DIGITAL NOTES

OPERATIONS RESEARCH
(R15A0333)

B.Tech IV Year I Semester

DEPARTMENT OF MECHANICAL ENGINEERING

MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

(An Autonomous Institution – UGC, Govt.of India)

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MALLA REDDY COLLEGE OF ENGINEERING & TECHNOLOGY

IV Year B. Tech, ME-I Sem

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(R15A0333) OPERATIONS RESEARCH
(CORE ELECTIVE–IV)

COURSE OBJECTIVES:

- To provide a comprehensive exposure to Production and operations management and its significance in Industries.
- To acquaint students with various activities of Production and operations management and to give insight into the ongoing & futuristic trends in the control of inventory.
- To appraise about need and benefits of functions related to products and processes.

UNIT – I

Development – Definition– Characteristics and Phases – Types of models - Operations Research models – applications.

Allocation: Linear Programming Problem Formulation – Graphical solution – Simplex method – Artificial variables techniques: Two–phase method, Big-M method-Duality principle.

UNIT – II

Transportation Problem - Formulation – Optimal solution, unbalanced transportation problem – Degeneracy.

Assignment problem – Formulation – Optimal solution - Variants of Assignment Problem- Traveling Salesman problem.

Sequencing - Introduction – Flow –Shop sequencing – n jobs through two machines – n jobs through three machines – Job shop sequencing – two jobs through 'm' machines

UNIT – III

Replacement: Introduction – Replacement of items that deteriorate with time – when money value is not counted and counted – Replacement of items that fail completely, Group Replacement.

Theory Of Games: Introduction – Terminology - Solution of games with saddle points and without saddle points – 2×2 games - dominance principle – m X 2 & 2 X n games - Graphical method.

UNIT-IV

Waiting Lines: Introduction – Terminology - Single Channel – Poisson arrivals and exponential service times – with infinite population and finite population models– Multichannel – Poisson arrivals and exponential service times with infinite population single channel – Poisson arrivals.

Inventory: Introduction – Single item – Deterministic models – Purchase inventory models with one price break and multiple price breaks – Stochastic models – demand may be discrete variable or continuous variable – Single period model and no setup cost.

UNIT – V:

Dynamic Programming: Introduction –Terminology - Bellman's Principle of optimality – Applications of dynamic programming - shortest path problem – linear programming problem.

Simulation: Introduction, Definition, types of simulation models, phases of simulation- applications of simulation-Inventory and Queuing problem- Advantages and Disadvantages-Simulation Languages.

COURSE OUTCOMES:

- The student will be able to Illustrate production planning functions and manage manufacturing functions in a better way.
- Develop competency in scheduling and sequencing in manufacturing operations and effect affordable manufacturing lead time.
- Manage and control inventory with cost effectiveness. Get conversant with various documents procedural aspects and preparation of orders for various manufacturing methods.

TEXT BOOKS:

Operations Research / S.D. Sharma
Introduction to O. R / Hiller and Libermann / TMH
Introduction to O.R / Taha / PHI

REFERENCES:

OperationsResearch/A.M.Natarajan,P.Balasubramani,A.Tamilarasi/Pearson Education.
Operations Research / R.Pannerselvam 2e.,PHI Publications
Operations Research /J. K.Sharma 4e. /MacMilan

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COURSE COVERAGE UNIT WISE

UNIT I

S.NO	TOPIC NAME	NAME OF THE TEXT BOOK	AUTHOR NAME	UNIT NUMBER IN TEXT BOOK
1	Development Definition– Characteristics and Phases	Operations Research	S.D. Sharma	UNIT 1 & 2
2	Types of models - Operations Research models – applications.	Operations Research	S.D. Sharma	UNIT 1 & 2
3	Linear Programming Problem Formulation – Graphical solution – Simplex method	Operations Research	S.D. Sharma	UNIT 1 & 2
4	Artificial variables techniques: Two–phase method, Big-M method-Duality principle.	Operations Research	S.D. Sharma	UNIT 1 & 2

UNIT II

S.NO	TOPIC NAME	NAME OF THE TEXT BOOK	AUTHOR NAME	UNIT NUMBER IN TEXT BOOK
1	**Transportation Problem -** Formulation–Optimal solution, Unbalancedtransportation problem–Degeneracy.	OperationsResearch Operations Research	A.M.Natarajan P.Balasubramani A.Tamilarasi S.D. Sharma	UNIT 5 & 6 UNIT 5 & 6
2	**Assignment problem** – Formulation – Optimal solution - Variants of Assignment Problem- Traveling Salesman problem	OperationsResearch Operations Research	A.M.Natarajan P.Balasubramani A.Tamilarasi S.D. Sharma	UNIT 5 & 6 UNIT 5 & 6
3	**Assignmentproblem** – Formulation – Optimal solution - Variants of Assignment Problem- Traveling Salesman problem.	OperationsResearch Operations Research	A.M.Natarajan P.Balasubramani A.Tamilarasi S.D. Sharma	UNIT 5 & 6 UNIT 5 & 6
4	Sequencing - Introduction – Flow –Shop sequencing – n jobs through two machines – n jobs through three machines – Job shop sequencing – two jobs through 'm' machines.	OperationsResearch Operations Research	A.M.Natarajan P.Balasubramani A.Tamilarasi S.D. Sharma	UNIT 7 & 8 UNIT 7 & 8

UNIT III

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1	**Replacement:** Introduction – Replacement of items that deteriorate with time – when money value is not counted and counted	Operations Research Operations Research	R.Pannerselvam S.D. Sharma	UNIT 9 & 10 UNIT 12 & 13
2	Replacement of items that failcompletely, Group Replacement.	Operations Research Operations Research	R.Pannerselvam S.D. Sharma	UNIT 9 & 10 UNIT 12 & 13
3	**Theory Of Games:** Introduction – Terminology.	Introduction to O.R Operations Research	Taha S.D. Sharma	UNIT 7 & 8 UNIT 14 & 15
4	Solution of games with saddle points and without saddle points – 2×2 games	Introduction to O.R Operations Research	Taha S.D. Sharma	UNIT 7 & 8 UNIT 14 & 15
5	m X 2 & 2 X n games - Graphical method.	O Introduction to O.R perations Research	Taha S.D. Sharma	UNIT 7 & 8 UNIT 14 & 15

UNIT IV

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1	**Waiting Lines:** Introduction – Terminology - Single Channel – Poisson arrivals and exponential service times	Operations Research Operations Research	R.Pannerselvam S.D. Sharma	UNIT 11 & 12 UNIT 16 & 17
2	with infinite population and finite population models– Multichannel –Poisson arrivals and exponential service times with infinite population single channel – Poisson arrivals.	Operations Research Operations Research	R.Pannerselvam S.D. Sharma	UNIT 11 & 12 UNIT 16 & 17
3	single channel – Poisson arrivals. **Inventory:** Introduction – Single item – Deterministic models – Purchase inventory models with one price break and multiple price breaks	Operations Research Operations Research	R.Pannerselvam S.D. Sharma	UNIT 11 & 12 UNIT 16 & 17
4	Stochastic models – demand may be discrete variable or continuous	Operations Research	R.Pannerselvam S.D. Sharma	UNIT 11 & 12 UNIT 16 & 17

	variable – Single period model and no setup cost.	Operations Research		

UNIT V

S.NO	TOPIC NAME	NAME OF THE TEXT BOOK	AUTHOR NAME	UNIT NUMBER IN TEXT BOOK
1	**Dynamic Programming:** Introduction –Terminology - Bellman's Principle of optimality	Operations Research Operations Research	S.D. Sharma J. K.Sharma	UNIT 19 & 20 UNIT 16 & 17
2	Applications of dynamic programming - shortest path problem – linear programming problem.	Operations Research Operations Research	S.D. Sharma J. K.Sharma	UNIT 19 & 20 UNIT 16 & 17
3	**Simulation:** Introduction, Definition, types of simulation models, phases of simulation-	Operations Research Operations Research	S.D. Sharma J. K.Sharma	UNIT 19 & 20 UNIT 16 & 17
4	applications of simulation- Inventory and Queuing problem- Advantages and Disadvantages- Simulation Languages.	Operations Research Operations Research	S.D. Sharma J. K.Sharma	UNIT 19 & 20 UNIT 16 & 17

UNIT - 1
BASICS OF OR AND LINEAR PROGRAMMING

1. INTRODUCTION TO OR

TERMINOLOGY

The British/Europeans refer to "operational research", the Americans to "operations research" - but both are often shortened to just "OR" (which is the term we will use). Another term which is used for this field is "management science" ("MS"). The Americans sometimes combine the terms OR and MS together and say "***OR/MS***" or "ORMS".

Yet other terms sometimes used are "industrial engineering" ("IE"), "decision science" ("DS"), and “problem solving”.

In recent years there has been a move towards a standardization upon a single term for the field, namely the term "OR".

“Operations Research (Management Science) is a scientific approach to decision making that seeks to best design and operate a system, usually under conditions requiring the allocation of scarce resources.”

A system is an organization of interdependent components that work together to accomplish the goal of the system.

THE METHODOLOGY OF OR

When OR is used to solve a problem of an organization, the following seven step procedure should be followed:

Step 1. Formulate the Problem

OR analyst first defines the organization's problem. Defining the problem includes specifying the organization's objectives and the parts of the organization (or system) that must be studied before the problem can be solved.

Step 2. Observe the System

Next, the analyst collects data to estimate the values of parameters that affect the organization's problem. These estimates are used to develop (in Step 3) and evaluate (in Step 4) a mathematical model of the organization's problem.

Step 3. Formulate a Mathematical Model of the Problem

The analyst, then, develops a mathematical model (in other words an idealized representation) of the problem. In this class, we describe many mathematical techniques that can be used to model systems.

Step 4. Verify the Model and Use the Model for Prediction

The analyst now tries to determine if the mathematical model developed in Step 3 is an accurate representation of reality. To determine how well the model fits reality, one determines how valid the model is for the current situation.

Step 5. Select a Suitable Alternative

Given a model and a set of alternatives, the analyst chooses the alternative (if there is one) that best meets the organization's objectives.Sometimes the set of alternatives is subject to certain restrictions and constraints. In many situations, the best alternative may be impossible or too costly to determine.

Step 6. Present the Results and Conclusions of the Study

In this step, the analyst presents the model and the recommendations from Step 5 to the decision making individual or group. In some situations, one might present several alternatives and let the organization choose the decision maker(s) choose the one that best meets her/his/their needs.

After presenting the results of the OR study to the decision maker(s), the analyst may find that s/he does not (or they do not) approve of the recommendations. This may result from incorrect definition of the problem on hand or from failure to involve decision maker(s) from the start of the project. In this case, the analyst should return to Step 1, 2, or 3.

Step 7. Implement and Evaluate Recommendation

If the decision maker(s) has accepted the study, the analyst aids in implementing the recommendations. The system must be constantly monitored (and updated dynamically as the environment changes) to ensure that the recommendations are enabling decision maker(s) to meet her/his/their objectives.

HISTORY OF OR

(Prof. Beasley's lecture notes)

OR is a relatively new discipline. Whereas 70 years ago it would have been possible to study mathematics, physics or engineering (for example) at university it would not have been possible to study OR, indeed the term OR did not exist then. It was only really in the late 1930's that operational research began in a systematic fashion, and it started in the UK.

Early in 1936 the British Air Ministry established Bawdsey Research Station, on the east coast, near Felixstowe, Suffolk, as the centre where all pre-war radar experiments for both the Air Force and the Army would be carried out. Experimental radar equipment was brought up to a high state of reliability and ranges of over 100 miles on aircraft were obtained.

It was also in 1936 that Royal Air Force (RAF) Fighter Command, charged specifically with the air defense of Britain, was first created. It lacked however any effective fighter aircraft - no Hurricanes or Spitfires had come into service - and no radar data was yet fed into its very elementary warning and control system.

It had become clear that radar would create a whole new series of problems in fighter direction and control so in late 1936 some experiments started at Biggin Hill in Kent into the effective use of such data. This early work, attempting to integrate radar data with ground based observer data for fighter interception, was the start of OR.

The first of three major pre-war air-defense exercises was carried out in the summer of 1937. The experimental radar station at Bawdsey Research Station was brought into operation and the information derived from it was fed into the general air-defense warning and control system. From the early warning point of view this exercise was encouraging, but the tracking information obtained from radar, after filtering and transmission through the control and display network, was not very satisfactory.

In July 1938 a second major air-defense exercise was carried out. Four additional radar stations had been installed along the coast and it was hoped that Britain now had an aircraft location and control system greatly improved both in coverage and effectiveness. Not so! The exercise revealed, rather, that a new and serious problem had arisen. This was the need to coordinate and correlate the additional, and often conflicting, information received from the additional radar stations. With the out-break of war apparently imminent, it was obvious that something new - drastic if necessary - had to be attempted. Some new approach was needed.

Accordingly, on the termination of the exercise, the Superintendent of Bawdsey Research Station, A.P. Rowe, announced that although the exercise had again demonstrated the technical feasibility of the radar system for detecting aircraft, its operational achievements still fell far short of requirements. He therefore proposed that a crash program of research into the operational - as opposed to the technical -

aspects of the system should begin immediately. The term "operational research" [RESEARCH into (military) OPERATIONS] was coined as a suitable description of this new branch of applied science. The first team was selected from amongst the scientists of the radar research group the same day.

In the summer of 1939 Britain held what was to be its last pre-war air defense exercise. It involved some 33,000 men, 1,300 aircraft, 110 antiaircraft guns, 700 searchlights, and 100 barrage balloons. This exercise showed a great improvement in the operation of the air defense warning and control system. The contribution made by the OR teams was so apparent that the Air Officer Commander-in-Chief RAF Fighter Command (Air Chief Marshal Sir Hugh Dowding) requested that, on the outbreak of war, they should be attached to his headquarters at Stanmore.

On May 15th 1940, with German forces advancing rapidly in France, Stanmore Research Section was asked to analyze a French request for ten additional fighter squadrons (12 aircraft a squadron) when losses were running at some three squadrons every two days. They prepared graphs for Winston Churchill (the British Prime Minister of the time), based upon a study of current daily losses and replacement rates, indicating how rapidly such a move would deplete fighter strength. No aircraft were sent and most of those currently in France were recalled.

This is held by some to be the most strategic contribution to the course of the war made by OR (as the aircraft and pilots saved were consequently available for the successful air defense of Britain, the Battle of Britain).

In 1941 an Operational Research Section (ORS) was established in Coastal Command which was to carry out some of the most well-known OR work in World War II.

Although scientists had (plainly) been involved in the hardware side of warfare (designing better planes, bombs, tanks, etc) scientific analysis of the operational use of military resources had never taken place in a systematic fashion before the Second World War. Military personnel, often by no means stupid, were simply not trained to undertake such analysis.

These early OR workers came from many different disciplines, one group consisted of a physicist, two physiologists, two mathematical physicists and a surveyor. What such people brought to their work were "scientifically trained" minds, used to querying assumptions, logic, exploring hypotheses, devising experiments, collecting data, analyzing numbers, etc. Many too were of high intellectual caliber (at least four wartime OR personnel were later to win Nobel prizes when they returned to their peacetime disciplines).

By the end of the war OR was well established in the armed services both in the UK and in the USA.

OR started just before World War II in Britain with the establishment of teams of scientists to study the strategic and tactical problems involved in military operations. The objective was to find the most effective utilization of limited military resources by the use of quantitative techniques.

Following the end of the war OR spread, although it spread in different ways in the UK and USA.

You should be clear that the growth of OR since it began (and especially in the last 30 years) is, to a large extent, the result of the increasing power and widespread availability of computers. Most (though not all) OR involves carrying out a large number of numeric calculations. Without computers this would simply not be possible.

2. BASIC OR CONCEPTS

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

We can also define a mathematical model as consisting of:

Decision variables, which are the unknowns to be determined by the solution to the model.

Constraints to represent the physical limitations of the system An *objective* function

An *optimal solution* to the model is the identification of a set of variable values which are feasible (satisfy all the constraints) and which lead to the optimal value of the objective function.

An optimization model seeks to find values of the decision variables that optimize (maximize or minimize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

Two Mines Example

The Two Mines Company own two different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade and 24 tons of low-grade ore per week. The two mines have different operating characteristics as detailed below.

Mine	Cost per day (£'000)	Production (tons/day)		
		High	Medium	Low
X	180	6	3	4
Y	160	1	1	6

Consider that mines cannot be operated in the weekend. How many days per week should each mine be operated to fulfill the smelting plant contract?

Guessing

To explore the Two Mines problem further we might simply guess (i.e. use our judgment) how many days per week to work and see how they turn out.

