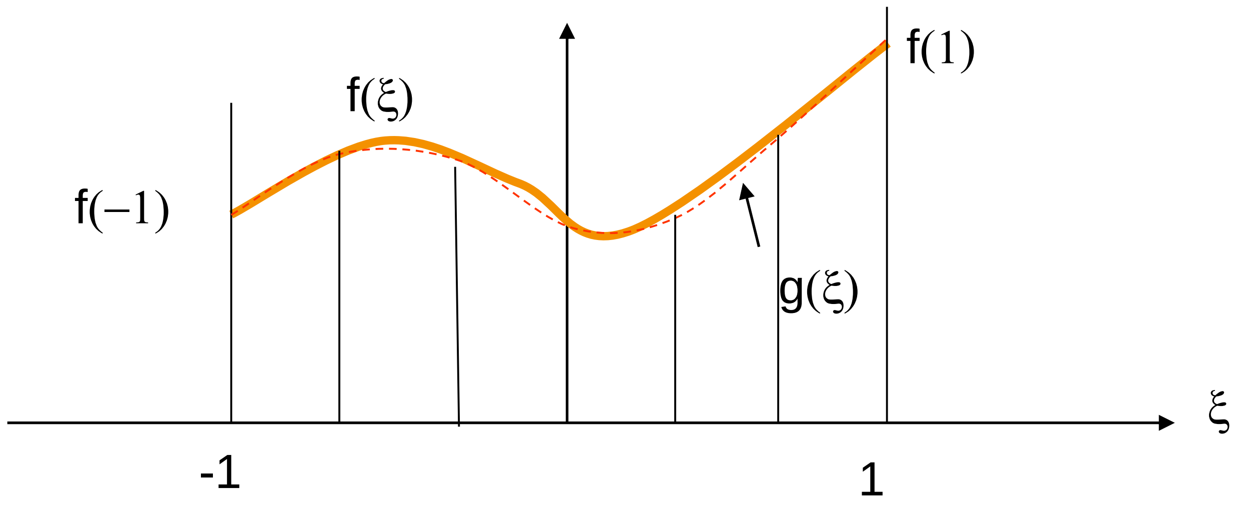
Simpson's rule:
M=3

$$W_1 = 1/3 \qquad \xi_1 = -1$$
$$W_2 = 4/3 \qquad \xi_2 = 0$$
$$W_2 = 1/3 \qquad \xi_2 = 1$$

Accurate for polynomial of degree at most 2 (=M-1)

Generalization of these two integration rules: **Newton-Cotes**

- Divide the interval (-1,1) into M-1 **equal** intervals using M points
- Pass a polynomial of degree M-1 through these M points (the value of this polynomial will be equal to the value of the function at these M-1 points)
- Integrate this polynomial to obtain an approximate value of the integral



With ‘M’ points we may integrate a polynomial of degree ‘M-1’ exactly.

Is this the best we can do ?

With ‘M’ integration points and ‘M’ weights, I should be able to integrate a polynomial of degree 2M-1 exactly!!

Gauss integration rule

See table 10-1 (p 405) of Logan

Gauss quadrature

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{M} W_i \; f(\xi_i)$$

Weight Integration point

How can we choose the integration points **and** weights to **exactly integrate a polynomial of degree 2M-1**?

Remember that now we do not know, a priori, the location of the integration points.

Example: M=1 (Midpoint qudrature)

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx W_1 \; f(\xi_1)$$

How can we choose W_1 and ξ_1 so that we may integrate a (2M-1=1) **linear polynomial** exactly?

$$f(\xi) = a_0 + a_1\xi$$

$$\int_{-1}^{1} f(\xi)\, d\xi = 2a_0$$

But we want

$$\int_{-1}^{1} f(\xi)\, d\xi = W_1 f(\xi_1) = a_0 W_1 + a_0 W \xi_1$$

Hence, we obtain the identity

$$2a_0 = a_0 W_1 + a_1 W_1 \xi_1$$

For this to hold for arbitrary a_0 and a_1 we need to satisfy 2 conditions

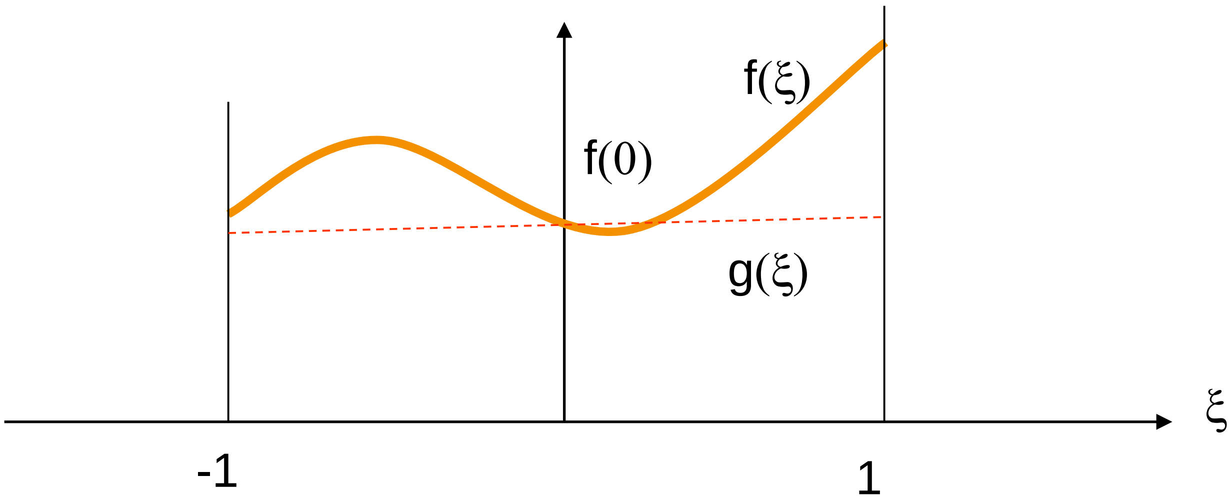
$$Condition\,1:\ W_1 = 2$$

$$Condition\,2:\ W_1 \xi_1 = 0$$

i.e., $$W_1 = 2;\ \xi_1 = 0$$

For M=1

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx 2\ f(0)$$



Midpoint quadrature rule:

- Only one evaluation of f(ξ) is required at the midpoint of the interval.
- Scheme is accurate for constants and linear polynomials (compare with Trapezoidal rule)

Example: M=2

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx W_1 \; f(\xi_1) + W_2 \; f(\xi_2)$$

How can we choose W_1 ,W_2 ξ_1 and ξ_2 so that we may integrate a **polynomial** of degree (2M-1=4-1=3) exactly?

$$f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$$

$$\int_{-1}^{1} f(\xi)\, d\xi = 2a_0 + \frac{2}{3}a_2$$

But we want

$$\int_{-1}^{1} f(\xi)\, d\xi = W_1 f(\xi_1) + W_2 \; f(\xi_2)$$

$$= a_0(W_1 + W_2) + a_1(W_1\xi_1 + W_2\xi_2) + a_2\left(W_1\xi_1^{\,2} + W_2\xi_2^{\,2}\right) + a_3\left(W_1\xi_1^{\,3} + W_2\xi_2^{\,3}\right)$$

Hence, we obtain the 4 conditions to determine the 4 unknowns (W_1 ,W_2 ξ_1 and ξ_2)

$$Condition\,1: \; W_1 + W_2 = 2$$

$$Condition 2: \; W_1 \xi_1 + W_2 \xi_2 = 0$$

$$Condition 3: \; W_1 \xi_1{}^2 + W_2 \xi_2{}^2 = \frac{2}{3}$$

$$Condition 4: \; W_1 \xi_1{}^3 + W_2 \xi_2{}^3 = 0$$

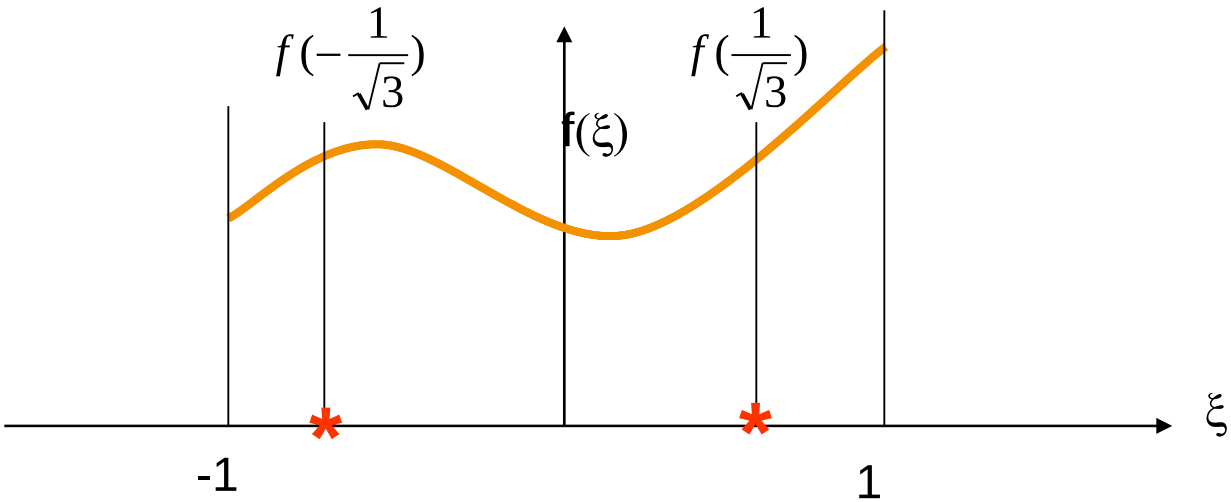
Check that the following is the solution

$$W_1 = W_2 = 1$$

$$\xi_1 = -\frac{1}{\sqrt{3}} ; \; \xi_2 = \frac{1}{\sqrt{3}}$$

For M=2

$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$



- Only two evaluations of f(ξ) is required.
- Scheme is accurate for polynomials of degree at most 3 (compare with Simpson's rule)

Exercise: Derive the 6 conditions required to find the integration points and weights for a 3-point Gauss quadrature rule

Newton-Cotes	Gauss quadrature
1. 'M' integration points are necessary to exactly integrate a polynomial of degree 'M-1' 2. More expensive	1. 'M' integration points are necessary to exactly integrate a polynomial of degree '2M-1' 2. Less expensive 3. Exponential convergence, error proportional to $\left(\frac{1}{2M}\right)^{2M}$

Example

$$I = \int_{-1}^{1} \mathrm{f}(\xi)\mathrm{d}\xi \quad where\ \mathrm{f}(\xi) = \xi^3 + \xi^2$$

Exact integration

$$I = \frac{2}{3}$$

Integrate and check!

Newton-Cotes

To exactly integrate this I need a 4-point Newton-Cotes formula. Why?

Gauss

To exactly integrate this I need a 2-point Gauss formula. Why?

Gauss quadrature:

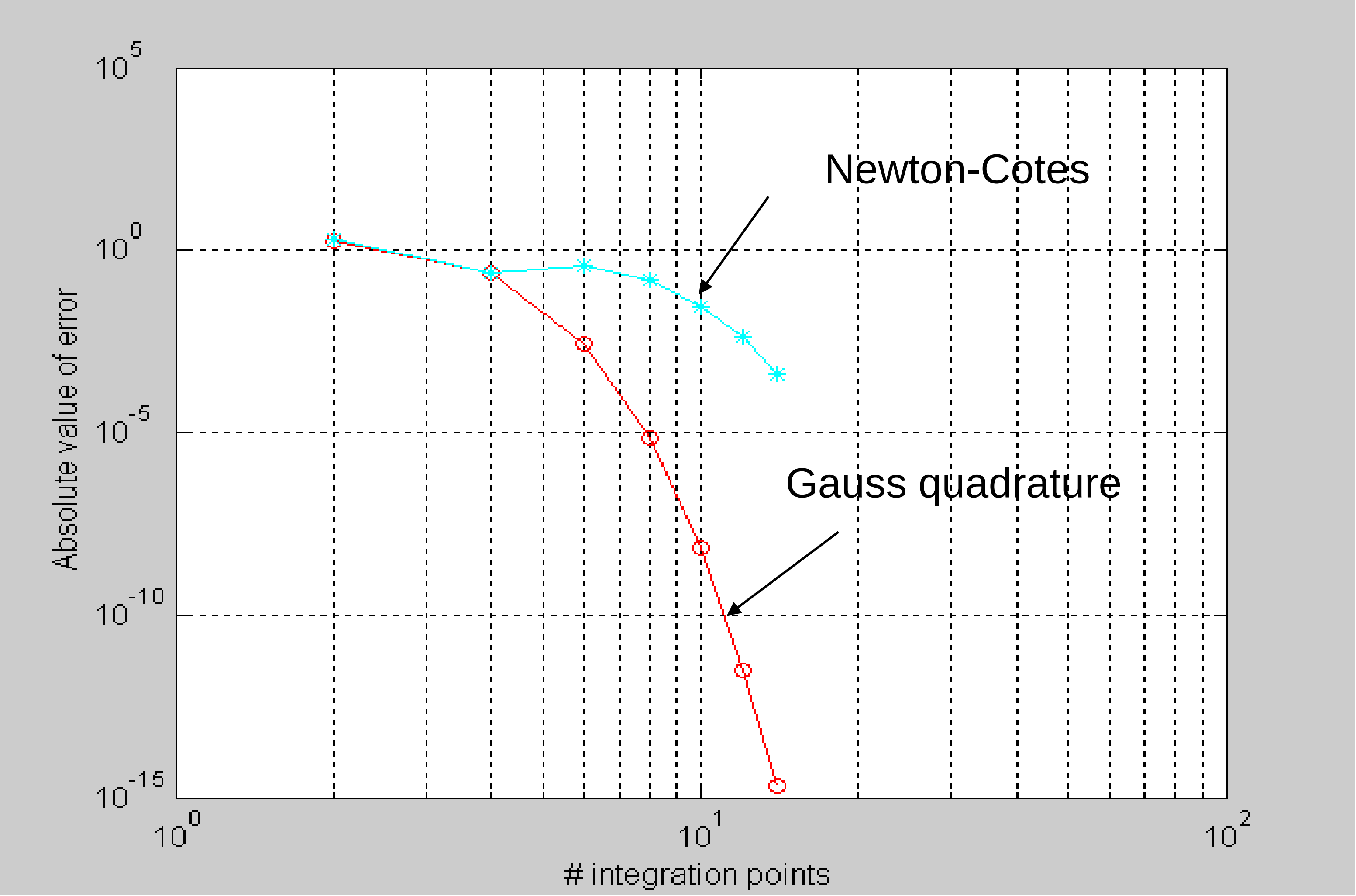
$$I = \int_{-1}^{1} \mathrm{f}(\xi)\mathrm{d}\xi$$

$$= \mathrm{f}(-\frac{1}{\sqrt{3}}) + \mathrm{f}(\frac{1}{\sqrt{3}})$$

$$= \frac{2}{3}$$ Exact answer!

Comparison of Gauss quadrature and Newton-Cotes for the integral

$$I = \int_{-1}^{1} \cos(2\pi x)\, dx$$



In FEM we ALWAYS use Gauss quadrature

Linear Element

Stiffness matrix

$$\underline{k} = \int_{\xi=-1}^{\xi=1} \underline{B}^T E \underline{B} \; A d\xi = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{-1}^{1} AEd\xi = \left(\int_{-1}^{1} AEd\xi \right) \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Nodal load vector

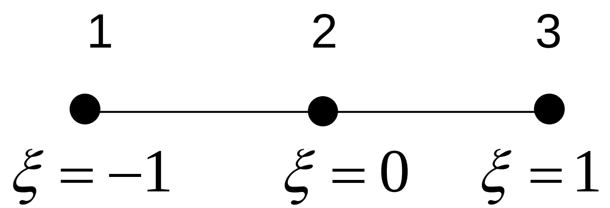
$$\underline{f_b} = \int_{-1}^{1} \underline{N}^T \; b \, d\xi$$

$$f_{bi} = \int_{-1}^{1} N_i \; b \, d\xi$$

Usually a 2-point Gauss integration is used. Note that if A, E and b are complex functions of x, they will not be accurately integrated

Quadratic Element

Nodal shape functions



$$N_1(\xi) = \frac{\xi}{2}(\xi - 1)$$

$$N_2(\xi) = \left(1 - \xi^2\right)$$

You should be able to derive these!

$$N_3(\xi) = \frac{\xi}{2}(\xi + 1)$$

Stiffness matrix

$$\underline{k} = \int_{-1}^{1} \underline{B}^{T} E \underline{B} \; A d\xi = AE \int_{-1}^{1} \underline{B}^{T} \underline{B} \; d\xi$$

Assuming E and A are constants

$$\underline{B} = \frac{d\underline{N}}{d\xi} = \left[\frac{dN_1}{d\xi} \; \frac{dN_2}{d\xi} \; \frac{dN_3}{d\xi} \right] = \left[\frac{1}{2}(2\xi - 1) \quad -2\xi \quad \frac{1}{2}(2\xi + 1) \right]$$

$$\underline{k} = \int_{-1}^{1} \underline{\mathrm{B}}^{\mathrm{T}} \mathrm{E} \underline{\mathrm{B}} \; A \mathrm{d}\xi = A\mathrm{E} \int_{-1}^{1} \underline{\mathrm{B}}^{\mathrm{T}} \underline{\mathrm{B}} \; \mathrm{d}\xi$$

$$= AE \int_{-1}^{1} \begin{bmatrix} (\xi - 1/2)^2 & -2\xi(\xi - 1/2) & (\xi^2 - 1/4) \\ -2\xi(\xi - 1/2) & 4\xi^2 & -2\xi(\xi + 1/2) \\ (\xi^2 - 1/4) & -2\xi(\xi + 1/2) & (\xi + 1/2)^2 \end{bmatrix} d\xi$$

Need to exactly integrate **quadratic** terms.
Hence we need a **2-point Gauss** quadrature scheme..Why?

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DEPARTMENT OF MECHANICAL ENGINEERING

INTRODUCTION TO FINITE ELEMENTS

NUMERICAL INTEGRATION IN 2D

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DEPARTMENT OF MECHANICAL ENGINEERING

Reading assignment:

Lecture notes, Logan 10.4

Summary:

- Gauss integration on a 2D square domain
- Integration on a triangular domain
- Recommended order of integration
- "Reduced" vs "Full" integration; concept of "spurious" zero energy modes/ "hour-glass" modes

1D quardrature rule recap

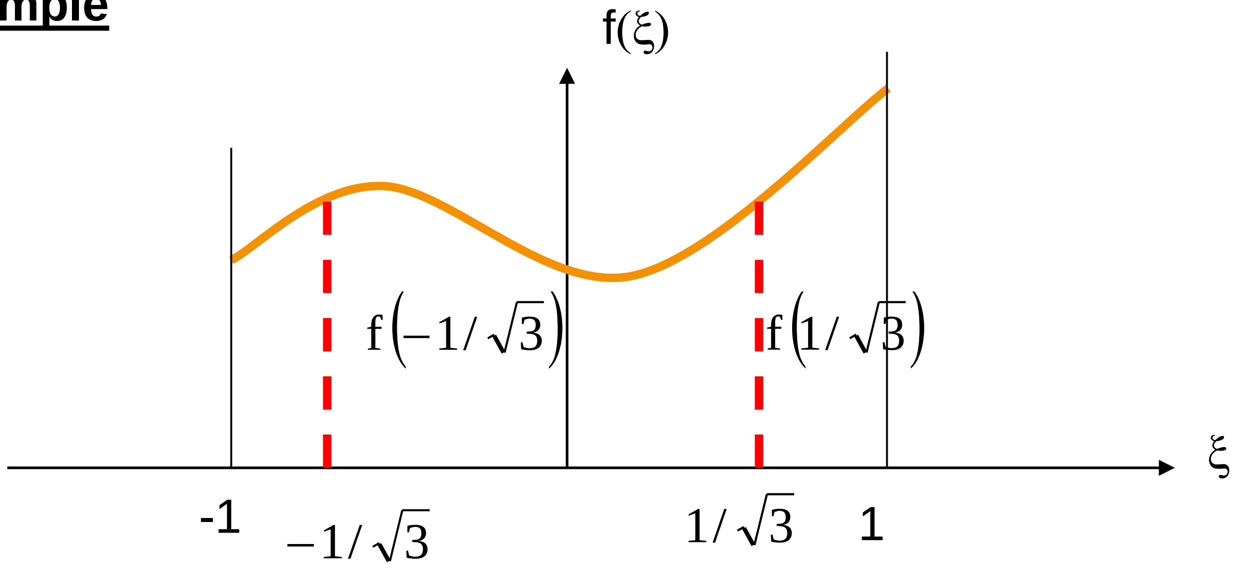
$$I = \int_{-1}^{1} f(\xi)\, d\xi \approx \sum_{i=1}^{M} W_i \; f(\xi_i)$$

Weight (W_i) — Integration point (ξ_i)

Choose the integration points **and** weights to maximize accuracy

Newton-Cotes	Gauss quadrature
1. 'M' integration points are necessary to exactly integrate a polynomial of degree 'M-1' 2. More expensive	1. 'M' integration points are necessary to exactly integrate a polynomial of degree '2M-1' 2. Less expensive 3. Exponential convergence, error proportional to $\left(\frac{1}{2M}\right)^{2M}$

Example

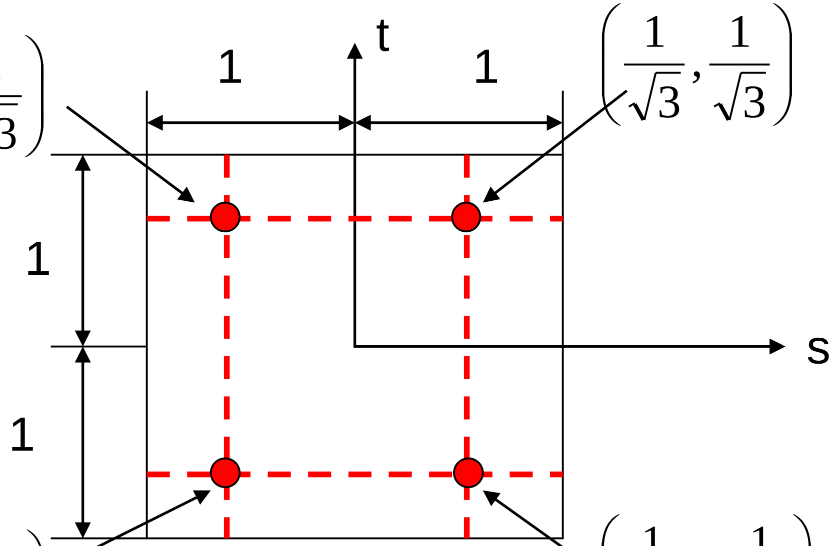


A 2-point Gauss quadrature rule

$$\int_{-1}^{1} f(\xi)\, d\xi \approx f(\frac{1}{\sqrt{3}}) + f(-\frac{1}{\sqrt{3}})$$

is **exact** for a polynomial of degree 3 or less

2D square domain



$$\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \quad \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) \quad \left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) \quad \left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, ds dt$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, ds dt$$

$$\approx \int_{-1}^{1}\left(\sum_{j=1}^{M} W_j f(s,t_j)\right) ds$$ Using 1D Gauss rule to integrate along ‘t’

$$\approx \sum_{i=1}^{M}\sum_{j=1}^{M} W_i W_j f(s_i,t_j)$$ Using 1D Gauss rule to integrate along ‘s’

$$= \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij} f(s_i,t_j)$$ Where $W_{ij} = W_i W_j$

For M=2

$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij}\, f(s_i, t_j)$$

$W_{ij} = W_i\, W_j = 1$

$$= f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$$

Number the Gauss points IP=1,2,3,4

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, ds\,dt \approx \sum_{IP=1}^{4} W_{IP}\, f_{IP}$$

The rule

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt \approx \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij}\, f(s_i, t_j)$$

Uses **M² integration points** on a nonuniform grid inside the parent element and is **exact for a polynomial of degree (2M-1)** i.e.,

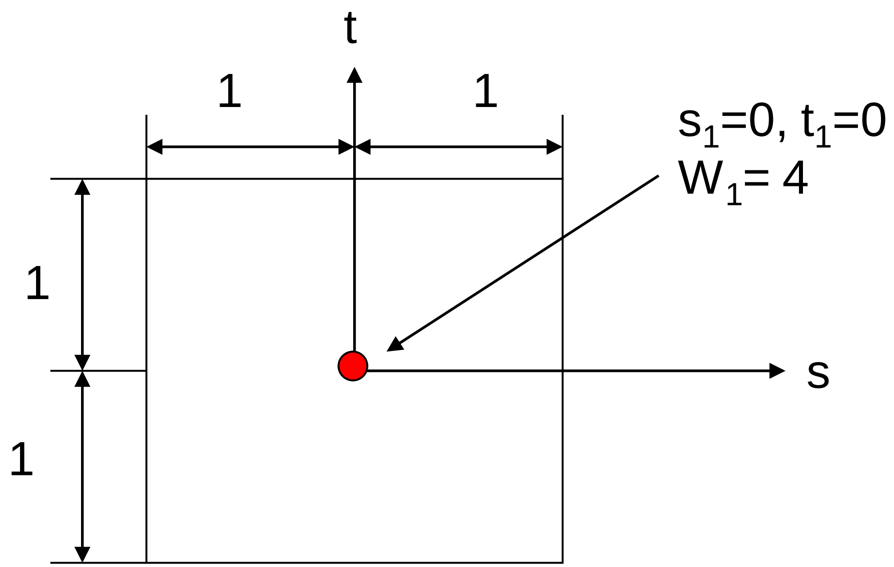
$$\int_{-1}^{1}\int_{-1}^{1} s^{\alpha} t^{\beta}\, dsdt \overset{exact}{=} \sum_{i=1}^{M}\sum_{j=1}^{M} W_{ij}\, s_i^{\ \alpha} t_j^{\ \beta} \quad for \;\; \alpha+\beta \le 2M-1$$

A **M² –point rule is exact for a complete polynomial of degree (2M-1)**

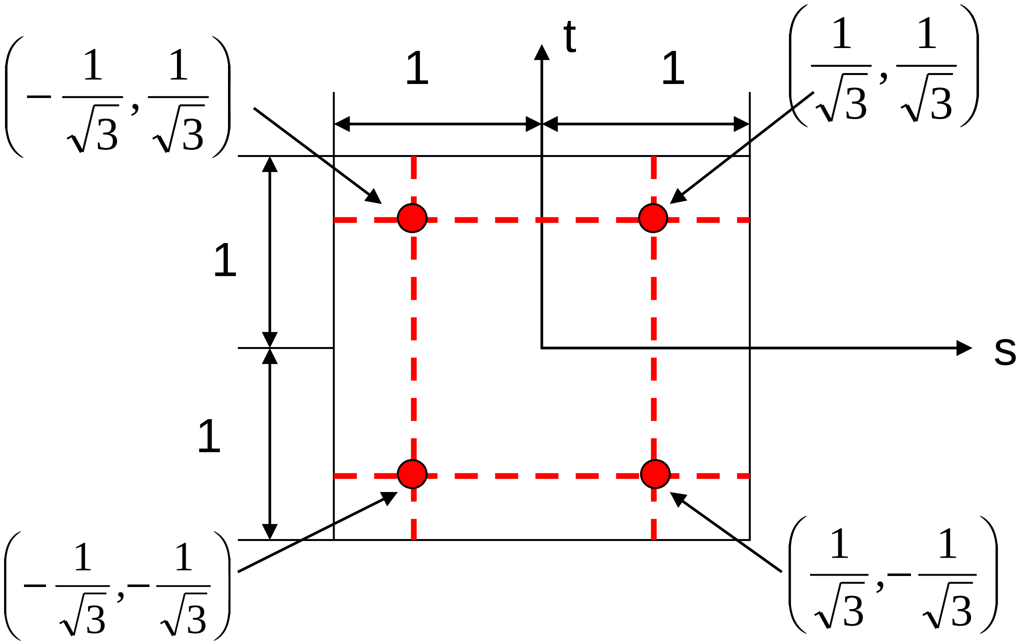
CASE I: M=1 (One-point GQ rule)

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt \approx 4 f(0,0)$$

is exact for a product of two linear polynomials



<u>CASE II:</u> M=2 (2x2 GQ rule)

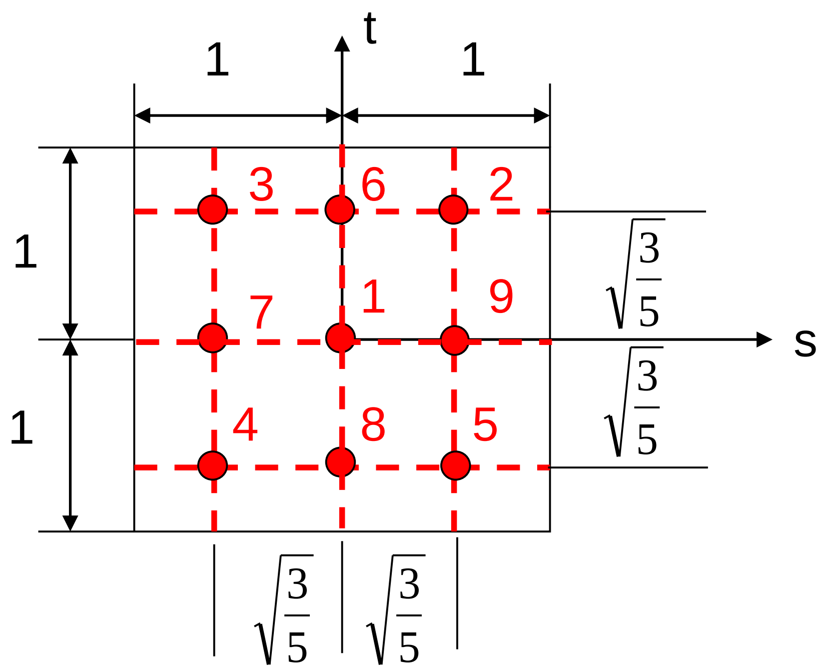


$$I \approx \sum_{i=1}^{2}\sum_{j=1}^{2} W_{ij} f(s_i, t_j)$$

$$= f\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

is exact for a product of two cubic polynomials

CASE III: M=3 (3x3 GQ rule)



$$W_1 = \frac{64}{81},$$

$$W_2 = W_3 = W_4 = W_5 = \frac{25}{81}$$

$$W_6 = W_7 = W_8 = W_9 = \frac{40}{81}$$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, ds dt \approx \sum_{i=1}^{3}\sum_{j=1}^{3} W_{ij}\, f(s_i, t_j)$$

is exact for a product of two 1D polynomials of degree 5

EXAMPLES

If *f(s,t)=1*

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = 4$$

A **1-point GQ scheme** is sufficient

If *f(s,t)=s*

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = 0$$

A **1-point GQ scheme** is sufficient

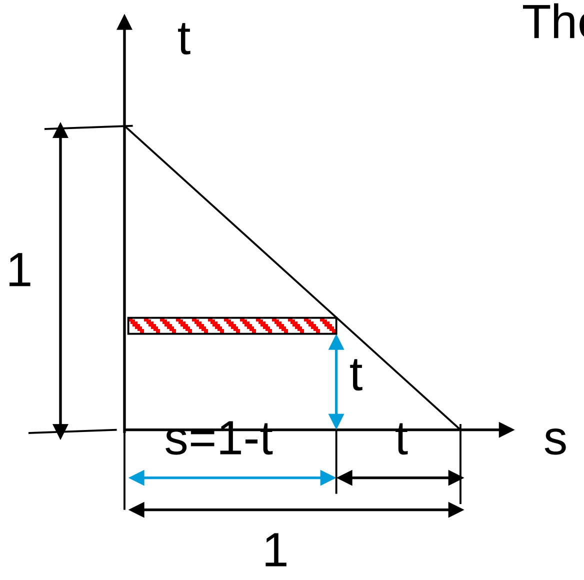
If $f(s,t)=s^2t^2$

$$I = \int_{-1}^{1}\int_{-1}^{1} f(s,t)\, dsdt = \frac{4}{9}$$

A **3x3 GQ scheme** is sufficient

2D Gauss quadrature for triangular domains

Remember that the **parent element** is a right angled triangle with unit sides



The type of integral encountered

$$I = \int_{t=0}^{1} \int_{s=0}^{1-t} \mathrm{f}(s,t) \;\; \mathrm{dsdt}$$

$$I = \int_{t=0}^{1} \int_{s=0}^{1-t} \mathrm{f}(s,t) \;\; \mathrm{dsdt}$$

$$\approx \sum_{IP=1}^{M} W_{IP} \, f_{IP}$$

Constraints on the weights

if f(s,t)=1

$$I = \int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \mathrm{dsdt} = \frac{1}{2}$$

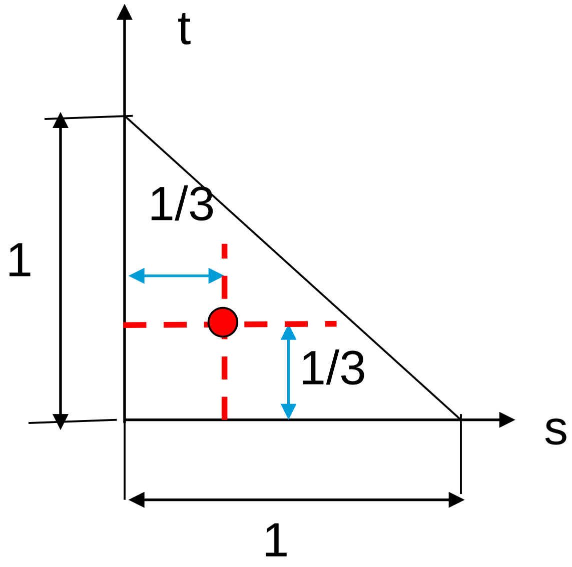
$$= \sum_{IP=1}^{M} W_{IP}$$

$$\therefore \sum_{IP=1}^{M} W_{IP} = \frac{1}{2}$$

Example 1. A M=1 point rule is exact for a polynomial

$$f(s,t) \sim \quad 1$$

$$s \quad t$$



$$I \approx \frac{1}{2} f\left(\frac{1}{3},\frac{1}{3}\right)$$

Why?

Assume

$$f(s,t) = \alpha_1 + \alpha_2 s + \alpha_3 t$$

Then

$$\int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \ \mathrm{dsdt} = \frac{1}{2}\alpha_1 + \frac{1}{3!}\alpha_2 + \frac{1}{3!}\alpha_3$$

But

$$\int_{t=0}^{1}\int_{s=0}^{1-t} \mathrm{f}(s,t)\ \ \mathrm{dsdt} = W_1 f(s_1, t_1)$$

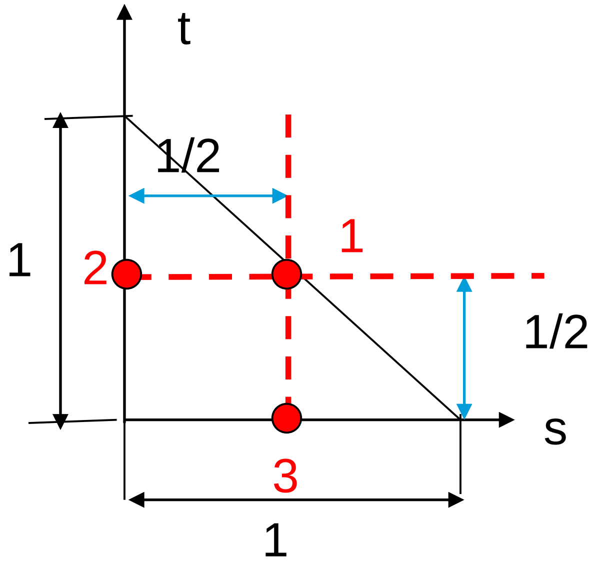
$$\therefore \frac{1}{2}\alpha_1 + \frac{1}{3!}\alpha_2 + \frac{1}{3!}\alpha_3 = W_1\left(\alpha_1 + \alpha_2 s_1 + \alpha_3 t_1\right)$$

Hence

$$W_1 = \frac{1}{2}; W_1 s_1 = \frac{1}{3!}; W_1 t_1 = \frac{1}{3!}$$

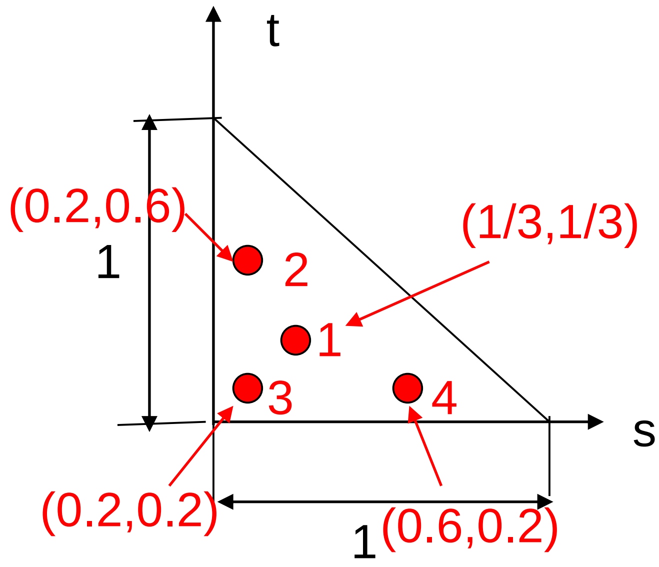
Example 2. A M=3 point rule is exact for a complete polynomial of degree 2

$$f(s,t) \sim \quad 1$$
$$s \quad t$$
$$s^2 \quad st \quad t^2$$



$$I \approx \frac{1}{6} f\left(\frac{1}{2},\frac{1}{2}\right) + \frac{1}{6} f\left(\frac{1}{2},0\right) + \frac{1}{6} f\left(0,\frac{1}{2}\right)$$

Example 4. A M=4 point rule is exact for a complete polynomial of degree 3

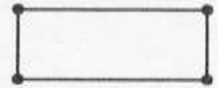
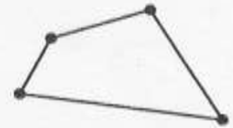
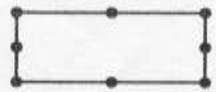
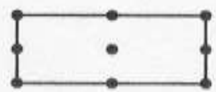
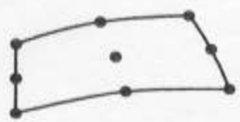
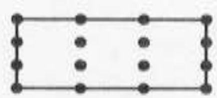


$$f(s,t) \sim \begin{array}{cccc} & & 1 & \\ & s & & t \\ s^2 & & st & & t^2 \\ s^3 & s^2t & st^2 & t^3 \end{array}$$

$$I \approx -\frac{27}{96} f\left(\frac{1}{3},\frac{1}{3}\right) + \frac{25}{96} f(0.2,0.6) + \frac{25}{96} f(0.2,0.2) + \frac{25}{96} f(0.6,0.2)$$

<u>Recommended order of integration</u>
“Finite Element Procedures”
by K. –J. Bathe

TABLE 5.9 *Recommended full Gauss numerical integration orders for the evaluation of isoparametric displacement-based element matrices (use of Table 5.7)*

Two-dimensional elements (plane stress, plane strain and axisymmetric conditions)		Integration order
4-node		2×2
4-node distorted		2×2
8-node		3×3
8-node distorted		3×3
9-node		3×3
9-node distorted		3×3
16-node		4×4
16-node distorted		4×4

“Reduced” vs “Full” integration

Full integration: Quadrature scheme sufficient to provide exact integrals of all terms of the stiffness matrix if the element is geometrically undistorted.
Reduced integration: An integration scheme of lower order than required by “full” integration.

Recommendation: Reduced integration is NOT recommended.

WHICH ORDER OF GQ TO USE FOR FULL INTEGRATION?