- work one day a week on *X*, one day a week on *Y*

This does not seem like a good guess as it results in only 7 tones a day of high-grade, insufficient to meet the contract requirement for 12 tones of high-grade a day. We say that such a solution is *infeasible*.

- work 4 days a week on *X*, 3 days a week on *Y*

This seems like a better guess as it results in sufficient ore to meet the contract. We say that such a solution is *feasible*. However it is quite expensive (costly).

We would like a solution which supplies what is necessary under the contract at minimum cost. Logically such a minimum cost solution to this decision problem must exist. However even if we keep guessing we can never be sure whether we have found this minimum cost solution or not. Fortunately our structured approach will enable us to find the minimum cost solution.

Solution

What we have is a verbal description of the Two Mines problem. What we need to do is to translate that verbal description into an *equivalent* mathematical description.

In dealing with problems of this kind we often do best to consider them in the order:

- Variables
- Constraints
- Objective

This process is often called *formulating* the problem (or more strictly formulating a mathematical representation of the problem).

Variables

These represent the "decisions that have to be made" or the "unknowns".

We have two decision variables in this problem:

x = number of days per week mine X is operated

y = number of days per week mine Y is operated

Note here that $x \geq 0$ and $y \geq 0$.

Constraints

It is best to first put each constraint into words and then express it in a mathematical form.

ore production constraints - balance the amount produced with the quantity required under the smelting plant contract

Ore	
High	$6x + 1y \geq 12$
Medium	$3x + 1y \geq 8$
Low	$4x + 6y \geq 24$

days per week constraint - we cannot work more than a certain maximum number of days a week e.g. for a 5 day week we have

$x \leq 5$

$y \leq 5$

Inequality constraints

Note we have an inequality here rather than an equality. This implies that we may produce more of some grade of ore than we need. In fact we have the general rule: given a choice between an equality and an inequality choose the inequality

For example - if we choose an equality for the ore production constraints we have the three equations $6x+y=12$, $3x+y=8$ and $4x+6y=24$ and there are no values of x and y which satisfy all three equations (the problem is therefore said to be "over-constrained"). For example the values of x and y which satisfy $6x+y=12$ and $3x+y=8$ are $x=4/3$ and $y=4$, but these values do not satisfy $4x+6y=24$.

The reason for this general rule is that choosing an inequality rather than an equality gives us more flexibility in optimizing (maximizing or minimizing) the objective (deciding values for the decision variables that optimize the objective).

Implicit constraints

Constraints such as days per week constraint are often called implicit constraints because they are implicit in the definition of the variables.

Objective

Again in words our objective is (presumably) to minimize cost which is given by $180x + 160y$

Hence we have the ***complete mathematical representation*** of the problem:

minimize

$180x + 160y$

subject to

$6x + y \geq 12$

$3x + y \geq 8$

$4x + 6y \geq 24$

$x \leq 5$

$y \leq 5$

$x, y \geq 0$

Some notes

The mathematical problem given above has the form

- all variables continuous (i.e. can take fractional values)
- a single objective (maximize or minimize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown (e.g. 24, $4x$, $6y$ are linear terms but xy or x^2 is a non-linear term)

Any formulation which satisfies these three conditions is called a *linear program* (LP).

We have (implicitly) assumed that it is permissible to work in fractions of days - problems where this is not permissible and variables must take integer values will be dealt with under *Integer Programming* (IP).

Discussion

This problem was a *decision problem.*

We have taken a real-world situation and constructed an equivalent mathematical representation - such a representation is often called a mathematical *model* of the real-world situation (and the process by which the model is obtained is called *formulating* the model).

Just to confuse things the mathematical model of the problem is sometimes called the *formulation* of the problem.

Having obtained our mathematical model we (hopefully) have some quantitative method which will enable us to numerically solve the model (i.e. obtain a numerical solution) - such a quantitative method is often called an *algorithm* for solving the model.

Essentially an algorithm (for a particular model) is a set of instructions which, when followed in a step-by-step fashion, will produce a numerical solution to that model.

Our model has an *objective*, that is something which we are trying to *optimize*. Having obtained the numerical solution of our model we have to translate that solution back into the real-world situation.

"OR is the representation of real-world systems by mathematical models together with the use of quantitative methods (algorithms) for solving such models, with a view to optimizing."

3. LINEAR PROGRAMMING

It can be recalled from the Two Mines example that the conditions for a mathematical model to be a linear program (LP) were:

- all variables continuous (i.e. can take fractional values)
- a single objective (minimize or maximize)
- the objective and constraints are linear i.e. any term is either a constant or a constant multiplied by an unknown.

LP's are important - this is because:

- many practical problems can be formulated as LP's
- there exists an algorithm (called the *simplex* algorithm) which enables us to solve LP's numerically relatively easily

We will return later to the simplex algorithm for solving LP's but for the moment we will concentrate upon formulating LP's.

Some of the major application areas to which LP can be applied are:

- Work scheduling
- Production planning & Production process
- Capital budgeting
- Financial planning
- Blending (e.g. Oil refinery management)
- Farm planning
- Distribution
- Multi-period decision problems
    - Inventory model
    - Financial models
    - Work scheduling

Note that the key to formulating LP's is practice. However a useful hint is that common objectives for LP's are maximize profit/minimize cost.

There are four basic assumptions in LP:

- Proportionality
  - The contribution to the objective function from each decision variable is proportional to the value of the decision variable (The contribution to the objective function from making four soldiers ($4\times\$3=\$12$) is exactly four times the contribution to the objective function from making one soldier ($3))
  - The contribution of each decision variable to the LHS of each constraint is proportional to the value of the decision variable (It takes exactly three times as many finishing hours ($2hrs\times3=6hrs$) to manufacture three soldiers as it takes to manufacture one soldier (2 hrs))
- Additivity
  - The contribution to the objective function for any decision variable is independent of the values of the other decision variables (No matter what the value of train (x_2), the manufacture of soldier (x_1) will always contribute $3x_1$ dollars to the objective function)
  - The contribution of a decision variable to LHS of each constraint is independent of the values of other decision variables (No matter what the value of x_1, the manufacture of x_2 uses x_2 finishing hours and x2 carpentry hours)

    1st implication: The value of objective function is the sum of the contributions from each decision variables.

    2nd implication: LHS of each constraint is the sum of the contributions from each decision variables.
- Divisibility
  - Each decision variable is allowed to assume fractional values. If we actually can not produce a fractional number of decision variables, we use IP (It is acceptable to produce 1.69 trains)
- Certainty
  - Each parameter is known with certainty

FORMULATING LP

Giapetto Example

(Winston 3.1, p. 49)

Giapetto's wooden soldiers and trains. Each soldier sells for $27, uses $10 of raw materials and takes $14 of labor & overhead costs. Each train sells for $21, uses $9 of raw materials, and takes $10 of overhead costs. Each soldier needs 2 hours finishing and 1 hour carpentry; each train needs 1 hour finishing and 1 hour carpentry. Raw materials are unlimited, but only 100 hours of finishing and 80 hours of carpentry are available each week. Demand for trains is unlimited; but at most 40 soldiers can be sold each week. How many of each toy should be made each week to maximize profits?

Answer

Decision variables completely describe the decisions to be made (in this case, by Giapetto). Giapetto must decide how many soldiers and trains should be manufactured each week. With this in mind, we define:

x_1 = the number of soldiers produced per week

x_2 = the number of trains produced per week

Objective function is the function of the decision variables that the decision maker wants to maximize (revenue or profit) or minimize (costs). Giapetto can concentrate on maximizing the total weekly profit (z).

Here profit equals to (weekly revenues) – (raw material purchase cost) – (other variable costs). Hence Giapetto’s objective function is:

Maximize $z = 3x_1 + 2x_2$

Constraints show the restrictions on the values of the decision variables. Without constraints Giapetto could make a large profit by choosing decision variables to be very large. Here there are three constraints:

Finishing time per week

Carpentry time per week

Weekly demand for soldiers

Sign restrictions are added if the decision variables can only assume nonnegative values (Giapetto can not manufacture negative number of soldiers or trains!)

All these characteristics explored above give the following ***Linear Programming*** (LP) model

max $z = 3x_1 + 2x_2$	(The Objective function)
s.t. $2x_1 + x_2 \le 100$	(Finishing constraint)
$x_1 + x_2 \le 80$	(Carpentry constraint)
$x_1 \le 40$	(Constraint on demand for soldiers)
$x_1, x_2 \ge 0$	(Sign restrictions)

A value of (x_1, x_2) is in the ***feasible region*** if it satisfies all the constraints and sign restrictions.

Graphically and computationally we see the solution is (x_1, x_2) = (20, 60) at which z = 180. (***Optimal solution***)

Report

The maximum profit is $180 by making 20 soldiers and 60 trains each week. Profit is limited by the carpentry and finishing labor available. Profit could be increased by buying more labor.

Advertisement Example

(Winston 3.2, p.61)

Dorian makes luxury cars and jeeps for high-income men and women. It wishes to advertise with 1 minute spots in comedy shows and football games. Each comedy spot costs $50K and is seen by 7M high-income women and 2M high-income men. Each football spot costs $100K and is seen by 2M high-income women and 12M high-income men. How can Dorian reach 28M high-income women and 24M high-income men at the least cost?

Answer

The decision variables are

x_1 = the number of comedy spots

x_2 = the number of football spots

The model of the problem:

$$\min z = 50x_1 + 100x_2$$
$$\text{st} \quad 7x_1 + 2x_2 \ge 28$$
$$2x_1 + 12x_2 \ge 24$$
$$x_1, x_2 \ge 0$$

The graphical solution is z = 320 when (x_1, x_2) = (3.6, 1.4). From the graph, in this problem rounding up to (x_1, x_2) = (4, 2) gives the best *integer* solution.

Report

The minimum cost of reaching the target audience is $400K, with 4 comedy spots and 2 football slots. The model is dubious as it does not allow for saturation after repeated viewings.

Diet Example

(Winston 3.4., p. 70)

Ms. Fidan's diet requires that all the food she eats come from one of the four "basic food groups". At present, the following four foods are available for consumption: brownies, chocolate ice cream, cola, and pineapple cheesecake. Each brownie costs 0.5$, each scoop of chocolate ice cream costs 0.2$, each bottle of cola costs 0.3$, and each pineapple cheesecake costs 0.8$. Each day, she must ingest at least 500 calories, 6 oz of chocolate, 10 oz of sugar, and 8 oz of fat. The nutritional content per unit of each food is shown in Table. Formulate an LP model that can be used to satisfy her daily nutritional requirements at minimum cost.

	Calories	Chocolate (ounces)	Sugar (ounces)	Fat (ounces)
Brownie	400	3	2	2
Choc. ice cream (1 scoop)	200	2	2	4
Cola (1 bottle)	150	0	4	1
Pineapple cheesecake (1 piece)	500	0	4	5

Answer

The decision variables:

x_1: number of brownies eaten daily

x_2: number of scoops of chocolate ice cream eaten daily

x_3: bottles of cola drunk daily

x_4: pieces of pineapple cheesecake eaten daily

The objective function (the total cost of the diet in cents):

$$\min w = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

Constraints:

$400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500$ (daily calorie intake)

$3x_1 + 2x_2 \geq 6$ (daily chocolate intake)

$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10$ (daily sugar intake)

$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8$ (daily fat intake)

$x_i \geq 0, i = 1, 2, 3, 4$ (Sign restrictions!)

Report

The minimum cost diet incurs a daily cost of 90 cents by eating 3 scoops of chocolate and drinking 1 bottle of cola ($w = 90$, $x_2 = 3$, $x_3 = 1$)

Post Office Example

(Winston 3.5, p.74)

A PO requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thur	Fri	Sat	Sun
Staff Needed	17	13	15	19	14	16	11

Answer

The decision variables are x_i (# of employees starting on day i)

Mathematically we must

$$
\begin{array}{llllllllll}
\min z = & x_1 & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & & \\
\text{s.t.} & x_1 & & & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 17 \\
& x_1 & + x_2 & & & + x_5 & + x_6 & + x_7 & \geq & 13 \\
& x_1 & + x_2 & + x_3 & & & + x_6 & + x_7 & \geq & 15 \\
& x_1 & + x_2 & + x_3 & + x_4 & & & + x_7 & \geq & 19 \\
& x_1 & + x_2 & + x_3 & + x_4 & + x_5 & & & \geq & 14 \\
& & + x_2 & + x_3 & + x_4 & + x_5 & + x_6 & & \geq & 16 \\
& & & + x_3 & + x_4 & + x_5 & + x_6 & + x_7 & \geq & 11 \\
& x_t \geq 0, \forall t & & & & & & & &
\end{array}
$$

The solution is $(x_i) = (4/3, 10/3, 2, 22/3, 0, 10/3, 5)$ giving $z = 67/3$.

We could round this up to $(x_i) = (2, 4, 2, 8, 0, 4, 5)$ giving $z = 25$ (may be wrong!).

However restricting the decision var.s to be integers and using Lindo again gives

$(x_i) = (4, 4, 2, 6, 0, 4, 3)$ giving $z = 23$.

Sailco Example

(Winston 3.10, p. 99)

Sailco must determine how many sailboats to produce in the next 4 quarters. The demand is known to be 40, 60, 75, and 25 boats. Sailco must meet its demands. At the beginning of the 1^{st} quarter Sailco starts with 10 boats in inventory. Sailco can produce up to 40 boats with regular time labor at $400 per boat, or additional boats at

\$450 with overtime labor. Boats made in a quarter can be used to meet that quarter's demand or held in inventory for the next quarter at an extra cost of \$20.00 per boat.

Answer

The decision variables are for t = 1,2,3,4

x_t = # of boats in quarter t built in regular time

y_t = # of boats in quarter t built in overtime

For convenience, introduce variables:

i_t = # of boats in inventory at the end quarter

t d_t = demand in quarter t

We are given that $d_1 = 40, d_2 = 60, d_3 = 75, d_4 = 25, i_0 = 10$

$x_t \leq 40, \forall t$

By logic $i_t = i_{t-1} + x_t + y_t - d_t, \forall t.$

Demand is met iff $i_t \geq 0, \forall t$

(Sign restrictions $x_t, y_t \geq 0, \forall t$)

We need to minimize total cost z subject to these three sets of conditions where z =

$$400\,(x_1 + x_2 + x_3 + x_4) + 450\,(y_1 + y_2 + y_3 + y_4) + 20\,(i_1 + i_2 + i_3 + i_4)$$

Report:

Lindo reveals the solution to be $(x_1, x_2, x_3, x_4) = (40, 40, 40, 25)$ and $(y_1, y_2, y_3, y_4) = (0, 10, 35, 0)$ and the minimum cost of \$78450.00 is achieved by the schedule

		Q_1	Q_2	Q_3	Q_4
Regular time (x_t)		40	40	40	25
Overtime (y_t)		0	10	35	0
Inventory (i_t)	10	10	0	0	0
Demand (d_t)		40	60	75	25

Customer Service Level Example

(Winston 3.12, p. 108)

CSL services computers. Its demand (hours) for the time of skilled technicians in the next 5 months is

t	Jan	Feb	Mar	Apr	May
d_t	6000	7000	8000	9500	11000

It starts with 50 skilled technicians at the beginning of January. Each technician can work 160 hrs/month. To train a new technician they must be supervised for 50 hrs by an experienced technician for a period of one month time. Each experienced technician is paid \$2K/mth and a trainee is paid \$1K/mth. Each month 5% of the skilled technicians leave. CSL needs to meet demand and minimize costs.

Answer

The decision variable is

x_t = # to be trained in month t

We must minimize the total cost. For convenience

let y_t = # experienced tech. at start of t^{th}

month d_t = demand during month t

Then we must

min $z = 2000\ (y_1 + ... + y_5) + 1000\ (x_1 + ... + x_5)$

subject to

$160y_t - 50x_t \geq d_t$ for $t = 1, ..., 5$

$y_1 = 50$, $d_1 = 6000$, $d_2 = 7000$, $d_3 = 8000$, $d_4 = 9500$, $d_5 =$ 11000

$y_t = .95y_{t-1} + x_{t-1}$ for $t = 2,3,4,5$

$x_t, y_t \geq 0$

SOLVING LP

LP Solutions: Four Cases

When an LP is solved, one of the following four cases will occur:

1. The LP has a **unique optimal solution**.
2. The LP has **alternative (multiple) optimal solutions**. It has more than one (actually an infinite number of) optimal solutions
3. The LP is **infeasible**. It has no feasible solutions (The feasible region contains no points).
4. The LP is **unbounded**. In the feasible region there are points with arbitrarily large (in a max problem) objective function values.