To computet the stiffness matrix we need to evaluate the following integral

$$\underline{k} = \int_{-1}^{1}\int_{-1}^{1} \underline{B}^T \underline{D}\ \underline{B} \det(\underline{J})\, ds dt$$

For an "undistroted" element det (J) =constant

Example : **4-noded parallelogram**

$$N_i \sim \begin{matrix} & 1 & \\ s & & t \\ & st & \end{matrix}$$

$$\underline{B} \sim \begin{matrix} & 1 & \\ s & & t \end{matrix}$$

$$\underline{B}^T \underline{D}\underline{B} \sim \begin{matrix} & 1 & \\ s & & t \\ s^2 & st & t^2 \end{matrix}$$

Hence, 2M-1=2
M=3/2

Hence we need at least a 2x2 GQ scheme

Example 2: **8-noded Serendipity element**

$N_i \sim$

1
s t
s^2 st t^2
s^2t st^2

$\underline{B} \sim$

1
s t
s^2 st t^2

$$\underline{B}^T \underline{D}\underline{B} \sim$$

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & s & & t & & & \\ & & s^2 & & st & & t^2 & & \\ & s^3 & & s^2t & & st^2 & & t^3 & \\ s^4 & & s^3t & & s^2t^2 & & st^3 & & t^4 \end{array}$$

Hence, 2M-1=4

M=5/2

Hence we need at least a 3x3 GQ scheme

Reduced integration leads to rank deficiency of the stiffness matrix and "spurious" zero energy modes

"Spurious" zero energy mode/ "hour-glass" mode

The strain energy of an element

$$U = \frac{1}{2}\underline{d}^T \underline{k}\underline{d} = \frac{1}{2}\int_{V^e} \underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\, dV$$

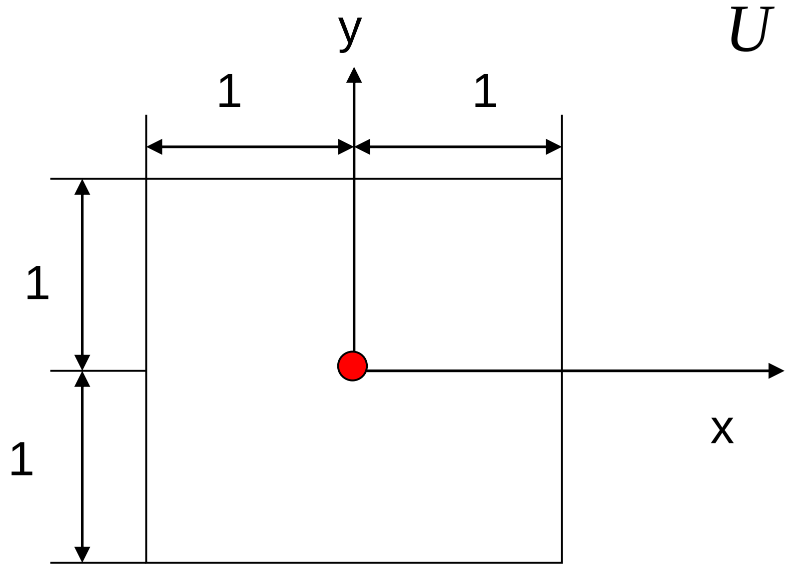
Corresponding to a rigid body mode, $\underline{\varepsilon} = \underline{0} \Rightarrow U = 0$

If *U=0* for a mode $\underline{d}$ that is different from a rigid body mode, then $\underline{d}$ is known as a "spurious" zero energy mode or "hour-glass" mode

Such a mode is **undesirable**

Example 1. 4-noded element

$$U = \frac{1}{2}\int_{V^e} \underline{\varepsilon}^T \underline{D}\,\underline{\varepsilon}\, dV \approx \sum_{i=1}^{NGAUSS} W_i \left(\underline{\varepsilon}^T \underline{D}\,\underline{\varepsilon}\right)_i$$



Full integration: NGAUSS=4
Element has 3 zero energy (rigid body) modes

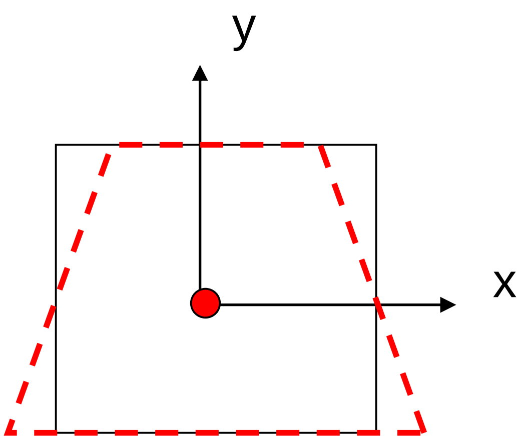
Reduced integration: e.g., NGAUSS=1

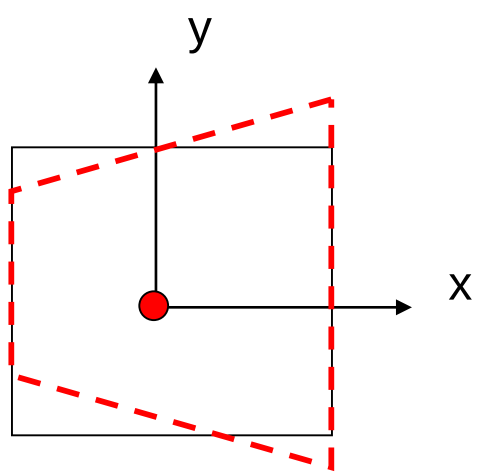
$$U \approx 4\left(\underline{\varepsilon}^T \underline{D}\,\underline{\varepsilon}\right)_{\substack{x=0\\y=0}}$$

Consider 2 displacement fields

$$u = C\,xy$$
$$v = 0$$

$$u = 0$$
$$v = C\,xy$$

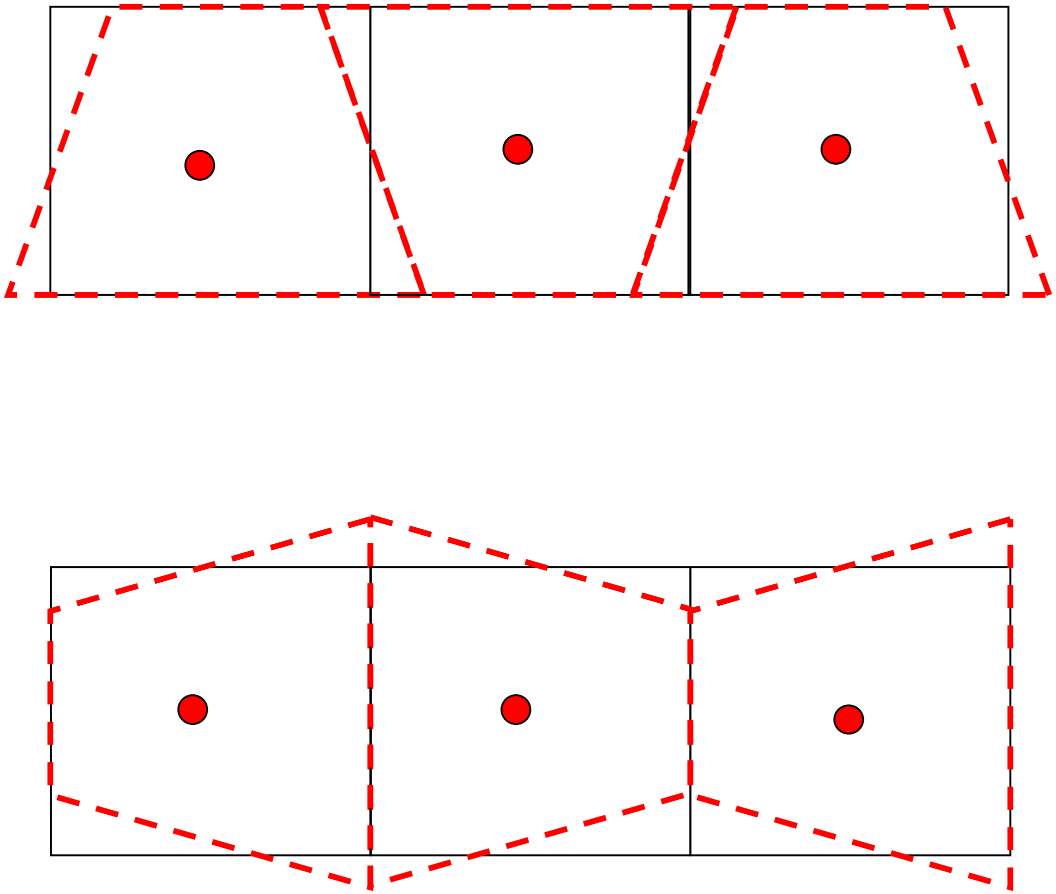




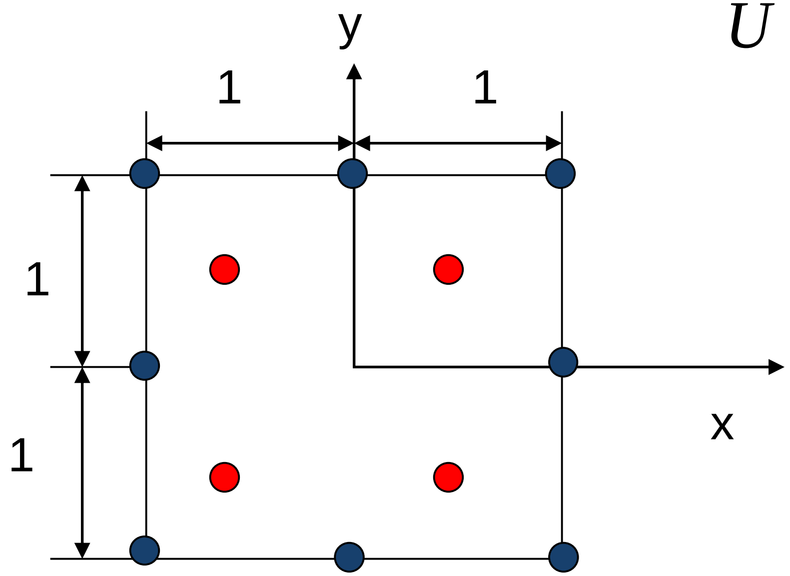
$$At\ x = y = 0 \qquad \varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$$
$$\Rightarrow U = 0$$

We have therefore 2 hour-glass modes.

Propagation of hour-glass modes through a mesh



Example 2. 8-noded serendipity element



$$U = \frac{1}{2}\int_{V^e} \underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\, dV \approx \sum_{i=1}^{NGAUSS} W_i \left(\underline{\varepsilon}^T \underline{D}\underline{\varepsilon}\right)_i$$

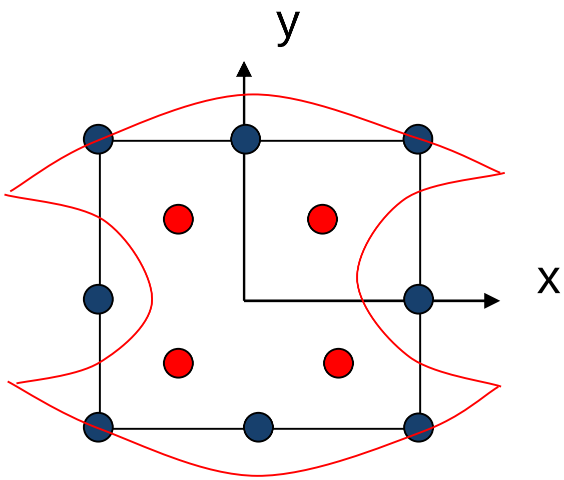
Full integration: NGAUSS=9
Element has 3 zero energy (rigid body) modes

Reduced integration: e.g., NGAUSS=4

Element has one spurious zero energy mode corresponding to the following displacement field

$$u = C\,x\,(y^2 - 1/3)$$

$$v = -C\,y\,(x^2 - 1/3)$$

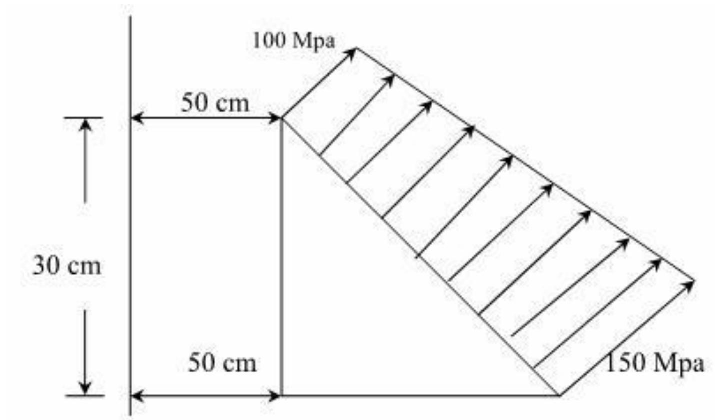


Show that the strains corresponding to this displacement field are all zero at the 4 Gauss points

<u>Elements with zero energy modes introduce uncontrolled errors and should NOT be used in engineering practice.</u>

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY
Subject : FINITE ELEMENT METHODS
UNIT – III
TUTORIAL - III

1. An axi-symmetric triangular element is subjected to the loading as shown in fig. the load is distributed throughout the circumference and normal to the boundary.
Derive all the necessary equations and derive the nodal point loads.



2 a.) Discuss in detail about 2D heat conduction in Composite slabs using FEA.
b.) Using the isoperimetric element, find the Jacobean and inverse of Jacobean matrix for the element shown in Fig.2, 3(a) & 3(b) for the following cases.
i) Determine the coordinate of a point P in x-y coordinate system for the ξ = 0.4 and η = 0.6.
ii) Determine the coordinate of the Q in ξ and η system for the x = 2.5 and y = 1.0.

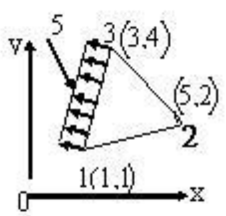


Fig. 2

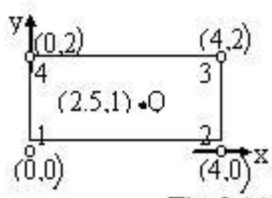


Fig. 3 (a)

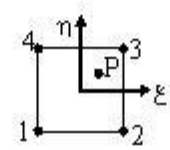


Fig. 3 (b)

3. Derive the shape functions of two dimensional four nodded iso-parametric element. Plot the shape functions
4. Explain the formulation of 4-noded Iso-parametric axi-symmetric element and derive the stiffness matrix.
5. Explain plane stress and plane strain conditions with suitable examples.

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

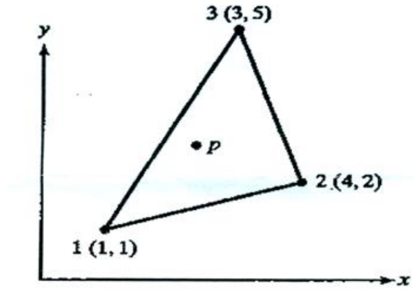
Subject : FINITE ELEMENT METHODS

UNIT – III

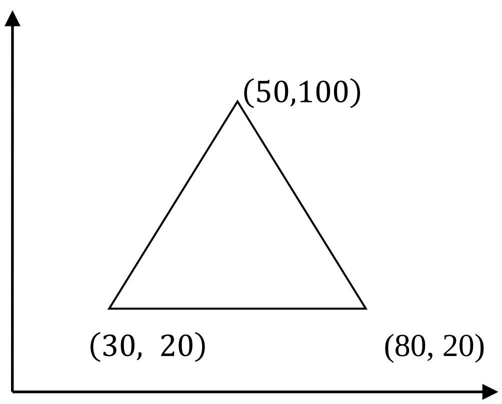
ASSIGNMENT- III

1. a.) Write the difference between CST and LST elements

b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P.



2. For the plane stress element shown in Fig, the nodal displacements are u_1 = 2.0 mm, v_1 = 1.0 mm , u_2 = 0.5 mm, v_2 = 0.0 mm, u_3 = 3.0 mm, v_3 = 1.0 mm and consider Young's Modulus E = 210 GPa, Poisson's ratio ν = 0.25 and uniform plate thickness t = 10 mm. Determine the element stresses σ_x , σ_y , τ_{xy} , σ_1 and σ_2 and the principal axis angle θ_p.



3. Differentiate between Axi- symmetric elements and symmetric elements with suitable examples.

4. a.) Derive the Shape Functions N_1, N_2 and N_3 for a plane triangular element.

b.) Compute the strain displacement matrix and also the strains of a axi-symmetric triangular element with the coordinates r_1 = 30 mm, z_1 = 40 mm, r_2 = 60 mm, z_2 = 50 mm, r_3= 50 mm, z_3 = 80 mm. The nodal displacement values are u_1 = 0.01 mm, w_1 = 0.01 mm, u_2 = 0.01 mm, w_2 = -0.04 mm, u_3 = -0.03 mm, w_3 = 0.07 mm

5 .a.)Explain the methodology to estimate the stiffness matrix of four nodded Quadrilateral element.

b.) Evaluate $\int [e^{2x} + x^3 + 1 / (x^2 + 2)]\, dx$ over the limits -1 and +1 using one point

UNIT 4

BEAM ANALYSIS & HEAT TRANSFER ANALYSIS

Syllabus:

Heat transfer analysis: One dimensional steady state analysis composite wall. One dimensional fin analysis and two dimensional analysis of thin plate.

BEAMS: Element matrices, assembling of global stiffness matrix, solution for displacements, reaction, stresses.

OBJECTIVE:

To learn the principles involved in the discretization of domains with various elements, polynomial and interpolation and assembly of global arrays.

To understand the application of FEM for Beams and Heat Transfer Problems.

OUTCOME:

Derive element matrices to find stresses in Beams and temperatures in Heat transfer problems.

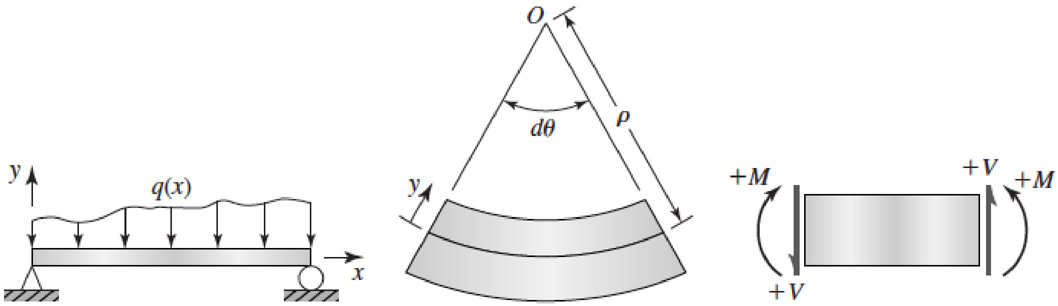
UNIT – IV

Flexural Elements or Beam Elements

The bar, one-dimensional elements discussed earlier will be now extended to general one-dimensional elements that can take bending.
• We develop here a flexural or beam element using the elementary beam theory.
• To begin with we consider one-dimensional beam that can bend in a plane.
• This will subsequently be extended to include two-plane bending, axial load and torsion.

Elementary Beam Theory

• The beam is loaded only in y - direction.
• Deflections of the beam are small compared to the characteristic dimensions of the beam.
• The material of the beam is linearly elastic, isotropic, and homogeneous.
• The beam is prismatic and has an axis of symmetry in the plane of bending.

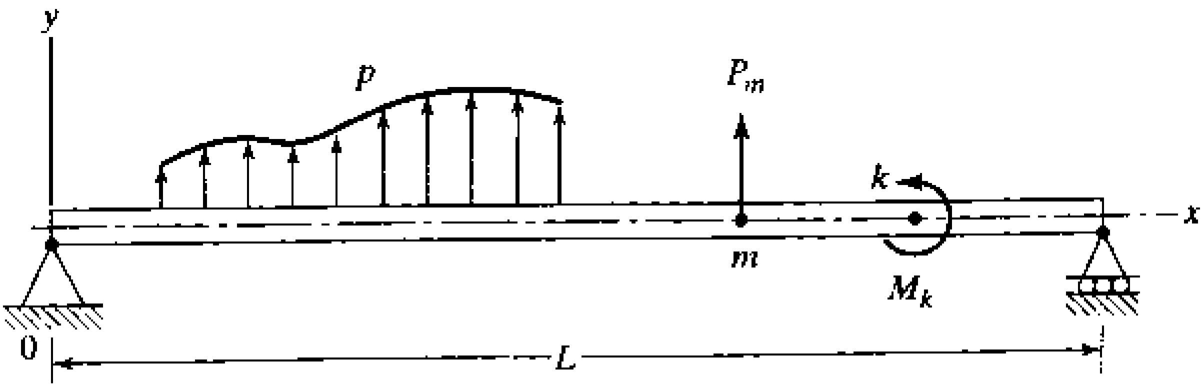


From elementary beam theory, the following assumptions are valid:
1. Each beam element is of length L and has two nodes
2. Each element is connected to other element only at nodes
3. Element loading occurs only at nodes
• Here the field variable of interest is the transverse displacement *v*(*x*) of the neutral surface away from its straight, undeflected position.
• The same end displacements, give rise to different beam configurations. Hence we need to take the slope v' (θ) into consideration.
The stresses and strains in a beam are given as Here σ and ε are the normal stress and normal strain respectively. M is the the bending moment at the section. v(x) is deflection of the neutral axis at x. I is the moment of inertia of the section about the neutral axis.

$$\sigma = -\frac{M}{I}y$$

$$\varepsilon = \frac{\sigma}{E}$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$



Beam Elements- PE Approach

- The potential Energy formulation

$$dU = \frac{1}{2}\int_A \sigma\varepsilon\, dA\, dx = \frac{1}{2}\left(\frac{M^2}{EI^2}\int_A y^2\, dA\right)dx$$

- In the above equation of elemental beam, we notice that

$$\int_A y^2\, dA = I, \therefore dU = \frac{1}{2}\frac{M^2}{EI}dx$$

- The total strain energy is obtained by integrating the above as

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx$$

- The potential energy of the beam is, where p is the load per unit length

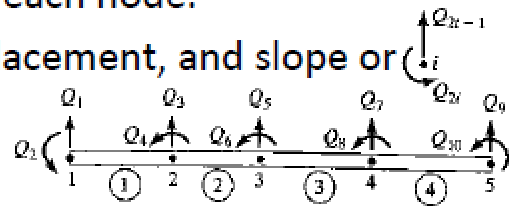
$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L pv\,dx - \sum_m P_m v_m - \sum_k P_k v_k'$$

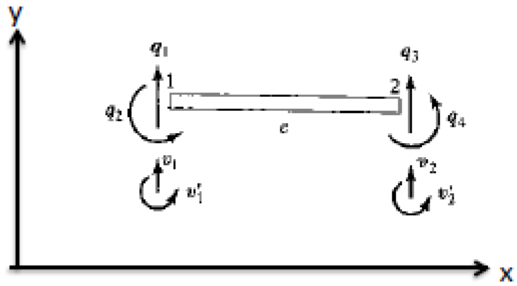
- The element length L is aligned with x-coordinate such that $x_1=0$ and $x_2=L$
- The beam is divided into number of flexural elements with typically two degrees of freedom at each node.
- The two dof are the transverse displacement, and slope or rotation (in radians)
- The global displacement vector is

Q=[Q1,Q2, ...Q2n-1] where n is number of nodes.

For an element the local dofs are

$$\{q\}=[q_1,q_2,q_3,q_4]^T=[v_1,v_1',v_2,v_2']^T$$

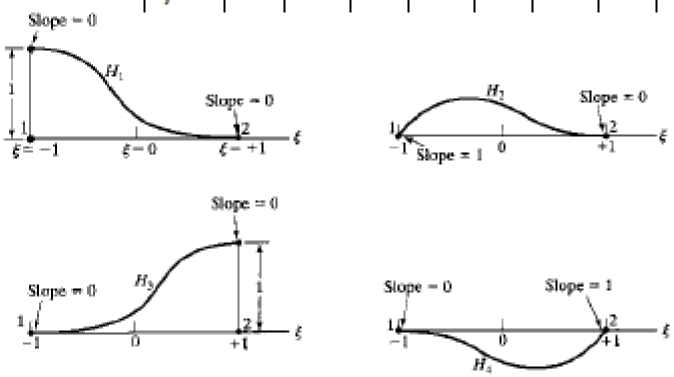




Beam Elements- Shape Functions

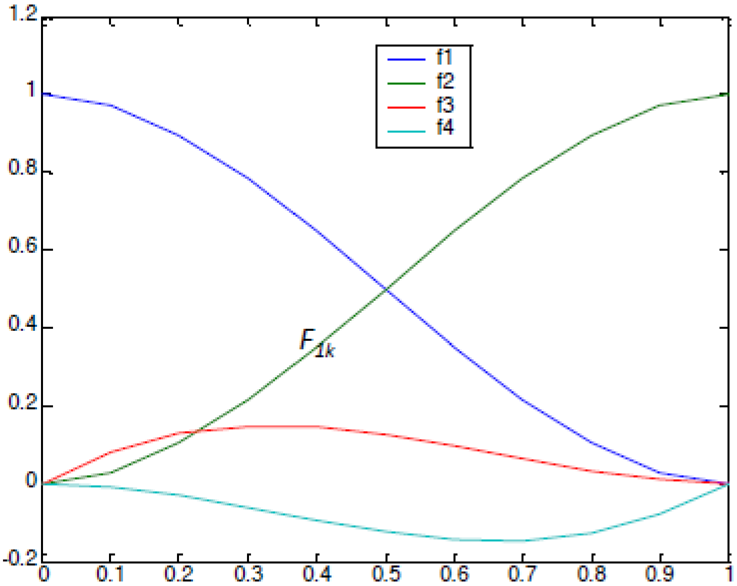
- The shape functions for interpolating v in an element are defined as $H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3 \qquad i = 1,2,3,4$
- As it is a cubic polynomial it will be c_2 continuous everywhere. The other conditions are summarized in the following table

	H_1	H'_1	H_2	H'_2	H_3	H'_3	H_4	H'_4
ξ=-1	1	0	0	1	0	0	0	0
ξ=-1	0	0	0	0	1	0	0	1



$$\sum_{i=1}^{4} H_i = 1$$

➢ Shape Functions



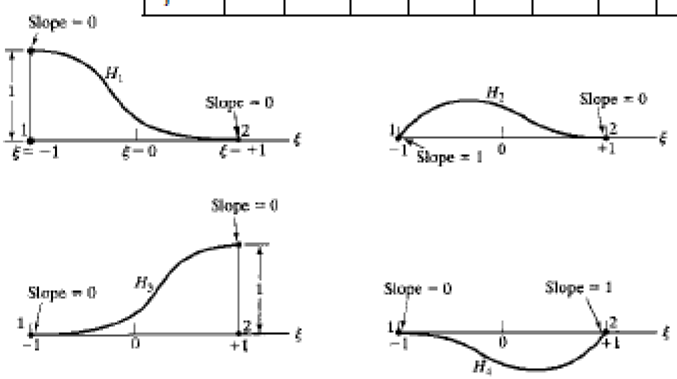
$$F_{1k}(\tau) = 2\tau^3 - 3\tau^2 + 1$$
$$F_{2k}(\tau) = -2\tau^3 + 3\tau^2$$
$$F_{3k}(\tau) = \tau(\tau^2 - 2\tau + 1)t_{k+1}$$
$$F_{4k}(\tau) = \tau(\tau^2 - \tau)t_{k+1}$$

$$\tau = 0 \Rightarrow \qquad F_{1k}(0) = 1,\ F_{2k}(0) = F_{3k}(0) = F_{4k}(0) = 0$$
$$\tau = 1 \Rightarrow \qquad F_{2k}(1) = 1,\ F_{1k}(1) = F_{3k}(1) = F_{4k}(1) = 0$$
$$F_{2k}(\tau) = 1 - F_{1k}(\tau)$$

- All 4 shape functions are given below

$$H_1 = \frac{1}{4}(1-\xi)^2(2+\xi) \quad = \frac{1}{4}(2 - 3\xi + \xi^2) \qquad H_2 = \frac{1}{4}(1-\xi)^2(\xi+1) \quad = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$$
$$H_3 = \frac{1}{4}(1+\xi)^2(2-\xi) \quad = \frac{1}{4}(2 + 3\xi - \xi^3) \qquad H_4 = \frac{1}{4}(1+\xi)^2(\xi-1) \quad = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$$

	H_1	H'_1	H_2	H'_2	H_3	H'_3	H_4	H'_4
ξ=-1	1	0	0	1	0	0	0	0
ξ=-1	0	0	0	0	1	0	0	1



$$\sum_{i=1}^{4} H_i = 1$$

Shape Functions and Displacements

- Derivatives of functions are given below differentiating w.r.t ξ

$$H_1' = \frac{1}{4}(-3+2\xi) \qquad H_2' = \frac{1}{4}(-1-2\xi+3\xi^2)$$

$$H_3' = \frac{1}{4}(3-3\xi^2) \qquad H_4' = \frac{1}{4}(-1+2\xi+3\xi^2)$$

- These shape functions are used to represent the displacements as

$$v(\xi) = H_1 v_1 + H_2\left(\frac{dv}{d\xi}\right)_1 + H_3 v_3 + H_4\left(\frac{dv}{d\xi}\right)_2$$

- The coordinates transform as $x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2 = \frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}\xi$
- Since x_2-x_1=l_e, $dx = \frac{l_e}{2}d\xi$

- Since x_2-x_1=l_e, $dx = \frac{l_e}{2}d\xi$

$$\frac{dv}{d\xi} = \frac{l_e}{2}\frac{dv}{dx}$$

- Noting that $\frac{dv}{dx}$ evaluated at nodes 1 and 2 are q_2 and q_4 respectively, we have

$$v(\xi) = H_1 q_1 + \frac{l_e}{2}H_2 q_2 + H_3 q_3 + \frac{l_e}{2}H_4 q_4 \qquad \{v\} = \{H\}\{q\}$$

- Where $[H] = [H_1, \frac{l_e}{2}H_2, H_3, \frac{l_e}{2}H_4]$

Displacements and its Derivatives

- Since $x_2 - x_1 = l_e$, $dx = \frac{l_e}{2} d\xi$ $\qquad \frac{dv}{d\xi} = \frac{l_e}{2}\frac{dv}{dx}$
- In total potential energy expression we have terms like $\frac{d^2v}{dx^2}$

$$\frac{dv}{dx} = \frac{2}{l_e}\frac{dv}{d\xi} \qquad \Rightarrow \frac{d^2v}{dx^2} = \frac{4}{l_e^2}\frac{d^2v}{d\xi^2}$$

- Using the expression $v = Hq$

$$\left(\frac{d^2v}{dx^2}\right)^2 = q^T \frac{16}{l_e^4}\left(\frac{d^2v}{d\xi^2}\right)^T\left(\frac{d^2v}{d\xi^2}\right)q$$

- Here $\left(\frac{d^2v}{d\xi^2}\right) = \left[\frac{3}{2}\xi, \frac{l_e}{2}\frac{-1+3\xi}{2}, -\frac{3}{2}\xi, \frac{l_e}{2}\frac{1+3\xi}{2}\right]$

Displacements and Potential Energy

$$\left(\frac{d^2v}{d\xi^2}\right) = \left[\frac{3}{2}\xi, \frac{l_e}{2}\frac{-1+3\xi}{2}, -\frac{3}{2}\xi, \frac{l_e}{2}\frac{1+3\xi}{2}\right]$$

- Substituting in total energy expression while noting $dx = \frac{l_e}{2} d\xi$

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx$$

$$U_e = \frac{1}{2} q^T \frac{8EI}{l_e^3}\int_0^L \begin{vmatrix} \frac{9}{4}\xi^2 & \frac{3}{8}\xi(-1+3\xi)l_e & -\frac{9}{4}\xi^2 & \frac{3}{8}\xi(1+3\xi)l_e \\ & \left(\frac{-1+3\xi^2}{4}\right)^2 l_e^2 & -\frac{3}{8}\xi(-1+3\xi)l_e & \frac{-1+9\xi^2}{16}l_e^2 \\ & & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(1+3\xi)l_e \\ & Symmetric & & \left(\frac{1+3\xi^2}{4}\right)^2 l_e^2 \end{vmatrix} d\xi q$$

- Each term is integrated noting that $\int_{-1}^{+1}\xi^2 d\xi = \frac{2}{3}$; $\quad \int_{-1}^{+1}\xi d\xi = 0$; $\quad \int_{-1}^{+1} d\xi = 2$

Potential Energy and Element Stiffness Matrix

Finally this gives the element strain energy as

$$U_e = \frac{1}{2} q^T k^e q$$

- Where the element stiffness matrix is

$$k^e = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

Beam Elements- Load Vector

- Assuming the distributed load to be uniform we write the following expression

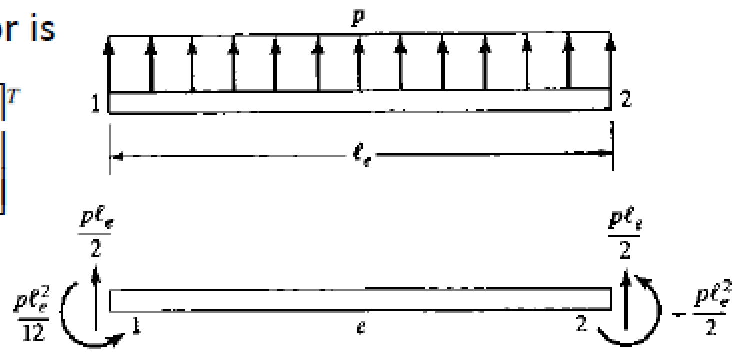
$$\int_{l_e} pv\,dx = \left(\frac{pl_e}{2}\int_{-1}^{1} H\,d\xi\right)q$$

- Substituting for H and integrating

$$\int_{l_e} pv\,dx = f^{eT} q$$

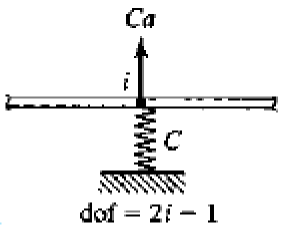
- Where the load vector is

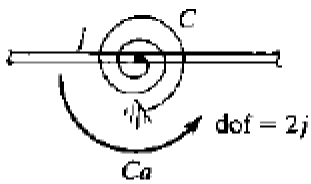
$$f^e = \begin{bmatrix} \frac{pl_e}{2} & \frac{pl_e^2}{12} & \frac{pl_e}{2} & -\frac{pl_e^2}{12} \end{bmatrix}^T$$



Beam Elements- BCs

- We implement the BCs using penalty approach
- Considering generalised displacement *a* for the dof *r*, ,
- We add $\frac{1}{2}C(Q_r - a)^2$ to the total potential energy Π and
- $\Psi_j C(Q_r - a)$ to the left hand side of of the Galerkin formulation
- Here C represents a large stiffness in comparsion with the beam stiffness terms
- This is equivalent of adding stiffness C to K_{rr} and Ca to F_r





Shear Force and Bending Moment

- We have the shear force and bending moment relations as

$$M = EI\frac{d^2v}{dx^2}; \quad V = \frac{dM}{dx}; \qquad v = Hq$$

- At the element level, the bending moment and shear force are respectively

$$M = \frac{EI}{l_e^2}[6\xi q_1 + (3\xi - 1)l_e q_2 - 6\xi q_3 + (3\xi + 1)l_e q_4]$$

$$V = \frac{6EI}{l_e^3}[2q_1 + l_e q_2 - 2q_3 + l_e q_4]$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} -\frac{pl_e}{2} \\ -\frac{pl_e^2}{12} \\ -\frac{pl_e}{2} \\ \frac{pl_e^2}{12} \end{Bmatrix}$$

Here the second term needs to be added only on those elements where there is distributed loads

Modeling Beams on Elastic Supports

- In some engineering applications, and in case of shafts there are elastic supports like walls or soil like medium. In case of shafts there are ball or journal bearings as supports.
- A node can be considered for the location of single row ball bearing with stiffness k_B added to the diagonal location of the vertical degree of freedom.
- Rotational (moment) degree of freedom is considered for journal and roller bearings.
- In wide journal bearings and Winkler foundations, *s*, stiffness per unit length of supporting medium is considered. This gives rise to the additional term in total potential energy as

$$\frac{1}{2}\int_0^l sv^2\,dx$$

Beam Elastic Supports

Because of the additional term in the total potential energy

$$\frac{1}{2}\int_0^l sv^2\,dx$$

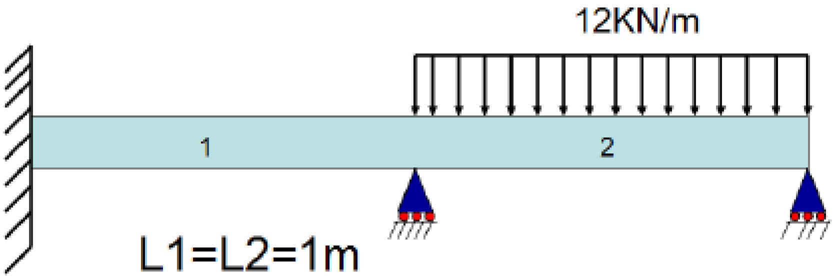
- We will have the modified discretized stiffness terms as follows $\frac{1}{2}\sum_e q^T s\int_e H^T H\,dx q$ The stiffness term is

$$k_s^e = s\int_e H^T H\,dx = \frac{sl_e}{2}\int_{-1}^{+1} H^T H\,d\xi$$

On integration we get

$$k_s^e = \frac{sl_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

EXAMPLE:



L1=L2=1m

E = 200GPa

$I = 4 \times 10^6 N/mm^4$

Solution:

Let's model the given system as 2 elements 3 nodes finite element model each node having 2 dof. For each element determine stiffness matrix.

$$K_1 = 8 \times 10^5 \begin{array}{c} \\ \end{array}\begin{array}{cccc|c} 1 & 2 & 3 & 4 & \\ \hline 12 & 6 & -12 & 6 & 1 \\ 6 & 4 & -6 & 2 & 2 \\ -12 & -6 & 12 & -6 & 3 \\ 6 & 4 & -6 & 4 & 4 \end{array} \qquad K_2 = 8 \times 10^5 \begin{array}{cccc|c} 3 & 4 & 5 & 6 & \\ \hline 12 & 6 & -12 & 6 & 3 \\ 6 & 4 & -6 & 2 & 4 \\ -12 & -6 & 12 & -6 & 5 \\ 6 & 4 & -6 & 4 & 6 \end{array}$$

Global stiffness matrix

$$K = 8 \times 10^5 \begin{array}{cccccc|c} 1 & 2 & 3 & 4 & 5 & 6 & \\ \hline 12 & 6 & -12 & 6 & 0 & 0 & 1 \\ 6 & 4 & -6 & 2 & 0 & 0 & 2 \\ -12 & -6 & 24 & 0 & -12 & 6 & 3 \\ 6 & 2 & 0 & 8 & -6 & 2 & 4 \\ 0 & 0 & -12 & -6 & 12 & -6 & 5 \\ 0 & 0 & 6 & 2 & -6 & 4 & 6 \end{array}$$

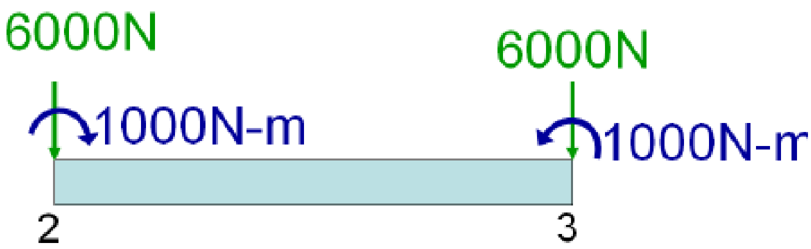
Load vector because of UDL

Element 1 do not contain any UDL hence all the force term for element 1 will be zero.

ie

$$F_1 = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

For element 2 that has UDL its equivalent load and moment are represented as



ie

$$F_2 = \begin{pmatrix} F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

Global load vector:

$$F = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

From KQ=F we write

$$8 \times 10^5 \begin{pmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

At node 1 since its fixed both q1=q2=0

node 2 because of roller q3=0

node 3 again roller ie q5= 0

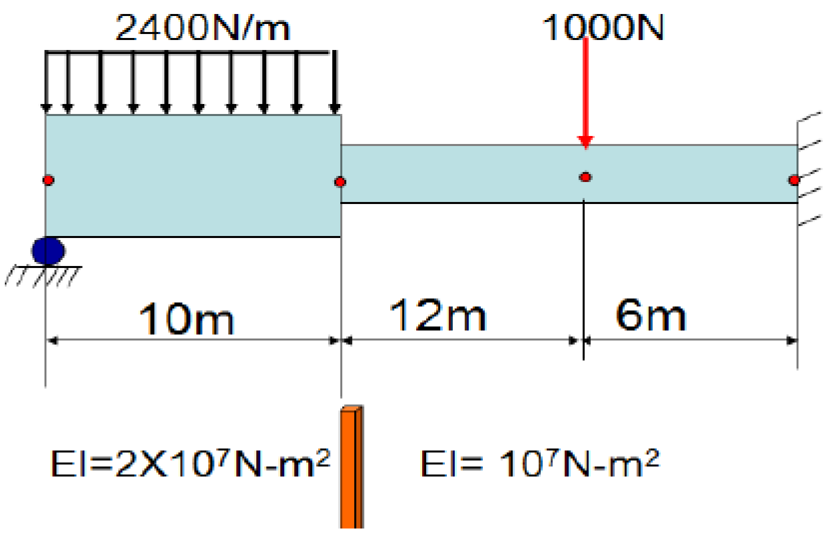
By elimination method the matrix reduces to 2 X 2 solving this we have Q4= -2.679×10^{-4}mm and Q6 = 4.464×10^{-4}mm

To determine the deflection at the middle of element 2 we can write the displacement function as

$$V(\xi) = H_1 q_3 + H_2 q_4 \frac{L_e}{2} + H_3 q_5 + H_4 q_6 \frac{L_e}{2}$$

= -0.089mm

Example:



Solution: Let's model the given system as 3 elements 4 nodes finite element model each node having 2 dof. For each element determine stiffness matrix. Q1, Q2……Q8 be nodal displacements for the entire system and F1……F8 be nodal forces.

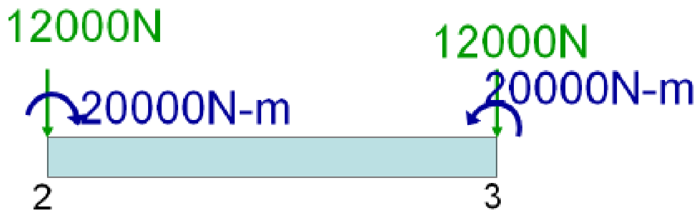
$$K_1 = \frac{2X\,10^7}{10^3}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 60 & -12 & 60 \\ 60 & 400 & -60 & 200 \\ -12 & -60 & 12 & -60 \\ 60 & 200 & -60 & 400 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad K_2 = \frac{10^7}{12^3}\begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 72 & -12 & 72 \\ 72 & 576 & -72 & 288 \\ -12 & -72 & 12 & -72 \\ 72 & 288 & -72 & 576 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$K_3 = \frac{10^7}{6^3}\begin{array}{c} \begin{array}{cccc} 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 12 & 36 & -12 & 36 \\ 36 & 14 & -36 & 72 \\ -12 & -36 & 12 & -36 \\ 36 & 72 & -36 & 144 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Global stiffness matrix:

$$K = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} \\ \\ & & & 8 \times 8 & & & \\ \\ \\ \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Load vector because of UDL:
For element 1 that is subjected to UDL we have load vector as



ie

$$F_1 = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \end{pmatrix} = \begin{pmatrix} -12000 \\ -20000 \\ -12000 \\ 20000 \end{pmatrix}$$

Element 2 and 3 does not contain UDL hence

$$F_2 = \begin{pmatrix} F3 \\ F4 \\ F5 \\ F6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad F_3 = \begin{pmatrix} F5 \\ F6 \\ F7 \\ F8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

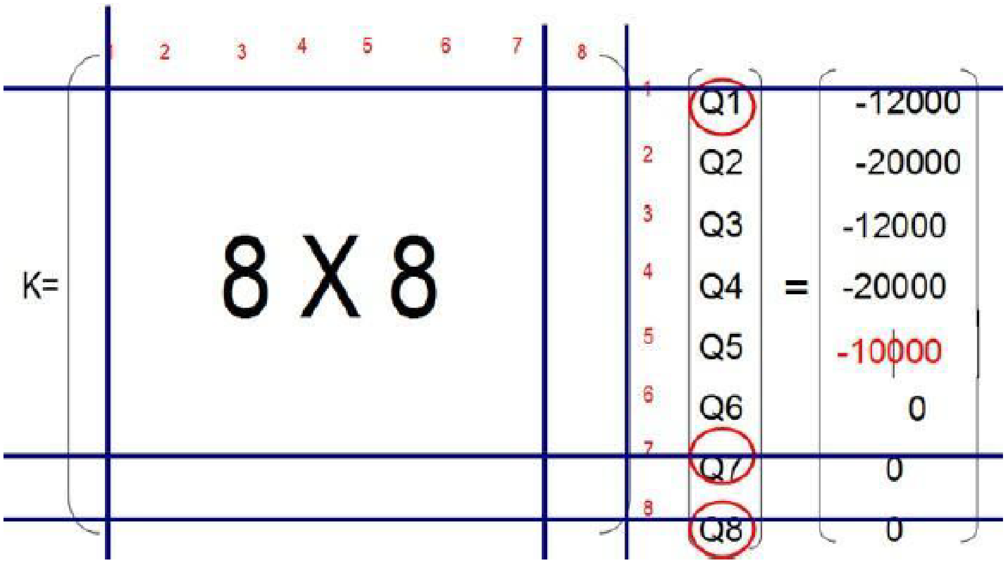
Global load vector:

$$F = \begin{pmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \\ F7 \\ F8 \end{pmatrix} = \begin{pmatrix} -12000 \\ -20000 \\ -12000 \\ -20000 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

And also we have external point load applied at node 3, it gets added to F5 term with negative sign since it is acting downwards. Now F becomes,

$$F=\begin{Bmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \\ F7 \\ F8 \end{Bmatrix} = \begin{Bmatrix} -12000 \\ -20000 \\ -12000 \\ -20000 \\ 0 \;\; -10000 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

From KQ=F



At node 1 because of roller support q1=0

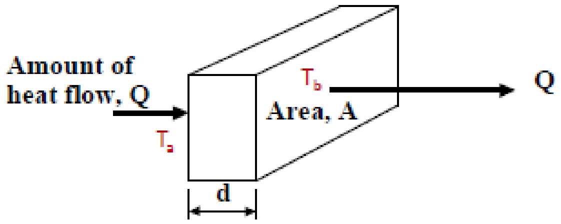
Node 4 since fixed q7=q8=0

After applying elimination and solving the matrix we determine the values of q2, q3, q4, q5 and q6.

Heat Transfer Analysis

Fourier law of heat conduction

A natural phenomenon is that heat flows in a solid is possible only with temperature gradients with heat from the locations at higher temperature to the locations with lower temperature. Consequently, heat will flow from the left side to the right side of the slab if we maintain the situation of $T_a > T_b$ with T_a and T_b being the temperature at left and right faces of the slab respectively as illustrated below:



$$Q \propto \frac{A(T_a - T_b)t}{d} \quad \Rightarrow \quad Q = k\frac{A(T_a - T_b)t}{d}$$

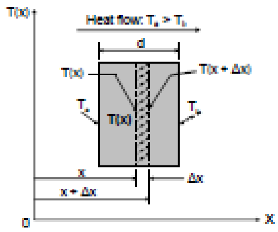
where K = Thermal conductivity of material with units of: Btu/in-s-°F in the traditional system, or w/m-°C in the SI system.

Fourier law of heat conduction-cont'd

Instead of total heat flow, a more commonly used terminology in engineering analysis is "heat flux" defined as "heat flow in solid per unit area and time." Mathematically, it is expressed as:

$$q = \frac{Q}{At} = k\frac{(T_a - T_b)}{d}$$ for heat flow in a solid slab – a vector quantity

For continuous variation of temperature between the two faces and let the coordinate along the length of the slab be x-axis, we will have the above expression in the form of:



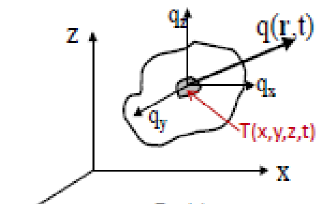
$$q = k\frac{T(x) - T(x+\Delta x)}{\Delta x} = -k\frac{T(x+\Delta x) - T(x)}{\Delta x}$$

with "contiguous" variation of temperature, the following expression prevails:

$$q = q(x) = -k\frac{dT(x)}{dx} \qquad (5.1)$$

Equation (5.1) is the mathematical expression of the "Fourier Law of Heat Conduction"

Fourier law of heat conduction in 3-D space



Position vector:
r: (x,y,z)

$$q(\mathbf{r},t) = -k\nabla T(\mathbf{r},t) \qquad (5.2)$$

with components: $$q_x = -k_x \frac{\partial T(x,y,z,t)}{\partial x} \qquad (5.3a)$$

$$q_y = -k_y \frac{\partial T(x,y,z,t)}{\partial y} \qquad (5.3b)$$

$$q_z = -k_z \frac{\partial T(x,y,z,t)}{\partial z} \qquad (5.3c)$$

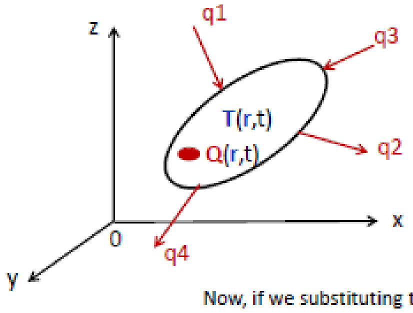
where k_x, k_y and k_z are the thermal conductivity of the material along the respective x-, y- and z-directions.

For isotropic materials, we will have $k = k_x = k_y = k_z$

The resultant total heat flux in the solid in Equation (5.2) is the vector sum of the components in Equation (5.3) To be:

$$q(x,y,z,t) = \sqrt{q_x^2 + q_y^2 + q_z^2}$$

Heat Conduction Equation in Solids



Given a solid situated in a space defined by a coordinate system (r,t) or (x,y,z,t)

Heat fluxes in and out of the solid by q1, q2, q3,........, and heat generated in the solid by the amount Q(x,y,z,t) per unit volume and unit time.

There will be induced temperature distribution (or temperature field) in the solid by T(r,t) or T(x,y,z,t) in the solid.

The heat conduction equation was derived using the Fourier law of heat conduction and on the basis of law of conservation of energies

$$-\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right) + Q = \rho c \frac{\partial T}{\partial t} \qquad (5.4)$$

Now, if we substituting the heat fluxes shown in Equations (5.2) and (5.3) into the above expression to yield:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right) + Q(x,y,z,t) = \rho c \frac{\partial T(x,y,z,t)}{\partial t} \qquad (5.5)$$

Equation (5.4) is the heat conduction equation for solids, in which ρ is the mass density and c is the specific heat of the material

For steady-state heat conduction in the solid:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right) + Q(x,y,z,t) = 0 \qquad (5.6)$$

The term Q(x,y,z,t) in both Equations (5.4) and (5.5) is the heat GENERATED by the solid, such as by Ohm's heating or nuclear fission

Heat Conduction Equation in Solids with specific conditions

The Heat conduction equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=\rho c\frac{\partial T(x,y,z,t)}{\partial t} \quad (5.5)$$

The boundary conditions:

(1) Specified temperature on the boundary surface S_1: $T_s = T_1(x,y,z,t)$ on S_1 (5.7a)

(2) Specified heat flow on the boundary surface S_2:

$q_x n_x + q_y n_y + q_z n_z = -q_s$ on S_2 (n_x=cosine to outward normal line in x-direction) (5.7b)

(3) Specified convective boundary condition on the boundary surface S_3:

$q_x n_x + q_y n_y + q_z n_z = h(T_s - T_f)$ on S_2 (5.7c)

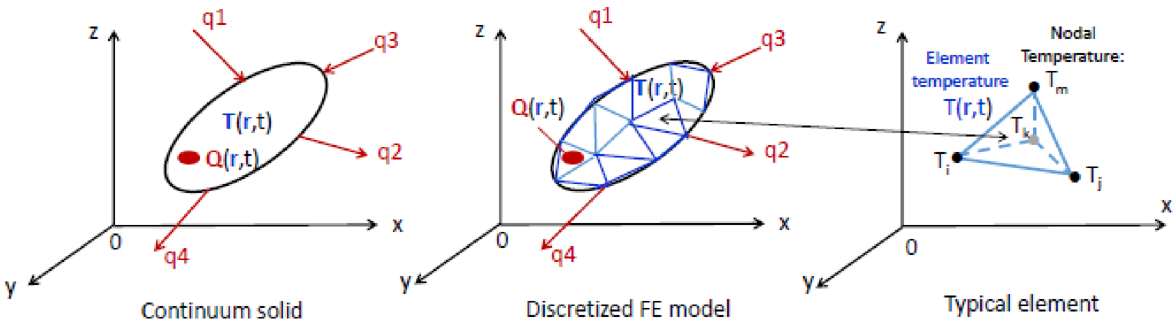
The initial conditions: $T(x,y,z,0) = T_o(x,y,z)$ (5.7d)

In the above boundary conditions, q_s in Equation (5.7b) is the heat flux across the boundary from external sources, and h is the heat transfer coefficient of the surrounding fluid at bulk fluid temperature T_f for convective boundary condition over surface S_3.

Finite element formulation of heat conduction in solid structures

The primary unknown quantity in finite element analysis of heat conduction in solid structures is the **TEMPERATURE** in the elements and NODES.

As usual, the very first step in FE analysis is to discretize the continuum structure into discretized FE model such as illustrated below:



Finite element formulation of heat conduction in solid structures – cont'd

The Interpolation Function, [N(x,y,z]:

The same definition of interpolation function for stress analysis is used for the heat conduction analysis, i.e.:

Element Temperature, T	=	Interpolation Function [N(x,y,z)]	×	Nodal Temperature {T}	(5.8)

where the interpolation function: $[N(x,y,z)] = \{ N_i \quad N_j \quad N_k \quad N_m \}$ (5.9)

The nodal temperature: $\{T\} = \{T_i \quad T_j \quad T_k \quad T_m\}^T$ (5.10)

The temperature gradients in the element may be obtained in terms of nodal temperature by differentiate the relationship in Equation (5.8) as:

$$\begin{Bmatrix} \frac{\partial T(x,y,z)}{\partial x} \\ \frac{\partial T(x,y,z)}{\partial y} \\ \frac{\partial T(x,y,z)}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_m}{\partial z} \end{bmatrix} \{T\} = [B]\{T\} \tag{5.11}$$

where the matrix [B] has the form:

$$[B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_m}{\partial z} \end{bmatrix} \tag{5.12}$$

Finite element formulation of heat conduction in solid structures – cont'd

The functional for deriving element equations:

Because the conduction of heat in solids can be completely described by simple differential equations such as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=\rho c\frac{\partial T(x,y,z,t)}{\partial t} \tag{5.5}$$

for transient state, and

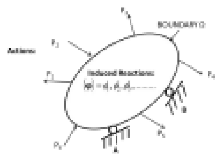
$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=0 \tag{5.6}$$

for steady-state, and the boundary and initial conditions expressed in Equations (5.7), Galerkin method such as described in Chapter 3 will be used to derive the element equation.

We will first review the Galerkin method in the next slide.

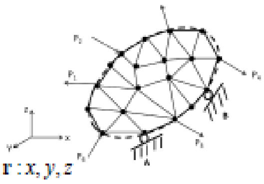
Step 4 – Chapter 3 Galerkin method

In contrast to the Rayleigh-Ritz method, this method is used to derive the element equations for the cases in which specific differential equations with appropriate mathematical expressions for the boundary conditions available for the analytical problems, such as heat conduction and fluid dynamic analyses



Real Situation on solids

Element Φ(r)
Nodal {Φ}
Φ(r) = N(r)Φ



Approximate situation: Discretized Situation with elements

Differential Equation: D(Φ) for the volume V (5.4)

Boundary condition: B(Φ) for the real situation on boundary S (5.5)

Mathematical model: $\int_v \overline{W}\, D(\phi) dv + \int_s \overline{W}\, B(\phi) ds = 0$

where W and $\overline{W}$ are arbitraly weighting functions

Differential Equation: D(N(r)Φ) for the element volume V

Boundary condition: B(N(r)Φ) for the real situation on element boundary

Mathematical model: $\int_v W_j D\left(\sum N_i(\mathbf{r})\phi_i\right) dv + \int_s \overline{W}_j B\left(\sum N_i(\mathbf{r})\phi_i\right) ds = R$

W_j and $\overline{W}_j$ are discretized weighting functions, and R is the Residual

[N(r)] in Equation (5.9)

Galerkin method lets W_j and $\overline{W}_j = N(\mathbf{r})$ and let R to be minimum, or R→ 0 for good discretization, resulting in:

The same element equation: $[K_e]\{q\} = \{Q\}$

Finite element formulation of heat conduction in solid structures – cont'd

Derivation of Element Equation using Galerkin Method

Using the Galerkin method, we can rewrite the basic heat conduction equation in the following form:

$$\int_v \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} - Q + \rho c \frac{\partial T}{\partial t} \right) N_i dv = 0$$

Equation (5.4) Equation (5.9)

By incorporate the boundary conditions in Equations (5.7) in the above equation will result in the element equation with the balanced of heat flus across the boundary and the induced temperature in the element in the following equation:

$$\int_v \rho c \frac{\partial T(x,y,z,t)}{\partial t} N_i dv - \int_v \left[\frac{\partial N_i}{\partial x} \quad \frac{\partial N_i}{\partial y} \quad \frac{\partial N_i}{\partial z} \right] \{q\} dv = \int_v Q N_i dv - \int_{s_1} \{q\}^T \{n\} N_i ds - \int_{s_2} q_s N_i ds - \int_{s_3} h(T - T_f) N_i ds \qquad (5.13)$$

with heat flux across boundaries $\{q\}^T = \{q_x \quad q_y \quad q_z\}$ and the direction cosine to outward normal $\{n\}^T = \{n_x \quad n_y \quad n_z\}$

Finite element formulation of heat conduction in solid structures – cont'd

Derivation of Element Equation using Galerkin Method – cont'd

The heat balance in Equation (5.13) may be lumped to the following element equation:

$$[C]\{\dot{T}\}+([K_c]+[K_h])\{T\}=\{R_T\}+\{R_q\}+\{R_h\} \quad (5.14)$$

where in the coefficient matrices:

$$\textit{The heat capcitnce matrix}:[C]=\int_v \rho c[N]^T[N]dv \quad (5.15a)$$

$$\textit{The condictivity matrix}:[K_c]=\int_v k[B]^T[B]dv \quad (5.15b)$$

$$\textit{The convective matrix}:[K_h]=\int_{S_3} h[N]^T[N]ds \quad (5.15c)$$

and the nodal thermal force matrices:

$$\textit{The heat flux across the boundary } S_1:\{R_T\}=-\int_{S_1}\{q\}^T\{n\}[N]^T ds \quad (5.16a)$$

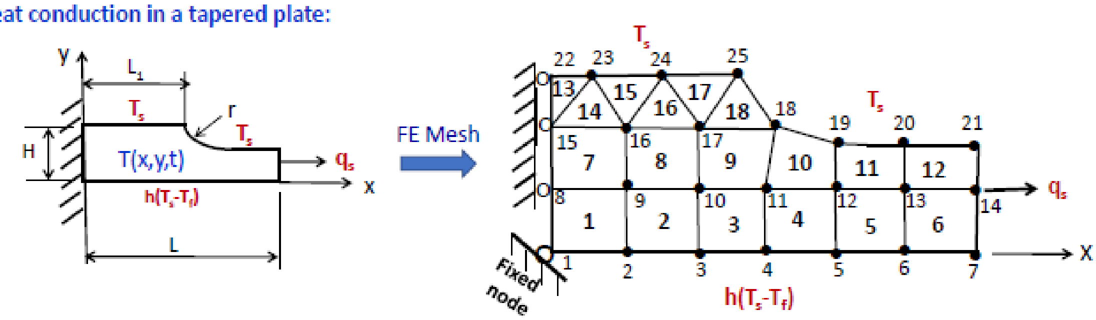
$$\textit{The heat generation matrix}:\{R_Q\}=\int_v Q[N]^T dv \quad (5.16b)$$

$$\textit{The heat flux across the boundary } S_2:\{R_q\}=\int_{S_2} q_s[N]^T ds \quad (5.16c)$$

$$\textit{The convective heat flux cross the boundary } S_3:\{R_h\}=\int_{S_3} hT_f[N]^T ds \quad (5.16d)$$

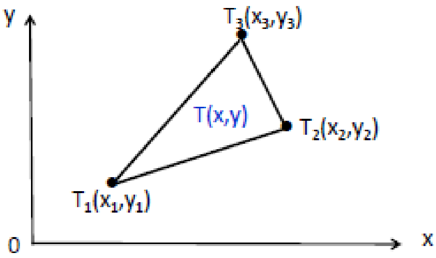
Finite element formulation of heat conduction in solid structures in planes

Heat conduction in a tapered plate:



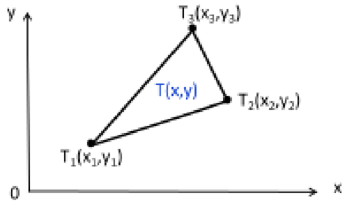
FE formulation in a triangular plate element:

Element temperature: $T(x,y)$
Nodal temperature:
$T_1(x_1,y_1)$; $T_2(x_2,y_2)$; $T_3(x_3,y_3)$



Finite element formulation of heat conduction in solid structures in planes

FE formulation in a triangular plate element-The interpolation function:



We assume the element temperature T(x,y) is represented by a simple linear polynomial function that:

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y = \{1 \quad x \quad y\}\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{Bmatrix} = \{R\}^T\{\alpha\} \tag{5.17}$$

with $\{R\}^T = \{1 \quad x \quad y\}$ (5.18)

where α_1, α_2, and α_3 are constants

Because the coordinates (x_1,y_1), (x_2,y_2) and (x_3,y_3) of the nodes in a FE mode are fixed. We may substitute these coordinates into Equation (5.17) and obtain the following expressions for the corresponding quantities at the three nodes:

$$T_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \quad for \quad Node\,1$$

$$T_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \quad for\ Node\ 2$$

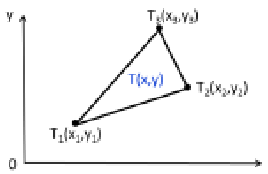
$$T_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \quad for\ Node\ 3$$

or in a matrix form for nodal temperatures: $\{T\} = [A]\{\alpha\}$ (5.19)

and the unknown coefficients $\{\alpha\} = [A]^{-1}\{T\} = [h]\{T\}$ (5.20)

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – the interpolation function - cont'd:



The matrix [A] in Equations (5.19) and (5.20) contains the coordinates of the three nodes as:

$$[A] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

The inversion of matrix $[A]^{-1}$ = [h] can be performed to give:

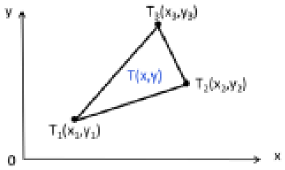
$$[h] = \frac{1}{|A|}\begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \tag{5.21}$$

where $|A|$ Is the determinant of the element of matrix [A] $= (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) = 2A$

with A= the area of triangle made by $(T_1T_2T_3)$

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – the interpolation function - cont'd:



By substituting (5.21) into (5.20) and then (5.19), the element quantity represented by T(x,y) can be made to equal the corresponding nodal quantities {T}: T_1, T_2, T_3 to be:

$$T(x,y) = \{R\}^T [h] \{T\} \qquad (5.22)$$

We will thus have the interpolation function: N(x,y) = $\{R\}^T[h]$ with $\{R\}^T = \{1 \quad x \quad y\}$ in Equation (5.18) and [h} given in Equation (5.21)

We thus have the relationship between the element quantity to the nodal quantifies by the following expression:
T(x,y) = {N(x,y)} {T}

or express the above equation in the form according to Equation (5.8) as:

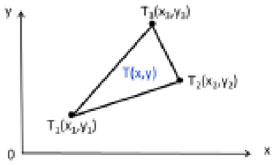
$$\{T(x,y)\} = \{N_1(x,y) \quad N_2(x,y) \quad N_3(x,y)\} \begin{Bmatrix} T_1(x_1,y_1) \\ T_2(x_2,y_2) \\ T_3(x_3,y_3) \end{Bmatrix} \qquad (5.23)$$

with $N_1 = \frac{1}{2A}(a_1 + b_1 x + c_1 y)$, $N_2 = \frac{1}{2A}(a_2 + b_2 x + c_2 y)$, $N_3 = \frac{1}{2A}(a_3 + b_3 x + c_3 y)$ (5.24)

and $2A = x_2 y_3 + x_3 y_1 + x_1 y_2 - x_2 y_1 - x_3 y_2 - x_1 y_3$

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – the interpolation function - cont'd:



$$\{T(x,y)\} = \{N_1(x,y) \quad N_2(x,y) \quad N_3(x,y)\} \begin{Bmatrix} T_1(x_1,y_1) \\ T_2(x_2,y_2) \\ T_3(x_3,y_3) \end{Bmatrix} \qquad (5.23)$$

with $N_1 = \frac{1}{2A}(a_1 + b_1 x + c_1 y)$, $N_2 = \frac{1}{2A}(a_2 + b_2 x + c_2 y)$, $N_3 = \frac{1}{2A}(a_3 + b_3 x + c_3 y)$

$$a_1 = (x_2 y_3 - x_3 y_2) \quad b_1 = (y_2 - y_3) \quad c_1 = (x_3 - x_2) \qquad (5.25a)$$

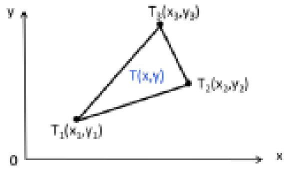
$$a_2 = (x_3 y_1 - x_1 y_3) \quad b_2 = (y_3 - y_1) \quad c_2 = (x_1 - x_3) \qquad (5.25b)$$

$$a_3 = (x_1 y_2 - x_2 y_1) \quad b_3 = (y_1 - y_2) \quad c_3 = (x_2 - x_1) \qquad (5.25c)$$

$$2A = (x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) = 2 \times \textit{the area of the element}(A) \textit{ made of triangle } (T_1 T_2 T_3)$$

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – The element coefficient matrix:



The conductivity matrix [Kc]:

By following Equation (5.15b), we have the conductivity matrix for a triangular plate element to be:

$$[K_c] = \int_A k[B]^T[B]dxdy \qquad (5.26)$$

The temperature gradient matrix [B] can be obtained by the following formulation:

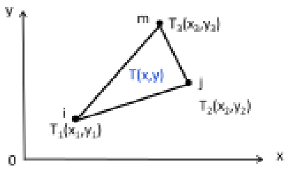
$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_{31}}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \qquad (5.27)$$

We may obtain the conductivity matrix by substituting Equation (5.27) into Equation (5.26), leading to:

$$[K_c] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix} \qquad (5.28)$$

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – The element equations:



As in the case of stress analysis in chapter 4, the element equations for heat conduction solids of plenary geometry may be shown to take the form:

$$[K_e]\{T\} = \{q\} \qquad (5.29)$$

where $[K_e]$ = coefficient matrix in Equation (4.28), {T} = nodal temperature, and {q} = thermal forces at the nodes

The thermal forces at nodes are: $\{q\} = \{f_Q\} + \{f_q\} = (f_h\}$ (5.30)

in which $\{f_Q\}$ = heat generation in the solid with $\{f_Q\} = \int_v [N]^T Q dv = Q\int_v [N]^T dv = \frac{Qv}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$ (5.31a)

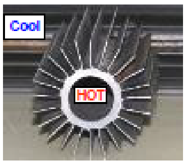
$\{f_q\}$ = heat flux across boundary with $\{f_q\} = \int_{S_2} [N]^T q ds = \int_{S_2} q \begin{Bmatrix} N_i \\ N_j \\ N_m \end{Bmatrix} ds$ (5.31b)

$$\frac{qL_{i-j}t}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix} \text{ for side i-j} \qquad \frac{qL_{j-m}t}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix} \text{ for side j-m} \qquad \frac{qL_{m-i}t}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix} \text{ for side m-i}$$

where t = thickness of the plane

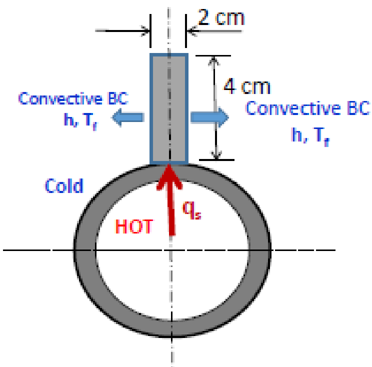
$\{f_h\}$ = convective heat flux across boundary with $\{f_h\} = \int_{S_3} [N]^T hT_f ds$ (5.31c)

Example 5.1



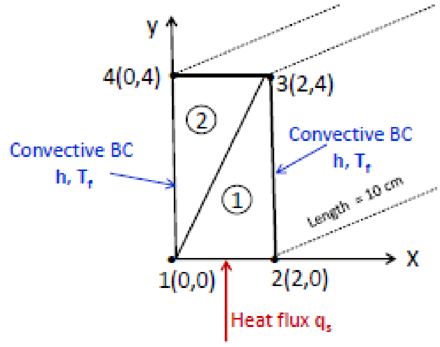
Use finite element method to determine the temperature variation across the thickness of longitudinal fins of a tubular heat exchanger as shown in the figure on the left. The heat exchanger is designed to heat up the cold fluid outside the tube by the hot fluid circulating inside the tube. The cross-section of a single fin is illustrated in the figure shown in lower-left of this slide.

The fin is made of aluminum with the properties: Mass density ρ = 2.7 g/cm³ , Specific heat c = 0.942 J/g-°C, and thermal conductivity k = 2.36 W/cm-°C



Tube with one longitudinal fin

The discretized FE model of the fin cross-section is shown below:



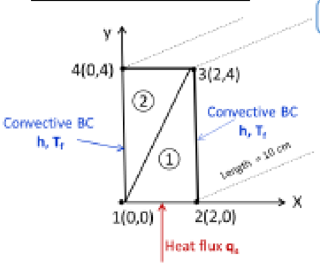
Boundary conditions:

q_s = 10 kW/m²

h = 20 W/m²-°C

T_f = 40°C

Example 5.1-cont'd



Interpolations functions for Elements:

We will use Equations (5.25a,b,c) to determine the constant coefficients a_i, b_i and c_i (I = 1,2,3) for each element. These coefficients will then be used to express the interpolation function of both Element 1 and 2, as in Equation (5.23).

We realize the following nodal coordinates in the FE model of the fin:

<u>For element 1 (Node 1, 2 and 3):</u>

$x_1 = 0$, $y_1 = 0$; $x_2 = 2$, $y_2 = 0$; $x_3 = 2$, $y_3 = 4$

The area A of the cross-section area of Element 1 is computed by using the expression:

$$2A = (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) = (0\times0 - 2\times0) + (2\times4 - 2\times0) + (2\times0 - 0\times4) = 8$$

This leads to A = 4 cm²

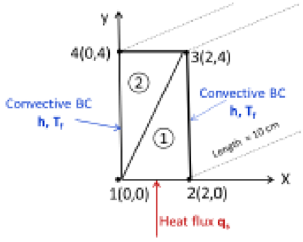
We will further compute the constant coefficients by the following expressions:

$$a_1 = (x_2y_3 - x_3y_2) = 2\times2 - 2\times0 = 4 \quad b_1 = (y_2 - y_3) = 0 - 4 = -4 \quad c_1 = (x_3 - x_2) = 2 - 2 = 0$$

$$a_2 = (x_3y_1 - x_1y_3) = 4\times0 - 0\times4 = 0 \quad b_2 = (y_3 - y_1) = 4 - 0 = 4 \quad c_2 = (x_1 - x_3) = 0 - 2 = -2$$

$$a_3 = (x_1y_2 - x_2y_1) = 0\times0 - 2\times0 = 0 \quad b_3 = (y_1 - y_2) = 0 - 0 = 0 \quad c_3 = (x_2 - x_1) = 2 - 0 = 2$$

Example 5.1-cont'd



Interpolations functions for Elements:

For Element 2 (Node 1, 3 and 4):

$x_1 = 0,\ y_1 = 0;\ x_3 = 3,\ y_3 = 4;\ x_4 = 0,\ y_4 = 4$

The area A of the cross-section area of Element 2 is the same as of Element 1 = 4 cm². the constant coefficients are determined the same way as for those in Element 1.

$$a_1 = (x_2y_3 - x_3y_2) = 3\times4 - 0\times4 = 12 \quad b_1 = (y_2 - y_3) = 4 - 4 = 0 \quad c_1 = (x_3 - x_2) = 0 - 3 = -3$$
$$a_2 = (x_3y_1 - x_1y_3) = 0\times0 - 0\times4 = 0 \quad b_2 = (y_3 - y_1) = 4 - 0 = 4 \quad c_2 = (x_1 - x_3) = 0 - 0 = 0$$
$$a_3 = (x_1y_2 - x_2y_1) = 0\times4 - 3\times0 = 0 \quad b_3 = (y_1 - y_2) = 0 - 4 = -4 \quad c_3 = (x_2 - x_1) = 3 - 0 = 3$$

We will thus have the interpolation functions for both element 1 and 2 by substituting the constant coefficients into Equation (5.23):

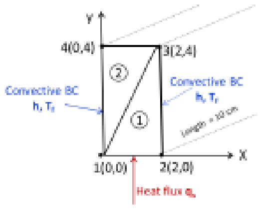
For Element 1:

$$N_1 = \frac{1}{2\times4}(4 + 4x + 0\times y) = 0.5 + 0.5x$$
$$N_2 = \frac{1}{2\times4}(0 + 4x - 2y) = x - 0.25y \quad \text{Leads to:} \quad \{T_e^1\} = \{(0.5+0.5x) \quad (x - 0.5y \quad 0.25y\}\begin{Bmatrix}T_1\\T_2\\T_3\end{Bmatrix} \qquad \text{(a)}$$
$$N_3 = \frac{1}{2\times4}(0 + 0\times x + 2y) = 0.25y$$

For Element 2:

$$N_1 = \frac{1}{2\times4}(12 + 0\times x - 3y) = 1.5 - 0.375y$$
$$N_3 = \frac{1}{2\times4}(0 + 4x + 0\times y) = 0.5x \quad \text{Leads to:} \quad \{T_e^2\} = \{(1.5 - 0.375y) \quad 0.5x \quad (-0.5x + 0.375y)\}\begin{Bmatrix}T_1\\T_3\\T_4\end{Bmatrix} \qquad \text{(b)}$$
$$N_4 = \frac{1}{2\times4}(0 - 4x + 3y) = -0.5x + 0.375y$$

Example 5.1-cont'd



Element coefficient matrices $[K_e]$:

We will use Equation (5.28) to derive these matrices.

$$[K_e] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix} \qquad (5.28)$$

For Element 1:

Node: 1 2 3

$$[K_e^1] = \frac{2.36}{4\times4^2}\begin{bmatrix} (-4)^2 + 0^2 & (-4)(4) + (0)(-2) & (-4)(0) + (0)(2) \\ (-4)(4) + (0)(-2) & 4^2 + (-2)^2 & (4)(0) + (-2)(2) \\ (-4)(0) + (0)(2) & (4)(0) + (-2)(2) & 0^2 + (2)^2 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.6 & 0 \\ -0.6 & 1.2 & -0.15 \\ 0 & -0.15 & 0.15 \end{bmatrix} \begin{matrix}1\\2\\3\end{matrix} \qquad \text{(C)}$$

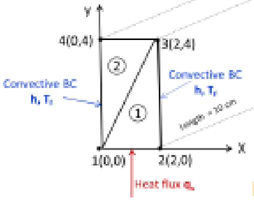
For Element 2:

Node: 1 3 4

$$[K_e^2] = \frac{2.36}{4\times4^2}\begin{bmatrix} 0^2 + (-2)^2 & (0)(4) = (-2)(0) & (0)(-4) + (-2)(2) \\ (0)(4) + (-2)(0) & 4^2 + 0^2 & 4(-4) + (0)(2) \\ (0)(-4) + (-2)(2) & (4)(-4) + (0)(2) & (-4)^2 + (2)^2 \end{bmatrix} = \begin{bmatrix} 0.1475 & 0 & -0.1475 \\ 0 & 0.6 & -0.6 \\ -0.1475 & -0.6 & 0.7375 \end{bmatrix} \begin{matrix}1\\3\\4\end{matrix} \qquad \text{(d)}$$

Example 5.1-cont'd

Assembly of element coefficient matrices for Overall coefficient (conductance) matrix



We need to assemble the element coefficient matrices to construct the overall structure coefficient matrix by summing up the two element coefficient matrices. We need to add the elements for the nodes that are shared by various elements. In the present case, we have Node 1 and 3 shared by both these two elements. We establish the following "map" for assembling the overall coefficient matrix [K]:

Elements in $[K_e^1]$

Node	1	2	3	4
1	●	●	●	
2	●	●	●	
3	●	●	●	
4				

+

Elements in $[K_e^2]$

Node	1	2	3	4
1	◊		◊	◊
2				
3	◊		◊	◊
4	◊		◊	◊

=

Node	1	2	3	4 for the $[K_c]$ matrix
1	●◊	●	●◊	◊
2	●	●	●	0
3	●◊	●	●◊	◊
4	◊	0	◊	◊

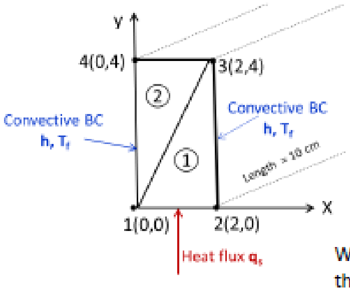
where ●=element in the matrix in Equation (c), and ◊ = elements in Equation (d)

We thus have the **overall coefficient (or conductance) matrix** in the form:

$$[K_c] = \begin{bmatrix} 0.7475 & -0.6 & 0 & -0.1473 \\ -0.6 & 1.2 & -0.15 & 0 \\ 0 & -0.15 & 0.75 & -0.6 \\ -0.1473 & 0 & -0.6 & 0.7375 \end{bmatrix} \qquad \text{(e)}$$

Example 5.1-cont'd

Set thermal forces at the nodes



We have the following heat across the boundaries of the fin:

(1) Heat flux entering the fin crossing the line 1-2 with q_s=10 W/cm²
(2) Heat leaving the fin crossing boundary line 2-3 by convection with h = 20 W/m²-°C = 20x10⁻⁴ W/cm²-°C
(3) Heat leaving the fin crossing boundary line 4-1 by convection with h = 20 W/m²-°C = 20x10⁻⁴ W/cm²-°C

The structure has a length, i.e. the thickness t = 10 cm

We will formulate the equivalent nodal thermal forces for the above specified boundary thermal forces according to the formulas of:

$\{f_q\} = \begin{Bmatrix} f_{iq} \\ f_{jq} \end{Bmatrix} = \frac{q_s L_{i-j} t}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ for heat flux cross line i-j line i-j, and $\{f_h\} = \int_{S_3} [N]^T h T_f ds \to \begin{Bmatrix} f_{ih} \\ f_{jh} \end{Bmatrix} = \{N_i \quad N_j\}^T (hT_f)(L_{i-j} t)$ for heat removal by convection

(1) Heat flux entering the fin crossing the line 1-2 with q_s=10 W/cm²:

$$f_{1q} = f_{2q} = q_s \frac{(L_{1-2})(t)}{2} = 10 \frac{2 \times 10}{2} = 100\,W$$

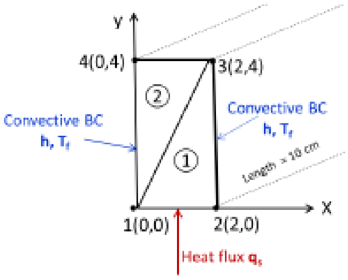
(2) Heat leaving the fin crossing boundary line 2-3 and line 4-1 by convection with h = 20 W/m²-°C = 20x10⁻⁴ W/cm²-°C

$$\begin{Bmatrix} f_{3h} \\ f_{2h} \end{Bmatrix} = \begin{Bmatrix} N_3 \\ N_2 \end{Bmatrix} (hT_f L_{21.0-3} t) = \begin{Bmatrix} (N_3)\big|_{x=2,\,y=0} (hT_f L_{2-3} t) \\ (N_2)\big|_{x=2,\,y=4} (hT_f L_{2-3} t) \end{Bmatrix} = \begin{Bmatrix} 1.5 \times 1.6 = 2.4 \\ 1.0 \times 1.6 = 1.6 \end{Bmatrix} W$$

and

$$\begin{Bmatrix} f_{4h} \\ f_{1h} \end{Bmatrix} = \begin{Bmatrix} N_4 \\ N_1 \end{Bmatrix} (hT_f L_{4-1} t) = \begin{Bmatrix} (N_4)\big|_{x=0,\,y=4} (hT_f L_{4-1} t) \\ (N_1)\big|_{x=0,\,y=0} (hT_f L_{4-1} t) \end{Bmatrix} = \begin{Bmatrix} 1.5 \times 1.6 = 2.4 \\ 0 \times 1.6 = 0 \end{Bmatrix} W$$

Example 5.1-cont'd | Set thermal forces at the nodes-cont'd



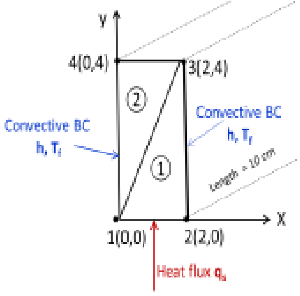
We thus have the thermal force matrix for the 4 nodes as:

$$\{q\}=\begin{Bmatrix} q_1\\ q_2\\ q_3\\ q_4 \end{Bmatrix}=\begin{Bmatrix} f_{1q}+f_{1h}\\ f_{2q}+f_{2h}\\ f_{3h}\\ f_{4h} \end{Bmatrix}=\begin{Bmatrix} 100\\ 100+1.6\\ 2.4\\ 2.4 \end{Bmatrix}=\begin{Bmatrix} 100\\ 101.6\\ 2.4\\ 2.4 \end{Bmatrix}$$

The overall structure heat conduction equation: [K]{T} = {q}

$$\begin{bmatrix} 0.7475 & -0.6 & 0 & -0.1473\\ -0.6 & 1.2 & -0.15 & 0\\ 0 & -0.15 & 0.75 & -0.6\\ -0.1473 & 0 & -0.6 & 0.7375 \end{bmatrix}\begin{Bmatrix} T_1\\ T_2\\ T_3\\ T_4 \end{Bmatrix}=\begin{Bmatrix} 100\\ 101.6\\ 2.4\\ 2.4 \end{Bmatrix} \qquad \text{(f)}$$

Example 5.1-cont'd | Set thermal forces at the nodes-cont'd



$$\begin{Bmatrix} T_1\\ T_2\\ T_3\\ T_4 \end{Bmatrix}=\begin{bmatrix} 0.7475 & -0.6 & 0 & -0.1473\\ -0.6 & 1.2 & -0.15 & 0\\ 0 & -0.15 & 0.75 & -0.6\\ -0.1473 & 0 & -0.6 & 0.7375 \end{bmatrix}^{-1}\begin{Bmatrix} 100\\ 101.6\\ 2.4\\ 2.4 \end{Bmatrix}$$

$$=\begin{bmatrix} 3.8178 & 2.2913 & 3.0596 & 3.2517\\ 2.2913 & 2.2728 & 2.3505 & 2.3699\\ 3.0596 & 2.3505 & 6.5654 & 5.9524\\ 3.2517 & 2.3699 & 5.9524 & 6.8480 \end{bmatrix}\begin{Bmatrix} 100\\ 101.6\\ 2.4\\ 2.4 \end{Bmatrix}=\begin{Bmatrix} 629.72\\ 471.38\\ 574.81\\ 595.47 \end{Bmatrix} \qquad \text{(g)}$$

We thus solve for the nodal temperatures to be:

T_1 = 629.72 °C, T_2 = 471.38 °C, T_3 = 574.81 °C and T_4 = 595.47 °C

Example 5.2 The same Example 13.6 of the textbook on "A First course in the Finite Element Method," 5th edition by Daryl Logan, published by Cenage Learning, 2012

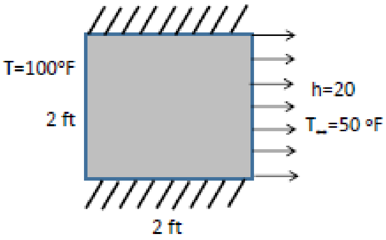


Figure 13-22 2-D body subjected to temperature variation and convection

Problem: "For the 2-D body shown in Figure 13-22, determine the temperature distribution. The temperature at the left side of the body is maintained at 100 °F. The edges on the top and bottom of the body are insulated. There is heat convection from the right side with convective coefficient h = 20 Btu/h-ft²-°F. The free stream temperature is T_∞=50 °F. The coefficients of thermal conductivity are $K_{xx}=K_{yy}$=25 Btu/h-ft-°F. The dimensions are shown in the figure. Assume the thickness to be 1 ft."