The Graphical Solution

Any LP with only two variables can be solved graphically

Example 1. Giapetto

(*Winston 3.1, p. 49*)

Since the Giapetto LP has two variables, it may be solved graphically.

Answer

The feasible region is the set of all points satisfying the constraints.

max $z = 3x_1 + 2x_2$

s.t.	$2x_1 + x_2 \le 100$	(Finishing constraint)
	$x_1 + x_2 \le 80$	(Carpentry constraint)
	$x_1 \le 40$	(Demand constraint)
	$x_1, x_2 \ge 0$	(Sign restrictions)

The set of points satisfying the LP is bounded by the five sided polygon DGFEH. Any point ***on*** or ***in*** the interior of this polygon (the shade area) is in the ***feasible region***. Having identified the feasible region for the LP, a search can begin for the ***optimal solution*** which will be the point in the feasible region with the *largest* z-value (maximization problem).

To find the optimal solution, a line on which the points have the same z-value is graphed. In a max problem, such a line is called an ***isoprofit*** line while in a min problem, this is called the ***isocost*** line. (*The figure shows the isoprofit lines for $z = 60$, $z = 100$, and $z = 180$*).

In the unique optimal solution case, isoprofit line last hits a point (vertex - corner) before leaving the feasible region.

The optimal solution of this LP is point G where (x_1, x_2) = (20, 60) giving z = 180.

A constraint is ***binding*** (active, tight) if the left-hand and right-hand side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint.

A constraint is ***nonbinding*** (inactive) if the left-hand side and the right-hand side of the constraint are unequal when the optimal values of the decision variables are substituted into the constraint.

In Giapetto LP, the finishing and carpentry constraints are binding. On the other hand the demand constraint for wooden soldiers is nonbinding since at the optimal solution $x_1 < 40$ ($x_1 = 20$).

Example 2. Advertisement

(*Winston 3.2, p. 61*)

Since the Advertisement LP has two variables, it may be solved graphically.

Answer

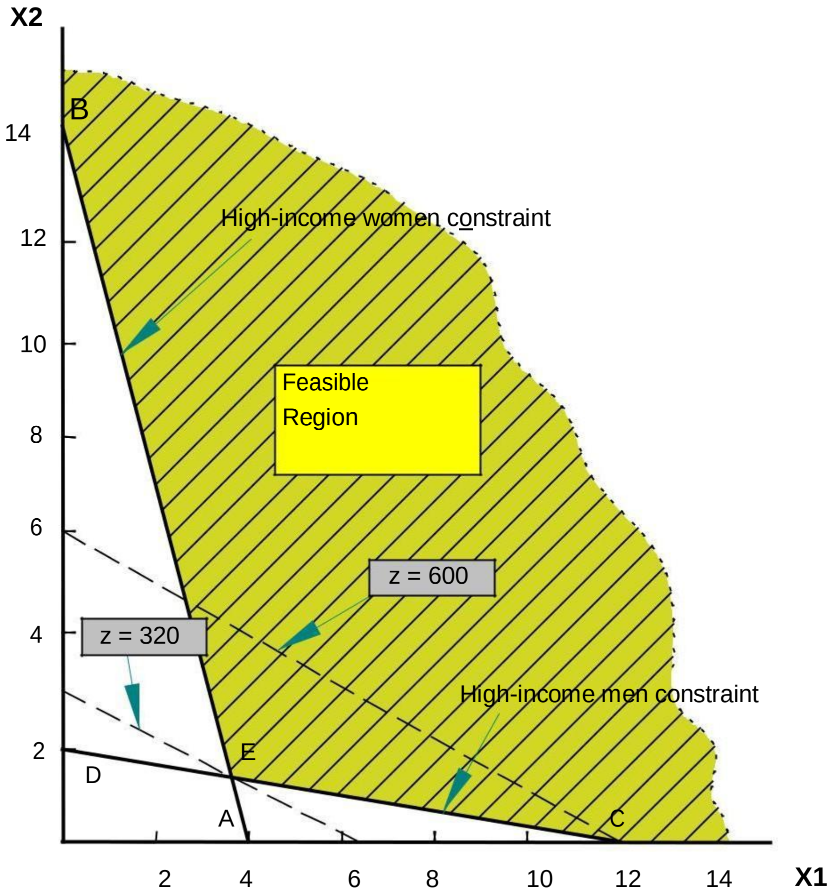
The feasible region is the set of all points satisfying the constraints.

$\min z = 50x_1 + 100x_2$

s.t. $7x_1 + 2x_2 \geq 28$ (high income women)

$2x_1 + 12x_2 \geq 24$ (high income men)

$x_1, x_2 \geq 0$



Since Dorian wants to minimize total advertising costs, the optimal solution to the problem is the point in the feasible region with the smallest z value.

An isocost line with the smallest z value passes through point E and is the optimal solution at $x_1 = 3.6$ and $x_2 = 1.4$ giving $z = 320$.

Both the high-income women and high-income men constraints are satisfied, both constraints are binding.

Example 3. Two Mines

min $180x + 160y$

st $6x + y \geq 12$

$3x + y \geq 8$

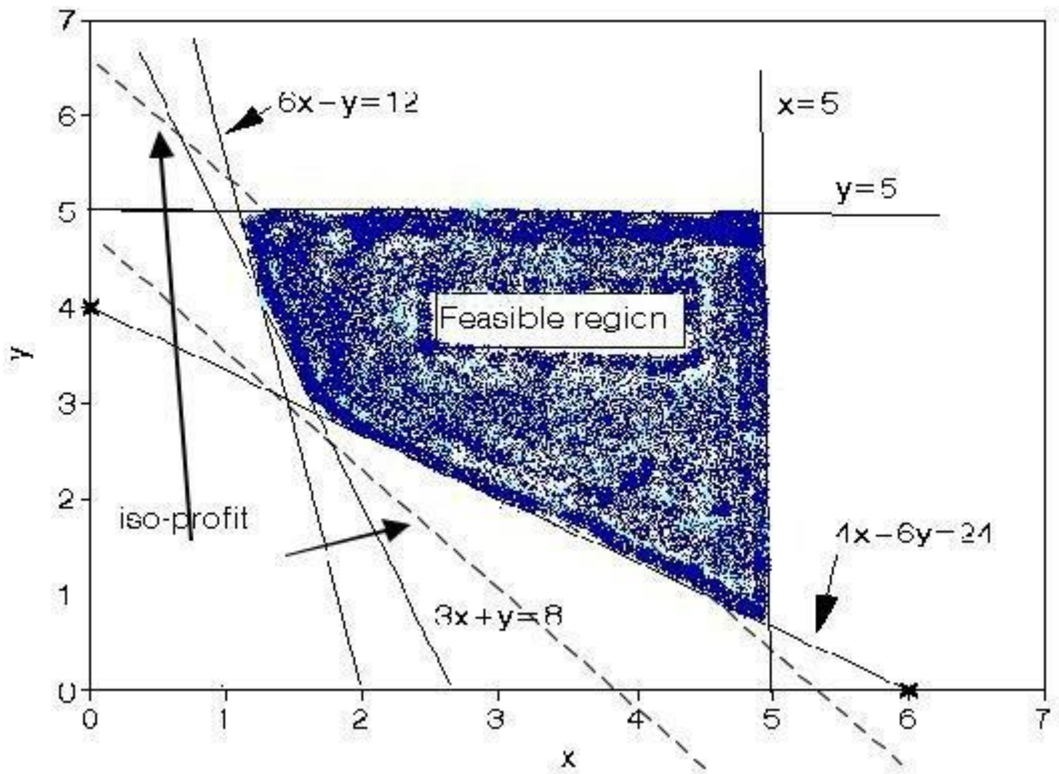
$4x + 6y \geq 24$

$x \leq 5$

$y \leq 5$

$x, y \geq 0$

Answer



Optimal sol'n is 765.71. 1.71 days mine *X* and 2.86 days mine *Y* are operated.

Example 4. Modified Giapetto

max $z = 4x_1 + 2x_2$

s.t.	$2x_1 + x_2 \leq 100$	(Finishing constraint)
	$x_1 + x_2 \leq 80$	(Carpentry constraint)
	$x_1 \leq 40$	(Demand constraint)
	$x_1, x_2 \geq 0$	(Sign restrictions)

Answer

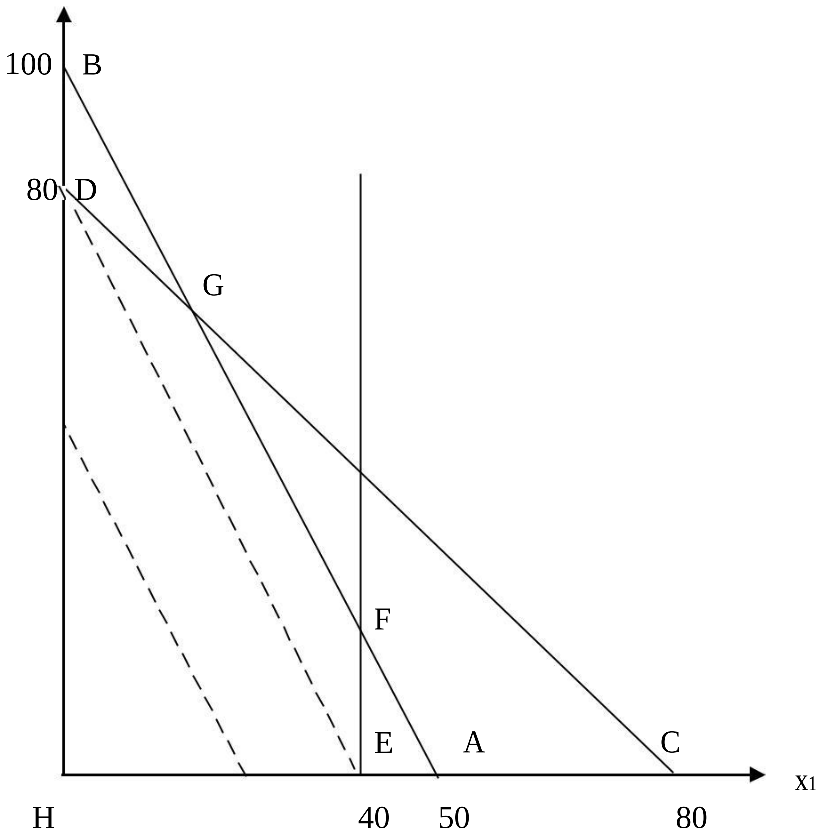
Points on the line between points G (20, 60) and F (40, 20) are the ***alternative optimal solutions*** (see figure below).

Thus, for $0 \leq c \leq 1$,

$$c\,[20\ 60] + (1 - c)\,[40\ 20] = \mathbf{[40 - 20c,\ 20 + 40c]}$$

will be optimal

For all optimal solutions, the optimal objective function value is 200.



Example 5. Modified Giapetto (v. 2)

Add constraint $x_2 \geq 90$ (Constraint on demand for trains).

Answer

No feasible region: ***Infeasible LP***

Example 6. Modified Giapetto (v. 3)

Only use constraint $x_2 \geq 90$

Answer

Isoprofit line never lose contact with the feasible region: ***Unbounded LP***

The Simplex Algorithm

Note that in the examples considered at the graphical solution, the unique optimal solution to the LP occurred at a vertex (corner) of the feasible region. In fact it is true that for *any* LP the optimal solution occurs at a vertex of the feasible region. This fact is the key to the simplex algorithm for solving LP's.

Essentially the simplex algorithm starts at one vertex of the feasible region and moves (at each iteration) to another (adjacent) vertex, improving (or leaving unchanged) the objective function as it does so, until it reaches the vertex corresponding to the optimal LP solution.

The simplex algorithm for solving linear programs (LP's) was developed by Dantzig in the late 1940's and since then a number of different versions of the algorithm have been developed. One of these later versions, called the *revised simplex* algorithm (sometimes known as the "product form of the inverse" simplex algorithm) forms the basis of most modern computer packages for solving LP's.

Steps

1. Convert the LP to standard form
2. Obtain a basic feasible solution (bfs) from the standard form
3. Determine whether the current bfs is optimal. If it is optimal, stop.
4. If the current bfs is not optimal, determine which nonbasic variable should become a basic variable and which basic variable should become a nonbasic variable to find a new bfs with a better objective function value
5. Go back to Step 3.

Related concepts:

- Standard form: all constraints are equations and all variables are nonnegative
- bfs: any basic solution where all variables are nonnegative
- Nonbasic variable: a chosen set of variables where variables equal to 0
- Basic variable: the remaining variables that satisfy the system of equations at the standard form

Example 1. Dakota Furniture

(Winston 4.3, p. 134)

Dakota Furniture makes desks, tables, and chairs. Each product needs the limited resources of lumber, carpentry and finishing; as described in the table. At most 5 tables can be sold per week. Maximize weekly revenue.

Resource	Desk	Table	Chair	Max Avail.
Lumber (board ft.)	8	6	1	48
Finishing hours	4	2	1.5	20
Carpentry hours	2	1.5	.5	8
Max Demand	unlimited	5	unlimited	
Price ($)	60	30	20	

LP Model:

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced.

Let the weekly profit be $z. Then, we must

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{aligned} \text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \\ & 4x_1 + 2x_2 + 1.5x_3 \le 20 \\ & 2x_1 + 1.5x_2 + .5x_3 \le 8 \\ & x_2 \le 5 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution with Simplex Algorithm

First introduce slack variables and convert the LP to the standard form and write a canonical form

$$\begin{aligned} R_0 \quad & z - 60x_1 - 30x_2 - 20x_3 = 0 \\ R_1 \quad & 8x_1 + 6x_2 + x_3 + s_1 = 48 \\ R_2 \quad & 4x_1 + 2x_2 + 1.5x_3 + s_2 = 20 \\ R_3 \quad & 2x_1 + 1.5x_2 + .5x_3 + s_3 = 8 \\ R_4 \quad & x_2 + s_4 = 5 \\ & x_1, x_2, x_3, s_1, s_2, s_3, s_4 \ge 0 \end{aligned}$$

Obtain a starting bfs.

As (x_1, x_2, x_3) = 0 is feasible for the original problem, the below given point where three of the variables equal 0 (the ***non-basic variables***) and the four other variables (the ***basic variables***) are determined by the four equalities is an obvious bfs:

$x_1 = x_2 = x_3 = 0$, $s_1 = 48$, $s_2 = 20$, $s_3 = 8$, $s_4 = 5$.

.

Determine whether the current bfs is optimal.

Determine whether there is any way that z can be increased by increasing some nonbasic variable.

If each nonbasic variable has a nonnegative coefficient in the objective function row (***row 0***), current bfs is optimal.

However, here all nonbasic variables have negative coefficients: It is not optimal.

Find a new bfs

- z increases most rapidly when x_1 is made non-zero; i.e. x_1 is the ***entering variable***.
- Examining R_1, x_1 can be increased only to 6. More than 6 makes $s_1 < 0$. Similarly R_2, R_3, and R_4, give limits of 5, 4, and no limit for x_1 (***ratio test***). The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. Thus by R_3, x_1 can only increase to $x_1 = 4$ when s_3 becomes 0. We say s_3 is the ***leaving variable*** and R_3 is the ***pivot equation***.
- Now we must rewrite the system so the values of the basic variables can be read off.

The new *pivot equation* ($R_3/2$) is

$$R_3' : x_1 + .75x_2 + .25x_3 + .5s_3 = 4$$

Then use R_3' to eliminate x_1 in all the other rows.

R0'=R0+60R3', R1'=R1-8R3', R2'=R2-4R3', R4'=R4

R_0'	z		$+ 15x_2$	$- 5x_3$			$+ 30s_3$	$= 240$	$z = 240$
R_1'				$- x_3$	$+ s_1$		$- 4s_3$	$= 16$	$s_1 = 16$
R_2'			$- x_2$	$+ .5x_3$		$+ s_2$	$- 2s_3$	$= 4$	$s_2 = 4$
R_3'		x_1	$+ .75x_2$	$+ .25x_3$			$+ .5s_3$	$= 4$	$x_1 = 4$
R_4'			x_2				$+ s_4$	$= 5$	$s_4 = 5$

The new bfs is $x_2 = x_3 = s_3 = 0$, $x_1 = 4$, $s_1 = 16$, $s_2 = 4$, $s_4 = 5$ making $z = 240$.

Check optimality of current bfs. Repeat steps until an optimal solution is reached

- We increase z fastest by making x_3 non-zero (i.e. x_3 enters).
- x_3 can be increased to at most $x_3 = 8$, when $s_2 = 0$ (i.e. s_2 leaves.)

Rearranging the pivot equation gives

R_2'' $- 2x_2 + x_3 + 2s_2 - 4s_3 = 8$ $(R_2' \times 2)$.

Row operations with R_2'' eliminate x_3 to give the new system

$R_0'' = R_0' + 5R_2''$, $R_1'' = R_1' + R_2''$, $R_3'' = R_3' - .5R_2''$, $R_4'' = R_4'$

The bfs is now $x_2 = s_2 = s_3 = 0$, $x_1 = 2$, $x_3 = 8$, $s_1 = 24$, $s_4 = 5$ making $z = 280$.

Each nonbasic variable has a nonnegative coefficient in row 0 ($5x_2$, $10s_2$, $10s_3$).

THE CURRENT SOLUTION IS OPTIMAL

Report: Dakota furniture's optimum weekly profit would be 280$ if they produce 2 desks and 8 chairs.

This was once written as a tableau.

(__Use tableau format for each operation in all HW and exams!!!__)

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\text{s.t.} \quad 8x_1 + 6x_2 + x_3 \le 48$$

$$4x_1 + 2x_2 + 1.5x_3 \le 20$$

$$2x_1 + 1.5x_2 + .5x_3 \le 8$$

$$x_2 \le 5$$

$$x_1, x_2, x_3 \ge 0$$

Initial tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	-60	-30	-20	0	0	0	0	0	z = 0	
0	8	6	1	1	0	0	0	48	$s_1 = 48$	6
0	4	2	1.5	0	1	0	0	20	$s_2 = 20$	5
0	2	1.5	0.5	0	0	1	0	8	$s_3 = 8$	4
0	0	1	0	0	0	0	1	5	$s_4 = 5$	-

First tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	15	-5	0	0	30	0	240	z = 240	
0	0	0	-1	1	0	-4	0	16	s_1 = 16	-
0	0	-1	0.5	0	1	-2	0	4	s_2 = 4	8
0	1	0.75	0.25	0	0	0.5	0	4	x_1 = 4	16
0	0	1	0	0	0	0	1	5	s_4 = 5	-

Second and optimal tableau:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	5	0	0	10	10	0	280	z = 280	
0	0	-2	0	1	2	-8	0	24	s_1 = 24	
0	0	-2	1	0	2	-4	0	8	x_3 = 8	
0	1	1.25	0	0	-0.5	1.5	0	2	x_1 = 2	
0	0	1	0	0	0	0	1	5	s_4 = 5	

Example 2. Modified Dakota Furniture

Dakota example is modified: $35/table

new $z = 60\, x_1 + 35\, x_2 + 20\, x_3$

Second and optimal tableau for the modified problem:

z	x_1	x_2 ⇓	x_3	s_1	s_2	s_3	s_4	RHS	BV	Ratio
1	0	**0**	0	0	10	10	0	280	z=280	
0	0	-2	0	1	2	-8	0	24	s_1=24	-
0	0	-2	1	0	2	-4	0	8	x_3=8	-
0	1	1.25	0	0	-0.5	1.5	0	2	x_1=2	2/1.25 ⇒
0	0	1	0	0	0	0	1	5	s_4=5	5/1

Another optimal tableau for the modified problem:

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	RHS	BV
1	0	0	0	0	10	10	0	280	z=280
0	1.6	0	0	1	1.2	-5.6	0	27.2	s_1=27.2
0	1.6	0	1	0	1.2	-1.6	0	11.2	x_3=11.2
0	0.8	1	0	0	-0.4	1.2	0	1.6	x_2=1.6
0	-0.8	0	0	0	0.4	-1.2	1	3.4	s_4=3.4

Therefore the optimal solution is as follows:

z = 280 and for $0 \le c \le 1$

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \end{vmatrix} = c \begin{vmatrix} 2 \\ 0 \\ 8 \end{vmatrix} + (1-c) \begin{vmatrix} 0 \\ 1.6 \\ 11.2 \end{vmatrix} = \begin{vmatrix} 2c \\ 1.6 - 1.6c \\ 11.2 - 3.2c \end{vmatrix}$$

Example 3. Unbounded LPs

z	x_1	x_2	x_3 ⇓	s_1	s_2	z	RHS	BV	Ratio
1	0	2	**-9**	0	12	4	100	z=100	
0	0	1	-6	1	6	-1	20	x_4=20	None
0	1	1	-1	0	1	0	5	x_1=5	None

Since ratio test fails, the LP under consideration is an unbounded LP.

The Big M Method

If an LP has any ≥ or = constraints, a starting bfs may not be readily apparent.
When a bfs is not readily apparent, the Big M method or the two-phase simplex method may be used to solve the problem.

The Big M method is a version of the Simplex Algorithm that first finds a bfs by adding "artificial" variables to the problem. The objective function of the original LP must, of course, be modified to ensure that the artificial variables are all equal to 0 at the conclusion of the simplex algorithm.

Steps

1. Modify the constraints so that the RHS of each constraint is nonnegative (This requires that each constraint with a negative RHS be multiplied by -1. Remember that if you multiply an inequality by any negative number, the direction of the inequality is reversed!). After modification, identify each constraint as a ≤, ≥ or = constraint.
2. Convert each inequality constraint to standard form (If constraint i is a ≤ constraint, we add a slack variable s_i; and if constraint i is a ≥ constraint, we subtract an excess variable e_i).
3. Add an artificial variable a_i to the constraints identified as ≥ or = constraints at the end of Step 1. Also add the sign restriction $a_i \geq 0$.
4. Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.
5. Since each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. Now solve the transformed problem by the simplex (In choosing the entering variable, remember that M is a very large positive number!).

If all artificial variables are equal to zero in the optimal solution, we have found the **optimal solution** to the original problem.

If any artificial variables are positive in the optimal solution, the original problem is **infeasible**!!!

Example 1. Oranj Juice

(Winston 4.10, p. 164)

Bevco manufactures an orange flavored soft drink called Oranj by combining orange soda and orange juice. Each ounce of orange soda contains 0.5 oz of sugar and 1 mg of vitamin C. Each ounce of orange juice contains 0.25 oz of sugar and 3 mg of vitamin C. It costs Bevco 2¢ to produce an ounce of orange soda and 3¢ to produce an ounce of orange juice. Marketing department has decided that each 10 oz bottle of Oranj must contain at least 20 mg of vitamin C and at most 4 oz of sugar. Use LP to determine how Bevco can meet marketing dept.'s requirements at minimum cost.

LP Model:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

min $z = 2x_1 + 3x_2$

s.t. $0.5\,x_1 + 0.25\,x_2 \leq 4$ (sugar const.)

$x_1 + 3\,x_2 \geq 20$ (vit. C const.)

$x_1 + x_2 = 10$ (10 oz in bottle)

$x_1, x_2 \geq 0$

Solving Oranj Example with Big M Method

1. Modify the constraints so that the RHS of each constraint is nonnegative The RHS of each constraint is nonnegative

2. Convert each inequality constraint to standard form

$z - 2x_1 - 3x_2 = 0$

$0.5x_1 + 0.25x_2 + s_1 = 4$

$x_1 + 3x_2 - e_2 = 20$

$x_1 + x_2 = 10$

all variables nonnegative

3. Add a_i to the constraints identified as > or = const.s

$z - 2x_1 - 3x_2 = 0$ Row

0 $0.5x_1 + 0.25x_2 + s_1 = 4$ Row 1

$x_1 + 3x_2 - e_2 + a_2 = 20$ Row 2

$x_1 + x_2 + a_3 = 10$ Row 3

all variables nonnegative

4. Add Ma_i to the objective function (min problem)

$$\min z = 2x_1 + 3x_2 + Ma_2 + Ma_3$$

Row 0 will change to

$$z - 2x_1 - 3x_2 - Ma_2 - Ma_3 = 0$$

5. Since each artificial variable are in our starting bfs, they must be eliminated from row 0

New Row 0 = Row 0 + M * Row 2 + M * Row 3 ⇒

$z + (2M-2)\, x_1 + (4M-3)\, x_2 - M\, e_2 = 30M$ New Row 0

Initial tableau:

z	x_1	x_2 ⇓	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	30M	z=30M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	20	a_2=20	**20/3***
0	1	1	0	0	0	1	10	a_3=10	10

In a min problem, entering variable is the variable that has the "most positive" coefficient in row 0!

First tableau:

z	x_1 ⇓	x_2	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	**(2M-3)/3**	0	0	(M-3)/3	(3-4M)/3	0	20+3.3M	z	
0	5/12	0	1	1/12	-1/12	0	7/3	s_1	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x_2	20
0	2/3	0	0	1/3	-1/3	1	10/3	a_3	**5***

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	0	0	0	-1/2	(1-2M)/2	(3-2M)/2	25	z=25
0	0	0	1	-1/8	1/8	-5/8	1/4	s_1=1/4
0	0	1	0	-1/2	1/2	-1/2	5	x_2=5
0	1	0	0	1/2	-1/2	3/2	5	x_1=5

Report:

In a bottle of Oranj, there should be 5 oz orange soda and 5 oz orange juice.

In this case the cost would be 25¢.

Example 2. Modified Oranj Juice

Consider Bevco's problem. It is modified so that 36 mg of vitamin C are required.

Related LP model is given as follows:

Let x_1 and x_2 be the quantity of ounces of orange soda and orange juice (respectively) in a bottle of Oranj.

min $z = 2x_1 + 3x_2$

s.t. $0.5\, x_1 + 0.25\, x_2 \leq 4$ (sugar const.)

$x_1 + 3\, x_2 \geq 36$ (vit. C const.)

$x_1 + x_2 = 10$ (10 oz in bottle)

$x_1, x_2 \geq 0$

Solving with Big M method:

Initial tableau:

z	x_1	x_2 ⇓	s_1	e_2	a_2	a_3	RHS	BV	Ratio
1	2M-2	**4M-3**	0	-M	0	0	46M	z=46M	
0	0.5	0.25	1	0	0	0	4	s_1=4	16
0	1	3	0	-1	1	0	36	a_2=36	36/3
0	1	**1**	0	0	0	1	10	a_3=10	**10** ⇒

Optimal tableau:

z	x_1	x_2	s_1	e_2	a_2	a_3	RHS	BV
1	1-2M	0	0	-M	0	3-4M	30+6M	z=30+6M
0	1/4	0	1	0	0	-1/4	3/2	s_1=3/2
0	-2	0	0	-1	1	-3	6	a_2=6
0	1	1	0	0	0	1	10	x_2=10

An artificial variable (a_2) is BV so the original LP has no feasible solution

Report:

It is impossible to produce Oranj under these conditions.

DUALITY

Primal – Dual

Associated with any LP is another LP called the ***dual***. Knowledge of the dual provides interesting economic and sensitivity analysis insights. When taking the dual of any LP, the given LP is referred to as the ***primal***. If the primal is a max problem, the dual will be a min problem and vice versa.

Finding the Dual of an LP

The dual of a ***normal max*** problem is a ***normal min*** problem.

Normal max problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≤ constraints.

Normal min problem is a problem in which all the variables are required to be nonnegative and all the constraints are ≥ constraints.

Similarly, the dual of a normal min problem is a normal max problem.

Finding the Dual of a Normal Max Problem PRIMAL

max z = $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$

… … … …

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m$

$x_j \ge 0$ (j = 1, 2, …, n)

DUAL

min w = $b_1y_1 + b_2y_2 + \ldots + b_my_m$

$_2 + \ldots + a_{m1}y_m \ge c_1$ $a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge$

c_2

… … … …

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n$

$y_i \ge 0$ (i = 1, 2, …, m)

Finding the Dual of a Normal Min Problem

PRIMAL

min w = $b_1y_1 + b_2y_2 + \ldots + b_my_m$

s.t. $a_{11}y_1 + a_{21}y_2 + \ldots + a_{m1}y_m \ge c_1$

$a_{12}y_1 + a_{22}y_2 + \ldots + a_{m2}y_m \ge c_2$

… … … …

$a_{1n}y_1 + a_{2n}y_2 + \ldots + a_{mn}y_m \ge c_n$

$y_i \ge 0$ (i = 1, 2, …, m)

DUAL

max z = $c_1x_1 + c_2x_2 + \ldots + c_nx_n$

s.t. $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m$

$x_j \geq 0$ ($j = 1, 2, \ldots, n$)

Finding the Dual of a Nonnormal Max Problem

- If the *i*th primal constraint is a ≥ constraint, the corresponding dual variable y_i must satisfy $y_i \leq 0$
- If the *i*th primal constraint is an equality constraint, the dual variable y_i is now unrestricted in sign (urs).
- If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

Finding the Dual of a Nonnormal Min Problem

- If the *i*th primal constraint is a ≤ constraint, the corresponding dual variable x_i must satisfy $x_i \leq 0$
- If the *i*th primal constraint is an equality constraint, the dual variable x_i is now urs.
- If the *i*th primal variable is urs, the *i*th dual constraint will be an equality constraint

The Dual Theorem

The primal and dual have equal optimal objective function values (if the problems have optimal solutions).

Weak duality implies that if for any feasible solution to the primal and an feasible solution to the dual, the *w*-value for the feasible dual solution will be at least as large as the *z*-value for the feasible primal solution $z \leq w$.

Consequences

- Any feasible solution to the dual can be used to develop a bound on the optimal value of the primal objective function.
- If the primal is unbounded, then the dual problem is infeasible.
- If the dual is unbounded, then the primal is infeasible.

- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a max problem:

  'optimal value of dual variable y_i'

  = 'coefficient of s_i in optimal row 0' (if const. i is a ≤ const.)

  = –'coefficient of e_i in optimal row 0' (if const. i is a ≥ const.)

  = 'coefficient of a_i in optimal row 0' – M (if const. i is a = const.)

- How to read the optimal dual solution from Row 0 of the optimal tableau if the primal is a min problem:

  'optimal value of dual variable x_i'

  = 'coefficient of s_i in optimal row 0' (if const. i is a ≤ const.)

  = –' coefficient of e_i in optimal row 0' (if const. i is a ≥ const.)

  = 'coefficient of a_i in optimal row 0' + M (if const. i is a = const.)

Economic Interpretation

When the primal is a normal max problem, the dual variables are related to the value of resources available to the decision maker. For this reason, dual variables are often referred to as ***resource shadow prices***.

Example

PRIMAL

Let x_1, x_2, x_3 be the number of desks, tables and chairs produced. Let the weekly profit be \$$z$. Then, we must

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$8x_1 + 6x_2 + x_3 \leq 48$ (Lumber constraint)

$4x_1 + 2x_2 + 1.5x_3 \leq 20$ (Finishing hour constraint)

$2x_1 + 1.5x_2 + 0.5x_3 \leq 8$ (Carpentry hour constraint)

$x_1, x_2, x_3 \geq 0$

DUAL

Suppose an entrepreneur wants to purchase all of Dakota's resources.

In the dual problem y_1, y_2, y_3 are the resource prices (price paid for one board ft of lumber, one finishing hour, and one carpentry hour). \$w is the cost of purchasing the resources.

Resource prices must be set high enough to induce Dakota to sell. i.e.

total purchasing cost equals total profit.

$$\min w = 48y_1 + 20y_2 + 8y_3$$

s.t. $8y_1 + 4y_2 + 2y_3 \geq 60$ (Desk constraint)

$6y_1 + 2y_2 + 1.5y_3 \geq 30$ (Table constraint)

$y_1 + 1.5y_2 + 0.5y_3 \geq 20$ (Chair constraint)

$y_1, y_2, y_3 \geq 0$

SENSITIVITY ANALYSIS

Reduced Cost

For any nonbasic variable, the reduced cost for the variable is the amount by which the nonbasic variable's objective function coefficient must be improved before that variable will become a basic variable in some optimal solution to the LP.

If the objective function coefficient of a nonbasic variable x_k is improved by its reduced cost, then the LP will have alternative optimal solutions at least one in which x_k is a basic variable, and at least one in which x_k is not a basic variable.

If the objective function coefficient of a nonbasic variable x_k is improved by more than its reduced cost, then any optimal solution to the LP will have x_k as a basic variable and $x_k > 0$.

Reduced cost of a basic variable is zero (see definition)!

Shadow Price

We define the shadow price for the ith constraint of an LP to be the amount by which the optimal z value is "improved" (increased in a max problem and decreased in a min problem) if the RHS of the ith constraint is increased by 1.

This definition applies only if the change in the RHS of the constraint leaves the current basis optimal!

A ≥ constraint will always have a nonpositive shadow price; a ≤ constraint will always have a nonnegative shadow price.