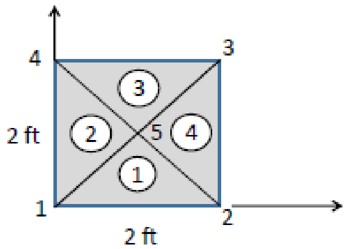


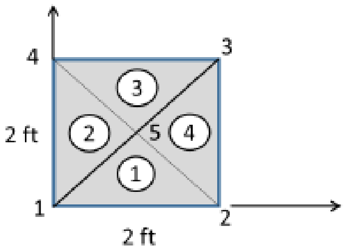
Figure 13-23 Discretized 2-D body of Figure 13-22

Solution: The discretized FE model of the body is shown in Figure 13-23 with 4 elements and 5 nodes. Nodal coordinates are:

$x_1 = 0$, $y_1 = 0$ for Node 1
$x_2 = 2$, $y_2 = 0$ for Node 2
$x_3 = 2$, $y_3 = 2$ for Node 3,
$x_4 = 0$, $y_4 = 2$ for Node 4, and
$x_5 = 1$, $y_5 = 1$ for Node 5

We will formulate the element coefficient matrices for all the 4 elements in Figure 13-23 using the equations (5.25a,b,c) and (5.28)

Example 5.2 – Cont'd



For Element 1: with Nodes 1,2 5

The area 2A is:

$$2A = x_2y_5 + x_5y_1 + x_1y_2 - x_2y_1 - x_5y_2 - x_1y_5 = 2$$ leads to: A = 1 ft².

To find the constant coefficients in Equation (5.25a,b,c):

From Equation (5.25a):

$$a_1 = x_2y_5 - x_5y_2 = 2\times1 - 1\times0 = 2$$
$$b_1 = y_2 - y_5 = 0 - 1 = -1$$
$$c_1 = x_5 - x_2 = 1 - 2 = -1$$

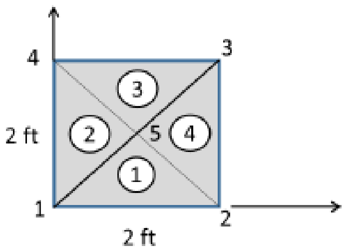
From Equation (5.25b):

$$a_2 = x_5y_1 - x_1y_5 = 1\times0 - 0\times1 = 0$$
$$b_2 = y_5 - y_1 = 1 - 0 = 1$$
$$c_2 = x_1 - x_5 = 0 - 1 = -1$$

From Equation (5.25c):

$$a_3 = x_1y_2 - x_2y_1 = 0\times0 - 2\times0 = 0$$
$$b_3 = y_1 - y_2 = 0 - 0 = 0$$
$$c_3 = x_2 - x_1 = 2 - 0 = 2$$

Example 5.2 – Cont'd



We will use Equation (5.28) to formulate the element coefficient matrix:

$$[K_e] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix} \qquad (5.28)$$

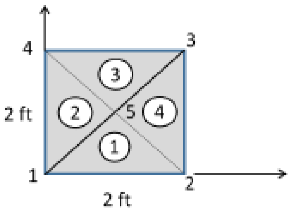
$$[K_e^1] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix}$$

$$= \frac{25}{4(1)^2}\begin{bmatrix} (-1)^2 + (-1)^2 & (-1)(1) + (-1)(-1) & (-1)(0) + (-1)(2) \\ (-1)(1) + (-1)(-1) & (1)^2 + (-1)^2 & (1)(0) + (-1)(2) \\ (-1)(0) + (-1)(2) & (1)(0) + (-1)(2) & (0)^2 + (2)^2 \end{bmatrix}$$

Node 1 2 5

$$= \begin{bmatrix} 12.5 & 0 & -12.5 \\ 0 & 12.5 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix}$$

Example 5.2 – Cont'd



Element coefficient matrices for Element 2, 3 and 4 following the similar approach as shown below:

For Element 2:

Node 1 5 4

$$[K_e^2] = \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 25 & -12.5 \\ 0 & -12.5 & 12.5 \end{bmatrix}$$

For Element 3:

Node 4 5 3

$$[K_e^3] = \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 25 & -12.5 \\ 0 & -12.5 & 12.5 \end{bmatrix}$$

A note from the Instructor:

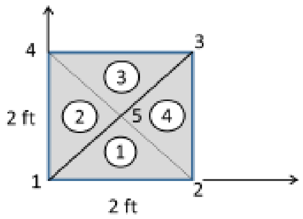
For Element 4:

Node 2 3 5

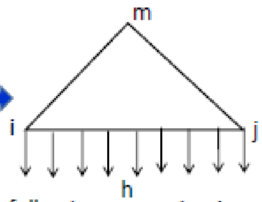
$$[K_e^4] = \begin{bmatrix} 12.5 & 0 & -12.5 \\ 0 & 12.5 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix} \qquad (a)$$

Example 5.2 – Cont'd

Additional heat conductance matrix for convective heat transfer in Element 4



$$[K_h] = \int_{S_3} h[N]^T[N]ds = h\int_{S_3}\begin{bmatrix} N_iN_i & N_iN_j & N_iN_m \\ N_jN_i & N_jN_j & N_jN_m \\ N_mN_i & N_mN_j & N_mN_m \end{bmatrix}$$



For the case with convective heat transfer from Edge i-j, the following expression is used:

$$[K_h] = \frac{h(L_{i-j})(t)}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (5.29)$$

For the current situation, the side that has convective heat transfer is Side 2-3, we will thus have:

$$[K_h^4] = \frac{(20(2)(1))}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

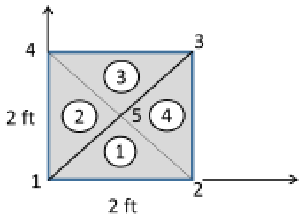
By adding this matrix to the conductance of Element 4 in Equation (a), we obtain the Conductance matrix of Element 4 to be:

Node: 2, 3, 5

$$[K_e^4] = \begin{bmatrix} 25.83 & 6.67 & -12.5 \\ 6.67 & 25.83 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix}$$

Example 5.2 – Cont'd

Assemble the element coefficient matrices for the Overall coefficient matrix by accounting the fact that Node 4 is shred by all 4 elements.



$$[K] = \begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} Btu/h-{}^{o}F \qquad (b)$$

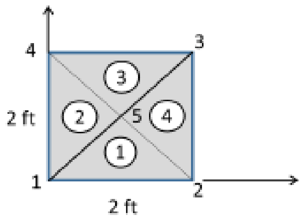
The thermal forces at nodes:

We already know that temperature at Node 1 and 4 are specified to be 100°F

The thermal forces across boundary 2-3 of element 4 is:

$$\{f^4\} = \begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \frac{h(T_\infty)(L_{2-3})(t)}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \frac{(20)(50)(2)(1)}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 1000 \\ 0 \end{Bmatrix} Btu/h$$

Example 5.2 – Cont'd



The overall heat conduction equation becomes:

$$\begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 1000 \\ 1000 \\ 100 \\ 5000^{**} \end{Bmatrix}$$

** = (-25)(100°F)+(-25)(100°F)=-5000°F on the left side of the fifth equation in the left-hand-side of the equation

We may sole the above equations and obtain:

T_2 = 69.33°F, T_3 = 59.33°F and T_5 = 84.62°F with specified T_1=T_4 = 100°F

Summary on Heat Conduction Analysis of Plane Structures by FE Method

1) An overview of heat conduction in 3-D solids was presented in this Chapter with heat conduction equation for the induced temperature distributions in the solids by the sources of: (a) heat generation by the solid, (b) the prescribed surface temperature, (c) specified heat flux across the boundary surfaces, and (d) the convective heat across the boundary surfaces.

2) Finite element formulation of heat conduction in solids is derived using the Galerkin method due to the fact that heat conduction in solids can e described by the heat conduction equations with prescribed boundary conditions by mathematical expressions.

3) Finite element formulations begin with the derivation of interpolation functions [N] = {N_i N_j N_m} for triangular plane elements with Nodes i, j and m. These functions relate the "element temperatures" and the "nodal temperatures."

4) The interpolation functions for the FE analysis were derive on the basis of linear polynomial function for the temperature variations in the element.

5) Special FE formulations of the aforementioned boundary conditions were presented.

6) This chapter only presents the FE formulation for steady-state heat conduction in solids of plane geometry.

UNIT-V
DYNAMIC ANALYSIS

Dynamics is a special branch of mechanics where inertia of accelerating masses must be considered in the force-deflection relationships. **In** order to describe motion of the mass system, a component with distributed mass is approximated by a finite number of mass points. Knowledge of certain principles of dynamics is essential to the formulation of these equations.
Every structure is associated with certain frequencies and mode shapes of free vibration (without continuous application of load), based on the distribution of mass and stiffness in the structure. Any time-dependent external load acting on the structure, whose frequency matches with the natural frequencies of the structure, causes resonance and produces large displacements leading to failure of the structure. Calculation of natural frequencies and mode shapes is there for every important.
In general, for a system with on' degrees of freedom, stiffness 'k' and mass 'm' are represented by stiffness matrix [K] and mass matrix [M] respectively.

Then

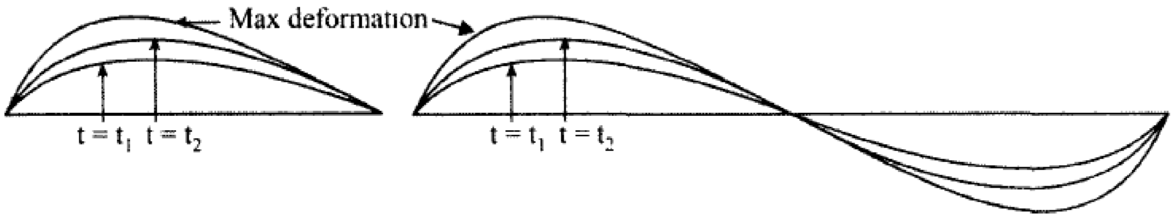
$$([K] - \omega^2 [M]) \{u\} = \{0\}$$
$$([M]^{-1}[K] - \omega^2 [I]) \{u\} = \{0\}$$

Here, [M] is the mass matrix of the entire structure and is of the same order, say n x n, as the stiffness matrix [K]. This is also obtained by assembling element mass matrices in a manner exactly identical to assembling element stiffness matrices. The mass matrix is obtained by two different approaches, as explained subsequently.

A structure with 'n' DOF will therefore have 'n' eigen values and 'n' eigenvectors. Some eigen values may be repeated and some eigen values maybe complex, in pairs. The equation can be represented in the standard form,
$[A]\{x\}_i = \lambda_i \{x\}_i$.
In dynamic analysis, ω_i, indicates ith natural frequency and $\{X\}_I$ indicates ith natural mode of vibration.

A natural mode is a *qualitative* plot of nodal displacements. In every natural mode of vibration, all the points on the component will reach their maximum values at the same time and will pass through zero displacements at the same time. Thus, in a particular mode, all the points of a component will vibrate with the same frequency and their relative displacements are indicated by

the components of the corresponding eigen vector. These relative (or proportional) displacements at different points
on structure remain same at every time instant for undamped free vibration.



Hence, without loss of generality, {u(t)} can be written as {u}.

Since {u} = {O} forms a trivial solution, the homogeneous system of equations

([A] - λ[I]) {u} = {O}

gives a non-trivial solution only when

([A]-λ[I])= {O},

which implies

Det([A] - λ[I]) = o.

This expression, called *characteristic equation,* results in nth order polynomial in A, and will therefore have n roots. For each λ, the corresponding eigenvector {u}. can be obtained from the n homogeneous equations represented by

([K] - λ[M]) {u} = {O}.

The mode shape represented by {u(t)}gives relatives values of displacements in various degrees of freedom.

NORMALIZATION

The equation of motion of free vibrations ([K] $-\omega^2$[M]) {u} = {O} is a system of homogeneous equations (right side vector zero) and hence does not give unique numerical solution.

Mode shape is a set of relative displacements in various degrees of freedom, while the structure is vibrating in a particular frequency and is usually expressed in normalized form, by following one of the
three normalization methods explained here.

(a) The maximum value of anyone component of the eigenvector is equated to 'I' and, so, all other components will have a value less than or equal to ' 1' .

(b) The length of the vector is equated to '1 ' and values of all components are divided by the length of this vector so that each component will have a value less than or equal to '1'.

(c) The eigenvectors are usually normalized so that

$$\{u\}_i^T [M] \{u\}_i = 1 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_i = \lambda_i$$

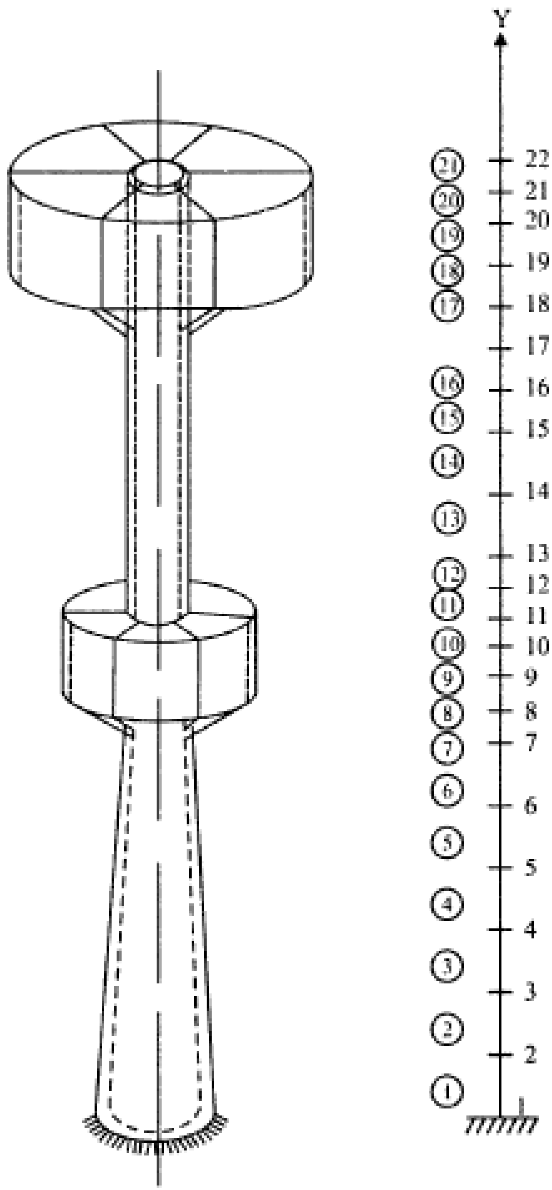
For a positive definite symmetric stiffness matrix of size n x n, the Eigen values are all real and eigenvectors are orthogonal

i.e.,

$$\{u\}_i^T [M] \{u\}_j = 0 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_j = 0 \quad \forall \quad i \neq j$$

MODELLING FOR DYNAMIC ANALYSIS

Solution for any dynamic analysis is an iterative process and, hence, is time -consuming. Geometric model of the structure for dynamic analysis can be significantly simplified, giving higher priority for proper representation of distributed mass. An example of a simplified model of a water storage tank is shown in Fig. Below, representing the central hollow shaft by long beam elements and water tanks at two levels by a few lumped masses and short beam elements of larger moment of inertia.



Actual structure Idealised structure

MASS MATRIX

Mass matrix [M] differs from the stiffness matrix in many ways:

(i) The mass of each element is equally distributed at all the nodes of that element

(ii) Mass, being a scalar quantity, has same effect along the three translational degrees of freedom (u, v and w) and is not shared

(iii) Mass, being a scalar quantity, is not influenced by the local or globa coordinate system. Hence, no transformation matrix is used for converting mass matrix from element (or local) coordinate system to structural (or global) coordinate system.

Two different approaches of evaluating mass matrix [M] are commonly considered.

(a) Lumped mass matrix

Total mass of the element is assumed equally distributed at all the nodes of the element in each of the translational degrees of freedom. Lumped mass is not used for rotational degrees of freedom. Off-diagonal elements of this matrix are all zero. This assumption *excludes dynamic coupling* that exists between different nodal displacements.

Lumped mass matrices [M] of some elements are given here.

Lumped mass matrix of truss element with 1 translational DOF per node along its local X-axis

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Lumped mass matrix of plane truss element in a 2-D plane with2 translational DOF per node (Displacements along X and Y coordinate axes)

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Please note that the same lumped mass is considered in each translational degree of freedom (without proportional sharing of mass between them) at each node.

Lumped mass matrix of a beam element in X-V plane, with its axis along x-axis and with two DOF per node (deflection along Y axis and slope about Z axis) is given below. Lumped mass is not considered in the rotational degrees of freedom.

$$[M]=\frac{\rho AL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

Note that lumped mass terms are not included in 2nd and 4th rows, as well as columns corresponding to rotational degrees of freedom.

Lumped mass matrix of a CST element with 2 DOF per node. In this case, irrespective of the shape of the element, mass is assumed equally distributed at the three nodes. It is distributed equally in all DOF at each node, without any sharing of mass between different DOF

$$[M]=\frac{\rho AL}{3}\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$$

(b) Consistent mass matrix

Element mass matrix is calculated here, *consistent* with the assumed displacement field or element stiffness matrix. [M] is a banded matrix of the same order as the stiffness matrix. This is evaluated using the same

interpolating functions which are used for approximating displacement field over the element. It yields more accurate results but with more computational cost. Consistent mass matrices of some elements are given here.

Consistent mass matrix of a Truss element along its axis (in local coordinate system)

$$\{u\}^T = [u \quad v]$$

$$[N]^T = [N_1 \quad N_2]$$

where, $N_1 = \frac{(1-\xi)}{2}$

and $N_2 = \frac{(1+\xi)}{2}$

$$[M]= \int_v [N]\rho[N]^T \, dV = \int_0^L A[N]\rho[N]^T$$

$$dx = \int_{-1}^{+1} A\,\rho[N][N]^T (\det J)(dx/d\xi)d\xi$$

Here, $x = N_1x_1 + N_2x_2 = \frac{(x_1+x_2)}{2} + \frac{(x_2-x_1)\xi}{2}$

and $dx = \frac{dx}{d\xi}.d\xi = \det J\, d\xi = \left(\frac{L}{2}\right)d\xi$

Using the values of integration in natural coordinate system,

$$[M] = \rho A\left(\frac{L}{2}\right)\int_{-1}^{+1}\begin{bmatrix}(1-\xi)/2\\(1+\xi)/2\end{bmatrix}\left[(1-\xi)/2 \qquad (1+\xi)/2\right]d\xi$$

$$= \frac{\rho AL}{8}\begin{bmatrix}\int(1-\xi)^2 d\xi & \int\left(1-\xi^2\right)d\xi\\ \int\left(1-\xi^2\right)d\xi & \int(1+\xi)^2\, d\xi\end{bmatrix}$$

$$= \frac{\rho AL}{8}\begin{bmatrix}\left(\xi-\xi^2+\xi^3/3\right) & \left(\xi-\xi^3/3\right)\\ \left(\xi-\xi^3/3\right) & \left(\xi+\xi^2+\xi^3/3\right)\end{bmatrix}$$

$$= \frac{\rho AL}{8}\begin{bmatrix}8/3 & 4/3\\ 4/3 & 8/3\end{bmatrix} = \frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Consistent mass *matrix (if a* **Plane** *Truss element,* inclined to global X-axis -Same elements of I-D mass matrix are repeated in two dimensions(along X and Y directions) without sharing mass between them. Mass terms in X and Y directions are uncoupled.

$$[M] = \frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 2\end{bmatrix}$$

Consistent mass matrix of a Space *Truss element,* inclined to X-Y plane) -Same elements of 1-0 mass matrix are repeated in three dimensions (along X, Y and Z directions) without sharing mass between them.

$$[M] = \frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 0 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 0 & 1\\ 1 & 0 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 0\\ 0 & 0 & 1 & 0 & 0 & 2\end{bmatrix}$$

Consistent *mass matrix of a Beam element*

$[M]=\rho A\left(\frac{L}{2}\right)\int\{H\}^T\{H\}d\xi$ with Hermite shape functions $\{H\}$ as used in a beam element.

$$=\frac{\rho AL}{128}\int\begin{bmatrix}2(2-3\xi+\xi^3)\\ L(1-\xi+\xi^2+\xi^3)\\ 2(2+3\xi-\xi^3)\\ L(-1-\xi+\xi^2+\xi^3)\end{bmatrix}\times$$

$$\left[2(2-3\xi+\xi^3)\quad L(1-\xi-\xi^2+\xi^3)\quad 2(2+3\xi-\xi^3)\quad L(-1-\xi+\xi^2+\xi^3)\right]d\xi$$

$$=\frac{\rho AL}{420}\begin{bmatrix}156 & 22L & 54 & -13L\\ 22L & 4L^2 & 13L & -3L^2\\ 54 & 13L & 156 & -22L\\ -13L & -3L^2 & -22L & 4L^2\end{bmatrix}$$

Consistent mass matrix of a CST element in a 2-D plane

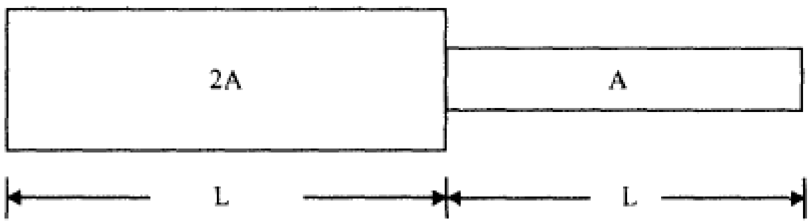
$$[N]^T=\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3\end{bmatrix}$$

$$[M]=\int[N]\rho[N]^T\,dV=t\int[N]\rho[N]^T\,dA$$

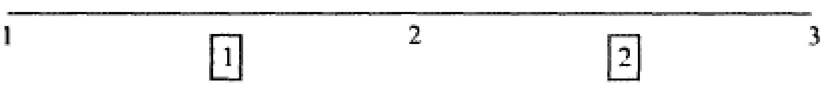
$$=\frac{\rho tA}{12}\begin{bmatrix}2 & 0 & 1 & 0 & 1 & 0\\ & 2 & 0 & 1 & 0 & 1\\ & & 2 & 0 & 1 & 0\\ & & & 2 & 0 & 1\\ & \text{Sym} & & & 2 & 0\\ & & & & & 2\end{bmatrix}$$

Note: Natural frequencies obtained using lumped mass matrix are LOWER than exact values.

Example 1 : Find the natural frequencies of longitudinal vibrations of the unconstrained stepped shaft of areas A and 2A and of equal lengths (L), as shown below.



Solution : Let the finite element model of the shaft be represented by 3 nodes and 2 truss elements (as only longitudinal vibrations are being considered) as shown below.



$$[K]_1 = (2A)\left(\frac{E}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left(\frac{AE}{L}\right)\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix};$$

$$[K]_2 = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Using consistent mass matrix approach

$$[M]_1 = \frac{\rho(2A)L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix};$$

$$[M]_2 = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assembling the element stiffness and mass matrices,

$$[K] = \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix};$$

$$[M] = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Eigenvalues of the equation ([K] – ω^2 [M]) {u} = {O} are the roots of the characteristic equation represented by

$$\begin{vmatrix} 2AE/L - \omega^2 4\rho\rho AL/ & -2AE/L - \omega^2 2\rho\rho AL/ & 0 \\ 2AE/L - \omega^2 2\rho\rho AL/ & 3AE/L - \omega^2 6\rho\rho AL/ & -1AE/L - \omega^2 \rho AL/6 \\ 0 & -AE/L - \omega^2 \rho AL/6 & AE/L - \omega^2 2\rho\rho AL/ \end{vmatrix} = 0$$

Multiplying all the terms by (L/AE)

$$\begin{vmatrix} 2(1-2\beta) & -2(1+\beta) & 0 \\ -2(1+\beta) & 3(1-2\beta) & -(1+\beta) \\ 0 & -(1+\beta) & (1-2\beta) \end{vmatrix} = 0$$

or $\quad 18\,\beta\,(\beta - 2)\,(1 - 2\beta) = 0$

$$\beta = \frac{\rho L^2 \omega^2}{6E}$$

The roots of this equation are

$$\beta = 0,\ 2 \text{ or } \frac{1}{2} \text{ or } \omega^2 = 0,\ \frac{12E}{\rho L^2} \text{ or } \frac{3E}{\rho L^2}$$

Corresponding eigenvectors are obtained from ([K] – ω^2 [M]) {u} = {0} for different values of ω^2 as $[1 \quad 1 \quad 1]^T$ for $\beta = 0$, $[1 \quad 0 \quad -2]^T$ for $\beta = \frac{1}{2}$ and $[1 \quad -1 \quad 1]^T$ for $\beta = 2$.

The first eigenvector implies rigid body motion of the shaft. One component (u I in this example) is equated to '1' and other displacement components (U2 and U3 in this example) are obtained as ratios w.r.t. that component,following one method of normalization. Alternatively, they may also be expressed in other normalized forms.

Note: Static solution for such an unconstrained bar, with rigid body motion, involves a singular [K] matrix and can not be solved for {u}, while dynamic analysis is mathematically possible.

SUMMARY

- A distributed mass system will have as many natural frequencies and mode shapes as the number of DOF, 'n'.
- Free undamped vibrations involve a set of n homogeneous equations. Such equations will not give a unique solution. A mode shape consists of relative displacement values at (n-l) DOF, obtained w.r.t. the chosen displacement value at one DoF. The mode shapes (Eigen vectors) are usually normalized.
- The n natural frequencies may be real or complex (in pairs). Some of them may be zero (indicating rigid body mode) or repeated.
- Only first few frequencies (lower values) are significant and are usually calculated by iterative methods. Hence, a coarse mesh is adequate for dynamic analysis.

UNIT IV

BEAM ELEMENTS & HEAT TRANSFER

LIVE TO LEARN & LEARN TO SHARE

MRCET CAMPUS

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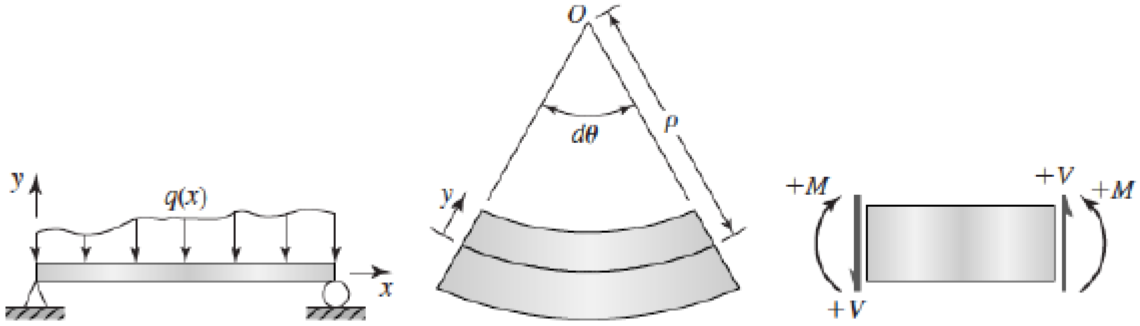
DEPARTMENT OF MECHANICAL ENGINEERING

Beam Elements

- The bar, one-dimensional elements discussed earlier will be now extended to general one-dimensional elements that can take bending.
- We develop here a flexural or beam element using the elementary beam theory.
- To begin with we consider one-dimensional beam that can bend in a plane.
- This will subsequently be extended to include two-plane bending, axial load and torsion

Elementary Beam Theory

- The beam is loaded only in y - direction.
- Deflections of the beam are small compared to the characteristic dimensions of the beam.
- The material of the beam is linearly elastic, isotropic, and homogeneous.
- The beam is prismatic and has an axis of symmetry in the plane of bending.



Beam Stresses and Strains

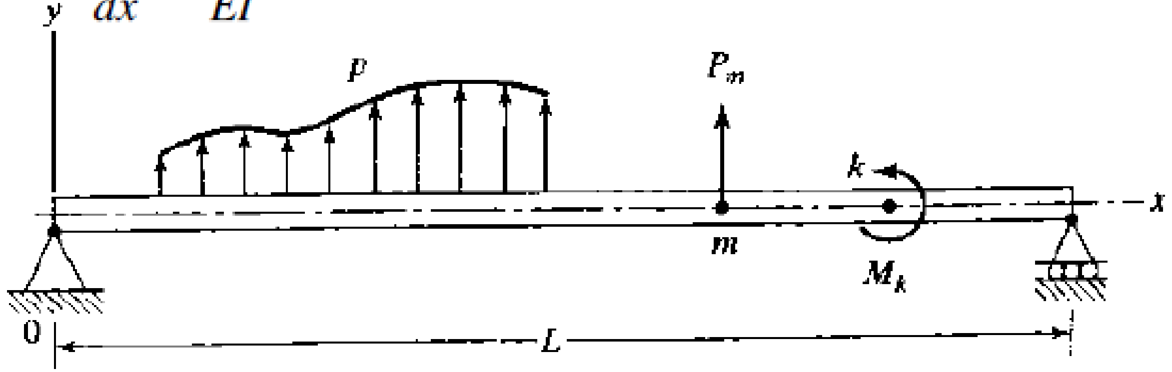
- The stresses and strains in a beam are given as

$$\sigma = -\frac{M}{I}y$$

$$\varepsilon = \frac{\sigma}{E}$$

$$\frac{d^2v}{dx^2} = \frac{M}{EI}$$

Here σ and ε are the normal stress and normal strain respectively. M is the the bending moment at the section. v(x) is deflection of the neutral axis at x. I is the moment of inertia of the section about the neutral axis



Beam Elements- PE Approach

- The potential Energy formulation

$$dU = \frac{1}{2}\int_A \sigma\varepsilon \, dA \, dx = \frac{1}{2}\left(\frac{M^2}{EI^2}\int_A y^2 \, dA\right)dx$$

- In the above equation of elemental beam, we notice that

$$\int_A y^2 \, dA = I, \quad \text{moment of inertia}$$

$$\therefore dU = \frac{1}{2}\frac{M^2}{EI}dx$$

- The total strain energy is obtained by integrating the above as

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx$$

- The potential energy of the beam is, where p is the load per unit length

$$U = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx - \int_0^L pv dx - \sum_m P_m v_m - \sum_k P_k v_k'$$

Beam Elements- Galerkin Approach

- Considering the equilibrium of elemental length, dx

$$\frac{dV}{dx} = p; \qquad \frac{dM}{dx} = V$$

- Combining all the equilibrium equations, we have

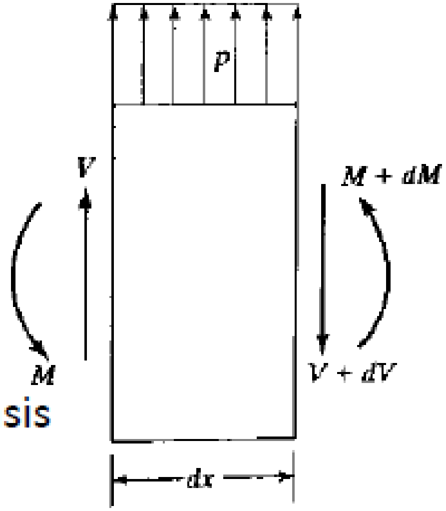
$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - p = 0$$

- Using FE shape functions, an approximate

Solution for v is obtained by Galerkin's as

$$\int_0^L \left[\frac{d}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - p\right]\phi dx = 0$$

Where φ is an arbitrary function with same basis as v.

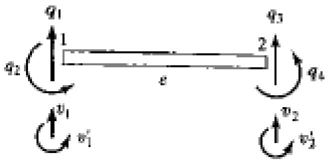


Beam: Finite Element Formulation

From elementary beam theory, the following assumptions are valid:

1. Each beam element is of length L and has two nodes
2. Each element is connected to other element only at nodes
3. Element loading occurs only at nodes

- Here the field variable of interest is the transverse displacement *v*(*x*) of the neutral surface away from its straight, undeflected position.
- The same end displacements, give rise to different beam configurations. Hence we need to take the slope v′ (θ) into consideration.



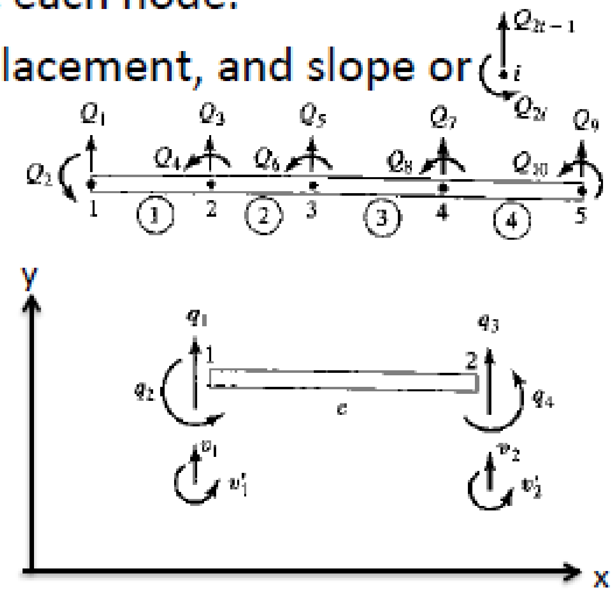
Beam: Finite Element Formulation

- The element length L is aligned with x-coordinate such that $x_1=0$ and $x_2=L$
- The beam is divided into number of flexural elements with typically two degrees of freedom at each node.
- The two dof are the transverse displacement, and slope or rotation (in radians)
- The global displacement vector is

Q=[Q1,Q2, ...Q2n-1] where n is number of nodes.

For an element the local dofs are

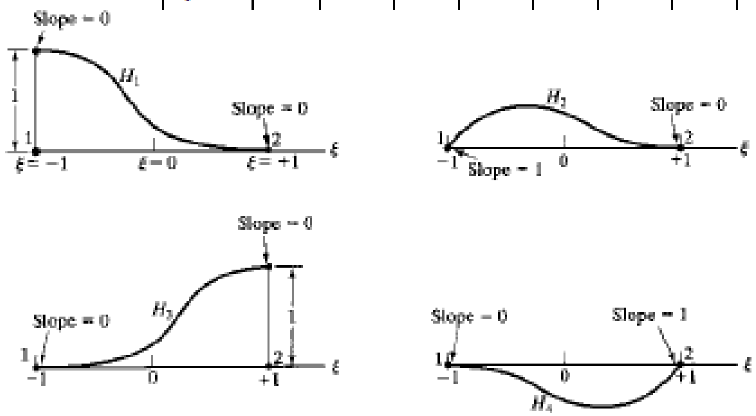
$$\{q\}=[q_1,q_2,q_3,q_4]^T=[v_1,v_1',v_2,v_2']^T$$



Beam Elements- Shape Functions

- The shape functions for interpolating v in an element are defined as $H_i = a_i + b_i\xi + c_i\xi^2 + d_i\xi^3 \qquad i = 1,2,3,4$
- As it is a cubic polynomial it will be c_2 continuous everywhere. The other conditions are summarized in the following table

	H_1	H'_1	H_2	H'_2	H_3	H'_3	H_4	H'_4
ξ=-1	1	0	0	1	0	0	0	0
ξ=-1	0	0	0	0	1	0	0	1



$$\sum_{i=1}^{4} H_i = 1$$

Beam Elements- Shape Functions

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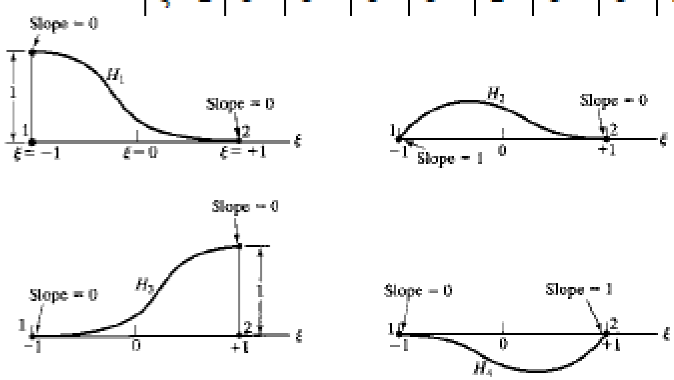
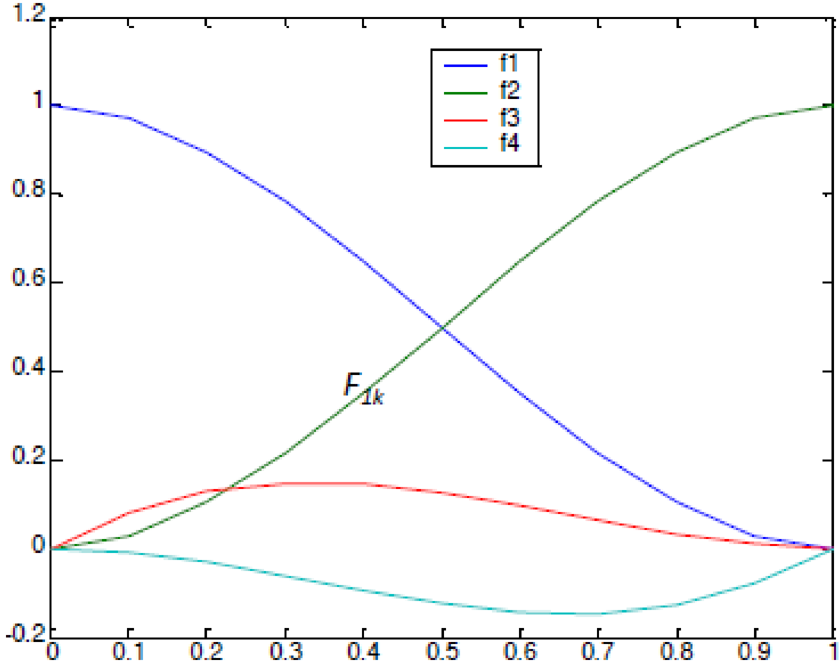
	H_1	H'_1	H_2	H'_2	H_3	H'_3	H_4	H'_4
ξ=-1	1	0	0	1	0	0	0	0
ξ=-1	0	0	0	0	1	0	0	1



$$\sum_{i=1}^{4} H_i = 1$$

Shape Functions



$$F_{1k}(\tau) = 2\tau^3 - 3\tau^2 + 1$$

$$F_{2k}(\tau) = -2\tau^3 + 3\tau^2$$

$$F_{3k}(\tau) = \tau(\tau^2 - 2\tau + 1)t_{k+1}$$

$$F_{4k}(\tau) = \tau(\tau^2 - \tau)t_{k+1}$$

$$\tau = 0 \Rightarrow \qquad F_{1k}(0) = 1,\ F_{2k}(0) = F_{3k}(0) = F_{4k}(0) = 0$$

$$\tau = 1 \Rightarrow \qquad F_{2k}(1) = 1,\ F_{1k}(1) = F_{3k}(1) = F_{4k}(1) = 0$$

$$F_{2k}(\tau) = 1 - F_{1k}(\tau)$$

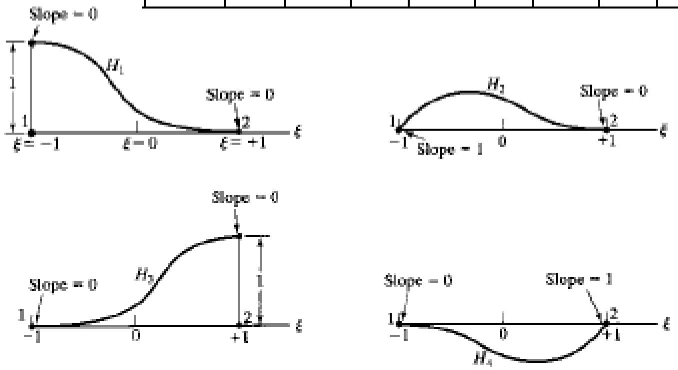
Beam Elements- Shape Functions

- All 4 shape functions are given below

$$H_1 = \frac{1}{4}(1-\xi)^2(2+\xi) \quad = \frac{1}{4}(2-3\xi+\xi^2) \qquad H_2 = \frac{1}{4}(1-\xi)^2(\xi+1) \quad = \frac{1}{4}(1-\xi-\xi^2+\xi^3)$$

$$H_3 = \frac{1}{4}(1+\xi)^2(2-\xi) \quad = \frac{1}{4}(2+3\xi-\xi^3) \qquad H_4 = \frac{1}{4}(1+\xi)^2(\xi-1) \quad = \frac{1}{4}(-1-\xi+\xi^2+\xi^3)$$

	H_1	H'_1	H_2	H'_2	H_3	H'_3	H_4	H'_4
ξ=-1	1	0	0	1	0	0	0	0
ξ=-1	0	0	0	0	1	0	0	1



$$\sum_{i=1}^{4} H_i = 1$$

Shape Functions and Displacements

- Derivatives of functions are given below differentiating w.r.t ξ

$$H_1' = \frac{1}{4}(-3+2\xi) \qquad H_2' = \frac{1}{4}(-1-2\xi+3\xi^2)$$

$$H_3' = \frac{1}{4}(3-3\xi^2) \qquad H_4' = \frac{1}{4}(-1+2\xi+3\xi^2)$$

- These shape functions are used to represent the displacements as

$$v(\xi) = H_1 v_1 + H_2 \left(\frac{dv}{d\xi}\right)_1 + H_3 v_3 + H_4 \left(\frac{dv}{d\xi}\right)_2$$

- The coordinates transform as $x = \frac{1-\xi}{2}x_1 + \frac{1+\xi}{2}x_2 = \frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}\xi$
- Since x_2-x_1=l_e, $dx = \frac{l_e}{2}d\xi$

Shape Functions and Displacements

- Since $x_2-x_1=l_e$, $dx = \frac{l_e}{2} d\xi$

$$\frac{dv}{d\xi} = \frac{l_e}{2}\frac{dv}{dx}$$

- Noting that $\frac{dv}{dx}$ evaluated at nodes 1 and 2 are q_2 and q_4 respectively, we have

$$v(\xi) = H_1 q_1 + \frac{l_e}{2} H_2 q_2 + H_3 q_3 + \frac{l_e}{2} H_4 q_4 \qquad \{v\} = \{H\}\{q\}$$

- Where $[H] = [H_1, \frac{l_e}{2} H_2, H_3, \frac{l_e}{2} H_4]$

Displacements and its Derivatives

- Since $x_2 - x_1 = l_e$, $dx = \frac{l_e}{2} d\xi$ $\qquad \frac{dv}{d\xi} = \frac{l_e}{2}\frac{dv}{dx}$
- In total potential energy expression we have terms like $\frac{d^2v}{dx^2}$

$$\frac{dv}{dx} = \frac{2}{l_e}\frac{dv}{d\xi} \qquad \Rightarrow \frac{d^2v}{dx^2} = \frac{4}{l_e^2}\frac{d^2v}{d\xi^2}$$

- Using the expression $v = Hq$

$$\left(\frac{d^2v}{dx^2}\right)^2 = q^T \frac{16}{l_e^4}\left(\frac{d^2v}{d\xi^2}\right)^T\left(\frac{d^2v}{d\xi^2}\right)q$$

- Here

$$\left(\frac{d^2v}{d\xi^2}\right) = \left[\frac{3}{2}\xi, \frac{l_e}{2}\frac{-1+3\xi}{2}, -\frac{3}{2}\xi, \frac{l_e}{2}\frac{1+3\xi}{2}\right]$$

Displacements and Potential Energy

$$\left(\frac{d^2v}{d\xi^2}\right)=\left[\frac{3}{2}\xi,\frac{l_e}{2}\frac{-1+3\xi}{2},-\frac{3}{2}\xi,\frac{l_e}{2}\frac{1+3\xi}{2}\right]$$

- Substituting in total energy expression while noting $dx=\frac{l_e}{2}d\xi$

$$U=\frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx$$

$$U_e=\frac{1}{2}q^T\frac{8EI}{l_e^3}\int_0^L\begin{bmatrix}\frac{9}{4}\xi^2 & \frac{3}{8}\xi(-1+3\xi)l_e & -\frac{9}{4}\xi^2 & \frac{3}{8}\xi(1+3\xi)l_e\\ & \left(\frac{-1+3\xi^2}{4}\right)^2 l_e^2 & -\frac{3}{8}\xi(-1+3\xi)l_e & \frac{-1+9\xi^2}{16}l_e^2\\ & & \frac{9}{4}\xi^2 & -\frac{3}{8}\xi(1+3\xi)l_e\\ & Symmetric & & \left(\frac{1+3\xi^2}{4}\right)^2 l_e^2\end{bmatrix}d\xi q$$

- Each term is integrated noting that $\int_{-1}^{+1}\xi^2 d\xi=\frac{2}{3};\quad \int_{-1}^{+1}\xi d\xi=0;\quad \int_{-1}^{+1}d\xi=2$

Potential Energy and Element Stiffness Matrix

Finally this gives the element strain energy as

$$U_e = \frac{1}{2} q^T k^e q$$

- Where the element stiffness matrix is

$$k^e = \frac{EI}{l_e^3} \begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}$$

BEAM ANALYSIS WITH FEM

Beam Elements- Load Vector

- Assuming the distributed load to be uniform we write the following expression

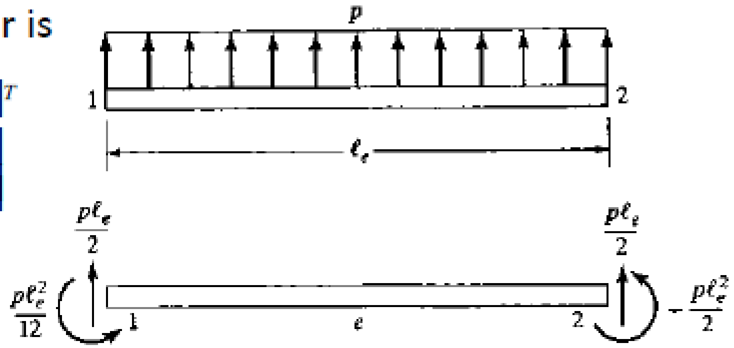
$$\int_{l_e} p v\, dx = \left(\frac{p l_e}{2} \int_{-1}^{1} H\, d\xi \right) q$$

- Substituting for H and integrating

$$\int_{l_e} p v\, dx = f^{eT} q$$

- Where the load vector is

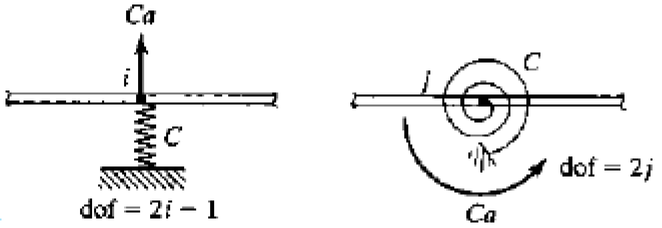
$$f^e = \left[\frac{p l_e}{2} \quad \frac{p l_e^2}{12} \quad \frac{p l_e}{2} \quad -\frac{p l_e^2}{12} \right]^T$$



BEAM ANALYSIS WITH FEM

Beam Elements- BCs

- We implement the BCs using penalty approach
- Considering generalised displacement *a* for the dof *r*, ,
- We add $\frac{1}{2}C(Q_r - a)^2$ to the total potential energy Π and
- $\Psi_j C(Q_r - a)$ to the left hand side of of the Galerkin formulation
- Here C represents a large stiffness in comparsion with the beam stiffness terms
- This is equivalent of adding stiffness C to K_{rr} and Ca to F_r



BEAM ANALYSIS WITH FEM

Shear Force and Bending Moment

- We have the shear force and bending moment relations as

$$M = EI\frac{d^2v}{dx^2}; \quad V = \frac{dM}{dx}; \qquad v = Hq$$

- At the element level, the bending moment and shear force are respectively

$$M = \frac{EI}{l_e^2}[6\xi q_1 + (3\xi - 1)l_e q_2 - 6\xi q_3 + (3\xi + 1)l_e q_4]$$

$$V = \frac{6EI}{l_e^3}[2q_1 + l_e q_2 - 2q_3 + l_e q_4]$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{Bmatrix} = \frac{EI}{l_e^3}\begin{bmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} + \begin{Bmatrix} -\frac{pl_e}{2} \\ -\frac{pl_e^2}{12} \\ -\frac{pl_e}{2} \\ \frac{pl_e^2}{12} \end{Bmatrix}$$

Here the second term needs to be added only on those elements where there is distributed loads

BEAM ANALYSIS WITH FEM

Modeling Beams on Elastic Supports

- In some engineering applications, and in case of shafts there are elastic supports like walls or soil like medium. In case of shafts there are ball or journal bearings as supports.
- A node can be considered for the location of single row ball bearing with stiffness k_B added to the diagonal location of the vertical degree of freedom.
- Rotational (moment) degree of freedom is considered for journal and roller bearings.
- In wide journal bearings and Winkler foundations, *s*, stiffness per unit length of supporting medium is considered. This gives rise to the additional term in total potential energy as

$$\frac{1}{2}\int_0^l sv^2\,dx$$

BEAM ANALYSIS WITH FEM

Beam Elastic Supports

Because of the additional term in the total potential energy

$$\frac{1}{2}\int_0^l sv^2\,dx$$

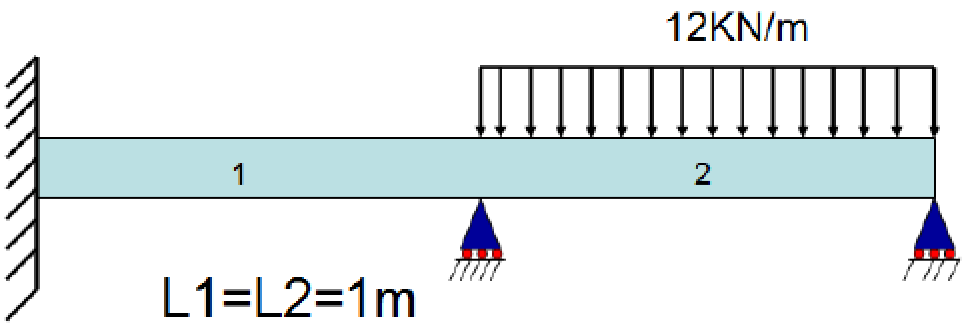
- We will have the modified discretized stiffness terms as follows $\frac{1}{2}\sum_e q^T s\int_e H^T H\,dxq$ The stiffness term is

$$k_s^e = s\int_e H^T H\,dx = \frac{sl_e}{2}\int_{-1}^{+1} H^T H\,d\xi$$

On integration we get

$$k_s^e = \frac{sl_e}{420}\begin{bmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e^2 & -22l_e & 4l_e^2 \end{bmatrix}$$

EXAMPLE PROBLEM



L1=L2=1m

E = 200GPa

$I = 4 \times 10^6 N/mm^4$

Let's model the given system as 2 elements 3 nodes finite element model each node having 2 dof. For each element determine stiffness matrix.

$$K_1 = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{bmatrix} \quad \text{(rows/columns 1, 2, 3, 4)}$$

$$K_2 = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 4 & -6 & 4 \end{bmatrix} \quad \text{(rows/columns 3, 4, 5, 6)}$$

Global stiffness matrix

$$K = 8 \times 10^5 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \quad \text{(rows/columns 1, 2, 3, 4, 5, 6)}$$

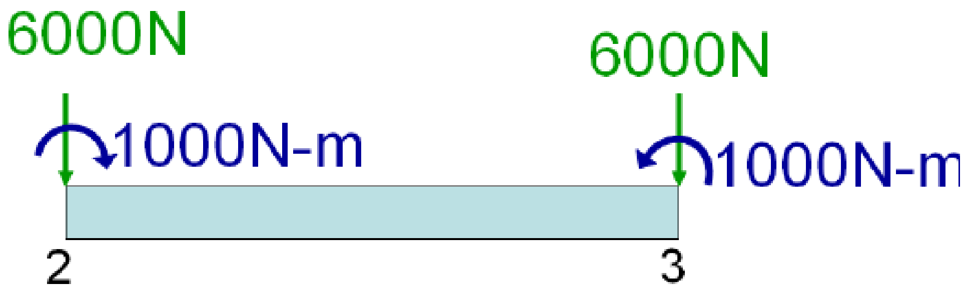
Load vector because of UDL

Element 1 do not contain any UDL hence all the force term for element 1 will be zero.

ie

$$F_1 = \begin{Bmatrix} F1 \\ F2 \\ F3 \\ F4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

For element 2 that has UDL its equivalent load and moment are represented as



ie

$$F_2 = \begin{Bmatrix} F3 \\ F4 \\ F5 \\ F6 \end{Bmatrix} = \begin{Bmatrix} -6000 \\ -1000 \\ -6000 \\ 1000 \end{Bmatrix}$$

Global load vector:

$$F = \begin{Bmatrix} F1 \\ F2 \\ F3 \\ F4 \\ F5 \\ F6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6000 \\ -1000 \\ -6000 \\ 1000 \end{Bmatrix}$$

From KQ=F we write

$$8 \times 10^5 \begin{pmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{pmatrix} \begin{pmatrix} Q1 \\ Q2 \\ Q3 \\ Q4 \\ Q5 \\ Q6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6000 \\ -1000 \\ -6000 \\ 1000 \end{pmatrix}$$

At node 1 since its fixed both q1=q2=0

node 2 because of roller q3=0

node 3 again roller ie q5= 0

By elimination method the matrix reduces to 2 X 2 solving this we have Q4= -2.679×10^{-4}mm and Q6 = 4.464×10^{-4}mm

To determine the deflection at the middle of element 2 we can write the displacement function as

$$V(\xi) = H_1 q_3 + H_2 q_4 \frac{L_e}{2} + H_3 q_5 + H_4 q_6 \frac{L_e}{2}$$

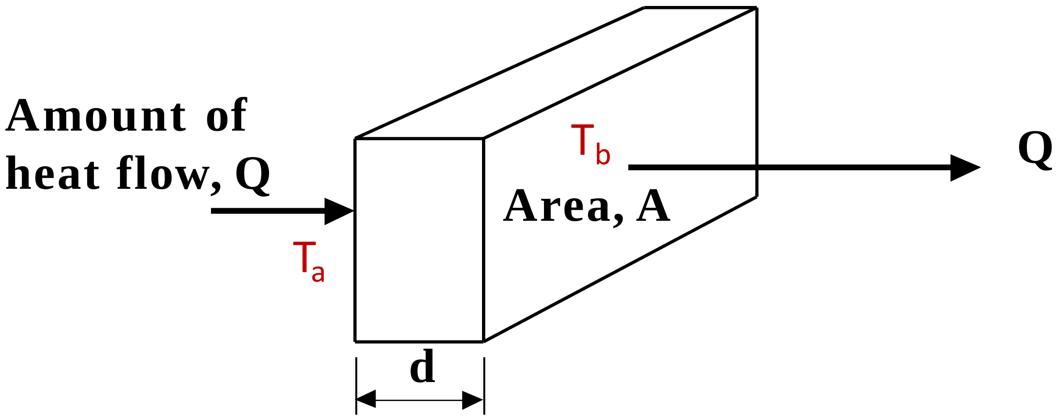
= -0.089mm

Heat Conduction Analysis of Solid Structures using FEM

Introduction to
Fundamentals of Heat Conduction in Solids

Fourier law of heat conduction

A natural phenomenon is that heat flows in a solid is possible only with temperature gradients with heat from the locations at higher temperature to the locations with lower temperature. Consequently, heat will flow from the left side to the right side of the slab if we maintain the situation of $T_a > T_b$ with T_a and T_b being the temperature at left and right faces of the slab respectively as illustrated below:



$$Q \propto \frac{A(T_a - T_b)t}{d} \quad \Rightarrow \quad Q = k\frac{A(T_a - T_b)t}{d}$$

where K = Thermal conductivity of material with units of: Btu/in-s-°F in the traditional system, or w/m-°C in the SI system.

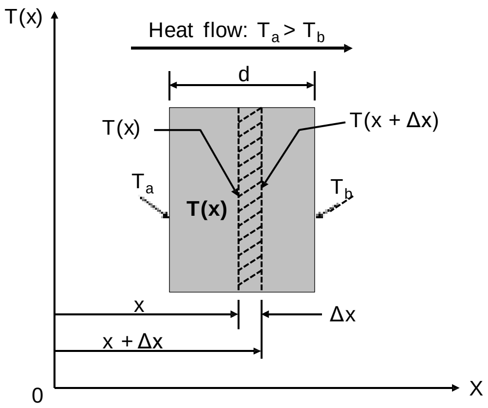
Fourier law of heat conduction-cont'd

Instead of total heat flow, a more commonly used terminology in engineering analysis is "heat flux" defined as "heat flow in solid per unit area and time." Mathematically, it is expressed as:

$$q = \frac{Q}{At} = k\,\frac{(T_a - T_b)}{d}$$

for heat flow in a solid slab – **a vector quantity**

For continuous variation of temperature between the two faces and let the coordinate along the length of the slab be x-axis, we will have the above expression in the form of:



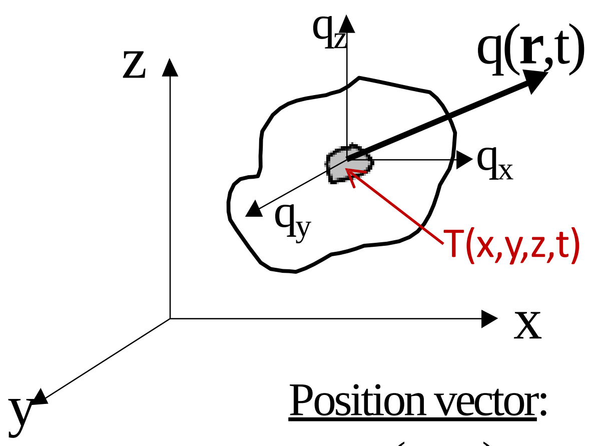
$$q = k\,\frac{T(x) - T(x+\Delta x)}{\Delta x} = -k\,\frac{T(x+\Delta x) - T(x)}{\Delta x}$$

with "**contiguous" variation** of temperature, the following expression prevails:

$$q = q(x) = -\,k\,\frac{dT(x)}{dx} \qquad (5.1)$$

Equation (5.1) is the mathematical expression of the "**Fourier Law of Heat Conduction**"

Fourier law of heat conduction in 3-D space



Position vector:

r: (x,y,z)

$$q(\mathbf{r},t) = -k\nabla T(\mathbf{r},t) \quad (5.2)$$

with components:

$$q_x = -k_x \frac{\partial T(x,y,z,t)}{\partial x} \quad (5.3a)$$

$$q_y = -k_y \frac{\partial T(x,y,z,t)}{\partial y} \quad (5.3b)$$

$$q_z = -k_z \frac{\partial T(x,y,z,t)}{\partial z} \quad (5.3c)$$

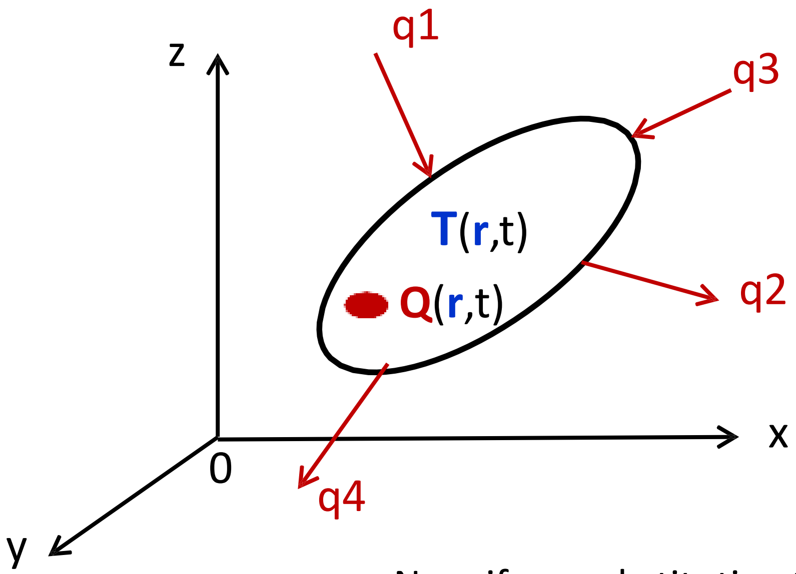
where k_x, k_y and k_z are the thermal conductivity of the material along the respective x-, y- and z-directions.

For isotropic materials, we will have $k = k_x=k_y=k_z$

The resultant total heat flux in the solid in Equation (5.2) is the vector sum of the components in Equation (5.3)
To be:

$$q(x,y,z,t) = \sqrt{q_x^2 + q_y^2 + q_z^2}$$

Heat Conduction Equation in Solids



Given a solid situated in a space defined by a coordinate system (**r**,t) or (x,y,z,t)

Heat fluxes in and out of the solid by q1, q2, q3,........, and heat generated in the solid by the amount Q(x,y,z,t) per unit volume and unit time.

There will be induced temperature distribution (or temperature field) in the solid by **T**(**r**,t) or **T**(x,y,z,t) in the solid.

The heat conduction equation was derived using the Fourier law of heat conduction and on the basis of law of conservation of energies

$$-\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\right)+Q=\rho c\frac{\partial T}{\partial t} \quad (5.4)$$

Now, if we substituting the heat fluxes shown in Equations (5.2) and (5.3) into the above expression to yield:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=\rho c\frac{\partial T(x,y,z,t)}{\partial t} \quad (5.5)$$

Equation (5.4) is the heat conduction equation for solids, in which ρ is the mass density and c is the specific heat of the material

For steady-state heat conduction in the solid:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=0 \quad (5.6)$$

The term Q(x,y,z,t) in both Equations (5.4) and (5.5) is the heat GENERATED by the solid, such as by Ohm's heating or nuclear fission

Heat Conduction Equation in Solids with specific conditions

The Heat conduction equation:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=\rho c\frac{\partial T(x,y,z,t)}{\partial t} \quad (5.5)$$

The boundary conditions:

(1) Specified temperature on the boundary surface S_1: $T_s = T_1(x,y,z,t)$ on S_1 (5.7a)

(2) Specified heat flow on the boundary surface S_2:

$q_x n_x + q_y n_y + q_z n_z = -q_s$ on S_2 (n_x=cosine to outward normal line in x-direction)

(5.7b)

(3) Specified convective boundary condition on the boundary surface S_3:

$q_x n_x + q_y n_y + q_z n_z = h(T_s - T_f)$ on S_2 (5.7c)

The initial conditions: $T(x,y,z,0) = T_o(x,y,z)$

(5.7d)

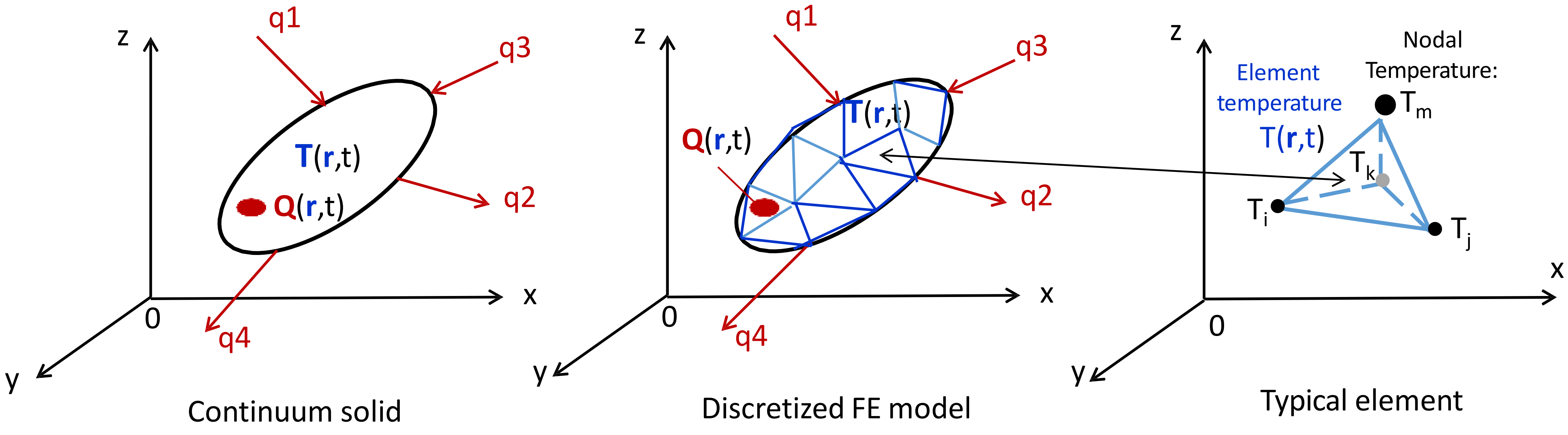
In the above boundary conditions, q_s in Equation (5.7b) is the heat flux across the boundary from external sources, and h is the heat transfer coefficient of the surrounding fluid at bulk fluid temperature T_f for convective boundary condition over surface S_3.

Finite Element Formulations

Finite element formulation of heat conduction in solid structures

The primary unknown quantity in finite element analysis of heat conduction in solid structures is the **TEMPERATURE** in the elements and NODES.

As usual, the very first step in FE analysis is to discretize the continuum structure into discretized FE model such as illustrated below:



Finite element formulation of heat conduction in solid structures – cont'd

The Interpolation Function, [N(x,y,z]:

The same definition of interpolation function for stress analysis is used for the heat conduction analysis, i.e.:

Element Temperature, **T**	=	Interpolation Function [N(x,y,z)]	×	Nodal Temperature {T}	(5.8)

where the interpolation function: $[N(x,y,z)] = \{ N_i \quad N_j \quad N_k \quad N_m \}$ (5.9)

The nodal temperature: $\{T\} = \{T_i \quad T_j \quad T_k \quad T_m\}^T$ (5.10)

The temperature gradients in the element may be obtained in terms of nodal temperature by differentiate the relationship in Equation (5.8) as:

$$\begin{Bmatrix} \frac{\partial T(x,y,z)}{\partial x} \\ \frac{\partial T(x,y,z)}{\partial y} \\ \frac{\partial T(x,y,z)}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_m}{\partial z} \end{bmatrix} \{T\} = [B]\{T\} \qquad (5.11)$$

where the matrix [B] has the form:

$$[B] = \begin{bmatrix} \frac{\partial N_i}{\partial x} & \frac{\partial N_j}{\partial x} & \frac{\partial N_k}{\partial x} & \frac{\partial N_m}{\partial x} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_j}{\partial y} & \frac{\partial N_k}{\partial y} & \frac{\partial N_m}{\partial y} \\ \frac{\partial N_i}{\partial z} & \frac{\partial N_j}{\partial z} & \frac{\partial N_k}{\partial z} & \frac{\partial N_m}{\partial z} \end{bmatrix} \qquad (5.12)$$

Finite element formulation of heat conduction in solid structures – cont'd

The functional for deriving element equations:

Because the conduction of heat in solids can be completely described by simple differential equations such as

$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=\rho c\frac{\partial T(x,y,z,t)}{\partial t} \quad (5.5)$$

for transient state, and

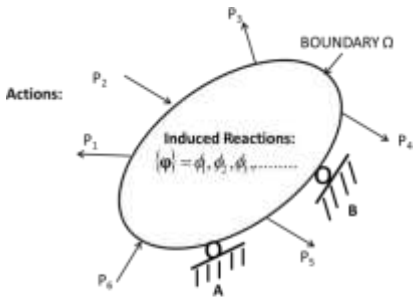
$$\frac{\partial}{\partial x}\left(k\frac{\partial T(x,y,z,t)}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T(x,y,z,t)}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T(x,y,z,t)}{\partial z}\right)+Q(x,y,z,t)=0 \quad (5.6)$$

for steady-state, and the boundary and initial conditions expressed in Equations (5.7), Galerkin method such as described in Chapter 3 will be used to derive the element equation.

We will first review the Galerkin method in the next slide.

Step 4 – Chapter 3 Galerkin method

In contrast to the Rayleigh-Ritz method, this method is used to derive the element equations for the cases in which specific differential equations with appropriate mathematical expressions for the boundary conditions available for the analytical problems, such as heat conduction and fluid dynamic analyses

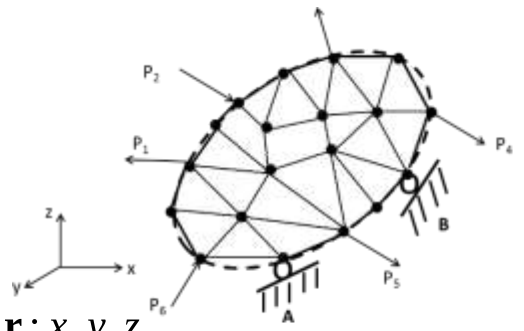


Real Situation on **solids**

Element Φ(**r**)

Nodal {Φ}

Φ(r) = N(r)Φ



$\mathbf{r}: x, y, z$

Approximate situation: Discretized Situation with **elements**

Differential Equation: D(Φ) for the volume V (5.4)

Boundary condition: B(Φ) for the real situation on boundary S (5.5)

Mathematical model: $\int_v W\, D(\phi) dv + \int_s \overline{W}\, B(\phi) ds = 0$

where W and $\overline{W}$ are arbitraly weighting functions

Differential Equation: D(N(**r**)Φ) for the element volume V

Boundary condition: B(N(**r**)Φ) for the real situation on element boundary

Mathematical model: $\int_v W_j D\left(\sum N_i(\mathbf{r})\phi_i\right) dv + \int_s \overline{W}_j B\left(\sum N_i(\mathbf{r})\phi_i\right) ds =$ **R**

W_j and $\overline{W}_j$ are discretized weighting functions, and R is the Residual

[N(**r**)] in Equation (5.9)

Galerkin method lets W_j and $\overline{W}_j = N(\mathbf{r})$ and let R to be minimum, or R→ 0 for good discretization, resulting in:

The same element equation: **[K$_e$] {q} = {Q}**

Finite element formulation of heat conduction in solid structures – cont'd

Derivation of Element Equation using **Galerkin** Method

Using the Galerkin method, we can rewrite the basic heat conduction equation in the following form:

$$\int_v \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} - Q + \rho c \frac{\partial T}{\partial t} \right) N_i dv = 0$$

Equation (5.4)   Equation (5.9)

By incorporate the boundary conditions in Equations (5.7) in the above equation will result in the element equation with the balanced of heat flus across the boundary and the induced temperature in the element in the following equation:

$$\int_v \rho c \frac{\partial T(x,y,z,t))}{\partial t} N_i dv - \int_v \left[\frac{\partial N_i}{\partial x} \quad \frac{\partial N_i}{\partial y} \quad \frac{\partial N_i}{\partial z} \right] \{q\} dv = \int_v Q N_i dv - \int_{s_1} \{q\}^T \{n\} N_i ds - \int_{s_2} q_s N_i ds - \int_{s_3} h(T - T_f) N_i ds \qquad (5.13)$$

with $heat\ flux\ across\ boundaries\ \{q\}^T = \{q_x \quad q_y \quad q_z\}\ and\ \ the\ direction\ cosine\ to\ outward\ norrmal\ \{n\}^T = \{n_x \quad n_y \quad n_z\}$

Finite element formulation of heat conduction in solid structures – cont'd

Derivation of Element Equation using **Galerkin** Method – cont'd

The heat balance in Equation (5.13) may be lumped to the following element equation:

$$[C]\{\dot{T}\}+([K_c]+[K_h])\{T\}=\{R_T\}+\{R_q\}+\{R_h\} \quad \textbf{(5.14)}$$

where in the coefficient matrices:

$$\text{The heat capcitnce matrix}: [C]=\int_v \rho c[N]^T[N]dv \quad (5.15a)$$

$$\text{The condictivity matrix}: [K_c]=\int_v k[B]^T[B]dv \quad (5.15b)$$

$$\text{The convective matrix}: [K_h]=\int_{S_3} h[N]^T[N]ds \quad (5.15c)$$

and the nodal thermal force matrices:

$$\text{The heat flux across the boundary } S_1: \{R_T\}=-\int_{S_1}\{q\}^T\{n\}[N]^T ds \quad (5.16a)$$

$$\text{The heat generation matrix}: \{R_Q\}=\int_v Q[N]^T dv \quad (5.16b)$$

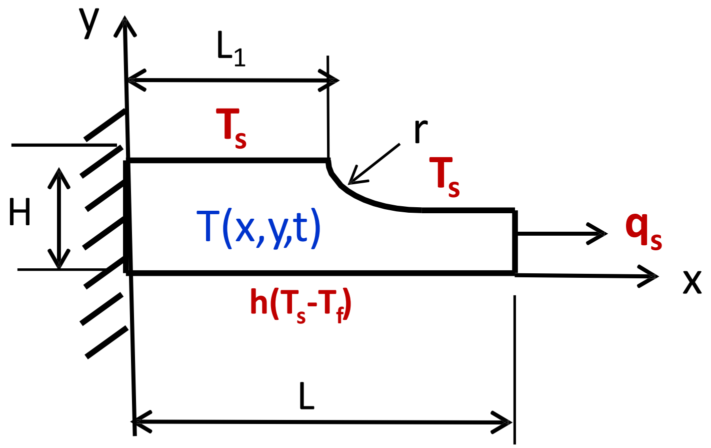
$$\text{The heat flux across the boundary } S_2: \{R_q\}=\int_{S_2} q_s[N]^T ds \quad (5.16c)$$

$$\text{The convective heat flux cross the boundary } S_3: \{R_h\}=\int_{S_3} hT_f[N]^T ds \quad (5.16d)$$

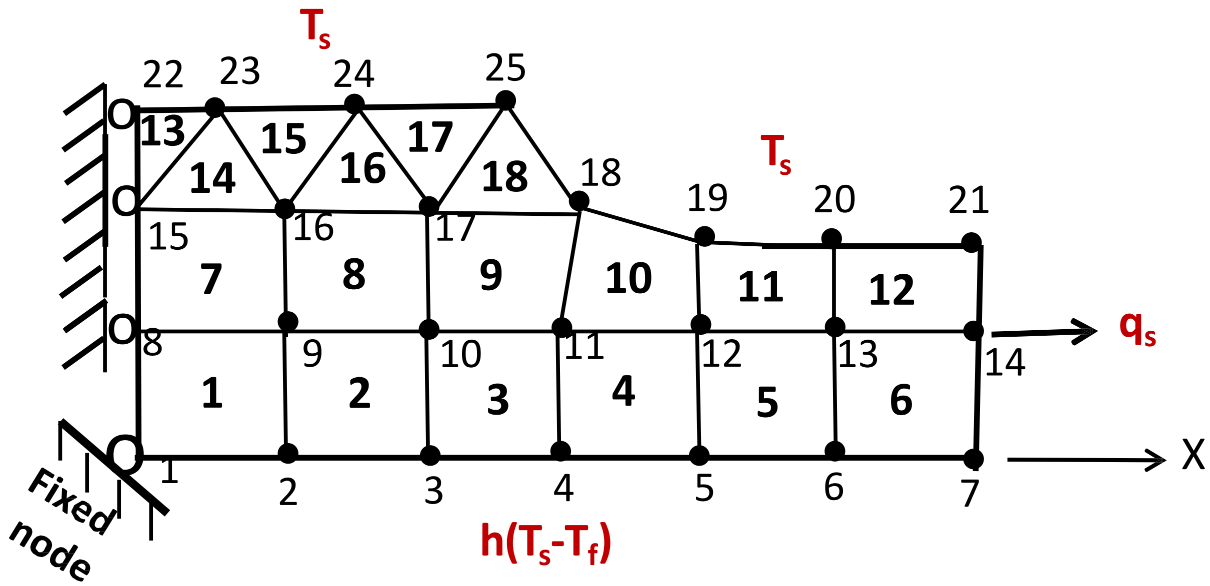
Heat Conduction in Planar Structures Using Finite Element Method

Finite element formulation of heat conduction in solid structures in planes

Heat conduction in a tapered plate:

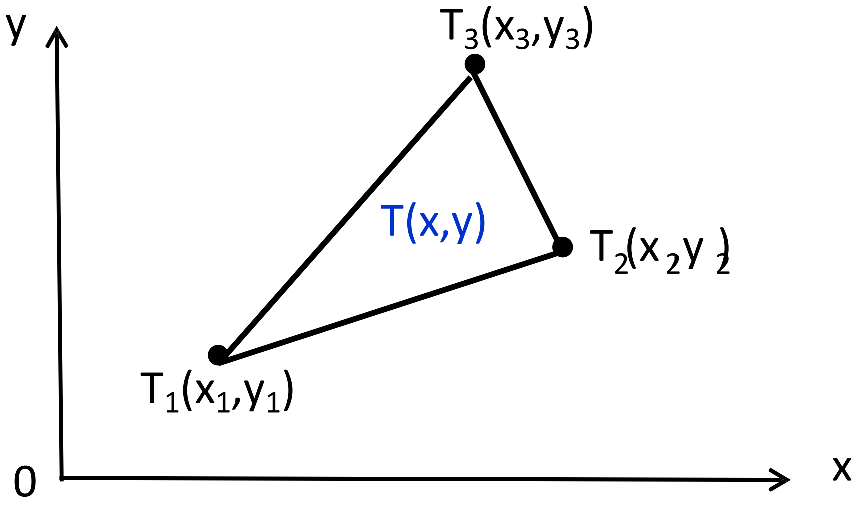


FE Mesh



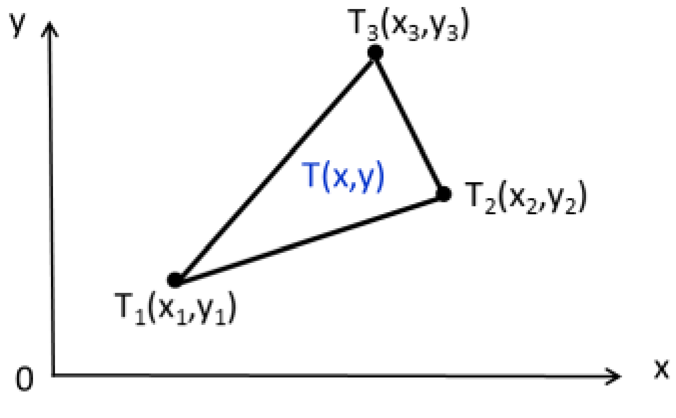
FE formulation in a triangular plate element:

Element temperature: T(x,y)
Nodal temperature:
$T_1(x_1,y_1)$; $T_2(x_2,y_2)$; $T_3(x_3,y_3)$



Finite element formulation of heat conduction in solid structures in planes

FE formulation in a triangular plate element-The interpolation function:



We assume the element temperature T(x,y) is represented by a simple linear polynomial function that:

$$T(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y = \{1 \quad x \quad y\}\begin{Bmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3\end{Bmatrix} = \{R\}^T\{\alpha\} \qquad (5.17)$$

with $\{R\}^T = \{1 \quad x \quad y\}$ (5.18)

where α_1, α_2, and α_3 are constants

Because the coordinates (x_1,y_1), (x_2,y_2) and (x_3,y_3) of the nodes in a FE mode are fixed. We may substitute these coordinates into Equation (5.17) and obtain the following expressions for the corresponding quantities at the three nodes:

$$T_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 \quad \text{for Node 1}$$

$$T_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 \quad \text{for Node 2}$$

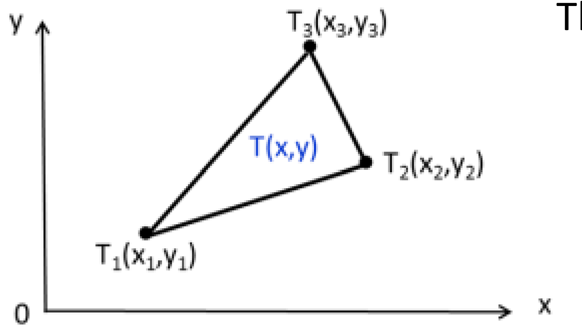
$$T_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 \quad \text{for Node 3}$$

or in a matrix form for nodal temperatures: $\{T\} = [A]\{\alpha\}$ (5.19)

and the unknown coefficients $\{\alpha\} = [A]^{-1}\{T\} = [h]\{T\}$ (5.20)

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – the interpolation function - cont'd:



The matrix [A] in Equations (5.19) and (5.20) contains the coordinates of the three nodes as:

$$[A] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}$$

The inversion of matrix $[A]^{-1}$ = [h] can be performed to give:

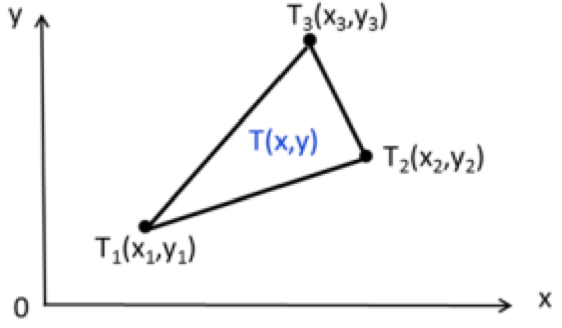
$$[h] = \frac{1}{|A|}\begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \tag{5.21}$$

where $|A|$ Is the determinant of the element of matrix [A] $= (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) = 2A$

with A= the area of triangle made by $(T_1T_2T_3)$

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – the interpolation function - cont'd:



By substituting (5.21) into (5.20) and then (5.19), the element quantity represented by T(x,y) can be made to equal the corresponding nodal quantities {T}: T_1, T_2, T_3 to be:

$$T(x,y) = \{R\}^T [h] \{T\} \qquad (5.22)$$

We will thus have the interpolation function: **N(x,y) = $\{R\}^T[h]$** with $\{R\}^T = \{1 \quad x \quad y\}$ in Equation (5.18) and [h} given in Equation (5.21)

We thus have the relationship between the element quantity to the nodal quantifies by the following expression:

T(x,y) = {N(x,y)} {T}

or express the above equation in the form according to Equation (5.8) as:

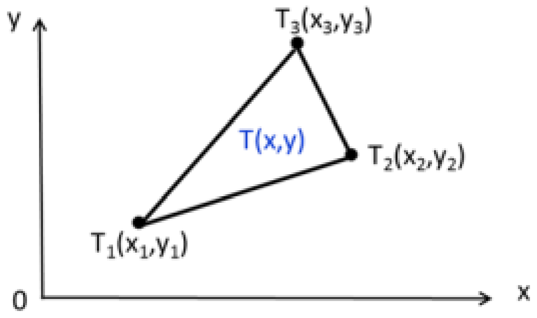
$$\{T(x,y)\} = \{N_1(x,y) \quad N_2(x,y) \quad N_3(x,y)\} \begin{Bmatrix} T_1(x_1,y_1) \\ T_2(x_2,y_2) \\ T_3(x_3,y_3) \end{Bmatrix} \qquad (5.23)$$

with $N_1 = \frac{1}{2A}(a_1 + b_1 x + c_1 y), \quad N_2 = \frac{1}{2A}(a_2 + b_2 x + c_2 y), \quad N_3 = \frac{1}{2A}(a_3 + b_3 x + c_3 y)$ (5.24)

and $2A = x_2 y_3 + x_3 y_1 + x_1 y_2 - x_2 y_1 - x_3 y_2 - x_1 y_3$

Finite element formulation of heat conduction in solid structures in planes – cont'd