Conceptualization

$$\max z = 5x_1 + x_2 + 10x_3$$

$$x_1 + x_3 \le 100$$

$$x_2 \le 1$$

All variables ≥ 0

This is a very easy LP model and can be solved manually without utilizing Simplex.

$x_2 = 1$ (This variable does not exist in the first constraint. In this case, as the problem is a maximization problem, the optimum value of the variable equals the RHS value of the second constraint).

$x_1 = 0$, $x_3 = 100$ (These two variables do exist only in the first constraint and as the objective function coefficient of x_3 is greater than that of x_1, the optimum value of x_3 equals the RHS value of the first constraint). Hence, the optimal solution is as follows:

z = 1001, [x_1, x_2, x_3] = [0, 1, 100]

Similarly, sensitivity analysis can be executed manually.

Reduced Cost

As x_2 and x_3 are in the basis, their reduced costs are 0.

In order to have x_1 enter in the basis, we should make its objective function coefficient as great as that of x_3. In other words, improve the coefficient as 5 (10-5). New objective function would be (max $z = 10x_1 + x_2 + 10x_3$) and there would be at least two optimal solutions for [x_1, x_2, x_3]: [0, 1, 100] and [100, 1, 0]. Therefore reduced cost of x_1 equals 5.

If we improve the objective function coefficient of x_1 more than its reduced cost, there would be a unique optimal solution: [100, 1, 0].

Shadow Price

If the RHS of the first constraint is increased by 1, new optimal solution of x_3 would be 101 instead of 100. In this case, new z value would be 1011.

If we use the definition: 1011 - 1001 = 10 is the shadow price of the first constraint. Similarly the shadow price of the second constraint can be calculated as 1 (please find it).

Utilizing Lindo Output for Sensitivity

NOTICE: The objective function which is regarded as Row 0 in Simplex is accepted as Row 1 in Lindo.

Therefore the first constraint of the model is always second row in Lindo!!!

```
MAX     5 X1 + X2 + 10 X3
SUBJECT TO
       2) X1 + X3 <= 100
       3) X2 <= 1
END

LP OPTIMUM FOUND AT STEP      1
```

OBJECTIVE FUNCTION VALUE
1) 1001.000

VARIABLE	VALUE	REDUCED COST
X1	0.000000	5.000000
X2	1.000000	0.000000
X3	100.000000	0.000000
2)	0.000000	10.000000
3)	0.000000	1.000000

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJ COEFFICIENT RANGES

VARIABLE	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
X1	5.000000	5.000000	INFINITY
X2	1.000000	INFINITY	1.000000
X3	10.000000	INFINITY	5.000000

RIGHTHAND SIDE RANGES

ROW	CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
2	100.000000	INFINITY	100.000000
3	1.000000	INFINITY	1.000000

Lindo output reveals the reduced costs of x_1, x_2, and x_3 as 5, 0, and 0 respectively.

In the maximization problems, the reduced cost of a non-basic variable can also be read from the allowable increase value of that variable at obj. coefficient ranges. Here, the corresponding value of x_1 is 5.

In the minimization problems, the reduced cost of a non-basic variable can also be read from the allowable decrease value of that variable at obj. coefficient ranges. The same Lindo output reveals the shadow prices of the constraints in the "dual price" section:

Here, the shadow price of the first constraint (Row 2) equals 10.

The shadow price of the second constraint (Row 3) equals 1.

Some important equations

If the change in the RHS of the constraint leaves the current basis optimal (within the allowable RHS range), the following equations can be used to calculate new objective function value:

for maximization problems

- new obj. fn. value = old obj. fn. value + (new RHS – old RHS) × shadow price for minimization problems

- new obj. fn. value = old obj. fn. value – (new RHS – old RHS) × shadow price

For Lindo example, as the allowable increases in RHS ranges are infinity for each constraint, we can increase RHS of them as much as we want. But according to allowable decreases, RHS of the first constraint can be decreased by 100 and that of second constraint by 1.

Lets assume that new RHS value of the first constraint is 60.

As the change is within allowable range, we can use the first equation (max. problem):

$$z_{new} = 1001 + (60 - 100)\ 10 = 601.$$

Utilizing Simplex for Sensitivity

In Dakota furniture example; x_1, x_2, and x_3 were representing the number of desks, tables, and chairs produced.

The LP formulated for profit maximization:

$$
\begin{array}{lllllllll}
\max z = & 60x_1 & 30x_2 & 20x_3 & & & & & \\
& 8x_1 & +6x_2 & +x_3 & +s_1 & & & = 48 & \text{Lumber} \\
& 4x_1 & +2x_2 & +1.5x_3 & & +s_2 & & = 20 & \text{Finishing} \\
& 2x_1 & +1.5x_2 & +.5x_3 & & & +s_3 & = 8 & \text{Carpentry} \\
& & x_2 & & & & +s_4 & = 5 & \text{Demand}
\end{array}
$$

The optimal solution was:

$$
\begin{array}{lllllllll}
z & & +5x_2 & & & +10s_2 & +10s_3 & & = 280 \\
& & -2x_2 & & +s_1 & +2s_2 & -8s_3 & & = 24 \\
& & -2x_2 & +x_3 & & +2s_2 & -4s_3 & & = 8 \\
& +x_1 & +1.25x_2 & & & -.5s_2 & +1.5s_3 & & = 2 \\
& & x_2 & & & & & +s_4 & = 5
\end{array}
$$

Analysis 1

Suppose available finishing time changes from 20 → 20+δ, then we have the system:

$$
\begin{array}{llllllll}
z' = 60x_1' & +30x_2' & +20x_3' & & & & & \\
8x_1' & +6x_2' & +x_3' & +s_1' & & & & = 48 \\
4x_1' & +2x_2' & +1.5x_3' & & +s_2' & & & = 20+\delta \\
2x_1' & +1.5x_2' & +.5x_3' & & & +s_3' & & = 8 \\
& +x_2' & & & & & +s_4' & = 5
\end{array}
$$

or equivalently:

$$z' = 60\,x_1' + 30\,x_2' + 20\,x_3'$$

$$\begin{aligned}
8\,x_1' + 6\,x_2' + x_3' + s_1' &= 48\\
4\,x_1' + 2\,x_2' + 1.5\,x_3' + (s_2'-\delta) &= 20\\
2\,x_1' + 1.5\,x_2' + .5\,x_3' + s_3' &= 8\\
+ x_2' + s_4' &= 5
\end{aligned}$$

That is z, x_1, x_2, x_3, x_4, s_1, $s_2-\delta$, s_3, s_4 satisfy the original problem, and hence (1) Substituting in:

$$\begin{aligned}
z' + 5\,x_2' + 10(s_2'-\delta) + 10\,s_3' &= 280\\
-2\,x_2' + s_1' + 2(s_2'-\delta) - 8\,s_3' &= 24\\
-2\,x_2' + x_3' + 2(s_2'-\delta) - 4\,s_3' &= 8\\
+ x_1' + 1.25\,x_2' - .5(s_2'-\delta) + 1.5\,s_3 &= 2\\
x_2' + s_4' &= 5
\end{aligned}$$

and thus

$$\begin{aligned}
z' + 5\,x_2' + 10\,s_2' + 10\,s_3' &= 280+10\delta\\
-2\,x_2' + s_1' + 2\,s_2' - 8\,s_3' &= 24+2\delta\\
-2\,x_2' + x_3' + 2\,s_2' - 4\,s_3' &= 8+2\delta\\
+ x_1' + 1.25\,x_2' - .5\,s_2' + 1.5\,s_3' &= 2-.5\delta\\
x_2' + s_4' &= 5
\end{aligned}$$

For $-4 \le \delta \le 4$, the new system maximizes z'. In this range RHS values are non-negative.

As δ increases, revenue increases by 10δ. Therefore, the ***shadow price*** of finishing labor is \$10 per hr. (This is valid for up to 4 extra hours or 4 fewer hours).

Analysis 2

What happens if revenue from desks changes to \$60+γ? For small γ, revenue increases by 2γ (as we are making 2 desks currently). But how large an increase is possible?

The new revenue is:

$$z' = (60+\gamma)x_1+30x_2+20x_3 = z+\gamma x_1$$

$$= (280 - 5x_2 - 10s_2 - 10s_3) + \gamma(2 - 1.25x_2 + .5s_2 - 1.5s_3)$$

$$= 280 + 2\gamma - (5 + 1.25\gamma)x_2 - (10-.5\gamma)s_2 - (10 + 1.5\gamma)s_3$$

So the top line in the final system would be:

$z' + (5 + 1.25\gamma)x_2 + (10 - .5\gamma)s_2 + (10 + 1.5\gamma)s_3 = 280 +$

2γ Provided all terms in this row are ≥ 0, we are still optimal.

For $-4 \leq \gamma \leq 20$, the current production schedule is still optimal.

Analysis 3

If revenue from a non-basic variable changes, the revenue is

$z' = 60x_1 + (30 + \gamma)x_2 + 20x_3 = z + \gamma x_2$

$= 280 - 5x_2 - 10s_2 - 10s_3 + \gamma x_2$

$= 280 - (5 - \gamma)x_2 - 10s_2 - 10s_3$

The current solution is optimal for $\gamma \leq 5$. But when $\gamma > 5$ or the revenue per table is increased past $35, it becomes better to produce tables. We say the **reduced cost** of tables is $5.00.

Duality and Sensitivity Analysis

Will be treated at the class.

The 100% Rule

Will be treated at the class.

UNIT - 2
TRANSPORTATION PROBLEMS

FORMULATING TRANSPORTATION PROBLEMS

In general, a transportation problem is specified by the following information:

- A set of m ***supply points*** from which a good/service is shipped. Supply point i can supply at most s_i units.
- A set of n ***demand points*** to which the good/service is shipped. Demand point j must receive at least d_j units.
- Each unit produced at supply point i and shipped to demand point j incurs a variable cost of c_{ij}.

The relevant data can be formulated in a ***transportation tableau***:

	Demand point 1	Demand point 2	.....	Demand point n	SUPPLY
Supply point 1	c_{11}	c_{12}		c_{1n}	s_1
Supply point 2	c_{21}	c_{22}		c_{2n}	s_2
.....					
Supply point m	c_{m1}	c_{m2}		c_{mn}	s_m
DEMAND	d_1	d_2		d_n	

If total supply equals total demand then the problem is said to be a ***balanced transportation problem***.

Let x_{ij} = number of units shipped from supply point i to demand point j

Decision variable x_{ij}: number of units shipped from supply point i to demand point j

then the general LP representation of a transportation problem is

$$\min \Sigma_i \Sigma_j c_{ij} x_{ij}$$

s.t. $\Sigma_j x_{ij} \leq s_i$ (i=1,2, ..., m) Supply constraints

$\Sigma_i x_{ij} > d_j$ (j=1,2, ..., n) Demand constraints

$x_{ij} \geq 0$

If a problem has the constraints given above and is a *maximization* problem, it is still a transportation problem.

Formulating Balanced Transportation

Problem Example 1. Powerco

Powerco has three electric power plants that supply the needs of four cities. Each power plant can supply the following numbers of kwh of electricity: plant 1, 35 million; plant 2, 50 million; and plant 3, 40 million. The peak power demands in these cities as follows (in kwh): city 1, 45 million; city 2, 20 million; city 3, 30 million; city 4, 30 million. The costs of sending 1 million kwh of electricity from plant to city is given in the table below. To minimize the cost of meeting each city's peak power demand, formulate a balanced transportation problem in a transportation tableau and represent the problem as a LP model.

	To			
From	City 1	City 2	City 3	City 4
Plant 1	\$8	\$6	\$10	\$9
Plant 2	\$9	\$12	\$13	\$7
Plant 3	\$14	\$9	\$16	\$5

Answer

<u>Representation of the problem as a LP model</u>

x_{ij}: number of (million) kwh produced at plant *i* and sent to city *j*.

min z = 8 x11 + 6 x12 + 10 x13 + 9 x14 + 9 x21 + 12 x22 + 13 x23 + 7 x24 + 14 x31 + 9 x32 + 16 x33 + 5 x34

x11 + x12 + x13 + x14 $\leq$ 35 (supply constraints)

x21 + x22 + x23 + x24 $\leq$ 50

x31 + x32 + x33 + x34 $\leq$ 40

x11 + x21 + x31 $\geq$ 45 (demand constraints)

x12 + x22 + x32 $\geq$ 20

x13 + x23 + x33 $\geq$ 30

x14 + x24 + x34 $\geq$ 30

$x_{ij} \geq 0$ (i = 1, 2, 3; j = 1, 2, 3, 4)

Formulation of the transportation problem

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
DEMAND	45	20	30	30	125

Total supply & total demand both equal 125: “balanced transport’n problem”.

Balancing an Unbalanced Transportation Problem

Excess Supply

If total supply exceeds total demand, we can balance a transportation problem by creating a ***dummy demand point*** that has a demand equal to the amount of excess supply. Since shipments to the dummy demand point are not real shipments, they are assigned a cost of zero. These shipments indicate unused supply capacity.

Unmet Demand

If total supply is less than total demand, actually the problem has no feasible solution. To solve the problem it is sometimes desirable to allow the possibility of leaving some demand unmet. In such a situation, *a penalty is often associated with unmet demand.* This means that a ***dummy supply point*** should be introduced.

Example 2. Modified Powerco for Excess Supply

Suppose that demand for city 1 is 40 million kwh. Formulate a balanced transportation problem.

Answer

Total demand is 120, total supply is 125.

To balance the problem, we would add a dummy demand point with a demand of 125 – 120 = 5 million kwh.

From each plant, the cost of shipping 1 million kwh to the dummy is 0.

For details see Table 4.

Table 4. Transportation Tableau for Excess Supply

	City 1	City 2	City 3	City 4	Dummy	SUPPLY
Plant 1	8	6	10	9	0	35
Plant 2	9	12	13	7	0	50
Plant 3	14	9	16	5	0	40
DEMAND	40	20	30	30	5	125

Example 3. Modified Powerco for Unmet Demand

Suppose that demand for city 1 is 50 million kwh. For each million kwh of unmet demand, there is a penalty of 80$. Formulate a balanced transportation problem.

Answer

We would add a dummy supply point having a supply of 5 million kwh representing shortage.

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8	6	10	9	35
Plant 2	9	12	13	7	50
Plant 3	14	9	16	5	40
Dummy (Shortage)	80	80	80	80	5
DEMAND	50	20	30	30	130

FINDING BFS FOR TRANSPORT'N PROBLEMS

For a balanced transportation problem, general LP representation may be written as:

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

s.t. $\sum_j x_{ij} = s_i$ (i=1,2, ..., m) Supply constraints

$\sum_i x_{ij} = d_j$ (j=1,2, ..., n) Demand constraints

$x_{ij} \geq 0$

To find a bfs to a balanced transportation problem, we need to make the following important observation:

If a set of values for the xij's satisfies all but one of the constraints of a balanced transportation problem, the values for the xij's will automatically satisfy the other constraint.

This observation shows that when we solve a balanced transportation, we may omit from consideration any one of the problem's constraints and solve an LP having $m+n-1$ constraints. We arbitrarily assume that the first supply constraint is omitted from consideration. In trying to find a bfs to the remaining $m+n-1$ constraints, you might think that any collection of $m+n-1$ variables would yield a basic solution. But this is not the case: If the $m+n-1$ variables yield a basic solution, the cells corresponding to this set contain ***no loop***.

An ordered sequence of at least four different cells is called a loop if

- Any two consecutives cells lie in either the same row or same column
- No three consecutive cells lie in the same row or column
- The last cell in the sequence has a row or column in common with the first cell in the sequence

There are three methods that can be used to find a bfs for a balanced transportation problem:

1. Northwest Corner method
2. Minimum cost method
3. Vogel's method

Northwest Corner Method

We begin in the upper left corner of the transportation tableau and set x_{11} as large as possible (clearly, x_{11} can be no larger than the smaller of s_1 and d_1).

- If $x_{11}=s_1$, cross out the first row of the tableau. Also change d_1 to d_1-s_1.
- If $x_{11}=d_1$, cross out the first column of the tableau. Change s_1 to s_1-d_1.
- If $x_{11}=s_1=d_1$, cross out either row 1 or column 1 (but not both!).
    - If you cross out row, change d_1 to 0.
    - If you cross out column, change s_1 to 0.

Continue applying this procedure to the most northwest cell in the tableau that does not lie in a crossed out row or column.

Eventually, you will come to a point where there is only one cell that can be assigned a value. Assign this cell a value equal to its row or column demand, and cross out both the cell's row or column.

A bfs has now been obtained.

Example 1.

For example consider a balanced transportation problem given below (We omit the costs because they are not needed to find a bfs!).