FE formulation in a triangular plate element – the interpolation function - cont'd:



$$\{T(x,y)\} = \{N_1(x,y) \quad N_2(x,y) \quad N_3(x,y)\}\begin{Bmatrix} T_1(x_1,y_1) \\ T_2(x_2,y_2) \\ T_3(x_3,y_3) \end{Bmatrix} \tag{5.23}$$

with $N_1 = \frac{1}{2A}(a_1 + b_1 x + c_1 y)$, $\quad N_2 = \frac{1}{2A}(a_2 + b_2 x + c_2 y)$, $\quad N_3 = \frac{1}{2A}(a_3 + b_3 x + c_3 y)$

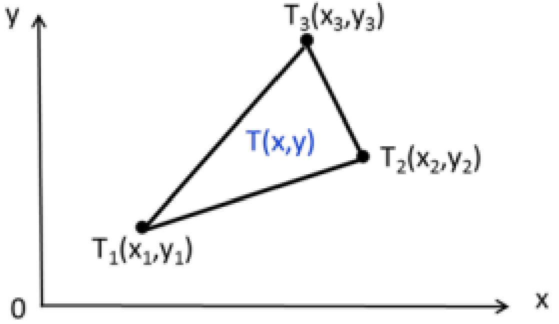
$$a_1 = (x_2 y_3 - x_3 y_2) \quad b_1 = (y_2 - y_3) \quad c_1 = (x_3 - x_2) \tag{5.25a}$$

$$a_2 = (x_3 y_1 - x_1 y_3) \quad b_2 = (y_3 - y_1) \quad c_2 = (x_1 - x_3) \tag{5.25b}$$

$$a_3 = (x_1 y_2 - x_2 y_1) \quad b_3 = (y_1 - y_2) \quad c_3 = (x_2 - x_1) \tag{5.25c}$$

$$2A = (x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) = 2 \times the\ area\ of\ the\ element(A)\ made\ of\ triangle\ (T_1 T_2 T_3)$$

FE formulation in a triangular plate element – The element coefficient matrix:



The conductivity matrix [Kc]:

By following Equation (5.15b), we have the conductivity matrix for a triangular plate element to be:

$$[K_c] = \int_A k[B]^T[B]dxdy \tag{5.26}$$

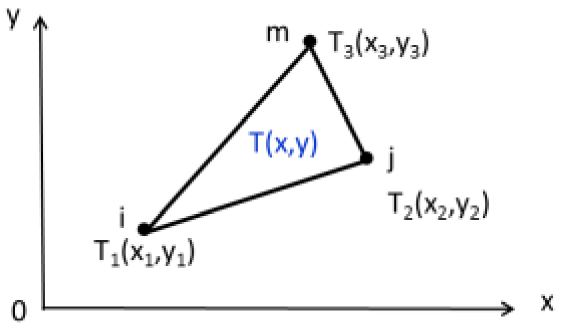
The temperature gradient matrix [B] can be obtained by the following formulation:

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_{31}}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix} = \frac{1}{2A}\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \tag{5.27}$$

We may obtain the conductivity matrix by substituting Equation (5.27) into Equation (5.26), leading to:

$$[K_c] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_1 b_2 + c_1 c_2 & b_2^2 + c_2^2 & b_2 b_3 + c_2 c_3 \\ b_1 b_3 + c_1 c_3 & b_2 b_3 + c_2 c_3 & b_3^2 + c_3^2 \end{bmatrix} \tag{5.28}$$

FE formulation in a triangular plate element – The element equations:



As in the case of stress analysis in chapter 4, the element equations for heat conduction solids of plenary geometry may be shown to take the form:

$$[K_e]\{T\} = \{q\} \qquad (5.29)$$

where $[K_e]$ = coefficient matrix in Equation (4.28), {T} = nodal temperature, and {q} = thermal forces at the nodes

The thermal forces at nodes are: $\{q\} = \{f_Q\} + \{f_q\} = (f_h\}$ (5.30)

in which $\{f_Q\}$ = heat generation in the solid with $\{f_Q\} = \int_v [N]\, Q dv = Q\int_v [N]^T dv = \frac{Qv}{3}\begin{Bmatrix}1\\1\\1\end{Bmatrix}$ (5.31a)

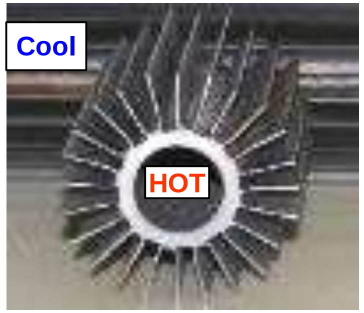
$\{f_q\}$ = heat flux across boundary with $\{f_q\} = \int_{S_2} [N]^T q ds = \int_{S_2} q\begin{Bmatrix}N_i\\N_j\\N_m\end{Bmatrix} ds$ (5.31b)

$\frac{qL_{i-j}t}{2}\begin{Bmatrix}1\\1\\0\end{Bmatrix}$ for side i-j $\qquad \frac{qL_{j-m}t}{2}\begin{Bmatrix}0\\1\\1\end{Bmatrix}$ for side j-m $\qquad \frac{qL_{m-i}t}{2}\begin{Bmatrix}1\\0\\1\end{Bmatrix}$ for side m-i

where t = thickness of the plane

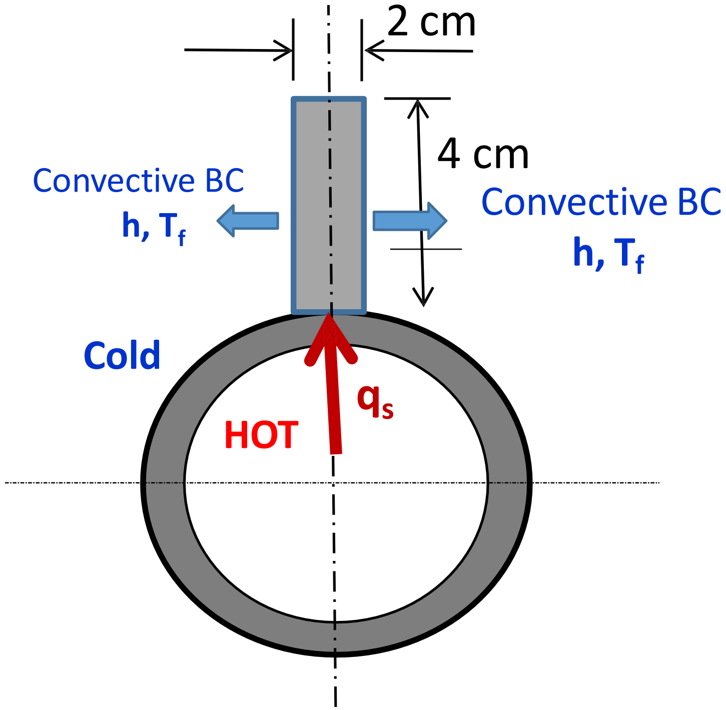
$\{f_h\}$ = convective heat flux across boundary with $\{f_h\} = \int_{S_3} [N]\, hT_f ds$ (5.31c)

Example 4.1



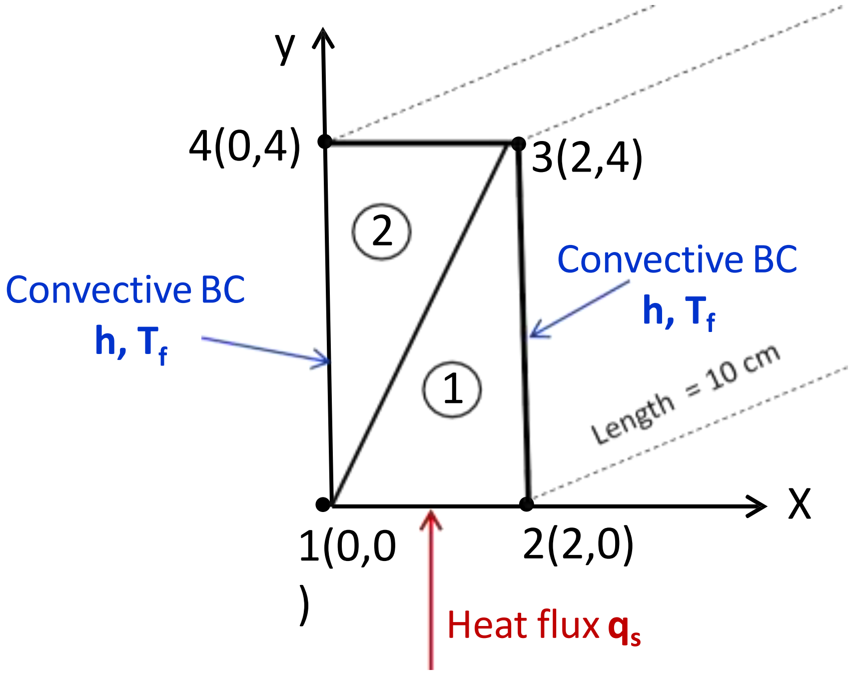
Use finite element method to determine the temperature variation across the thickness of longitudinal fins of a tubular heat exchanger as shown in the figure on the left. The heat exchanger is designed to heat up the cold fluid outside the tube by the hot fluid circulating inside the tube. The cross-section of a single fin is illustrated in the figure shown in lower-left of this slide.

The fin is made of aluminum with the properties: Mass density $\rho = 2.7$ g/cm^3 , Specific heat $c = 0.942$ J/g-^{o}C, and thermal conductivity $k = 2.36$ W/cm-^{o}C



Tube with one longitudinal fin

The discretized FE model of the fin cross-section is shown below:



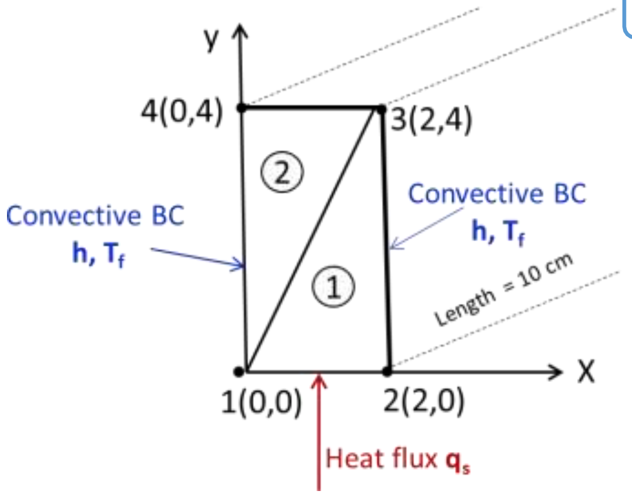
Boundary conditions:

q_s = 10 kW/m^2

h = 20 W/m^2-^{o}C

T_f = 40^{o}C

Example 4.1-cont'd



Interpolations functions for Elements:

We will use Equations (5.25a,b,c) to determine the constant coefficients a_i, b_i and c_i (I = 1,2,3) for each element. These coefficients will then be used to express the interpolation function of both Element 1 and 2, as in Equation (5.23).

We realize the following nodal coordinates in the FE model of the fin:

<u>For element 1 (Node 1, 2 and 3):</u>

$x_1 = 0$, $y_1 = 0$; $x_2 = 2$, $y_2 = 0$; $x_3 = 2$, $y_3 = 4$

The area A of the cross-section area of Element 1 is computed by using the expression:

$$2A = (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) = (0\times0 - 2\times0) + (2\times4 - 2\times0) + (2\times0 - 0\times4) = 8$$

This leads to A = 4 cm^2

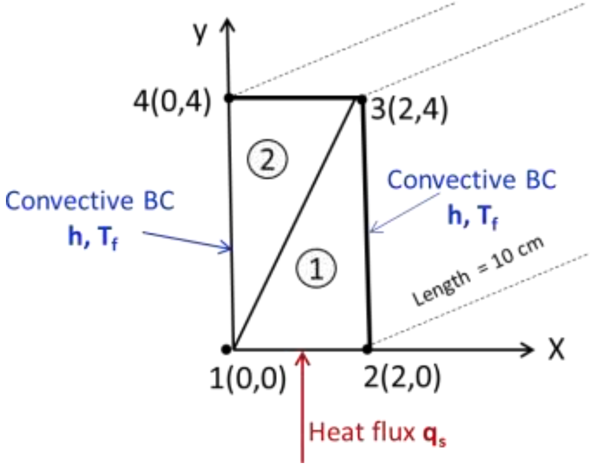
We will further compute the constant coefficients by the following expressions:

$$a_1 = (x_2y_3 - x_3y_2) = 2\times2 - 2\times0 = 4 \qquad b_1 = (y_2 - y_3) = 0 - 4 = -4 \qquad c_1 = (x_3 - x_2) = 2 - 2 = 0$$

$$a_2 = (x_3y_1 - x_1y_3) = 4\times0 - 0\times4 = 0 \qquad b_2 = (y_3 - y_1) = 4 - 0 = 4 \qquad c_2 = (x_1 - x_3) = 0 - 2 = -2$$

$$a_3 = (x_1y_2 - x_2y_1) = 0\times0 - 2\times0 = 0 \qquad b_3 = (y_1 - y_2) = 0 - 0 = 0 \qquad c_3 = (x_2 - x_1) = 2 - 0 = 2$$

Example 4.1-cont'd



Interpolations functions for Elements:

For Element 2 (Node 1, 3 and 4):

$x_1 = 0,\ y_1 = 0;\ x_3 = 3,\ y_3 = 4;\ x_4 = 0,\ y_4 = 4$

The area A of the cross-section area of Element 2 is the same as of Element 1 = 4 cm². the constant coefficients are determined the same way as for those in Element 1.

$$a_1 = (x_2y_3 - x_3y_2) = 3\times4 - 0\times4 = 12 \quad b_1 = (y_2 - y_3) = 4-4=0 \quad c_1 = (x_3 - x_2) = 0-3 = -3$$

$$a_2 = (x_3y_1 - x_1y_3) = 0\times0 - 0\times4 = 0 \quad b_2 = (y_3 - y_1) = 4-0=4 \quad c_2 = (x_1 - x_3) = 0-0 = 0$$

$$a_3 = (x_1y_2 - x_2y_1) = 0\times4 - 3\times0 = 0 \quad b_3 = (y_1 - y_2) = 0-4=-4 \quad c_3 = (x_2 - x_1) = 3-0 = 3$$

We will thus have the interpolation functions for both element 1 and 2 by substituting the constant coefficients into Equation (5.23):

For Element 1:

$$N_1 = \frac{1}{2\times4}(4+4x+0\times y) = 0.5+0.5x$$

$$N_2 = \frac{1}{2\times4}(0+4x-2y) = x-0.25y$$

$$N_3 = \frac{1}{2\times4}(0+0\times x+2y) = 0.25y$$

Leads to:

$$\{T_e^1\} = \{(0.5+0.5x) \quad (x-0.5y \quad 0.25y\}\begin{Bmatrix}T_1\\T_2\\T_3\end{Bmatrix} \qquad \text{(a)}$$

For Element 2:

$$N_1 = \frac{1}{2\times4}(12+0\times x-3y) = 1.5-0.375y$$

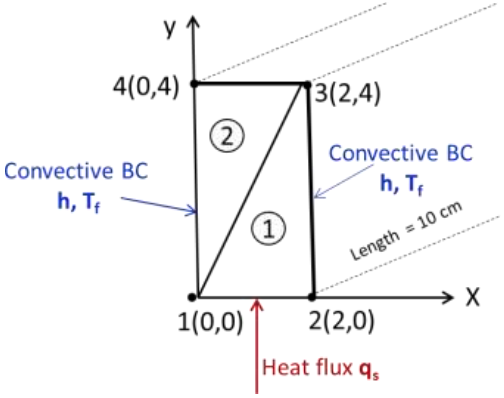
$$N_3 = \frac{1}{2\times4}(0+4x+0\times y) = 0.5x$$

$$N_4 = \frac{1}{2\times4}(0-4x+3y) = -0.5x+0.375y$$

Leads to:

$$\{T_e^2\} = \{(1.5-0.375y) \quad 0.5x \quad (-0.5x+0.375y)\}\begin{Bmatrix}T_1\\T_3\\T_4\end{Bmatrix} \qquad \text{(b)}$$

Example 4.1-cont'd



Element coefficient matrices [K_e]:

We will use Equation (5.28) to derive these matrices.

$$[K_e] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix} \tag{5.28}$$

For Element 1:

Node: 1 2 3

$$[K_e^1] = \frac{2.36}{4\times 4^2}\begin{bmatrix} (-4)^2 + 0^2 & (-4)(4) + (0)(-2) & (-4)(0) + (0)(2) \\ (-4)(4) + (0)(-2) & 4^2 + (-2)^2 & (4)(0) + (-2)(2) \\ (-4)(0) + (0)(2) & (4)(0) + (-2)(2) & 0^2 + (2)^2 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.6 & 0 \\ -0.6 & 1.2 & -0.15 \\ 0 & -0.15 & 0.15 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \tag{C}$$

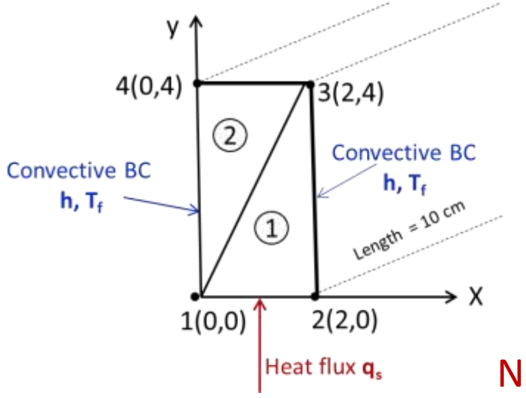
For Element 2:

Node: 1 3 4

$$[K_e^2] = \frac{2.36}{4\times 4^2}\begin{bmatrix} 0^2 + (-2)^2 & (0)(4) = (-2)(0) & (0)(-4) + (-2)(2) \\ (0)(4) + (-2)(0) & 4^2 + 0^2 & 4(-4) + (0)(2) \\ (0)(-4) + (-2)(2) & (4)(-4) + (0)(2) & (-4)^2 + (2)^2 \end{bmatrix} = \begin{bmatrix} 0.1475 & 0 & -0.1475 \\ 0 & 0.6 & -0.6 \\ -0.1475 & -0.6 & 0.7375 \end{bmatrix} \begin{matrix} 1 \\ 3 \\ 4 \end{matrix} \tag{d}$$

Example 4.1-cont'd

Assembly of element coefficient matrices for Overall coefficient (conductance) matrix



We need to assemble the element coefficient matrices to construct the overall structure coefficient matrix by summing up the two element coefficient matrices. We need to add the elements for the nodes that are shared by various elements. In the present case, we have Node 1 and 3 shared by both these two elements. We establish the following "map" for assembling the overall coefficient matrix *K+:

Elements in $[K_e^1]$

Node	1	2	3	4
1	●	●	●	
2	●	●	●	
3	●	●	●	
4				

+

Elements in $[K_e^2]$

Node	1	2	3	4
1	◊		◊	◊
2				
3	◊		◊	◊
4	◊		◊	◊

=

Node	1	2	3	4 for the $[K_c]$ matrix
1	●◊	●	●◊	◊
2	●	●	●	0
3	●◊	●	●◊	◊
4	◊	0	◊	◊

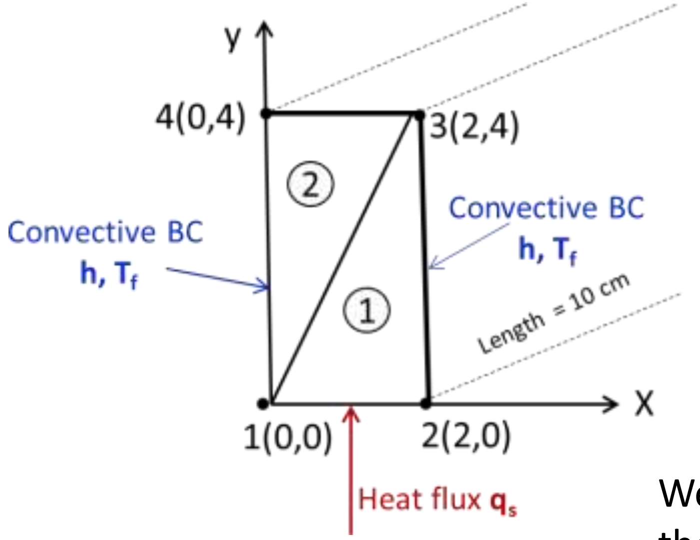
where ●=element in the matrix in Equation (c), and ◊ = elements in Equation (d)

We thus have the **overall coefficient (or conductance) matrix** in the form:

$$[K_c]=\begin{bmatrix} 0.7475 & -0.6 & 0 & -0.1473 \\ -0.6 & 1.2 & -0.15 & 0 \\ 0 & -0.15 & 0.75 & -0.6 \\ -0.1473 & 0 & -0.6 & 0.7375 \end{bmatrix} \quad \text{(e)}$$

Example 4.1-cont'd

Set thermal forces at the nodes



We have the following heat across the boundaries of the fin:

(1) Heat flux entering the fin crossing the line 1-2 with q_s=10 W/cm^2
(2) Heat leaving the fin crossing boundary line 2-3 by convection with h = 20 W/m^2-^{o}C = 20x10^{-4} W/cm^2-^{o}C
(3) Heat leaving the fin crossing boundary line 4-1 by convection with h = 20 W/m^2-^{o}C = 20x10^{-4} W/cm^2-^{o}C

The structure has a length, i.e. the thickness t = 10 cm

We will formulate the equivalent nodal thermal forces for the above specified boundary thermal forces according to the formulas of:

$$\{f_q\} = \begin{Bmatrix} f_{iq} \\ f_{jq} \end{Bmatrix} = \frac{q_s L_{i-j}\ t}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

for heat flux cross line i-j line i-j, and $\{f_h\} = \int_{S_3} [N]^T h T_f ds \rightarrow \begin{Bmatrix} f_{ih} \\ f_{jh} \end{Bmatrix} = \{N_i \quad N_j\}^T (hT_f)(L_{i-j} t)$ for heat removal by convection

(1) Heat flux entering the fin crossing the line 1-2 with q_s=10 W/cm^2:

$$f_{1q} = f_{2q} = q_s \frac{(L_{1-2})(t)}{2} = 10\frac{2\times 10}{2} = 100W$$

(2) Heat leaving the fin crossing boundary line 2-3 and line 4-1 by convection with h = 20 W/m^2-^{o}C = 20x10^{-4} W/cm^2-^{o}C

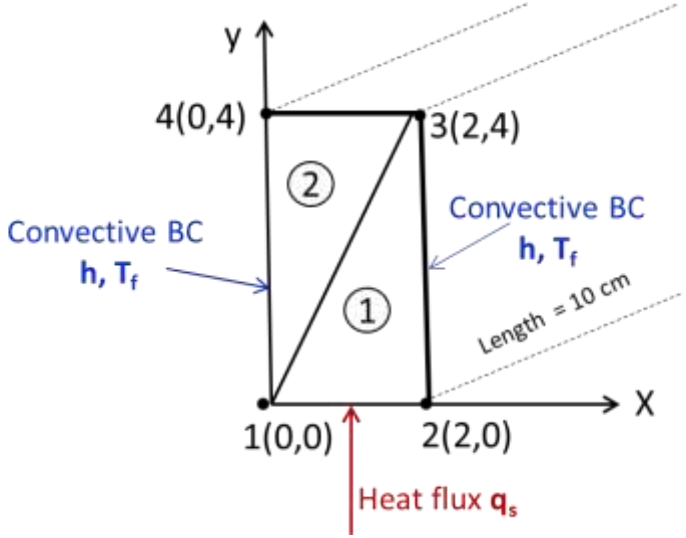
$$\begin{Bmatrix} f_{3h} \\ f_{2h} \end{Bmatrix} = \begin{Bmatrix} N_3 \\ N_2 \end{Bmatrix}(hT_f L_{21.0-3} t) = \begin{Bmatrix} (N_3)_{\substack{x=2\\y=0}}(hT_f L_{2-3} t) \\ (N_2)_{\substack{x=2\\y=4}}(hT_f L_{2-3} t) \end{Bmatrix} = \begin{Bmatrix} 1.5\times 1.6 = 2.4 \\ 1.0\times 1.6 = 1.6 \end{Bmatrix} W$$

and

$$\begin{Bmatrix} f_{4h} \\ f_{1h} \end{Bmatrix} = \begin{Bmatrix} N_4 \\ N_1 \end{Bmatrix}(hT_f L_{4-1} t) = \begin{Bmatrix} (N_4)_{\substack{x=0\\y=4}}(hT_f L_{4-1} t) \\ (N_1)_{\substack{x=0\\y=0}}(hT_f L_{4-1} t) \end{Bmatrix} = \begin{Bmatrix} 1.5\times 1.6 = 2.4 \\ 0\times 1.6 = 0 \end{Bmatrix} W$$

Example 4.1-cont'd

Set thermal forces at the nodes-cont'd



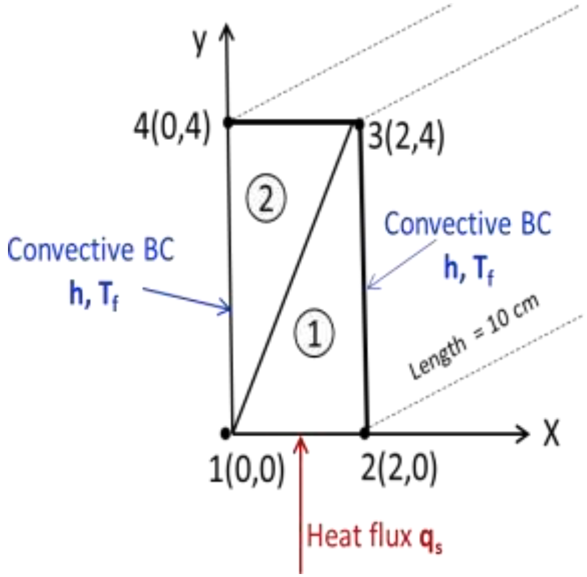
We thus have the thermal force matrix for the 4 nodes as:

$$\{q\}=\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} f_{1q}+f_{1h} \\ f_{2q}+f_{2h} \\ f_{3h} \\ f_{4h} \end{Bmatrix} = \begin{Bmatrix} 100 \\ 100+1.6 \\ 2.4 \\ 2.4 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 101.6 \\ 2.4 \\ 2.4 \end{Bmatrix}$$

The overall structure heat conduction equation: [K]{T} = {q}

$$\begin{bmatrix} 0.7475 & -0.6 & 0 & -0.1473 \\ -0.6 & 1.2 & -0.15 & 0 \\ 0 & -0.15 & 0.75 & -0.6 \\ -0.1473 & 0 & -0.6 & 0.7375 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 101.6 \\ 2.4 \\ 2.4 \end{Bmatrix} \qquad \text{(f)}$$

Example 4.1-cont'd



$$\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{bmatrix} 0.7475 & -0.6 & 0 & -0.1473 \\ -0.6 & 1.2 & -0.15 & 0 \\ 0 & -0.15 & 0.75 & -0.6 \\ -0.1473 & 0 & -0.6 & 0.7375 \end{bmatrix}^{-1} \begin{Bmatrix} 100 \\ 101.6 \\ 2.4 \\ 2.4 \end{Bmatrix}$$

$$= \begin{bmatrix} 3.8178 & 2.2913 & 3.0596 & 3.2517 \\ 2.2913 & 2.2728 & 2.3505 & 2.3699 \\ 3.0596 & 2.3505 & 6.5654 & 5.9524 \\ 3.2517 & 2.3699 & 5.9524 & 6.8480 \end{bmatrix} \begin{Bmatrix} 100 \\ 101.6 \\ 2.4 \\ 2.4 \end{Bmatrix} = \begin{Bmatrix} 629.72 \\ 471.38 \\ 574.81 \\ 595.47 \end{Bmatrix} \qquad \text{(g)}$$

We thus solve for the nodal temperatures to be:

T_1 = 629.72 ^oC, T_2 = 471.38 ^oC, T_3 = 574.81 ^oC and T_4 = 595.47 ^oC

Example 4.2

The same Example 13.6 of the textbook on "A First course in the Finite Element Method," 5th edition by Daryl Logan, published by Cenage Learning, 2012

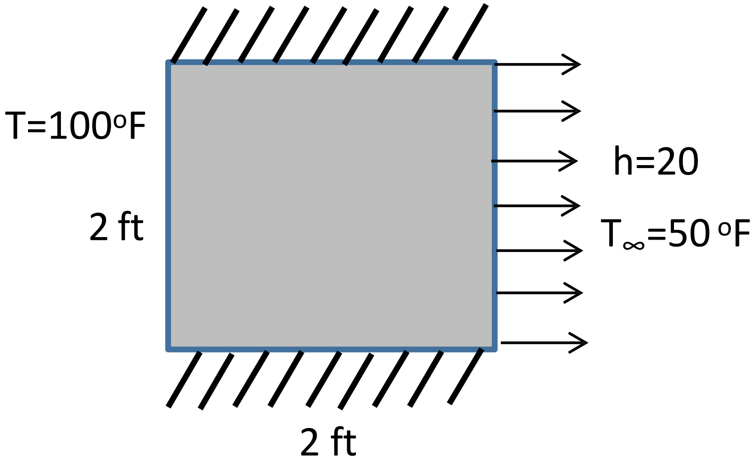


Figure 13-22 2-D body subjected to temperature variation and convection

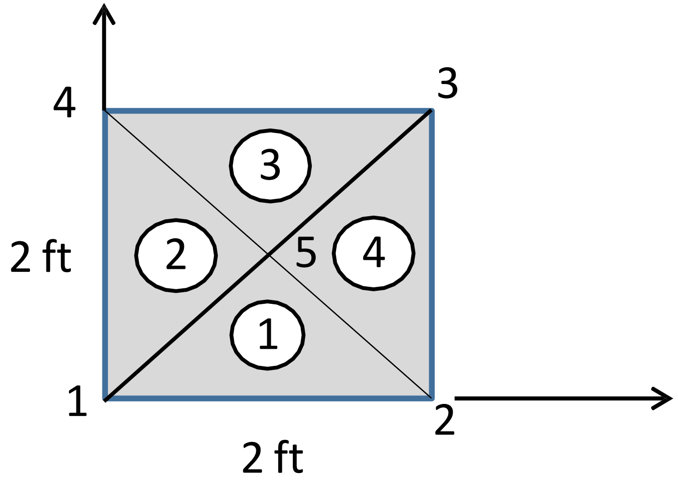


Figure 13-23 Discretized 2-D body of Figure 13-22

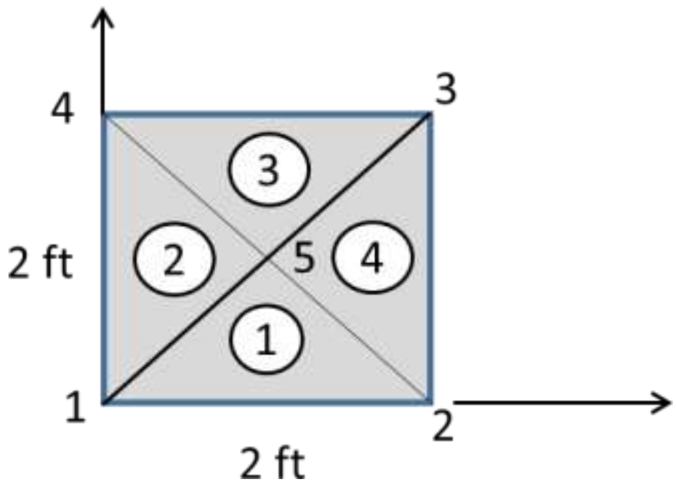
Problem: "For the 2-D body shown in Figure 13-22, determine the temperature distribution. The temperature at the left side of the body is maintained at 100 °F. The edges on the top and bottom of the body are insulated. There is heat convection from the right side with convective coefficient h = 20 Btu/h-ft^2-°F. The free stream temperature is T_∞ = 50 °F. The coefficients of thermal conductivity are $K_xx=K_{yy}$=25 Btu/h-ft-°F. The dimensions are shown in the figure. Assume the thickness to be 1 ft."

Solution: The discretized FE model of the body is shown in Figure 13-23 with 4 elements and 5 nodes. Nodal coordinates are:

$x_1 = 0$, $y_1 = 0$ for Node 1
$x_2 = 2$, $y_2 = 0$ for Node 2
$x_3 = 2$, $y_3 = 2$ for Node 3,
$x_4 = 0$, $y_4 = 2$ for Node 4, and
$x_5 = 1$, $y_5 = 1$ for Node 5

We will formulate the element coefficient matrices for all the 4 elements in Figure 13-23 using the equations (5.25a,b,c) and (5.28)

Example 4.2 – Cont'd



For Element 1: with Nodes 1,2 5

The area 2A is:

$$2A = x_2 y_5 + x_5 y_1 + x_1 y_2 - x_2 y_1 - x_5 y_2 - x_1 y_5 = 2$$

leads to: A = 1 ft^2.

To find the constant coefficients in Equation (5.25a,b,c):

From Equation (5.25a):

$$a_1 = x_2 y_5 - x_5 y_2 = 2 \times 1 - 1 \times 0 = 2$$

$$b_1 = y_2 - y_5 = 0 - 1 = -1$$

$$c_1 = x_5 - x_2 = 1 - 2 = -1$$

From Equation (5.25b):

$$a_2 = x_5 y_1 - x_1 y_5 = 1 \times 0 - 0 \times 1 = 0$$

$$b_2 = y_5 - y_1 = 1 - 0 = 1$$

$$c_2 = x_1 - x_5 = 0 - 1 = -1$$

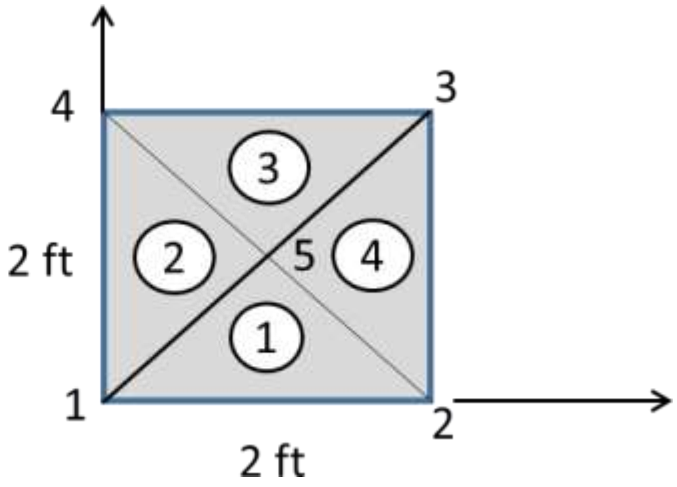
From Equation (5.25c):

$$a_3 = x_1 y_2 - x_2 y_1 = 0 \times 0 - 2 \times 0 = 0$$

$$b_3 = y_1 - y_2 = 0 - 0 = 0$$

$$c_3 = x_2 - x_1 = 2 - 0 = 2$$

Example 4.2 – Cont'd



We will use Equation (5.28) to formulate the element coefficient matrix:

$$[K_e] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix} \quad (5.28)$$

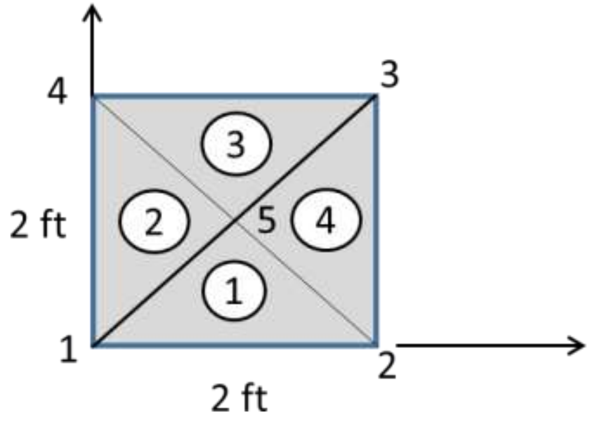
$$[K_e^1] = \frac{k}{4A^2}\begin{bmatrix} b_1^2 + c_1^2 & b_1b_2 + c_1c_2 & b_1b_3 + c_1c_3 \\ b_1b_2 + c_1c_2 & b_2^2 + c_2^2 & b_2b_3 + c_2c_3 \\ b_1b_3 + c_1c_3 & b_2b_3 + c_2c_3 & b_3^2 + c_3^2 \end{bmatrix}$$

$$= \frac{25}{4(1)^2}\begin{bmatrix} (-1)^2 + (-1)^2 & (-1)(1) + (-1)(-1) & (-1)(0) + (-1)(2) \\ (-1)(1) + (-1)(-1) & (1)^2 + (-1)^2 & (1)(0) + (-1)(2) \\ (-1)(0) + (-1)(2) & (1)(0) + (-1)(2) & (0)^2 + (2)^2 \end{bmatrix}$$

Node 1 2 5

$$= \begin{bmatrix} 12.5 & 0 & -12.5 \\ 0 & 12.5 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix}$$

Example 4.2 – Cont'd



Element coefficient matrices for Element 2, 3 and 4 following the similar approach as shown below:

For Element 2:

Node 1, 5, 4

$$[K_e^2] = \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 25 & -12.5 \\ 0 & -12.5 & 12.5 \end{bmatrix}$$

For Element 3:

Node 4, 5, 3

$$[K_e^3] = \begin{bmatrix} 12.5 & -12.5 & 0 \\ -12.5 & 25 & -12.5 \\ 0 & -12.5 & 12.5 \end{bmatrix}$$

A note from the Instructor:

For Element 4:

Node 2, 3, 5

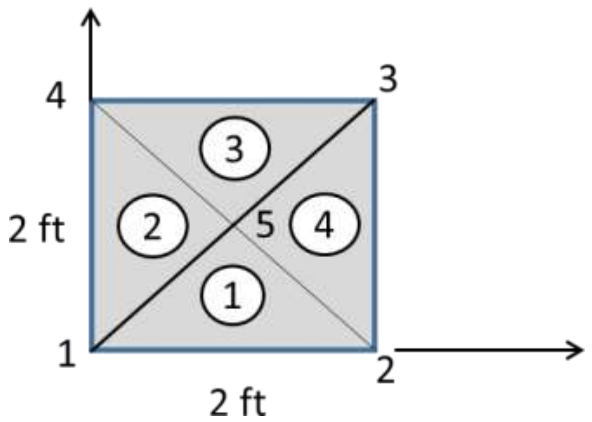
$$[K_e^4] = \begin{bmatrix} 12.5 & 0 & -12.5 \\ 0 & 12.5 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix} \quad \text{(a)}$$

A note from the Instructor:

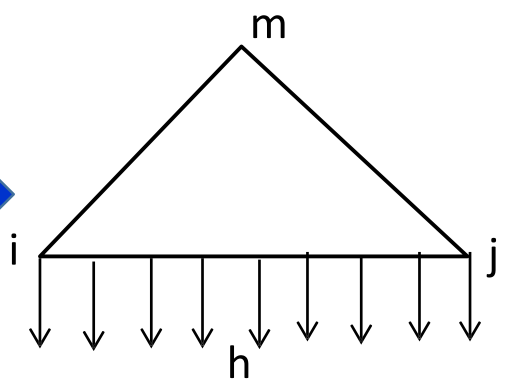
The author of this example (Daryl Logan) attributes the convective heat leaving the body in Element 4 as part of heat conduction activity in this element. So he derives another conductance coefficient matrix [K_h] for this Element 4, as will be shown in the next slide.