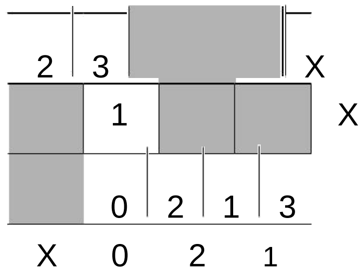
				5
				1
				3
2	4	2	1	

Total demand equals total supply (9): this is a balanced transport'n problem.

2				3
				1
				3
X	4	2	1	

2	3			X
				1
				3
X	1	2	1	

2	3			X
	1			X
				3
X	0	2	1	



NWC method assigned values to $m+n-1$ (3+4-1 = 6) variables. The variables chosen by NWC method can not form a loop, so a bfs is obtained.

Minimum Cost Method

Northwest Corner method does not utilize shipping costs, so it can yield an initial bfs that has a very high shipping cost. Then determining an optimal solution may require several pivots.

To begin the minimum cost method, find the variable with the smallest shipping cost (call it x_{ij}). Then assign x_{ij} its largest possible value, min $\{s_i, d_j\}$.

As in the NWC method, cross out row i or column j and reduce the supply or demand of the noncrossed-out of row or column by the value of x_{ij}.

Continue like NWC method (instead of assigning upper left corner, the cell with the minimum cost is assigned). See Northwest Corner Method for the details!

Example 2.

2	3	5	6	5
2	1	3	5	10
3	8	4	6	15
12	8	4	6	

2	3	5	6	5
2	1 8	3	5	2
3	8	4	6	15
12	X	4	6	

				Supply
[2]	[3]	[5]	[6]	5
2 [**2**]	8 [1]	[3]	[5]	X
[3]	[8]	[4]	[6]	15
10	X	4	6	

				Supply
5 [**2**]	[3]	[5]	[6]	X
2 [2]	8 [1]	[3]	[5]	X
[3]	[8]	[4]	[6]	15
5	X	4	6	

				Supply
5 [2]	[3]	[5]	[6]	X
2 [2]	8 [1]	[3]	[5]	X
5 [3]	[8]	4 [4]	6 [6]	10
5	X	4	6	

Vogel's Method

Begin by computing for each row and column a penalty equal to the difference between the two smallest costs in the row and column. Next find the row or column with the largest penalty. Choose as the first basic variable the variable in this row or column that has the smallest cost. As described in the NWC method, make this variable as large as possible, cross out row or column, and change the supply or demand associated with the basic variable (See Northwest Corner Method for the details!). Now recomputed new penalties (using only cells that do not lie in a crossed out row or column), and repeat the procedure until only one uncrossed cell remains. Set this variable equal to the supply or demand associated with the variable, and cross out the variable's row and column.

Example 3.

				Supply	Row penalty
	6	7	8	10	7-6=1
	15	80	78	15	78-15=63
Demand	15	5	5		
Column penalty	15-6=9	80-7=**73**	78-8=70		

				Supply	Row penalty
	6	**7** 5	8	5	8-6=2
	15	80	78	15	78-15=63
Demand	15	X	5		
Column penalty	15-6=9	-	78-8=**70**		

				Supply	Row penalty
	6	7 5	**8** 5	X	-
	15	80	78	15	-
Demand	15	X	0		
Column penalty	15-6=9	-	-		

	6	7 5	8 5	X
	15 15	80	78 0	15
Demand	15	X	0	

THE TRANSPORTATION SIMPLEX METHOD

Steps of the Method

1. If the problem is unbalanced, balance it
2. Use one of the methods to find a bfs for the problem
3. Use the fact that $u_1 = 0$ and $u_i + v_j = c_{ij}$ for all basic variables to find the *u*'s and *v*'s for the current bfs.
4. If $u_i + v_j - c_{ij} \leq 0$ for all nonbasic variables, then the current bfs is optimal. If this is not the case, we enter the variable with the most positive $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

For a maximization problem, proceed as stated, but replace Step 4 by the following step:

If $u_i + v_j - c_{ij} \geq 0$ for all nonbasic variables, then the current bfs is optimal. Otherwise, enter the variable with the most negative $u_i + v_j - c_{ij}$ into the basis using the *pivoting procedure*. This yields a new bfs. Return to Step 3.

Pivoting procedure

1. Find the loop (there is only one possible loop!) involving the entering variable (determined at step 4 of the transport'n simplex method) and some or all of the basic variables.
2. Counting *only cells in the loop*, label those that are an even number (0, 2, 4, and so on) of cells away from the entering variable as *even cells*. Also label those that are an odd number of cells away from the entering variable as *odd cells*.
3. Find the odd cell whose variable assumes the smallest value. Call this value Φ. The variable corresponding to this odd cell will leave the basis. To perform the pivot, decrease the value of each odd cell by Φ and increase the value of each even cell by Φ. The values of variables not in the loop remain unchanged. The pivot is now complete. If $\Phi = 0$, the entering variable will equal 0, and odd variable that has a current value of 0 will leave the basis.

Example 1. Powerco

The problem is balanced (total supply equals total demand).

When the NWC method is applied to the Powerco example, the bfs in the following table is obtained (check: there exist $m+n-1=6$ basic variables).

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 35	6	10	9	35
Plant 2	9 10	12 20	13 20	7	50
Plant 3	14	9	16 10	5 30	40
DEMAND	45	20	30	30	125

$u1 = 0$

$u1 + v1 = 8$ yields $v1 = 8$

$u2 + v1 = 9$ yields $u2 = 1$

$u2 + v2 = 12$ yields $v2 = 11$

$u2 + v3 = 13$ yields $v3 = 12$

$u3 + v3 = 16$ yields $u3 = 4$

$u3 + v4 = 5$ yields $v4 = 1$

$\hat{c}_{12} = 0 + 11 - 6 = 5$

$\hat{c}_{13} = 0 + 12 - 10 = 2$

$\hat{c}_{14} = 0 + 1 - 9 = -8$

$\hat{c}_{24} = 1 + 1 - 7 = -5$

$\hat{c}_{31} = 4 + 8 - 14 = -2$

$\hat{c}_{32} = 4 + 11 - 9 = 6$

Since $\hat{c}_{32}$ is the most positive one, we would next enter x_{32} into the basis: Each unit of x_{32} that is entered into the basis will decrease Powerco's cost by $6.

The loop involving x_{32} is (3,2)-(3,3)-(2,3)-(2,2). $\Phi = 10$ (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 35	6	10	9	35
Plant 2	9 10	12 **20–Φ**	13 **20+Φ**	7	50
Plant 3	14	9 **Φ**	16 **10–Φ**	5 30	40
DEMAND	45	20	30	30	125

x_{33} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	11	12	7	SUPPLY
0	8 35	6	10	9	35
1	9 10	12 10	13 30	7	50
-2	14		9	16 5	40
	10			30	
DEMAND	45	20	30	30	125

$\hat{c}_{12} = 5$, $\hat{c}_{13} = 2$, $\hat{c}_{14} = -2$, $\hat{c}_{24} = 1$, $\hat{c}_{31} = -8$, $\hat{c}_{33} = -6$

Since $\hat{c}_{12}$ is the most positive one, we would next enter x_{12} into the basis.

The loop involving x_{12} is (1,2)-(2,2)-(2,1)-(1,1). $\Phi = 10$ (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **35–Φ**	6 **Φ**	10	9	35
Plant 2	9 **10+Φ**	12 **10–Φ**	13 30	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{22} would leave the basis. New bfs is shown at the following table:

u_i/v_j	8	6	12	2	SUPPLY
0	8 25	6 10	10	9	35
1	9		12 7	13	50
3	20 14		30 9	16	40
DEMAND	45 10	20	30 5	30 30	125

$\hat{c}_{13} = 2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -5$, $\hat{c}_{24} = -4$, $\hat{c}_{31} = -3$, $\hat{c}_{33} = -1$

Since $\hat{c}_{13}$ is the most positive one, we would next enter x_{13} into the basis.

The loop involving x_{13} is (1,3)-(2,3)-(2,1)-(1,1). $\Phi = 25$ (see table)

	City 1	City 2	City 3	City 4	SUPPLY
Plant 1	8 **25**–**Φ**	6 10	10 **Φ**	9	35
Plant 2	9 **20**+**Φ**	12	13 **30**–**Φ**	7	50
Plant 3	14	9 10	16	5 30	40
DEMAND	45	20	30	30	125

x_{11} would leave the basis. New bfs is shown at the following table:

u_i/v_j	6	6	10	2	SUPPLY
0	8	6 10	10	9 25	35
3	9		12 7	13	50
3	45 14		5 9	16	40
DEMAND	45 10	20	30 5	30 30	125

$\hat{c}_{11} = -2$, $\hat{c}_{14} = -7$, $\hat{c}_{22} = -3$, $\hat{c}_{24} = -2$, $\hat{c}_{31} = -5$, $\hat{c}_{33} = -3$

Since all $\hat{c}_{ij}$'s are negative, an optimal solution has been obtained.

Report

45 million kwh of electricity would be sent from plant 2 to city 1.

10 million kwh of electricity would be sent from plant 1 to city 2. Similarly, 10 million kwh of electricity would be sent from plant 3 to city 2.

25 million kwh of electricity would be sent from plant 1 to city 3. 5 million kwh of electricity would be sent from plant 2 to city 3.

30 million kwh of electricity would be sent from plant 3 to city 4 and

Total shipping cost is:

$$z = .9\ (45) + 6\ (10) + 9\ (10) + 10\ (25) + 13\ (5) + 5\ (30) = \$\ 1020$$

TRANSSHIPMENT PROBLEMS

Sometimes a point in the shipment process can both receive goods from other points and send goods to other points. This point is called as **transshipment point** through which goods can be transshipped on their journey from a supply point to demand point.

Shipping problem with this characteristic is a transshipment problem.

The optimal solution to a transshipment problem can be found by converting this transshipment problem to a transportation problem and then solving this transportation problem.

Remark

As stated in “Formulating Transportation Problems”, we define a **supply point** to be a point that can send goods to another point but cannot receive goods from any other point.

Similarly, a **demand point** is a point that can receive goods from other points but cannot send goods to any other point.

Steps

1. If the problem is unbalanced, balance it
   Let *s* = total available supply (or demand) for balanced problem
2. Construct a transportation tableau as follows
   A row in the tableau will be needed for each supply point and transshipment point A column will be needed for each demand point and transshipment point
   Each supply point will have a supply equal to its original supply Each demand point will have a demand equal to its original demand
   Each transshipment point will have a supply equal to “that point’s original supply + *s*”
   Each transshipment point will have a demand equal to “that point’s original demand + *s*”
3. Solve the transportation problem

Example 1. Bosphorus

(Based on Winston 7.6.)

Bosphorus manufactures LCD TVs at two factories, one in Istanbul and one in Bruges. The Istanbul factory can produce up to 150 TVs per day, and the Bruges factory can produce up to 200 TVs per day. TVs are shipped by air to customers in London and Paris. The customers in each city require 130 TVs per day. Because of the deregulation of air fares, Bosphorus believes that it may be cheaper to first fly some TVs to Amsterdam or Munchen and then fly them to their final destinations.

The costs of flying a TV are shown at the table below. Bosphorus wants to minimize the total cost of shipping the required TVs to its customers.

€	To					
From	Istanbul	Bruges	Amsterdam	Munchen	London	Paris
Istanbul	0	-	8	13	25	28
Bruges	-	0	15	12	26	25
Amsterdam	-	-	0	6	16	17
Munchen	-	-	6	0	14	16
London	-	-	-	-	0	-
Paris	-	-	-	-	-	0

Answer:

In this problem Amsterdam and Munchen are *transshipment points*. **Step 1.** Balancing the problem

Total supply = 150 + 200 = 350

Total demand = 130 + 130 = 260

Dummy's demand = 350 – 260 = 90

s = 350 (total available supply or demand for balanced problem)

Step 2. Constructing a transportation tableau

Transshipment point's demand = Its original demand + *s* = 0 + 350 = 350

Transshipment point's supply = Its original supply + *s* = 0 + 350 = 350

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8	13	25	28	0	150
Bruges	15	12	26	25	0	200
Amsterdam	0	6	16	17	0	350
Munchen	6	0	14	16	0	350
Demand	350	350	130	130	90	

Step 3. Solving the transportation problem

	Amsterdam	Munchen	London	Paris	Dummy	Supply
Istanbul	8 130	13	25	28	0 20	150
Bruges	15		12 0	26	25	200
Amsterdam				130		350
Munchen	0		6	70 16	17	350
Demand	220 6	350	130 0 0	130 14	90 16	1050

350

Report:

Bosphorus should produce 130 TVs at Istanbul, ship them to Amsterdam, and transship them from Amsterdam to London.

The 130 TVs produced at Bruges should be shipped directly to Paris.

The total shipment is 6370 Euros.

ASSIGNMENT PROBLEMS

There is a special case of transportation problems where each supply point should be assigned to a demand point and each demand should be met. This certain class of problems is called as “assignment problems”. For example determining which employee or machine should be assigned to which job is an assignment problem.

LP Representation

An assignment problem is characterized by knowledge of the cost of assigning each supply point to each demand point: c_{ij}

On the other hand, a 0-1 integer variable x_{ij} is defined as follows

$x_{ij} = 1$ if supply point i is assigned to meet the demands of demand point

j $x_{ij} = 0$ if supply point i is not assigned to meet the demands of point j

In this case, the general LP representation of an assignment problem is

$$\min \sum_i \sum_j c_{ij} x_{ij}$$

s.t. $\sum_j x_{ij} = 1$ (i=1,2, ..., m) Supply constraints

$\sum_i x_{ij} = 1$ (j=1,2, ..., n) Demand constraints

$x_{ij} = 0$ or $x_{ij} = 1$

Hungarian Method

Since all the supplies and demands for any assignment problem are integers, all variables in optimal solution of the problem must be integers. Since the RHS of each constraint is equal to 1, each x_{ij} must be a nonnegative integer that is no larger than 1, so each x_{ij} must equal 0 or 1.

Ignoring the $x_{ij} = 0$ or $x_{ij} = 1$ restrictions at the LP representation of the assignment problem, we see that we confront with a balanced transportation problem in which each supply point has a supply of 1 and each demand point has a demand of 1.

However, the high degree of degeneracy in an assignment problem may cause the Transportation Simplex to be an inefficient way of solving assignment problems.
For this reason and the fact that the algorithm is even simpler than the Transportation Simplex, the Hungarian method is usually used to solve assignment problems.

Remarks

1. To solve an assignment problem in which the goal is to maximize the objective function, multiply the profits matrix through by –1 and solve the problem as a **minimization** problem.
2. If the number of rows and columns in the cost matrix are unequal, the assignment problem is **unbalanced**. Any assignment problem should be balanced by the addition of one or more dummy points before it is solved by the Hungarian method.

Steps

1. Find the minimum cost each row of the *m**m* cost matrix.
2. Construct a new matrix by subtracting from each cost the minimum cost in its row
3. For this new matrix, find the minimum cost in each column
4. Construct a new matrix (reduced cost matrix) by subtracting from each cost the minimum cost in its column
5. Draw the minimum number of lines (horizontal and/or vertical) that are needed to cover all the zeros in the reduced cost matrix. If *m* lines are required, an optimal solution is available among the covered zeros in the matrix. If fewer than *m* lines are needed, proceed to next step
6. Find the smallest cost (*k*) in the reduced cost matrix that is uncovered by the lines drawn in Step 5
7. Subtract *k* from each uncovered element of the reduced cost matrix and add *k* to each element that is covered by two lines. Return to Step 5

Example 1. Flight Crew

(Based on Winston 7.5.)

Four captain pilots (CP1, CP2, CP3, CP4) has evaluated four flight officers (FO1, FO2, FO3, FO4) according to perfection, adaptation, morale motivation in a 1-20 scale (1: very good, 20: very bad). Evaluation grades are given in the table. Flight

Company wants to assign each flight officer to a captain pilot according to these evaluations. Determine possible flight crews.

	FO1	FO2	FO3	FO4
CP1	2	4	6	10
CP2	2	12	6	5
CP3	7	8	3	9
CP4	14	5	8	7

Answer:

Step 1. For each row in the table we find the minimum cost: 2, 2, 3, and 5 respectively

Step 2 & 3. We subtract the row minimum from each cost in the row. For this new matrix, we find the minimum cost in each column

	0	2	4	8
	0	10	4	3
	4	5	0	6
	9	0	3	2
Column minimum	0	0	0	2

Step 4. We now subtract the column minimum from each cost in the column obtaining reduced cost matrix.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 5. As shown, lines through row 3, row 4, and column 1 cover all the zeros in the reduced cost matrix. The minimum number of lines for this operation is 3. Since fewer than four lines are required to cover all the zeros, solution is not optimal: we proceed to next step.

0	2	4	6
0	10	4	1
4	5	0	4
9	0	3	0

Step 6 & 7. The smallest uncovered cost equals 1. We now subtract 1 from each uncovered cost, add 1 to each twice-covered cost, and obtain

0	1	3	5
0	9	3	0
5	5	0	4
10	0	3	0

Four lines are now required to cover all the zeros: An optimal s9olution is available. Observe that the only covered 0 in column 3 is x_{33}, and in column 2 is x_{42}. As row 5 can not be used again, for column 4 the remaining zero is x_{24}. Finally we choose x_{11}.

Report:

CP1 should fly with FO1; CP2 should fly with FO4; CP3 should fly with FO3; and CP4 should fly with FO4.

Example 2. Maximization problem

	F	G	H	I	J
A	6	3	5	8	10
B	2	7	6	3	2
C	5	8	3	4	6
D	6	9	3	1	7
E	2	2	2	2	8

Report:

Optimal profit = 36

Assigments: A-I, B-H, C-G, D-F, E-J

Alternative optimal sol'n: A-I, B-H, C-F, D-G, E-J

5. INTEGER PROGRAMMING

When formulating LP's we often found that, strictly, certain variables should have been regarded as taking integer values but, for the sake of convenience, we let them take fractional values reasoning that the variables were likely to be so large that any fractional part could be neglected.

While this is acceptable in some situations, in many cases it is not, and in such cases we must find a numeric solution in which the variables take integer values.

Problems in which this is the case are called ***integer programs*** (***IP***'s) and the subject of solving such programs is called ***integer programming*** (also referred to by the initials ***IP***).

IP's occur frequently because many decisions are essentially discrete (such as yes/no, do/do not) in that one or more options must be chosen from a finite set of alternatives.

An IP in which all variables are required to be integers is called a ***pure IP*** problem. If some variables are restricted to be integer and some are not then the problem is a ***mixed IP*** problem.

The case where the integer variables are restricted to be 0 or 1 comes up surprising often. Such problems are called ***pure (mixed) 0-1 programming*** problems or ***pure (mixed) binary IP*** problems.

For any IP we can generate an LP by taking the same objective function and same constraints but with the requirement that variables are integer replaced by appropriate continuous constraints:

"$x_i \geq 0$ and integer" can be replaced by $x_i \geq 0$

"$x_i = 0$ or 1" can be replaced by $x_i \geq 0$ and $x_i \leq 1$

The LP obtained by omitting all integer or 0-1 constraints on variables is called ***LP Relaxation of the IP*** (***LR***).

FORMULATING IP

Practical problems can be formulated as IPs. For instance budgeting problems, knapsack problems, fixed charge production and location problems, set covering problems, etc.

Budgeting Problems Example

Capital Budgeting

(Winston 9.2, p. 478 – modified)

Stock is considering four investments

Each investment yields a determined NPV ($8,000, $11,000, $6,000, $4,000)

Each investment requires at certain cash flow at the present time ($5,000, $7,000, $4,000, $3,000)

Currently Stock has $14,000 available for investment.

Formulate an IP whose solution will tell Stock how to maximize the NPV obtained from the four investments.

Answer

Begin by defining a variable for each decision that Stockco must make.

In this case, we will use a 0-1 variable x_j for each investment:

If x_j is 1 then Stock will make investment j.

If it is 0, Stock will not make the investment.

This leads to the 0-1 programming problem:

$$\max z = 8x_1 + 11x_2 + 6x_3 + 4x_4$$

$$\text{s.t.} \quad 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$$

$$x_j = 0 \text{ or } 1 \ (j = 1,2,3,4)$$

Comment

Now, a straightforward "bang for buck" (taking ratios of objective coefficient over constraint coefficient) suggests that investment 1 is the best choice.

Ignoring integrality constraints, the optimal linear programming solution is:

$x_1 = x_2 = 1$, $x_3 = 0.5$, and $x_4 = 0$ for a value of $22K

Unfortunately, this solution is not integral. Rounding x_3 down to 0:

$x_1 = x_2 = 1$, $x_3 = x_4 = 0$ for a value of $19K

There is a better integer solution (optimal solution):

$x_1 = 0$, $x_2 = x_3 = x_4 = 1$ for a value of $21K

This example shows that rounding does not necessarily give an optimal value.

Example 1.b. Multiperiod

There are four possible projects, which each run for three years and have the following characteristics:

Which projects would you choose in order to maximize the total return?

		Capital requirements		
Project	Return	Year1	Year2	Year3
1	0.2	0.5	0.3	0.2
2	0.3	1	0.5	0.2
3	0.5	1.5	1.5	0.3
4	0.1	0.1	0.4	0.1
Available capital		3.1	2.5	0.4

Answer

We will use a 0-1 variable x_j for each project:

x_j is 1 if we decide to do project j;

x_j is 0 otherwise (i.e. not do project j).

This leads to the 0-1 programming problem:

$$\max \quad 0.2x_1 + 0.3x_2 + 0.5x_3 + 0.1x_4$$

$$\begin{aligned} \text{s.t.} \quad & 0.5x_1 + 1x_2 + 1.5x_3 + 0.1x_4 \leq 3.1 \\ & 0.3x_1 + 0.5x_2 + 1.5x_3 + 0.4x_4 \leq 2.5 \\ & 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.1x_4 \leq 0.4 \\ & x_j = 0 \text{ or } 1 \quad j = 1, \dots 4 \end{aligned}$$

Example 1.c. Capital Budgeting Extension

There are a number of additional constraints Stock might want to add.

Logical restrictions can be enforced using 0-1 variables:

Stock can only make two investments

$x_1 + x_2 + x_3 + x_4 \leq 2$

Any choice of three or four investments will have $x_1 + x_2 + x_3 + x_4 \geq$ 3

If investment 2 is made, investment 4 must also be made

$x_2 < x_4$ or $x_2 - x_4 \leq 0$

If x_2 is 1, then x_4 is also 1 as Stock desires; if x_2 is 0, then there is no restriction for x_4 (x_4 is 0 or 1)

If investment 1 is made, investment 3 cannot be made $x_1 + x_3 \leq 1$

If x_1 is 1, then x_3 is 0 as Stock desires; if x_1 is 0, then there is no restriction for x_3 (x_3 is 0 or 1)

Either investment 1 or investment 2 must be done

$x_1 + x_2 = 1$

If x_1 is 1, then x_2 is 0 (only investment 1 is done); if x_1 is 0, then x_2 is 1 (only investment 2 is done)

Knapsack Problems

Any IP that has only one constraint is referred to as a knapsack problem. Furthermore, the coefficients of this constraint and the objective are all non-negative. The traditional story is that: There is a knapsack. There are a number of items, each with a size and a value. The objective is to maximize the total value of the items in the knapsack.

Knapsack problems are nice because they are (usually) easy to solve.

Example 2. Knapsack

For instance, the following is a knapsack problem:

Maximize $8x_1 + 11x_2 + 6x_3 + 4x_4$

Subject to $5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$

$x_j = 0$ or $1 \quad j = 1, \dots 4$

Fixed Charge Problems

There is a cost associated with performing an activity at a nonzero level that does not depend on the level of the activity.

An important trick can be used to formulate many production and location problems involving the idea of a fixed charge as IP.

Example 3.a. Gandhi

(Winston 9.2, p. 480)

Gandhi Co makes shirts, shorts, and pants using the limited labor and cloth described below.

In addition, the machinery to make each product must be rented.

	Shirts	Shorts	Pants	Total Avail.
Labor (hrs/wk)	3	2	6	150
Cloth (m^2/wk)	4	3	4	160
Rent for machine ($/wk)	200	150	100	
Variable unit cost	6	4	8	
Sale Price	12	8	15	

Answer

Let x_j be number of clothing produced.

Let y_j be 1 if any clothing j is manufactured and 0 otherwise.

Profit = Sales revenue – Variable Cost – Costs of renting machinery For example the profit from shirts is

$$z_1 = (12 - 6)\, x_1 - 200\, y_1$$

Since supply of labor and cloth is limited, Gandhi faces two constraints.

To ensure $x_j > 0$ forces $y_j = 1$, we include the additional constraints

$$x_j \le M_j\, y_j$$

From the cloth constraint at most 40 shirts can be produced (M_1=40), so the additional constraint for shirts is not an additional limit on x_1 (If M_1 were not chosen large (say M_1=10), then the additional constraint for shirts would unnecessarily restrict the value of x_1).

From the cloth constraint at most 53 shorts can be produced (M_2=53)

From the labor constraint at most 25 pants can be produced (M_3=25)

We thus get the mixed (binary) integer problem:

max $6 x_1 + 4 x_2 + 7 x_3 - 200 y_1 - 150 y_2 - 100 y_3$

s.t. $3 x_1 + 2 x_2 + 6 x_3 \le 150$ (Labor constraint)

$4 x_1 + 3 x_2 + 4 x_3 \le 160$ (Cloth constraint)

$x_1 \le 40 y_1$ (Shirt production constraint)

$x_2 \le 53 y_2$ (Short production constraint)

$x_3 \le 25 y_3$ (Pant production constraint)

$x_1, x_2, x_3 \ge 0$ and integer

$y_1, y_2, y_3 = 0$ or 1

Example 3.b. Lockbox

(Winston 9.2, p. 483)

Consider a national firm that receives checks from all over the United States.

There is a variable delay from when the check is postmarked (and hence the customer has met her obligation) and when the check clears (the firm can use the money).

It is in the firm's interest to have the check clear as quickly as possible since then the firm can use the money.

To speed up this clearing, firms open offices (lockboxes) in different cities to handle the checks.

Suppose firm receives payments from four regions (West, Midwest, East, and South). The average daily value from each region is as follows: $70,000 from the West, $50,000 from the Midwest, $60,000 from the East, and $40,000 from the South. Firm is considering opening lockboxes in L.A., Chicago, New York, and/or Atlanta.

Operating a lockbox costs $50,000 per year.

Assume that each region must send all its money to a single city.

Also assume that investment rate is 20%.

The average days from mailing to clearing is given in the table:

	LA	Chicago	NY	Atlanta
West	2	6	8	8
Midwest	6	2	5	5
East	8	5	2	5
South	8	5	5	2

Which lockboxes should firm open? (Formulate an IP that firm can use to minimize the sum of costs due to lost interest and lockbox operations.)

Answer

First we must calculate the losses due to lost interest for each possible assignment. For instance, if the West sends to New York, then on average there will be $560,000 (=8×$70.000) in process on any given day. Assuming an investment rate of 20%, this corresponds to a yearly loss of $112,000.

We can calculate the losses for the other possibilities in a similar fashion to get the following table:

	LA	Chicago	NY	Atlanta
West	28	84	112	112
Midwest	60	20	50	50
East	96	60	24	60
South	64	40	40	16

Let y_j be a 0-1 variable that is 1 if lockbox *j* is opened and 0 if it is not.

Let x_{ij} be 1 if region i sends to lockbox *j*; and 0 otherwise.

Our objective is to minimize our total yearly costs. This is:

$$28\,x_{11} + 84\,x_{12} + \ldots + 50\,y_1 + 50\,y_2 + 50\,y_3 + 50\,y_4$$

One set of constraint is that each region must be assigned to one lockbox:

$$\sum_j x_{ij} = 1 \qquad \text{for all } i$$

($\sum_j$ should be read as "sum over all integer values of j from 1 to n inclusive")

A region can only be assigned to an open lockbox:

M is any number that should be at least 4 as there are four regions.

(Suppose we do not open LA lockbox; then y_1 is 0, so all of x_{11}, x_{21}, x_{31}, and x_{41} must also be 0. If y_1 is 1, then there is no restriction on the *x* values.)

$$\begin{aligned}
\min \quad & 28\,x_{11} + 84\,x_{12} + 112\,x_{13} + 112\,x_{14} \\
& + 60\,x_{21} + 20\,x_{22} + 50\,x_{23} + 50\,x_{24} \\
& + 96\,x_{31} + 60\,x_{32} + 24\,x_{33} + 60\,x_{34} \\
& + 64\,x_{41} + 40\,x_{42} + 40\,x_{43} + 16\,x_{44} \\
& + 50\,y_1 + 50\,y_2 + 50\,y_3 + 50\,y_4
\end{aligned}$$

$$\begin{aligned}
\text{s.t. } & x_{11} + x_{12} + x_{13} + x_{14} = 1 \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 1 \\
& x_{41} + x_{42} + x_{43} + x_{44} = 1 \\
& x_{11} + x_{21} + x_{31} + x_{41} \le 4y_1 \\
& x_{12} + x_{22} + x_{32} + x_{42} \le 4y_2 \\
& x_{13} + x_{23} + x_{33} + x_{43} \le 4y_3 \\
& x_{14} + x_{24} + x_{34} + x_{44} \le 4y_4
\end{aligned}$$

All x_{ij} and $y_j = 0$ or 1

Membership in Specified Subsets

Using decision variables that equal 1 if an object is part of a solution and 0 otherwise, set covering, set packing, and set partitioning models formulate problems where the core issue is membership in specifed subsets.

Many applications in areas such as location problems (fire/police station, warehouse, facility), scheduling (crew, airline, truck, bus), political districting

- Set covering constraints

  Require that at least one member of subcollection J belongs to a solution:

  $\sum_{j \in J} x_j \geq 1$

- Set packing constraints

  Require that at most one member of subcollection J belongs to a solution:

  $\sum_{j \in J} x_j \leq 1$

- Set partitioning constraints

  Require that exactly one member of subcollection J belongs to a solution:

  $\sum_{j \in J} x_j = 1$

Set Covering Problems

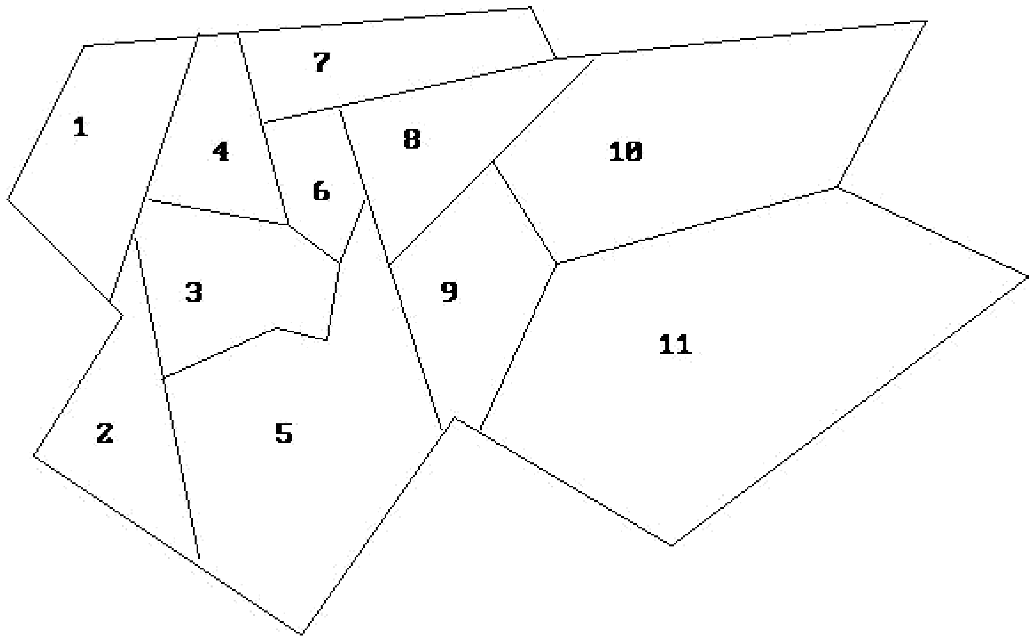
Each member of a given set (call it set 1) must be “covered” by an acceptable member of some set (call it set 2).

The objective of a set-covering problem is to minimize the number of elements in set 2 that are required to cover all the elements in set 1.

Example 4. Fire Station

A county is reviewing the location of its fire stations.

The county is made up of a number of cities:



A fire station can be placed in any city.

It is able to handle the fires for both its city and any adjacent city (any city with a non-zero border with its home city).

How many fire stations should be built and where?

Answer

We can create one variable x_j for each city *j* (1 if we place a station in the city, 0 otherwise):

Each constraint should state that there must be a station either in city *j* or in some adjacent city.

The jth column of the constraint matrix represents the set of cities that can be served by a fire station in city *j*.