Example 4.2 – Cont'd

Additional heat conductance matrix for convective heat transfer in Element 4



$$[K_h] = \int_{S_3} h[N]^T[N]ds = h\int_{S_3}\begin{bmatrix} N_iN_i & N_iN_j & N_iN_m \\ N_jN_i & N_jN_j & N_jN_m \\ N_mN_i & N_mN_j & N_mN_m \end{bmatrix}$$



For the case with convective heat transfer from Edge i-j, the following expression is used:

$$[K_h] = \frac{h(L_{i-j})(t)}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (5.29)$$

For the current situation, the side that has convective heat transfer is Side 2-3, we will thus have:

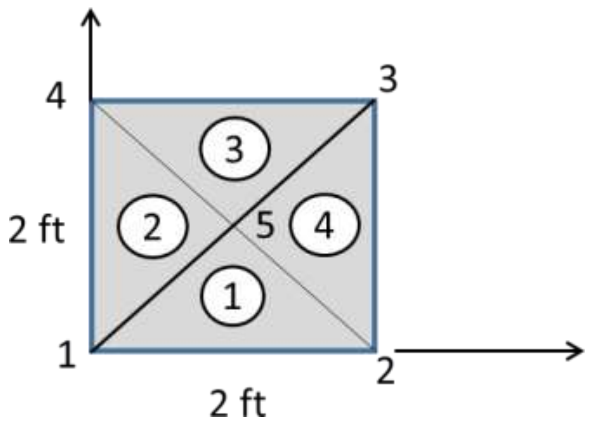
$$[K_h^4] = \frac{(20(2)(1))}{6}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

By adding this matrix to the conductance of Element 4 in Equation (a), we obtain the Conductance matrix of Element 4 to be:

Node 2 3 5

$$[K_e^4] = \begin{bmatrix} 25.83 & 6.67 & -12.5 \\ 6.67 & 25.83 & -12.5 \\ -12.5 & -12.5 & 25 \end{bmatrix}$$

Example 4.2 – Cont'd



Assemble the element coefficient matrices for the Overall coefficient matrix by accounting the fact that Node 4 is shred by all 4 elements.

$$[K]=\begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} Btu/h-{}^{o}F \qquad \text{(b)}$$

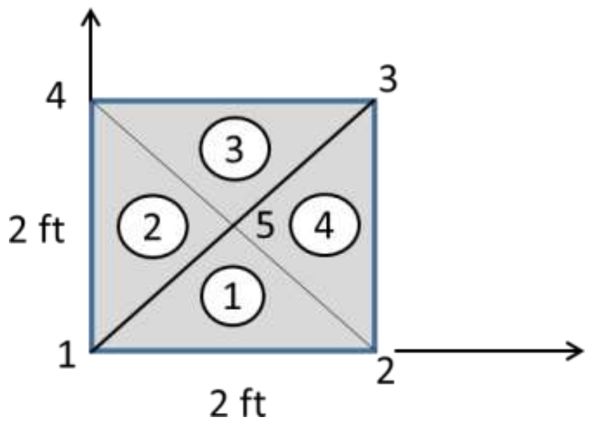
The thermal forces at nodes:

We already know that temperature at Node 1 and 4 are specified to be 100°F

The thermal forces across boundary 2-3 of element 4 is:

$$\{f^4\}=\begin{Bmatrix} f_2 \\ f_3 \\ f_5 \end{Bmatrix} = \frac{h(T_\infty)(L_{2-3})(t)}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \frac{(20)(50)(2)(1)}{2}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1000 \\ 1000 \\ 0 \end{Bmatrix} Btu/h$$

Example 4.2 – Cont'd



The overall heat conduction equation becomes:

$$\begin{bmatrix} 25 & 0 & 0 & 0 & -25 \\ 0 & 38.33 & 6.67 & 0 & -25 \\ 0 & 6.67 & 38.33 & 0 & -25 \\ 0 & 0 & 0 & 25 & -25 \\ -25 & -25 & -25 & -25 & 100 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 1000 \\ 1000 \\ 100 \\ 5000^{**} \end{Bmatrix}$$

** = (-25)(100°F)+(-25)(100°F)=-5000°F on the left side of the fifth equation in the left-hand-side of the equation

We may sole the above equations and obtain:

T_2 = 69.33°F, T_3 = 59.33°F and T_5 = 84.62°F with specified T_1=T_4 = 100°F

Summary on Heat Conduction Analysis of Plane Structures by FE Method

1) An overview of heat conduction in 3-D solids was presented in this Chapter with heat conduction equation for the induced temperature distributions in the solids by the sources of: (a) heat generation by the solid, (b) the prescribed surface temperature, (c) specified heat flux across the boundary surfaces, and (d) the convective heat across the boundary surfaces.

2) Finite element formulation of heat conduction in solids is derived using the Galerkin method due to the fact that heat conduction in solids can e described by the heat conduction equations with prescribed boundary conditions by mathematical expressions.

3) Finite element formulations begin with the derivation of interpolation functions [N} = {N_i N_j N_m} for triangular plane elements with Nodes i, j and m. These functions relate the “element temperatures” and the “nodal temperatures.”

4) The interpolation functions for the FE analysis were derive on the basis of linear polynomial function for the temperature variations in the element.

5) Special FE formulations of the aforementioned boundary conditions were presented.

6) This chapter only presents the FE formulation for steady-state heat conduction in solids of plane geometry.

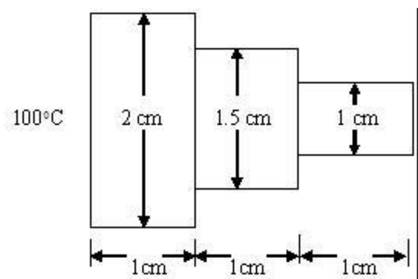
MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject : FINITE ELEMENT METHODS

UNIT – IV

TUTORIAL - IV

1. Calculate the temperature distribution and the heat dissipating capacity of a fin shown in Figure. The thermal conductivity of the material is 200 W/mk. The surface transfer coefficient is 0.5 W/m^2K. The ambient temperature is 30^0C the thickness of the fin is 1 cm.



2. a) A 20-cm thick wall of an industrial furnace is constructed using fireclay bricks that have a thermal conductivity of k = 2 W/m-°C. During steady state operation, the furnace wall has a temperature of 800°C on the inside and 300°C on the outside. If one of the walls of the furnace has a surface area of 2 m^2 (with 20-cm thickness), find the rate of heat transfer and rate of heat loss through the wall.

3. Establish the Hermite shape functions for a beam element Derive the equivalent nodal point loads for a **UDL** acting on the beam element in the transverse direction and also determine stiffness matrix

4. Estimate the temperature distribution in a fin whose cross section is 15mm X 15mm and 500mm long. Take Thermal conductivity as 50W/m-k and convective heat transfer coefficient as 75 W/m^2-k at 25°C. The base temperature is assumed to be constant and its value may be taken as 900°C. And also calculate the heat transfer rate?

5.) A metal pipe of 10-cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20°C while the outer surface of the pipe is at a temperature of 250°C. If the heat transfer coefficient for free convection from the pipe to the air is h = 20 W/m^2-°C find the rate of heat loss from the pipe.

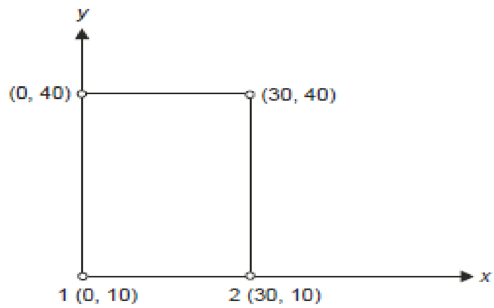
MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject : FINITE ELEMENT METHODS

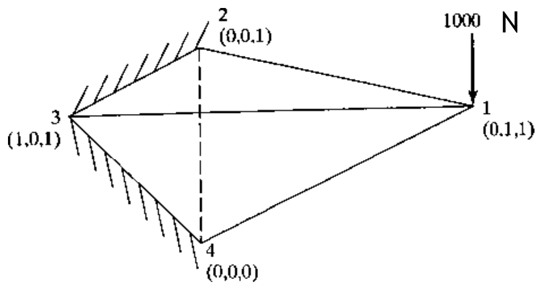
UNIT – IV

ASSIGNMENT - IV

1. Consider a quadrilateral element as shown in fig, the local coordinates are ξ =0.5, η = 0.5. Evaluate Jacobean matrix and strain- Displacement matrix.



2.Compute the Strain displacement matrix for the tetrahedral element shown in Figure.



3.a.) Define lumped mass and consistent mass

b.) What do you mean by steady state heat transfer analysis?

4.Estimate the temperature profile in **a** fin of diameter 25 mm, whose length is 500mm. The thermal conductivity of the fin material is 50 W/m K and heat transfer coefficient over the surface of the fin is 40 W/m^2 K at 30^{0}C. The tip is insulated and the base is exposed to a temperature of 150 ^{0}C. Evaluate the temperatures at points separated by 100 mm each.

5.a.) What are different thermal applications of finite element analysis? Compare the structural analysis with thermal analysis.

b.) Calculate the temperature distribution in the fin of 10 mm diameter, which is exposed to the convective boundary conditions of 40 W/m^2 K with 30^0 C. The base of the fin is exposed to a heat flux of 450 KW/m^3 and the thermal conductivity of fin material is 30W/m K

UNIT 5

DYNAMIC ANALYSIS

Syllabus:

Dynamic Analysis: Formulation of finite element model, element matrices, evaluation of Eigen values and Eigen vectors for a stepped bar and a beam. Overview of commercial softwares like Ansys, Abaqus etc.

OBJECTIVE:

To Learn the Applications of FEM for dynamic problems with emphasis on undamped vibration system.

OUTCOME:

Solve dynamic problems where the effects of mass matters during the analysis.

UNIT-V
DYNAMIC ANALYSIS

Dynamics is a special branch of mechanics where inertia of accelerating masses must be considered in the force-deflection relationships. **In** order to describe motion of the mass system, a component with distributed mass is approximated by a finite number of mass points. Knowledge of certain principles of dynamics is essential to the formulation of these equations.

Every structure is associated with certain frequencies and mode shapes of free vibration (without continuous application of load), based on the distribution of mass and stiffness in the structure. Any time-dependent external load acting on the structure, whose frequency matches with the natural frequencies of the structure, causes resonance and produces large displacements leading to failure of the structure. Calculation of natural frequencies and mode shapes is there for every important.

In general, for a system with on' degrees of freedom, stiffness 'k' and mass 'm' are represented by stiffness matrix [K] and mass matrix [M] respectively.

Then

$$([K] - \omega^2 [M]) \{u\} = \{0\}$$
$$([M]^{-1}[K] - \omega^2 [I]) \{u\} = \{0\}$$

Here, [M] is the mass matrix of the entire structure and is of the same order, say n x n, as the stiffness matrix [K]. This is also obtained by assembling element mass matrices in a manner exactly identical to assembling element stiffness matrices. The mass matrix is obtained by two different approaches, as explained subsequently.

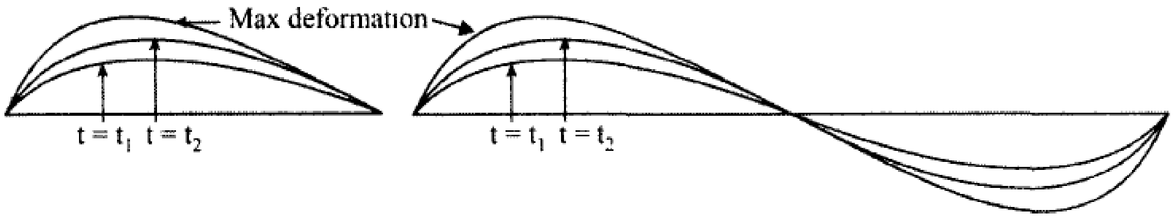
A structure with 'n' DOF will therefore have 'n' eigen values and 'n' eigenvectors. Some eigen values may be repeated and some eigen values maybe complex, in pairs. The equation can be represented in the standard form,

$[A]\{x\}_i = \lambda_i \{x\}_i$.

In dynamic analysis, ω_i, indicates ith natural frequency and $\{X\}_I$ indicates ith natural mode of vibration.

A natural mode is a *qualitative* plot of nodal displacements. In every natural mode of vibration, all the points on the component will reach their maximum values at the same time and will pass through zero displacements at the same time. Thus, in a particular mode, all the points of a component will vibrate with the same frequency and their relative displacements are indicated by

the components of the corresponding eigen vector. These relative (or proportional) displacements at different points
on structure remain same at every time instant for undamped free vibration.



Hence, without loss of generality, {u(t)} can be written as {u}.

Since {u} = {O} forms a trivial solution, the homogeneous system of equations

([A] - λ[I]) {u} = {O}

gives a non-trivial solution only when

([A]-λ[I])= {O},

which implies

Det([A] - λ[I]) = o.

This expression, called *characteristic equation,* results in nth order polynomial in A, and will therefore have n roots. For each λ, the corresponding eigenvector {u}. can be obtained from the n homogeneous equations represented by

([K] - λ[M]) {u} = {O}.

The mode shape represented by {u(t)}gives relatives values of displacements in various degrees of freedom.

NORMALIZATION

The equation of motion of free vibrations ([K] $-\omega^2$[M]) {u} = {O} is a system of homogeneous equations (right side vector zero) and hence does not give unique numerical solution.

Mode shape is a set of relative displacements in various degrees of freedom, while the structure is vibrating in a particular frequency and is usually expressed in normalized form, by following one of the
three normalization methods explained here.

(a) The maximum value of anyone component of the eigenvector is equated to 'I' and, so, all other components will have a value less than or equal to ' 1' .

(b) The length of the vector is equated to '1 ' and values of all components are divided by the length of this vector so that each component will have a value less than or equal to '1'.

(c) The eigenvectors are usually normalized so that

$$\{u\}_i^T [M] \{u\}_i = 1 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_i = \lambda_i$$

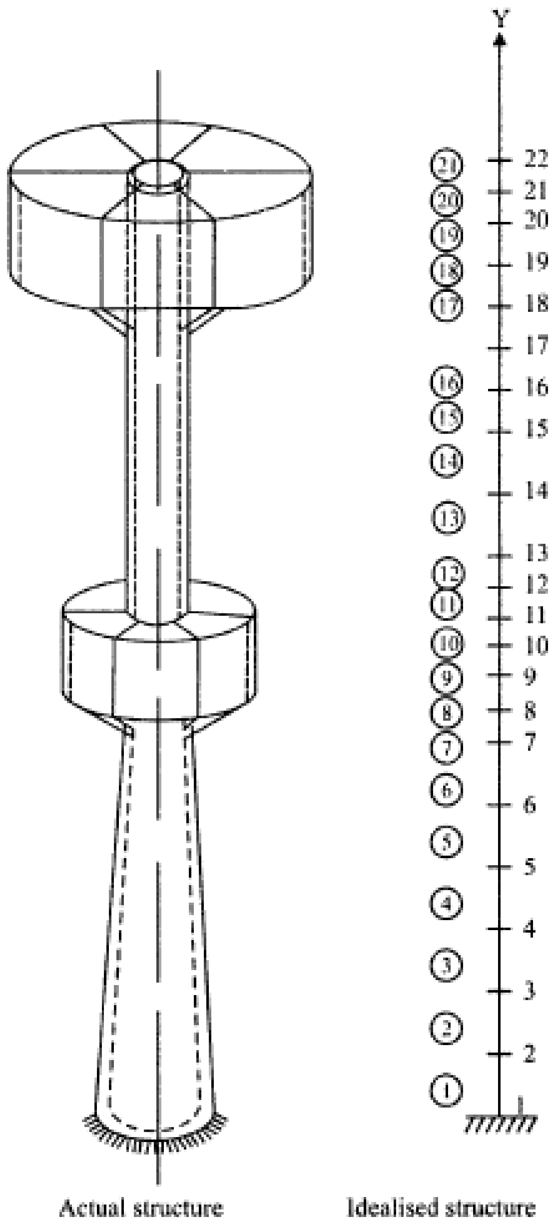
For a positive definite symmetric stiffness matrix of size n x n, the Eigen values are all real and eigenvectors are orthogonal

i.e.,

$$\{u\}_i^T [M] \{u\}_j = 0 \quad \text{and} \quad \{u\}_i^T [K] \{u\}_j = 0 \quad \forall \quad i \neq j$$

MODELLING FOR DYNAMIC ANALYSIS

Solution for any dynamic analysis is an iterative process and, hence, is time -consuming. Geometric model of the structure for dynamic analysis can be significantly simplified, giving higher priority for proper representation of distributed mass. An example of a simplified model of a water storage tank is shown in Fig. Below, representing the central hollow shaft by long beam elements and water tanks at two levels by a few lumped masses and short beam elements of larger moment of inertia.



MASS MATRIX

Mass matrix [M] differs from the stiffness matrix in many ways:

(i) The mass of each element is equally distributed at all the nodes of that element

(ii) Mass, being a scalar quantity, has same effect along the three translational degrees of freedom (u, v and w) and is not shared

(iii) Mass, being a scalar quantity, is not influenced by the local or globa coordinate system. Hence, no transformation matrix is used for converting mass matrix from element (or local) coordinate system to structural (or global) coordinate system.

Two different approaches of evaluating mass matrix [M] are commonly considered.

(a) Lumped mass matrix

Total mass of the element is assumed equally distributed at all the nodes of the element in each of the translational degrees of freedom. Lumped mass is not used for rotational degrees of freedom. Off-diagonal elements of this matrix are all zero. This assumption *excludes dynamic coupling* that exists between different nodal displacements.

Lumped mass matrices [M] of some elements are given here.

Lumped mass matrix of truss element with 1 translational DOF per node along its local X-axis

$$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lumped mass matrix of plane truss element in a 2-D plane with2 translational DOF per node (Displacements along X and Y coordinate axes)

$$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Please note that the same lumped mass is considered in each translational degree of freedom (without proportional sharing of mass between them) at each node.

Lumped mass matrix of a beam element in X-V plane, with its axis along x-axis and with two DOF per node (deflection along Y axis and slope about Z axis) is given below. Lumped mass is not considered in the rotational degrees of freedom.

$$[M] = \frac{\rho AL}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note that lumped mass terms are not included in 2nd and 4th rows, as well as columns corresponding to rotational degrees of freedom.

Lumped mass matrix of a CST element with 2 DOF per node. In this case, irrespective of the shape of the element, mass is assumed equally distributed at the three nodes. It is distributed equally in all DOF at each node, without any sharing of mass between different DOF

$$[M]= \frac{\rho AL}{3}\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Consistent mass matrix

Element mass matrix is calculated here, *consistent* with the assumed displacement field or element stiffness matrix. [M] is a banded matrix of the same order as the stiffness matrix. This is evaluated using the same

interpolating functions which are used for approximating displacement field over the element. It yields more accurate results but with more computational cost. Consistent mass matrices of some elements are given here.

Consistent mass matrix of a Truss element along its axis (in local coordinate system)

$$\{u\}^T = [u \quad v]$$

$$[N]^T = [N_1 \quad N_2]$$

where, $N_1 = \frac{(1-\xi)}{2}$

and $N_2 = \frac{(1+\xi)}{2}$

$$[M]= \int_v [N]\rho[N]^T \, dV = \int_0^L A[N]\rho[N]^T$$

$$dx = \int_{-1}^{+1} A\,\rho[N][N]^T (\det J)(dx/d\xi)d\xi$$

Here, $x = N_1x_1 + N_2x_2 = \frac{(x_1+x_2)}{2} + \frac{(x_2-x_1)\xi}{2}$

and $dx = \frac{dx}{d\xi}.d\xi = \det J \, d\xi = \left(\frac{L}{2}\right)d\xi$

Using the values of integration in natural coordinate system,

$$[M]=\rho A\left(\frac{L}{2}\right)\int_{-1}^{+1}\begin{bmatrix}(1-\xi)/2\\(1+\xi)/2\end{bmatrix}\left[(1-\xi)/2 \qquad (1+\xi)/2\right]d\xi$$

$$=\frac{\rho AL}{8}\begin{bmatrix}\int(1-\xi)^2 d\xi & \int\left(1-\xi^2\right)d\xi\\ \int\left(1-\xi^2\right)d\xi & \int(1+\xi)^2\,d\xi\end{bmatrix}$$

$$=\frac{\rho AL}{8}\begin{bmatrix}\left(\xi-\xi^2+\xi^3/3\right) & \left(\xi-\xi^3/3\right)\\ \left(\xi-\xi^3/3\right) & \left(\xi+\xi^2+\xi^3/3\right)\end{bmatrix}$$

$$=\frac{\rho AL}{8}\begin{bmatrix}8/3 & 4/3\\ 4/3 & 8/3\end{bmatrix}=\frac{\rho AL}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Consistent mass *matrix (if a* **Plane** *Truss element,* inclined to global X-axis -Same elements of I-D mass matrix are repeated in two dimensions(along X and Y directions) without sharing mass between them. Mass terms in X and Y directions are uncoupled.

$$[M]=\frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 1 & 0\\ 0 & 2 & 0 & 1\\ 1 & 0 & 2 & 0\\ 0 & 1 & 0 & 2\end{bmatrix}$$

Consistent mass matrix of a Space *Truss element,* inclined to X-Y plane) -Same elements of 1-0 mass matrix are repeated in three dimensions (along X, Y and Z directions) without sharing mass between them.

$$[M]=\frac{\rho AL}{6}\begin{bmatrix}2 & 0 & 0 & 1 & 0 & 0\\ 0 & 2 & 0 & 0 & 1 & 0\\ 0 & 0 & 2 & 0 & 0 & 1\\ 1 & 0 & 0 & 2 & 0 & 0\\ 0 & 1 & 0 & 0 & 2 & 0\\ 0 & 0 & 1 & 0 & 0 & 2\end{bmatrix}$$

Consistent *mass matrix of a Beam element*

$[M]=\rho A\left(\frac{L}{2}\right)\int\{H\}^T\{H\}d\xi$ with Hermite shape functions $\{H\}$ as used in a beam element.

$$=\frac{\rho AL}{128}\int\begin{bmatrix}2\left(2-3\xi+\xi^3\right)\\ L\left(1-\xi+\xi^2+\xi^3\right)\\ 2\left(2+3\xi-\xi^3\right)\\ L\left(-1-\xi+\xi^2+\xi^3\right)\end{bmatrix}\times$$

$$\left[2\left(2-3\xi+\xi^3\right)\quad L\left(1-\xi-\xi^2+\xi^3\right)\quad 2\left(2+3\xi-\xi^3\right)\quad L\left(-1-\xi+\xi^2+\xi^3\right)\right]d\xi$$

$$=\frac{\rho AL}{420}\begin{bmatrix}156 & 22L & 54 & -13L\\ 22L & 4L^2 & 13L & -3L^2\\ 54 & 13L & 156 & -22L\\ -13L & -3L^2 & -22L & 4L^2\end{bmatrix}$$

Consistent mass matrix of a CST element in a 2-D plane

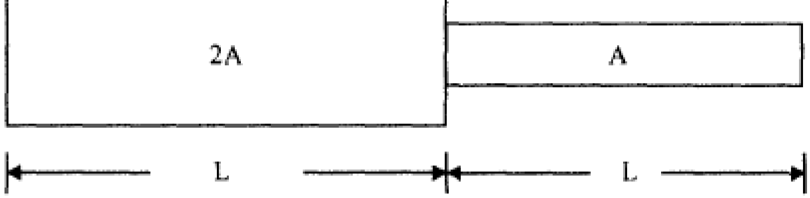
$$[N]^T=\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3\end{bmatrix}$$

$$[M]=\int[N]\rho[N]^T\,dV=t\int[N]\rho[N]^T\,dA$$

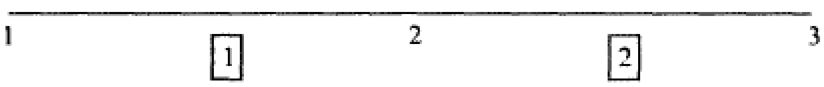
$$=\frac{\rho tA}{12}\begin{bmatrix}2 & 0 & 1 & 0 & 1 & 0\\ & 2 & 0 & 1 & 0 & 1\\ & & 2 & 0 & 1 & 0\\ & & & 2 & 0 & 1\\ & \text{Sym} & & & 2 & 0\\ & & & & & 2\end{bmatrix}$$

Note: Natural frequencies obtained using lumped mass matrix are LOWER than exact values.

Example 1 : Find the natural frequencies of longitudinal vibrations of the unconstrained stepped shaft of areas A and 2A and of equal lengths (L), as shown below.



Solution : Let the finite element model of the shaft be represented by 3 nodes and 2 truss elements (as only longitudinal vibrations are being considered) as shown below.



$$[K]_1 = (2A)\left(\frac{E}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \left(\frac{AE}{L}\right)\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix};$$

$$[K]_2 = \left(\frac{AE}{L}\right)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Using consistent mass matrix approach

$$[M]_1 = \frac{\rho(2A)L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix};$$

$$[M]_2 = \frac{\rho AL}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Assembling the element stiffness and mass matrices,

$$[K] = \frac{AE}{L}\begin{bmatrix} 2 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix};$$

$$[M] = \frac{\rho AL}{6}\begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Eigenvalues of the equation ([K] – ω^2 [M]) {u} = {O} are the roots of the characteristic equation represented by

$$\begin{vmatrix} 2AE/L - \omega^2 4\rho\rho AL/ & -2AE/L - \omega^2 2\rho\rho AL/ & 0 \\ 2AE/L - \omega^2 2\rho\rho AL/ & 3AE/L - \omega^2 6\rho\rho AL/ & -1AE/L - \omega^2 \rho AL/6 \\ 0 & -AE/L - \omega^2 \rho AL/6 & AE/L - \omega^2 2\rho\rho AL/ \end{vmatrix} = 0$$

Multiplying all the terms by (L/AE)

$$\begin{vmatrix} 2(1-2\beta) & -2(1+\beta) & 0 \\ -2(1+\beta) & 3(1-2\beta) & -(1+\beta) \\ 0 & -(1+\beta) & (1-2\beta) \end{vmatrix} = 0$$

or $\quad 18\,\beta\,(\beta - 2)\,(1 - 2\beta) = 0$

$$\beta = \frac{\rho L^2 \omega^2}{6E}$$

The roots of this equation are

$$\beta = 0,\ 2 \text{ or } \frac{1}{2} \text{ or } \omega^2 = 0,\ \frac{12E}{\rho L^2} \text{ or } \frac{3E}{\rho L^2}$$

Corresponding eigenvectors are obtained from ([K] – ω^2 [M]) {u} = {0} for different values of ω^2 as $[1 \quad 1 \quad 1]^T$ for $\beta = 0$, $[1 \quad 0 \quad -2]^T$ for $\beta = \frac{1}{2}$ and $[1 \quad -1 \quad 1]^T$ for $\beta = 2$.

The first eigenvector implies rigid body motion of the shaft. One component (u I in this example) is equated to '1' and other displacement components (U2 and U3 in this example) are obtained as ratios w.r.t. that component,following one method of normalization. Alternatively, they may also be expressed in other normalized forms.

Note: Static solution for such an unconstrained bar, with rigid body motion, involves a singular [K] matrix and can not be solved for {u}, while dynamic analysis is mathematically possible.

SUMMARY

• A distributed mass system will have as many natural frequencies and mode shapes as the number of DOF, 'n'.

• Free undamped vibrations involve a set of n homogeneous equations. Such equations will not give a unique solution. A mode shape consists of relative displacement values at (n-l) DOF, obtained w.r.t. the chosen displacement value at one DoF. The mode shapes (Eigen vectors) are usually normalized.

• The n natural frequencies may be real or complex (in pairs). Some of them may be zero (indicating rigid body mode) or repeated.

• Only first few frequencies (lower values) are significant and are usually calculated by iterative methods. Hence, a coarse mesh is adequate for dynamic analysis.

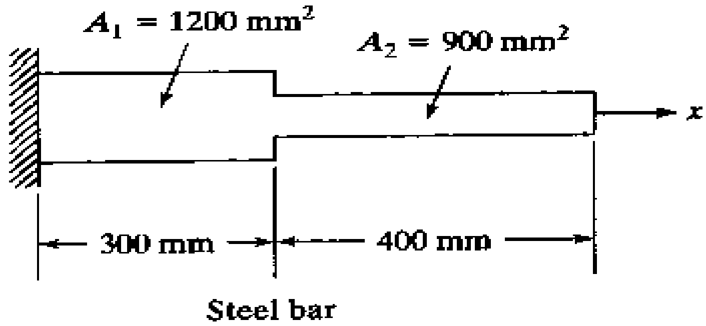
MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

Subject : FINITE ELEMENT METHODS

UNIT – V

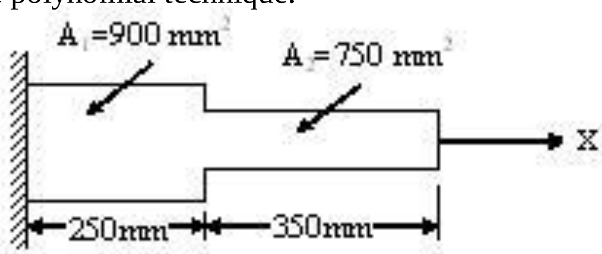
TUTORIAL - V

1. Determine natural frequencies for a Steel bar as shown in figure.

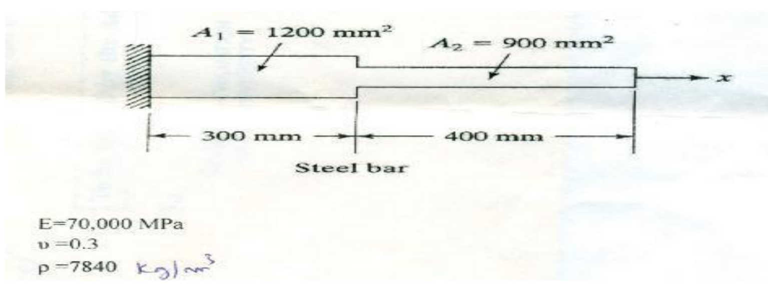


2. a.) Write a short note on damping.

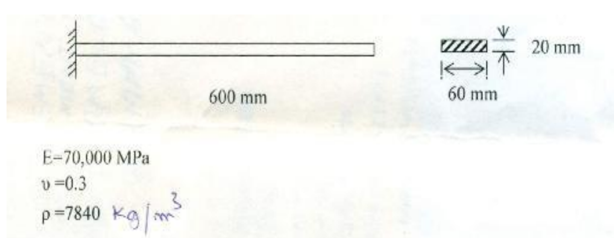
   b.) Consider axial vibration of the steel bar shown in Figure., Develop the global stiffness and mass matrices Determine the natural frequencies and mode shapes using the characteristic polynomial technique.



3. Consider axial vibration of the steel bar shown in Fig. a) Develop the global stiffness and mass matrices b) By hand calculations, determine the lowest natural frequency and mode shape 1and 2



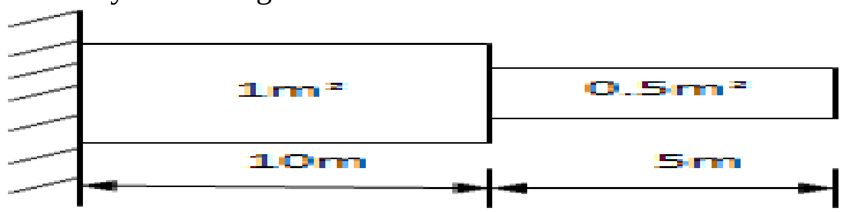
4.Write the step by step procedure to determine the frequencies and nodal displacements of the steel cantilever beam shown in Figure.



5. Explain the Overview of Commercial software's like ANSYS, ABAQUES .

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY
Subject : FINITE ELEMENT METHODS
UNIT – V
ASSIGNMENT - V

1. Determine the Eigen values and Eigen Vectors for the stepped bar as shown in Figure, take density as 7850 kg/m^3 and E= 30X10^{6}N/m^2?

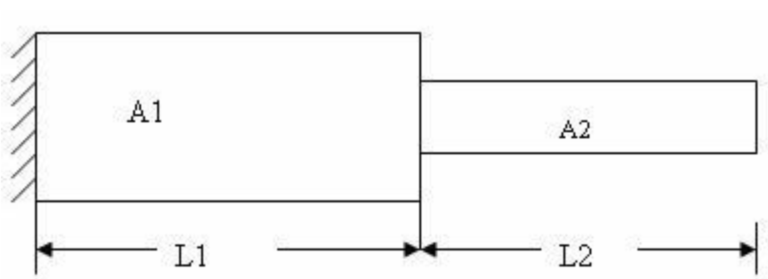


2. Define a.) Eigen value and Eigenvector

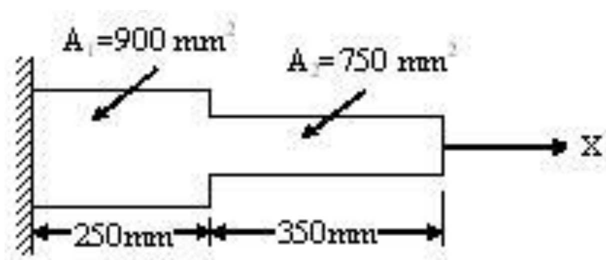
   b.) Dynamic analysis

3. Determine natural frequencies and corresponding mode shapes for the figure

   Take L_1=1m, L_2=2m, A_1=2m^2, A_2=1m^2, ρ = 7850 kg/m^3, E = 200Gpa



4. Consider axial vibration of the steel bar shown in Figure.6,
   i) Develop the global stiffness and mass matrices
   ii) Determine the natural frequencies and mode shapes using the characteristic polynomial technique.



5.) Write short note on a.) Eigen vectors for a stepped beam b.) Evaluation of Eigen values

PREVIOUS QUESTION PAPERS

R15

Code No: R15A0322

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester Regular/supplementary Examinations, April/May 2019

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

***

PART-A (25 Marks)

1). a What is meant by finite Element method **[2M]**
b Name the weighted residual techniques? **[3M]**
c Write down the expression of stiffness matrix for a truss element. **[2M]**
d Define plane strain problem. **[3M]**
e What is CST element? **[2M]**
f Write down the shape functions for an axisymmetric triangular element. **[3M]**
g Write the governing equation for a steady flow heat conduction. **[2M]**
h Write down the expression of stiffness matrix for a beam element. **[3M]**
i What is meant by discretization and assembling? **[2M]**
j What is the difference between static and dynamic analysis? **[3M]**

PART-B (50 MARKS)

SECTION-I

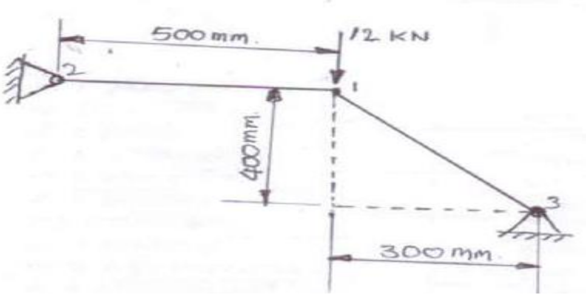
2 Describe advantages, disadvantages and applications of finite element analysis. **[10M]**

OR

3 The following equation is available for a physical phenomena
$\frac{d^2y}{dx^2}$ -10x^2=5; 0<x<1, Boundary Conditions; y(0) =0, y(1) =0, Using Galarkin method of weighted residual find an approximate solution of the above differential equation. **[10M]**

SECTION-II

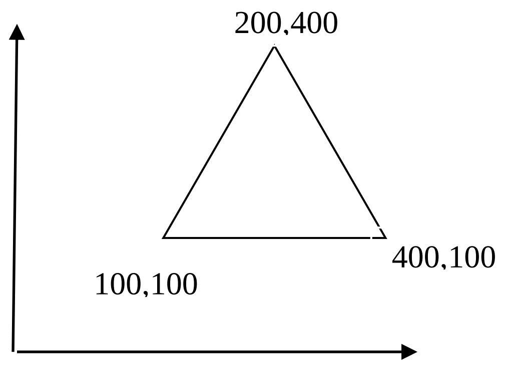
4 For the two bar truss shown in figure, determine the displacement at node 1 and stresses in element2, Take E=70GPa, A= 200mm^2. **[10M]**



OR

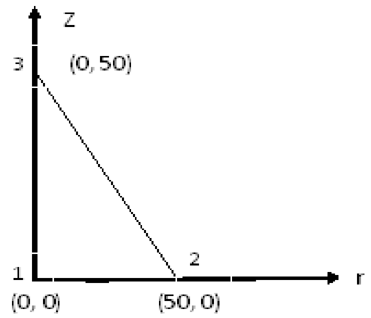
5 For the plane stress element shown in figure the nodal displacements are **[10M]**
U_1= 2.0mm, V_1=1.0mm
U_2= 1.0 mm,V_2= 1.5mm, U_3= 2.5mm,V_3=0.5mm, Take E= 210GPa, ν= 0.25,

t=10mm. Determine the strain-Displacement matrix [B].



SECTION-III

6 For axisymmetric element shown in figure, determine the strain-displacement matrix. Let E = $2.1 \times 10^5 N/mm^2$ and ν= 0.25. The co-ordinates shown in figure are in millimeters. **[10M]**



OR

7 Evaluate the following integral using Gaussian quadrature, so that the result is exact. **[10M]**

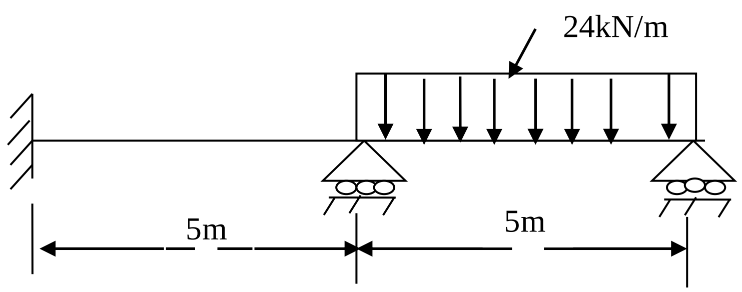
$f(r) = \int_{-1}^{1} (\frac{1}{1+x^2}$ +2x –sinx) dx

SECTION-IV

8 Estimate the temperature distribution in a fin whose cross section is 15mm X 15mm and 500mm long. Take Thermal conductivity as 50W/m-k and convective heat transfer coefficient as 75 W/m^2-k at 25^oC. The base temperature is assumed to be constant and its value may be taken as 900^oC. And also calculate the heat transfer rate? **[10M]**

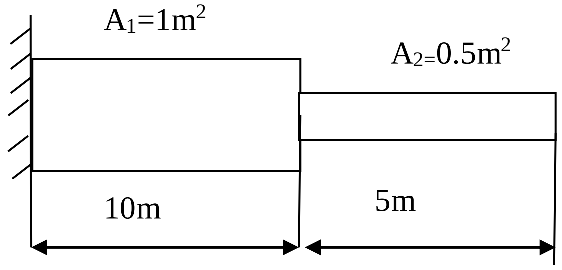
OR

9 For the beam loaded as shown in figure, determine the slope at the simple supports. Take E=200GPa, $I=4 \times 10^6 m^4$. **[10M]**



SECTION-V

10 Determine the Eigen values and Eigen vectors for the beam shown in figure



$E=30x10^5 N/m^2$
$\rho=0.283 kg/m^3$

[10M]

OR

11 Write short note on **[10M]**

(a) Eigen vectors for a stepped beam
(b) Evaluation of Eigen values.

****

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

B. Tech III Year II Semester

FINITE ELEMENT METHODS

- - -

1.a) Derive the interpolation functions at all nodes for the quadratic serendipity element.

b) Evaluate the integral by using one and two-point Gaussian quadrature and compare with exact value.

$$I = \int_{-1}^{+1}\int_{-1}^{+1} (x^3 + x^2 y + xy^2 + \sin 2x + \cos 2y)dx\,dy$$

2.a) Clearly explain the finite element formulation for an axisymmetric shell with an axisymmetric loading. Determine the matrix relating strains and nodal displacements for an axisymmetric triangular element.

b) Establish the Hermite shape functions for a beam element Derive the equivalent nodal point loads for a u.d.l. acting on the beam element in the transverse direction and also determine stiffness matrix.

3.a) Write about different boundary considerations in beams.

b) Determine the support reactions and maximum vertical deflection for the continuous beam shown in Figure.1.

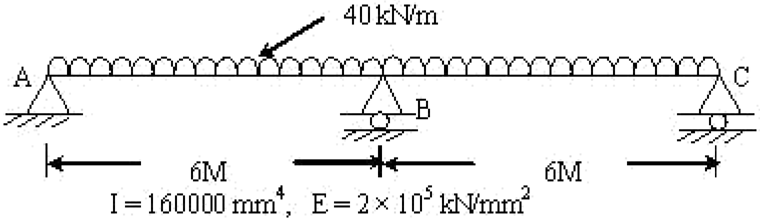


Figure.1

4.a) Discuss in detail about 2D heat conduction in Composite slabs using FEA.

b) Using the isoparametric element, find the Jacobian and inverse of Jacobian matrix for the element shown in Fig.2, 3(a) & 3(b) for the following cases.

i) Determine the coordinate of a point P in x-y coordinate system for the ξ = 0.4 and η = 0.6.

ii) Determine the coordinate of the Q in ξ and η system for the x = 2.5 and y = 1.0.

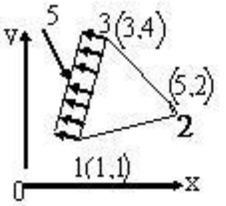


Fig. 2

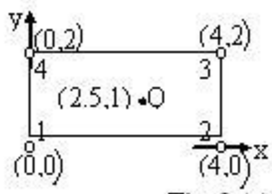


Fig. 3 (a)

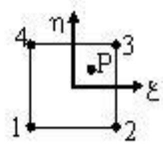


Fig. 3 (b)

5. Calculate the temperature distribution and the heat dissipating capacity of a fin shown in Figre.4. The thermal conductivity of the material is 200 W/ m.K. The surface transfer coefficient is 0.5 W/m^2K. The ambient temperature is 30 °C. the thickness of the fin is 1 cm.

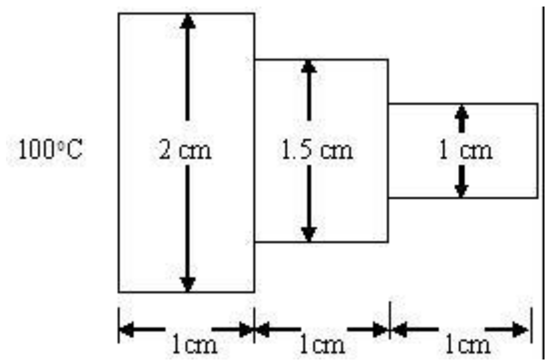


Figure.4

6.a) Write the steps involved in finite-element analysis of a typical problem.

b) Determine the nodal displacements, element stresses and support reactions for the bar as shown in Figure 5. Take E = 200 × 10^9 N/m^2.

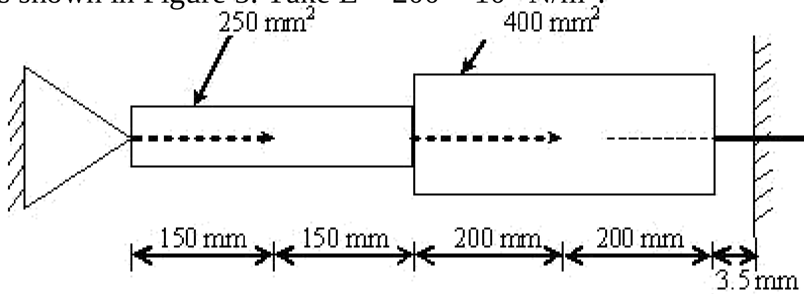


Figure.5

7.a) Derive the equilibrium equation for an elastic continuum using potential energy by displacement approach.

b) Explain the following methods used for the formulation of element characteristics and load matrices:

i) Variational approach ii) Galerkin approach

8.a) With an example differentiate Between Lumped mass, Consistent mass and Hybrid mass matrix and derive for truss element.

b) Consider axial vibration of the steel bar shown in Figure.6,

i) Develop the global stiffness and mass matrices

ii) Determine the natural frequencies and mode shapes using the characteristic polynomial technique.

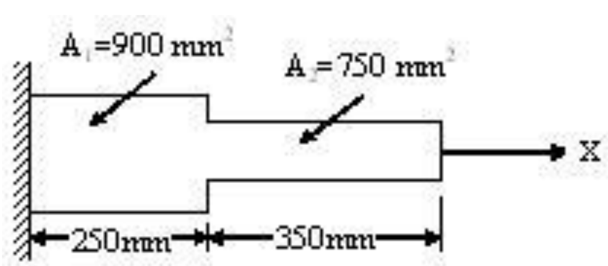


Figure.6

MALLA REDDY COLLEGE OF ENGINEERING AND TECHNOLOGY

B. Tech III Year II Semester
FINITE ELEMENT METHODS
---

1)a) Discuss in detail about the concepts of FEM formulation .How is that FEM emerged as powerful tool. Discuss in detail about applications of finite element method.

b)Derive an equation for finding out the potential energy by Rayleigh –Ritz method. Using Rayleigh – Ritz method, find the displacement of the midpoint of the rod shown in Fig.1. Assume E = 1, A = 1, ρ g = 1 by using linear and quadratic shape functions concept.

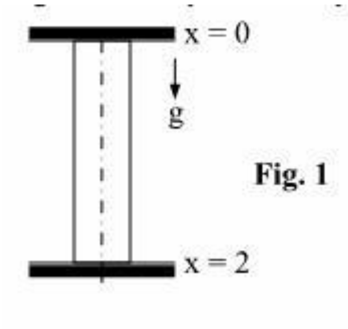


Fig. 1

2. a) Discuss in detail about Linear and Quadratic shape functions with examples.
   b) For the truss shown in fig.2 determine the displacements at point B and stresses in the bars by considering linear and quadratic shape functions.

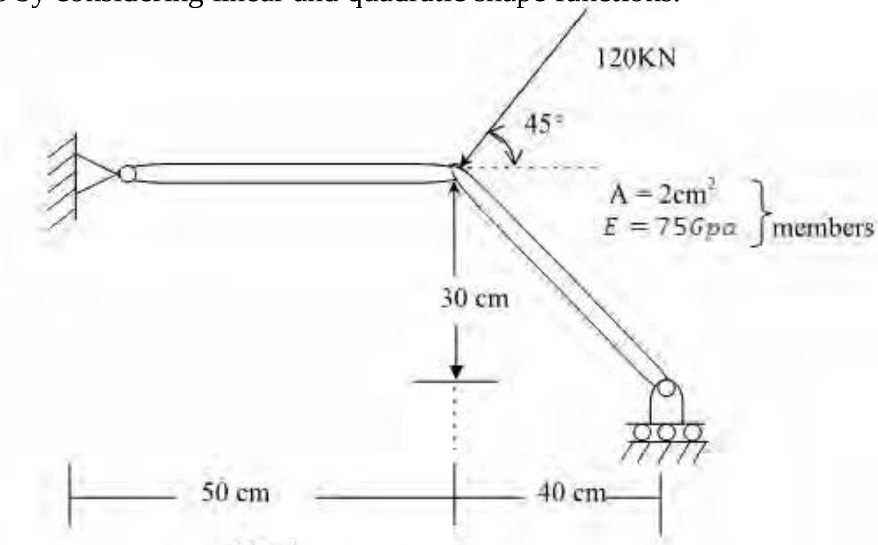


Fig. 2

3. a) Consider axial vibration of the Aluminum bar shown in Fig.3, (i) develop the global stiffness and (ii) determine the nodal displacements and stresses using elimination approach and with help of linear and quadratic shape function concept. Assume Young's Modulus E = 70Gpa.

b) Determine the mass matrix for truss element with an example.

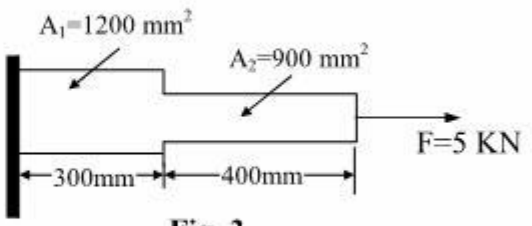


Fig. 3

4. a) Establish the shape functions for a 3 – noded triangular element.

b) Find the deformed configuration, and the maximum stress and minimum stress locations for the rectangular plate loaded as shown in the fig.4. Solve the problem using 2 triangular elements. Assume thickness = 10cm; E = 70 Gpa, and υ = 0.33.

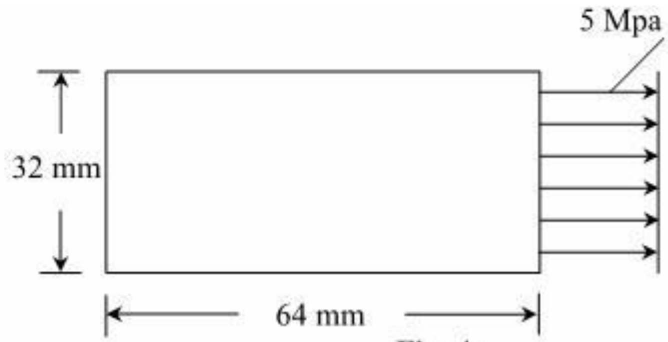


Fig. 4

5. a) Determine the shape functions for 4 – nodded quadrilateral element.

b) For a beam and loading shown in fig.5, determine the slopes at 2 and 3 and the vertical deflection at the midpoint of the distributed load.

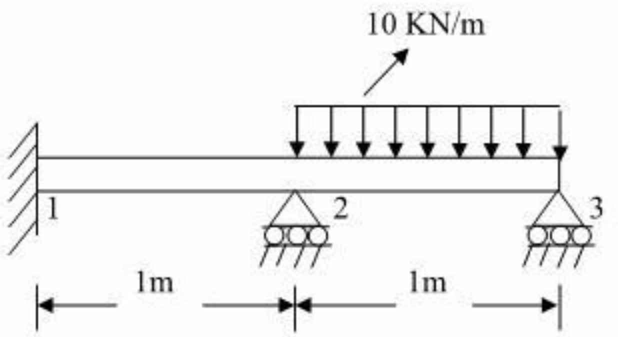


Fig.5

6. a) Clearly explain the finite element formulation for an axisymmetric shell with an axisymmetric loading.Determine the matrix relating strains and nodal displacements for an axisymmetric triangular element.

b) Determine the temperature distribution in a straight fin of circular c/s. Use three one dimensional linear elements and consider the tip is insulated. Diameter of fin is 1 cm, length is 6 cm, h = 0.6 W/cm^2 –C, φ_∞ = 25^{0}C and base temperature is φ_1 =80^{0}C.

7. a) Determine the element stresses, strains and support reactions for the given bar problem as shown in Fig. 6

=1.2 mm ; L =150 mm; P = 60000 N ; $E = 2\ X\ 10^4\ \frac{N}{m}$; A = 250 mm.

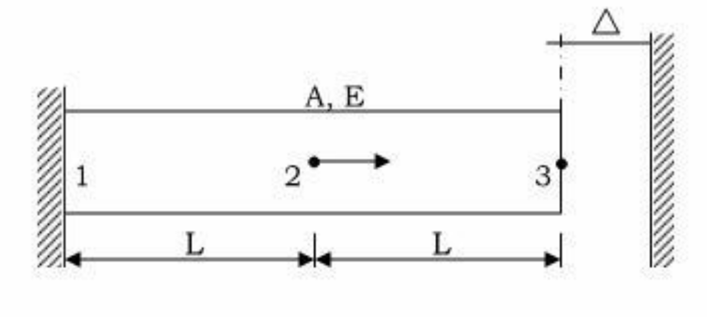


Fig. 6

b) What are shape functions? Indicate briefly the role of shape functions in FEM analysis.

8. a) Derive one dimensional steady state heat conduction equation.

b) An axisymmetric triangular element is subjected to the loading as shown in fig.7 the load is distributed throughout the circumference and normal to the boundary. Derive all the necessary equations and derive the nodal point loads.

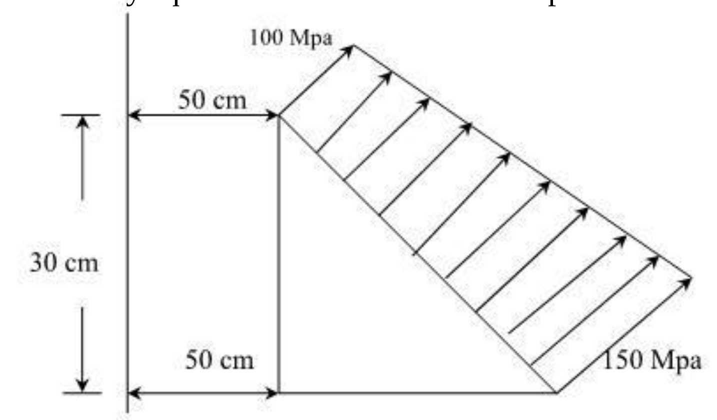


Fig.7

--ooOoo--

Code No: 07A70303 **R07**

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY, HYDERABAD

B. Tech IV Year I Semester Examinations, November/December-2012

FINITE ELEMENT METHODS

(Common to ME, AME)

Time: 3 hours **Max. Marks: 80**

Answer any five questions

All questions carry equal marks

\- - -

1. Determine the total potential energy functional for a beam resting on an elastic foundation, with one end rigidly fixed and the other end propped on an elastic support as shown in figure 1 below. Assume K_f is the subgrade reaction-spring constant per unit length and K_s is the spring constant for the flexible end support. Using the variational principles, obtain the governing differential equation and boundary conditions for this beam. [16]

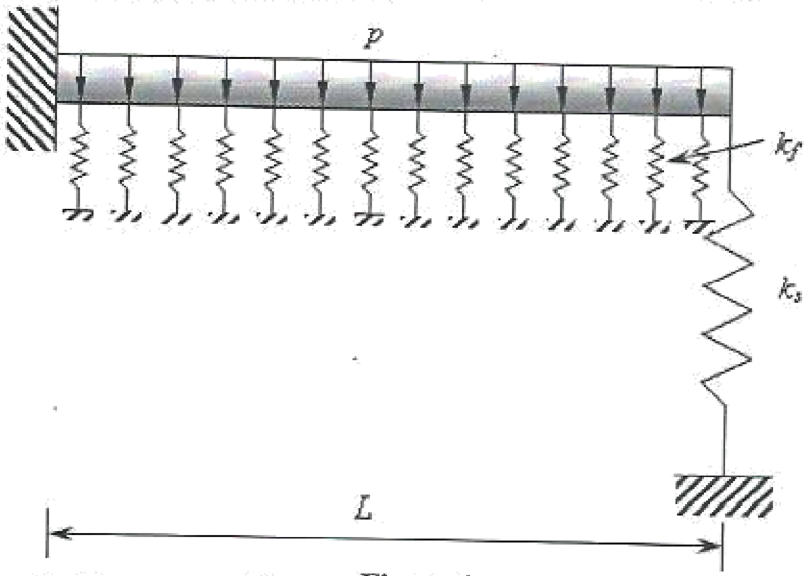


Figure 1

2. An axially loaded bar carries a distributed load over one quarter of its length and has a small gap at one end as shown in the following figure 2. Divide the bar into appropriate number of elements and compute displacements and stresses in the bar. Note that if the load is large enough to close the gap then the gap can be treated as a known displacement boundary condition. Assume $L = 500$ mm, $A = 25$ mm^2, $E = 20{,}000$ N/mm^2, $q = 400$ N/mm, (a) gap = 1 mm, (b) gap = 20 mm. [16]

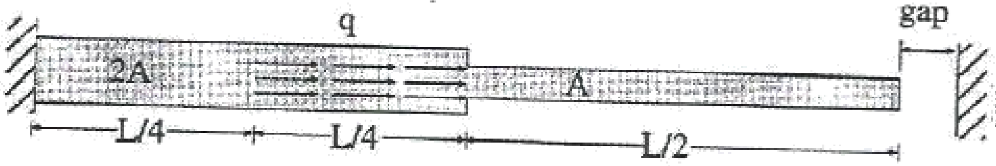


Figure 2

3. Using beam element with two degrees of freedom per node, calculate the end moments and shear forces of a beam shown in figure below subjected to a load of 5 kN/m (Figure 3). [16]

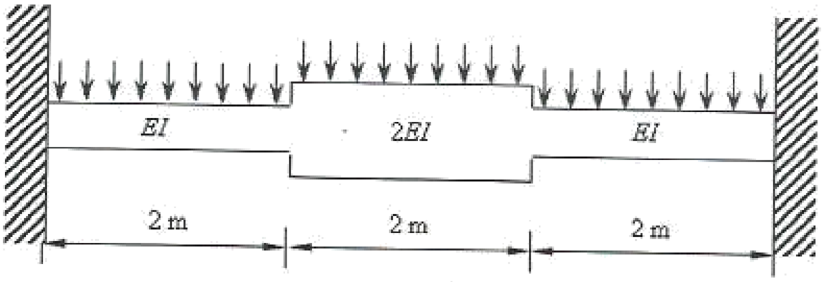


Figure 3

4. For a linear triangular element shown in figure 4 obtain matrix B and also determine the strain vector ε at the point P. [16]

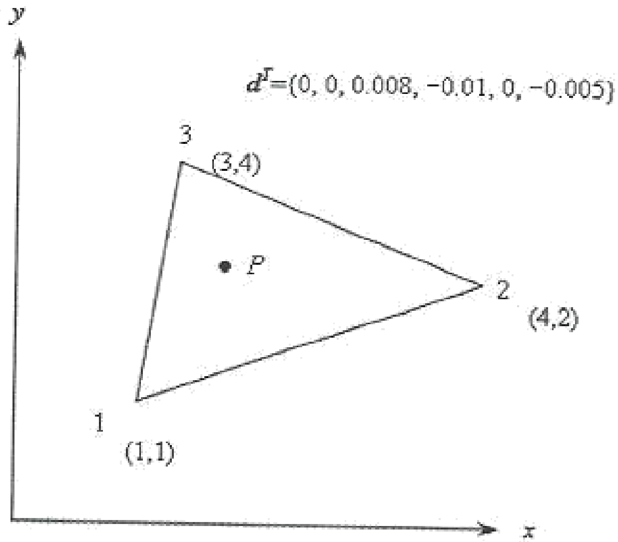


Figure 4

5. Compute stresses in the axisymmetric triangular ring shown in the following figure 5, using two linear triangular elements. $E = 2\times10^7\ N/cm^2$, $\nu = 0.25$. Also compute the principal stresses and the von-Mises effective stress. [16]

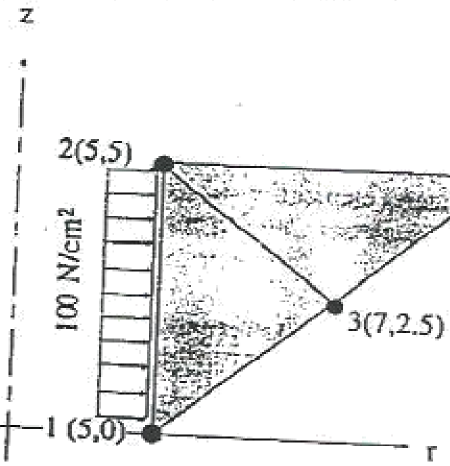


Figure 5

6. Evaluate the following integral by a 2 × 2 Gauss quadrature rule: [16]

$$\int_{-1}^{1}\int_{-1}^{1} \frac{2+x}{3+xy}\,dxdy$$

7. Using 2 quadratic elements, determine temperature distribution in a composite fin consisting Aluminum (K_{xx} = 210 W/m ^{0}C), and Bronze (K_{xx} = 58 W/m ^{0}C) segments as shown in the following figure 6. The thickness is $L/20$. The top and bottom are insulated. The left end is at a temperature of 40^{0}C and the right end is at a temperature of 250^{0}C. Consider a unit width perpendicular to plane of paper and treat the problem as one dimensional. Take $L = 0.5$ m. [16]

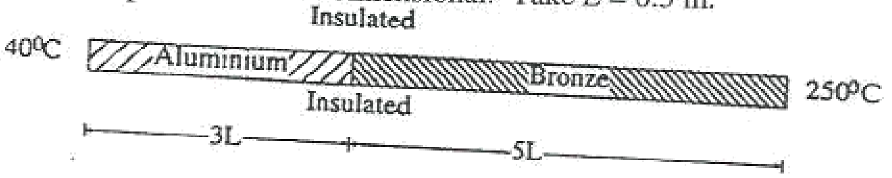


Figure 6

8. For the one-dimensional bar shown in the figure 7, determine the natural frequencies of vibration, using two linear elements of equal length. Use the consistent-mass approach. Let the bar have modulus of elasticity E, mass density ρ, and cross-sectional area A. [16]

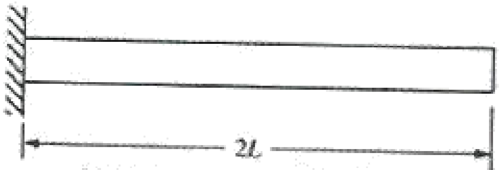


Figure 7

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Code No: 09A60302

R09

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY HYDERABAD

B. Tech III Year II Semester Examinations, December - 2012

FINITE ELEMENT METHODS

(Common to ME, AE)

Time: 3 hours **Max. Marks: 75**

Answer any five questions

All questions carry equal marks

1.a) Write down six 3D strain - displacement equations.

b) Describe Rayleigh – Ritz Mehod. [15]

2.a) Explain about Natural Co-ordinates system.

b) With the help of a suitable example, derive the finite element equations of elastic axial bar using direct stiffness method. [15]

3. Determine the stiffness matrix, stresses and reactions in the truss structure shown in figure 1. [15]

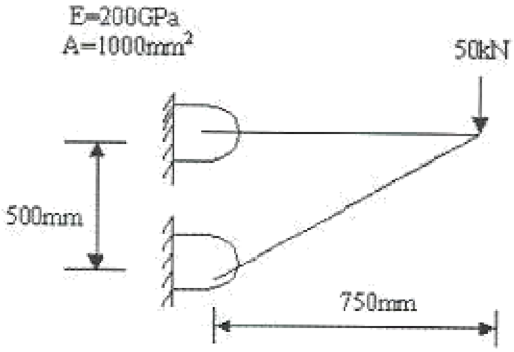

Figure 1

4. A beam of 4 m length is subjected to point loads at the distances of 2 m and 4 m from the fixed end of 10 kN and 20 kN respectively. Calculate the deflection at the center of the beam, if $E = 2.1 \times 10^5$ N/m^2 and $A = 450$ mm^2 as shown in the figure 2. [15]

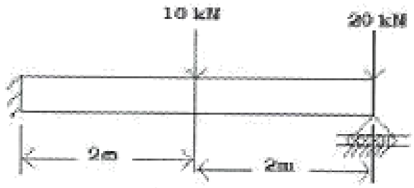

Figure 2

5.a) What is a constant strain triangular element? State its properties and applications.

b) The nodal coordinates of the triangular element are shown in figure 3. At the interior Point P.

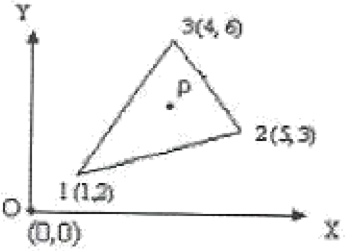

Figure 3

The x coordinate is 3.3 and the shape function at node 1 is N1 is 0.3. Determine the shape functions at nodes 2 and 3 and also the y coordinate of the point P.[15]

6. A uniform aluminum circular fin of diameter 3 cm is extruded from the surface whose temperature is 100^0C. The convection takes place from the lateral surface and tip of the fin. Assuming k = 30 W/m K, h = 1200 W/m^2 K and $T_\infty = 30^0$C, determine the temperature distribution in the fin using two element idealization? [15]

7.a) How do you calculate the element stresses for 3-dimensional body?
b) Derive the element stiffness term and force term for four noded tetrahedral elements. [15]

8.a) From first principles, derive the general equation for elemental mass matrix?
b) Determine the natural frequencies and mode shapes of a stepped bar as shown in figure 4 using the characteristic polynomial technique. Assume E = 300 GPa and density is 7800 kg/m^3. [15]

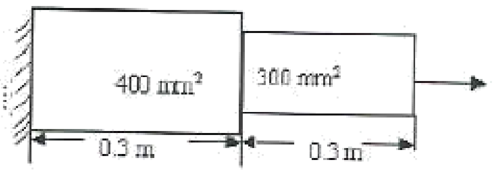

Figure 4

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Code No: 09A60302

R09

JAWAHARLAL NEHRU TECHNOLOGICAL UNIVERSITY, HYDERABAD
B. Tech III Year II Semester Examinations, May/June, 2013
Finite Element Methods
(Common to ME, AE)

Time: 3 hours **Max. Marks: 75**

Answer any five questions
All questions carry equal marks

1.a) Differentiate between planar frame element and space frame element.
b) Use finite element method to calculate displacements and stresses of the bar shown in the Figure.1. [15]

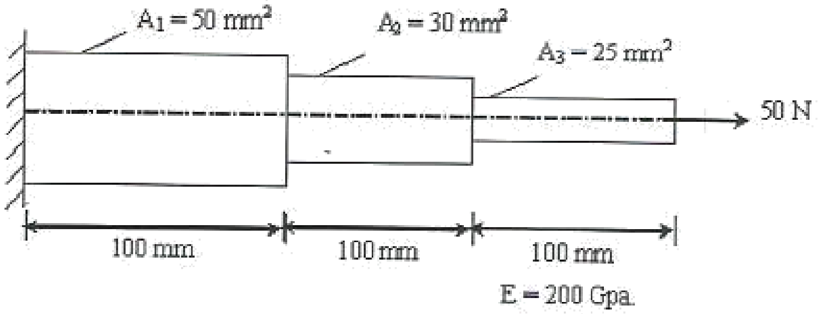

Figure.1

2.a) Write about different boundary considerations in beams.
b) For a beam and loading shown in Figure.2, determine the slopes at 2 and 3 and the vertical deflection at the midpoint of the distributed load. [15]

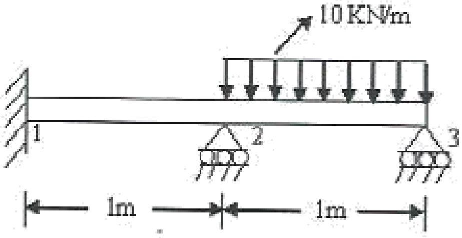

Figure.2

3. Determine the deflection at the free end under its own weight using three elements shown in figure.3, $E = 200Gpa$ and $\rho = 7800\ kg/m^3$. [15]

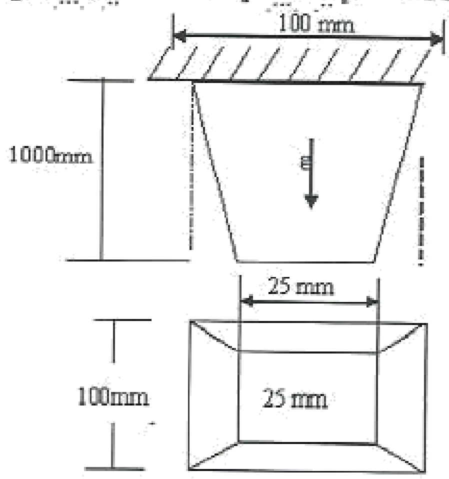

Figure.3

4. Calculate the nodal displacement, stresses and support reactions for the truss shown in Figure.4. [15]

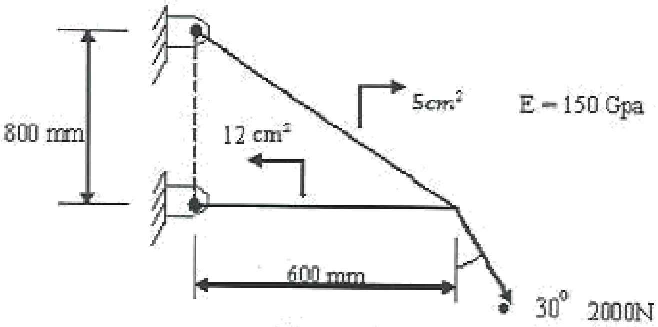

Figure.4

5.a) An axisymmetric triangular element is subjected to the loading as shown in Figure.5.the load is distributed throughout the circumference and normal to the boundary. Derive all the necessary equations and derive the nodal point loads.

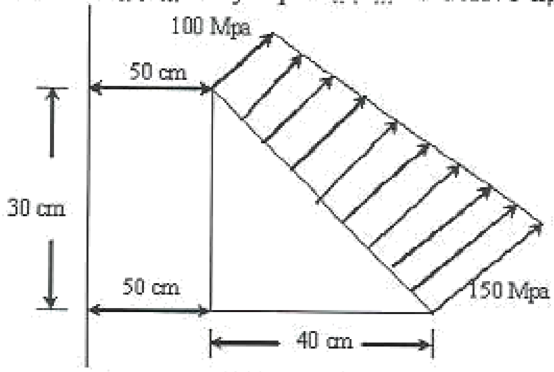

Figure.5

b) Discuss in detail about Hexahedron element and their higher order element. Derive the [B] matrix for any one type of Hexahedron element. [15]

6.a) Write the steps involved with finite-element analysis of a typical problem.

b) Determine the nodal displacements, element stresses and support reactives for the bar as shown in Figure.6. Take $E = 200 \times 10^9$ N/m^2. [15]

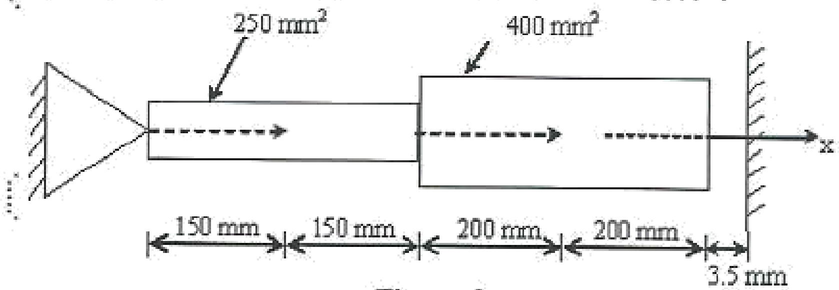

Figure.6

7.a) Distinguish between consistent mass matrix and Lumped mass matrix.

b) Consider axial vibration of the steel bar shown in Figure.7.

i) develop the global stiffness and mass matrices and

ii) determine the natural frequencies and mode shapes using the characteristic polynomial technique. [15]

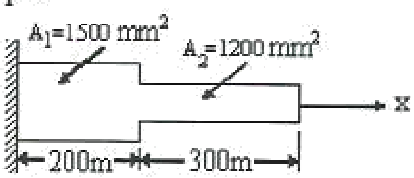

Figure.7

8.a) Derive one dimensional Conduction problem using Weighted Residual Method.

b) Evaluate the integral

$$I = \int_{y=c=4}^{y=d=6} \int_{x=a=-2}^{x=b=2} (1-x)^2 (4-y)^2 \, dxdy$$

Using two point Gaussian Quadrature function values at four points (1, 1), (1, 2), (2, 1) and (2,2) as shown in Figure.8. [15]

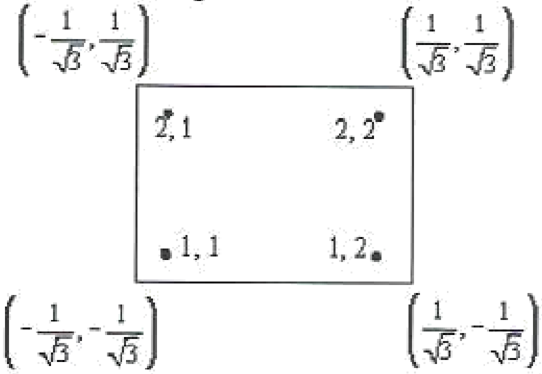

Figure.8

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Finite Element Methods

1.a) Write the strain stress relations based on generalized Hooke's law and derive the elasticity matrix for 3-D field problems.

c) Describe the standard procedure to be followed for understanding the finite element method step by step with suitable example.

2.a) Derive the stiffness matrix of axial bar element with quadratic shape functions based on first principles.

c) Calculate the nodal displacements and forces for the stepped bar with the stiffness values of 10 kN/m and 18 kN/m and a load of 32 kN is subjected at the end of the stepped bar and other end of the bar is fixed.

3.a) Derive the shape functions and stiffness matrix of a two nodded beam element.

c) Derive the load vector for the beam element when a uniformly distributed load is applied.

4.a) For a plane strain problem, the nodal displacements are u_1 = 4.4 μm, u_2 = 2.2 μm, u_3=2.2 μm, v_1 = 3.8 μm, v_2 = 2.9 μm, v_3 = 4.5 μm. Take E=200 GPa, μ = 0.3 and t=10mm. Find the stresses, principal stresses. The coordinates of triangular element are 1(5,25), 2(15,5) and 3(25,15). All dimensions are in millimeters.

c) Show that the stiffness for a triangular element is $[B]^T[D][B]$ At using variational principle. Where A=area of the triangle and t= thickness.

5.a) Compute the strain displacement matrix and also the strains of a axisymmetric triangular element with the coordinates r_1 = 3 cm, z_1 = 4 cm, r_2 = 6 cm, z_2 = 5 cm, r_3= 5 cm, z_3 = 8 cm. The nodal displacement values are u_1 = 0.01 mm, w_1 = 0.01 mm, u_2 = 0.01 mm, w_2 = -0.04 mm, u_3 = -0.03 mm, w_3 = 0.07 mm

b) Differentiate between Axi symmetric elements and symmetric elements with suitable examples.

6.a) Explain the methodology to estimate the stiffness matrix of four noded quadrilateral element.

b) Evaluate $\int [e^{2x} + x^3 + 1 / (x^2 + 2)]\, dx$ over the limits -1 and +1 using one point and three point quadrature formula and compare with exact solution.

7.a) What are different thermal applications of finite element analysis? Compare the structural analysis with thermal analysis.

b) Calculate the temperature distribution in the fin of 10 mm diameter, which is exposed to the convective b.c. of 40 W/m^2 K with 30^0 C. The base of the fin is exposed to a heat flux of 450 kW/m^3 and the thermal conductivity of fin material is 30 W/m K.

8. Determine natural frequencies and corresponding mode shapes for the figure 8.

Take L_1=1m, L_2=2m, A_1=2m^2, A_2=1m^2, ρ = 7850 kg/m^3, E = 200Gpa

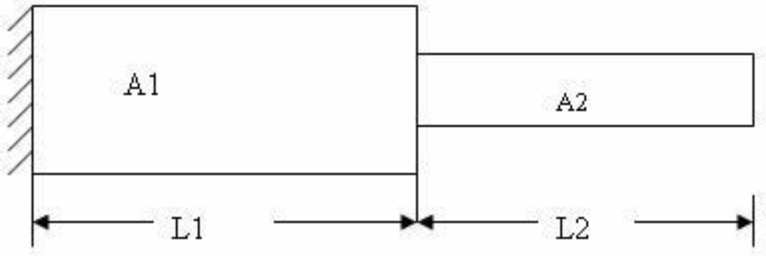


Fig: 8

Code No: R15A0322

R15

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(Autonomous Institution – UGC, Govt. of India)

III B.Tech II Semester supplementary Examinations, Nov/Dec 2018

Finite Element Methods

(ME)

Roll No										

Time: 3 hours **Max. Marks: 75**

Note: This question paper contains two parts A and B
Part A is compulsory which carriers 25 marks and Answer all questions.
Part B Consists of 5 SECTIONS (One SECTION for each UNIT). Answer FIVE Questions, Choosing ONE Question from each SECTION and each Question carries 10 marks.

PART – A

1.a. Briefly discuss weighted residual method for giving approximate solutions for complicated domains [2M]
b. Write the stiffness matrix for 1-d element with linear interpolation functions [3M]
c. Differentiate iso-parametric, sub-parametric, and super parametric elements? [2M]
d. What is the difference between plane truss and space truss? [3M]
e. What are the uses of natural coordinates in 2d- Quadrilateral elements [2M]
f. What are the suitable applications of axi-symmetric elements in FEM? [3M]
g. Write the governing equation for FEA formulation for a fin [2M]
h Express the stiffness matrix for a 1-D conduction problem [3M]
i. What do you understand by mode shapes? [2M]
j. How principle of minimum potential energy is useful in dynamic analysis of systems [3M]

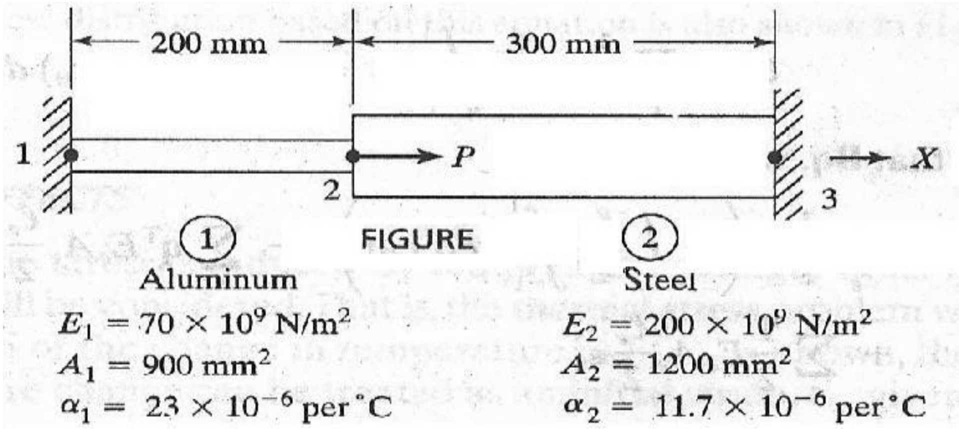
PART – B 10 * 5 = 50 Marks

SECTION-I

2. Derive the equations equilibriums for 3-D body [10M]

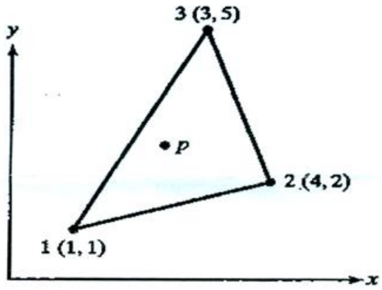
OR

3. An axial load P=300X103N is applied at 200 C to the rod as shown in Figure below. The temperature is the raised to 600 C . [10M]
a) Assemble the K and F matrices.
b) Determine the nodal displacements and stresses.



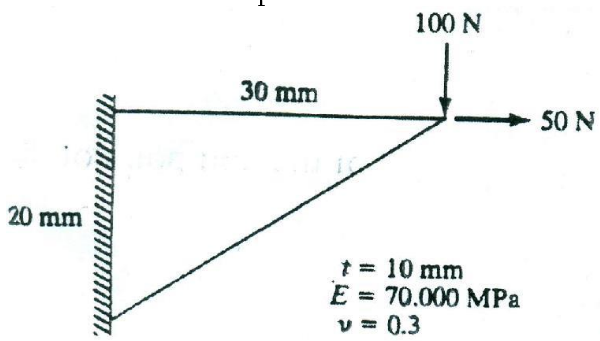
SECTION-II

4. a) Write the difference between CST and LST elements [3M]
b) For point P located inside the triangle shown in the figure below the shape functions N1 and N2 are 0.15 and 0.25, respectively. Determine the x and y coordinates of point P. [7M]



OR

5 For the configuration shown in Fig. determine the deflection at the point of load application using a one-element model. If a mesh of several triangular elements is used, comment on the stress values in the elements close to the tip [10M]



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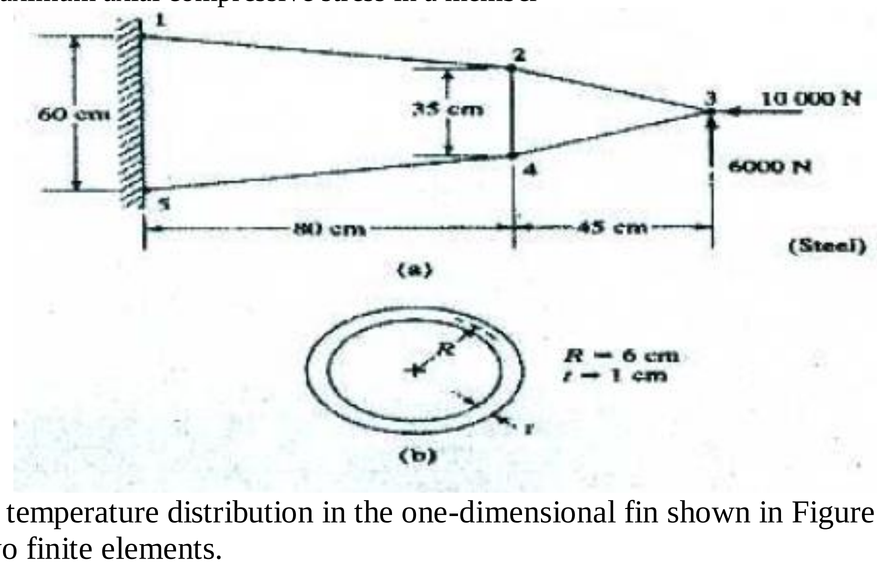
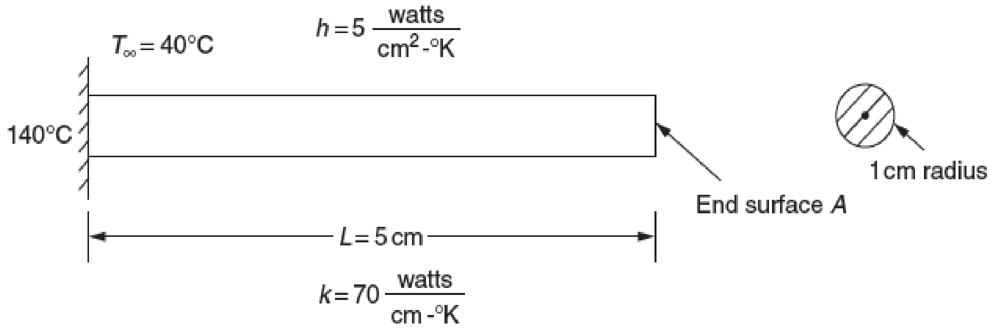
6. Derive the strain displacement matrix for axisymmetric triangular element Discuss advantages of axisymmetric modelling in FEM [10M]

OR

7. Figure shows a five – member steel frame subjected to loads at the free end. The cross section of each member is a tube of wall thickness t=1 cm and mean radius=6cm. Determine the following: [10M]
a) The displacement of node 3 and
b) The maximum axial compressive stress in a member



8. Find the temperature distribution in the one-dimensional fin shown in Figure below using two finite elements. [10M]



OR

9. (a) A 20-cm thick wall of an industrial furnace is constructed using fireclay bricks that have a thermal conductivity of k = 2 W/m-°C. During steady state operation, the furnace wall has a temperature of 800°C on the inside and 300°C on the outside. If one of the walls of the furnace has a surface area of 2 m^2 (with 20-cm thickness), find the rate of heat transfer and rate of heat loss through the wall. [5M]

(b) A metal pipe of 10-cm outer diameter carrying steam passes through a room. The walls and the air in the room are at a temperature of 20°C while the outer surface of the pipe is at a temperature of 250°C. If the heat transfer coefficient for free convection from the pipe to the air is h = 20 W/m^2-°C find the rate of heat loss from the pipe. [5M]

10. For the two-bar truss shown in Figure below, determine the nodal displacements, element stresses and support reactions. A force of P=1000kN is applied at node-1. Assume E=210GPa and A=600mm2 for each element. [10M]

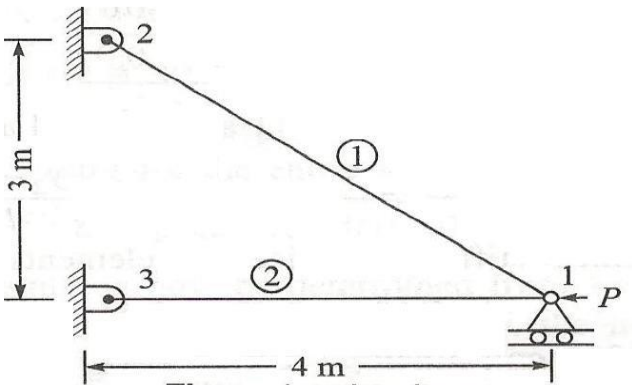


Figure two-bar truss.

OR

11. A bar of length 1 m; cross sectional area 100 mm2; density of 7 gm/cc and Young's modulus 200Gpa is fixed at both the ends. Consider the bar as three bar elements and determine the first two natural frequencies and the corresponding mode shapes. Discuss on the accuracy of the obtained solution [10M]

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