We are asked to find a set of such subsets *j* that *covers* the set of all cities in the sense that every city appears in the service subset associated with *at least* one fire station

$$\begin{array}{lll}
\text{min} & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} & \\
\text{s.t.} & x_1 + x_2 + x_3 + x_4 & \geq 1 \text{ (city 1)} \\
 & x_1 + x_2 + x_3 + x_5 & \geq 1 \text{ (city 2)} \\
 & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 & \geq 1 \text{ (city 3)} \\
 & x_1 + x_3 + x_4 + x_6 + x_7 & \geq 1 \text{ (city 4)} \\
 & x_2 + x_3 + x_5 + x_6 + x_8 + x_9 & \geq 1 \text{ (city 5)} \\
 & x_3 + x_4 + x_5 + x_6 + x_7 + x_8 & \geq 1 \text{ (city 6)}
\end{array}$$

$$x_4 + x_6 + x_7 + x_8 \geq 1 \text{ (city 7)}$$
$$x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \geq 1 \text{ (city 8)}$$
$$x_5 + x_8 + x_9 + x_{10} + x_{11} \geq 1 \text{ (city 9)}$$
$$x_8 + x_9 + x_{10} + x_{11} \geq 1 \text{ (city 10)}$$
$$x_9 + x_{10} + x_{11} \geq 1 \text{ (city 11)}$$

All $x_j = 0$ or 1

Either-Or Constraints

Given two constraints

$f(x_1, x_2,\ldots, x_n) \leq 0$ (1)

$g(x_1, x_2,\ldots, x_n) \leq 0$ (2)

ensure that at least one is satisfied (1 or 2) by adding either-or-constraints:

$f(x_1, x_2,\ldots, x_n) \leq M\, y$

$g(x_1, x_2,\ldots, x_n) \leq M\,(1 - y)$

Here y is a 0-1 variable, and M is a number chosen large enough to ensure that both constraints are satisfied for all values of decision variables that satisfy the other constraints in the problem:

- If $y = 0$, then (1) and possibly (2) must be satisfied.
- If $y = 1$, then (2) and possibly (1) must be satisfied.

Example 5. Fire Station

Suppose 1.5 tons of steel and 30 hours of labor are required for production of one compact car.

At present, 6,000 tons of steel and 60,000 hours of labor are available.

For an economically feasible production, at least 1,000 cars of compact car must be produced.

- Constraint: $x \leq 0$ or $x \geq 1000$
- Sign restriction: $x \geq 0$ and Integer

Answer

For $f(x) = x$; $g(x) = 1000 - x$

We can replace the constraint by the following pair of linear

constraints: $x \leq M\, y$

$1000 - x \leq M\,(1 - y)$

$y = 0$ or 1

M = min (6.000/1.5, 60.000/30) = 2000

If-Then Constraints

Suppose we want to ensure that

a constraint $f(x_1, x_2,\ldots, x_n) > 0$ implies

the constraint $g(x_1, x_2,\ldots, x_n) \geq 0$

$-g(x_1, x_2,\ldots, x_n) \leq M\,y$ (1)

$f(x_1, x_2,\ldots, x_n) \leq M\,(1 - y)$ (2)

Here y is a 0-1 variable, and M is a large positive number, chosen large enough so that f<M and $-g$<M hold for all values of decision variables that satisfy the other constraints in the problem.

If $f > 0$, then (2) can be satisfied only if $y = 0$. (1) implies $-g \leq 0$ or $g \geq 0$, which is the desired result

Example 6. Modified Lockbox

(*Winston 9.2, p. 490*)

Suppose we add the following constraint

If customers in region 1 send their payments to city 1, no other customers may send their payments to city 1:

If $x_{11} = 1$, then $x_{21} = x_{31} = x_{41} = 0$

If $x_{11} > 0$, then $x_{21} + x_{31} + x_{41} \leq 0$

Answer

For $f = x_{11}$ and $g = -x_{21} - x_{31} - x_{41}$

We can replace the implication by the following pair of linear constraints:

$x_{21} + x_{31} + x_{41} \leq My$

$x_{11} \leq M\,(1 - y)$

$y = 0$ or 1

$-g$ and f can never exceed 3, we can choose M as 3.

Traveling Salesperson Problems

“Given a number of cities and the costs of traveling from any city to any other city, what is the cheapest round-trip route (tour) that visits each city once and then returns to the starting city?”

This problem is called the traveling salesperson problem (TSP), not surprisingly.

An itinerary that begins and ends at the same city and visits each city once is called a ***tour***.

Suppose there are *N* cities.

Let c_{ij} = Distance from city *i* to city *j* (for $i \neq j$) and

Let c_{ii} = M (a very large number relative to actual distances)

Also define x_{ij} as a 0-1 variable as follows:

$x_{ij} = 1$ if s/he goes from city *i* to city *j*;

$x_{ij} = 0$ otherwise

The formulation of the TSP is:

min $\sum_i \sum_j c_{ij} x_{ij}$

s.t. $\sum_i x_{ij} = 1$ for all *j*

$\sum_j x_{ij} = 1$ for all *i*

$u_i - u_j + N x_{ij} \leq N - 1$ for $i \neq j$; $i, j > 1$

All x_{ij} = 0 or 1, All $u_i \geq 0$

The first set of constraints ensures that s/he arrives once at each city.

The second set of constraints ensures that s/he leaves each city once.

The third set of constraints ensure the following:

Any set of x_{ij}’s containing a subtour will be infeasible

Any set of x_{ij}’s that forms a tour will be feasible

$u_i - u_j + N x_{ij} \leq N - 1$ for $i \neq j$; $i, j > 1$

Assume *N*=5

Subtours: 1-5-2-1, 3-4-3 ???

Choose the subtour that does not contain city 1:

$u_3 - u_4 + 5\, x_{34} \leq 4$

$u_4 - u_3 + 5\, x_{43} \leq 4$

$5\,(x_{34} + x_{43}) \leq 8$

This rules out the possibility that $x_{34} = x_{43} = 1$

The formulation of an IP whose solution will solve a TSP becomes unwieldy and inefficient for large TSPs.

When using branch and bound methods to solve TSPs with many cities, large amounts of computer time may be required. For this reason, heuristics, which quickly lead to a good (but not necessarily optimal) solution to a TSP, are often used.

SOLVING IP

We have gone through a number of examples of IPs at the “Formulating IP Problems” section.

“How can we get solutions to these models?” There are two common approaches:

The technique based on dividing the problem into a number of smaller problems in a *tree search* method called ***branch and bound***.

The method based on ***cutting planes*** (adding constraints to force integrality). Solving IP

Actually, all these approaches involve solving a series of LP.

For solving LP’s we have *general purpose* (independent of the LP being solved) and *computationally effective* (able to solve large LP's) algorithms (simplex or interior point).

For solving IP's *no* similar general purpose and computationally effective algorithms exist

Categorization Categorization

(w.r.t. Purpose)

- General purpose methods will solve any IP but potentially computationally ineffective (will only solve relatively small problems); or
- Special purpose methods are designed for one particular type of IP problem but potentially computationally more effective.

Categorization (w.r.t. Algorithm)

- Optimal algorithms mathematically guarantee to find the optimal solution

- Heuristic algorithms are used to solve a problem by trial and error when an optimal algorithm approach is impractical. They hopefully find a good feasible solution that, in objective function terms, is close to the optimal solution.

Why Heuristics?

Because the size of problem that we want to solve is beyond the computational limit of known optimal algorithms within the computer time we have available.
We could solve optimally but feel that this is not worth the effort (time, money, etc) we would expend in finding the optimal solution.
In fact it is often the case that a well-designed heuristic algorithm can give good quality (near-optimal) results.

Solution Algorithms Categories

- General Purpose, Optimal
  Enumeration, branch and bound, cutting plane
- General Purpose, Heuristic
  Running a general purpose optimal algorithm and terminating after a specified time
- Special Purpose, Optimal
  Tree search approaches based upon generating bounds via dual ascent, lagrangean relaxation
- Special Purpose, Heuristic
  Bound based heuristics, tabu search, simulated annealing, population heuristics (e.g. genetic algorithms), interchange

LP Relaxation

For any IP we can generate an LP by taking the same objective function and same constraints but with the requirement that variables are integer replaced by appropriate continuous constraints:

"x_i = 0 or 1" $x_i \geq 0$ and $x_i \leq 1$

"$x_i \geq 0$ and integer" $x_i \geq 0$

The LP obtained by omitting all integer and 0-1 constraints on variables is called the ***LP Relaxation of the IP*** (***LR***). We can then solve this LR of the original IP.

Naturally Integer LP

If LR is optimized by integer variables then that solution is feasible and optimal for IP. In other words, if the solution is turned out to have all variables taking integer values at the optimal solution, it is also optimal solution for IP:

LR – IP Relation

Since LR is less constrained than IP:

- If IP is a maximization problem, the optimal objective value for LR is greater than or equal to that of IP.
- If IP is a minimization problem, the optimal objective value for LR is less than or equal to that of IP.
- If LR is infeasible, then so is IP.

So solving LR does give some information. It gives a bound on the optimal value, and, if we are lucky, may give the optimal solution to IP.

Enumeration

Unlike LP (where variables took continuous values) in IP's (where all variables are integers) each variable can only take a finite number of discrete (integer) values. Hence the obvious solution approach is simply to ***enumerate*** all these possibilities - calculating the value of the objective function at each one and choosing the (feasible) one with the optimal value.

Example 1. Multi-period Capital Budgeting

Maximize $0.2\,x_1 + 0.3\,x_2 + 0.5\,x_3 + 0.1\,x_4$

Subject to $0.5\,x_1 + 1\,x_2 + 1.5\,x_3 + 0.1\,x_4 \leq 3.1$

$0.3\,x_1 + 0.8\,x_2 + 1.5\,x_3 + 0.4\,x_4 \leq 2.5$

$0.2\,x_1 + 0.2\,x_2 + 0.3\,x_3 + 0.1\,x_4 \leq 0.4$

$x_j = 0$ or 1 $\quad j = 1, \ldots 4$

Possible Solutions

There are 2^4=16 possible solutions:

0 0 0 0 do no projects
0 0 0 1 do one project
0 0 1 0
0 1 0 0
1 0 0 0
0 0 1 1 do two projects
0 1 0 1
1 0 0 1
0 1 1 0
1 0 1 0
1 1 0 0
1 1 1 0 do three projects
1 1 0 1
1 0 1 1
0 1 1 1
1 1 1 1 do four projects

Review

Hence for our example, we merely have to examine 16 possibilities before we know precisely what the best possible solution is. This example illustrates a general truth about integer programming.

What makes solving the problem easy when it is small is precisely what makes it hard very quickly as the problem size increases.

This is simply illustrated: suppose we have 100 integer variables each with two possible integer values then there are 2x2x2x ... x2 = 2^{100} (approximately 10^{30}) possibilities which we have to enumerate (obviously many of these possibilities will be infeasible, but until we generate one we cannot check it against the constraints to see if it is feasible).

For 100 integer variable - ***conceptually*** there is not a problem - simply enumerate all possibilities and choose the best one. But ***computationally*** (numerically) this is just impossible.

The Branch-and-Bound Method

The most effective general purpose optimal algorithm is an LP-based tree search approach called as ***branch and bound*** (B&B).

The method was first put forward in the early 1960's by Land and Doig.

This is *a way of systematically (implicitly) enumerating* feasible solutions such that the optimal integer solution is found.

Where this method differs from the enumeration method is that *not all* the feasible solutions are enumerated but only a part (hopefully a small part) of them. However we can still *guarantee* that we will find the optimal integer solution.

By solving a single sub-problem, many possible solutions may be eliminated from consideration.

Sub-problems are generated by branching on an appropriately chosen fractional-valued variable.

Suppose that in a given sub-problem (call it subp.1), assumes a fractional value between the integers *i* and *i*+1. Then the two newly generated sub-problems:

Subp.2 = Subp.1 + Constraint "$x_i \geq i+1$"

Subp.3 = Subp.1 + Constraint "$x_i \leq I$"

If all variables have integer values in the optimal solution to the sub-problem then the solution is a ***feasible*** solution for the original IP.

If the current feasible solution for the IP has a better *z*-value than any previously obtained feasible solution, then it becomes a ***candidate solution***, and its *z*-value becomes the current ***Lower Bound*** (***LB***) on the optimal *z*-value (for a max problem).

If it is unnecessary to branch on a sub-problem, we say that it is fathomed (inactive):

- The sub-problem is infeasible
- The sub-problem yields an optimal solution in which all variables have integer values
- The optimal *z*-value for the sub-problem does not exceed the current LB, so it cannot yield the optimal solution of the IP

Two general approaches are used to determine which sub-problem should be solved next:

- Backtracking (LIFO)
  Leads us down one side of the B&B tree and finds a candidate solution. Then we backtrack our way up to the top of the other side of the tree.
- Jumptracking
  Solves all the problems created by branching. Then it branches again on the node with the best *z*-value. Often jumps from one side of the tree to the other.

A display of the sub-problems that have been created is called a ***tree***.

Each sub-problem is referred to as a ***node*** of the tree.

Each additional constraint is referred to as a line (***arc***) connecting two nodes (old sub-problem and one of the new sub-problems) of the tree.

B&B for Solving Pure IP Problems Example 2.

Pure IP

(*Winston 9.3., p. 513*)

max $z = 8x_1 + 5x_2$

s.t. $x_1 + x_2 \leq 6$

$9x_1 + 5x_2 \leq$

45

$x_1, x_2 \geq 0$ and integer

Answer

Suppose that we were to solve the LR of the problem [replace "$x_1, x_2 \geq 0$ and integer" by "$x_1, x_2 \geq 0$"]

Then using any LP package or utilizing simplex or graphical solution method we

get $z = 165/4$, $x_1 = 15/4$, $x_2 = 9/4$

As a result of this we now know something about the optimal integer solution, namely that it is ≤ 165/4, i.e. this value of 165/4 is an ***Upper Bound*** on the optimal integer solution

This is because when we relax the integrality constraint we (as we are maximizing) end up with a solution value at least that of the optimal integer solution (and maybe better)

We arbitrarily choose a variable that is fractional in the optimal solution to the LR (subp.1): say x_1.

We need x_1 to be integer. We branch on x_1 and create two new sub-problems:

Subp.2: LR + "$x_1 \geq 4$"

Subp.3: LR + "$x_1 \leq 3$"

Observe that neither subp.2 nor subp.3 includes any points with $x_1 = 15/4$. This means that the optimal solution to LR can not recur when we solve these new sub-problems.

We now arbitrarily choose to solve subp.2.

We see that the optimal solution to subp.2 is

$z = 41$, $x_1 = 4$, $x_2 = 9/5$

We choose x_2 that is fractional in the optimal solution to subp.2.

We need x_2 to be integer. We branch on x_2 and create two new sub-problems:

Subp.4: LR + $x_1 \geq 4$ and $x_2 \geq 2$ = Subp.2 + $x_2 \geq 2$

Subp.5: LR + $x_1 \geq 4$ and $x_2 \leq 1$ = Subp.2 + $x_2 \leq 1$

The set of unsolved sub-problems are consists of subp.3, 4, and 5.

We choose to solve the most recently created sub-problem (This is called LIFO):

The LIFO rule implies that we should next solve subp.4 or 5. We now arbitrarily choose to solve subp.4.

We see that subp.4 is infeasible. Thus subp.4 can not yield the optimal solution to the IP.

Because any branches emanating from subp.4 will yield no useful information, it is fruitless to create them.

LIFO rule implies that we should next solve subp.5.

The optimal solution to subp.5 is

$z = 365/9$, $x_1 = 40/9$, $x_2 = 1$

We branch on fractional-valued x_1:

Subp.6: Subp.5 + $x_1 \geq 5$

Subp.7: Subp.5 + $x_1 \leq 4$

Subp.3, 6, and 7 are now unsolved.

The LIFO rule implies that we next solve subp.6 or 7.

We now arbitrarily choose to solve subp.7.

The optimal solution to subp.7 is

$z = 37$, $x_1 = 4$, $x_2 = 1$

As both variables assume integer values, this solution is ***feasible*** for the original IP this solution is a ***candidate solution***

We must keep this candidate solution until a better feasible solution to the IP (if any exists) is found.

We may conclude that the optimal z-value for the IP ≥ 37 ***Lower Bound*** (LB)

LIFO rule implies that we should next solve subp.6.

The optimal solution to subp.6 is

$z = 40$, $x_1 = 5$, $x_2 = 0$

Its z-value of 40 is larger than LB.

Thus subp.7 cannot yield the optimal solution of the IP.

We update our LB to 40.

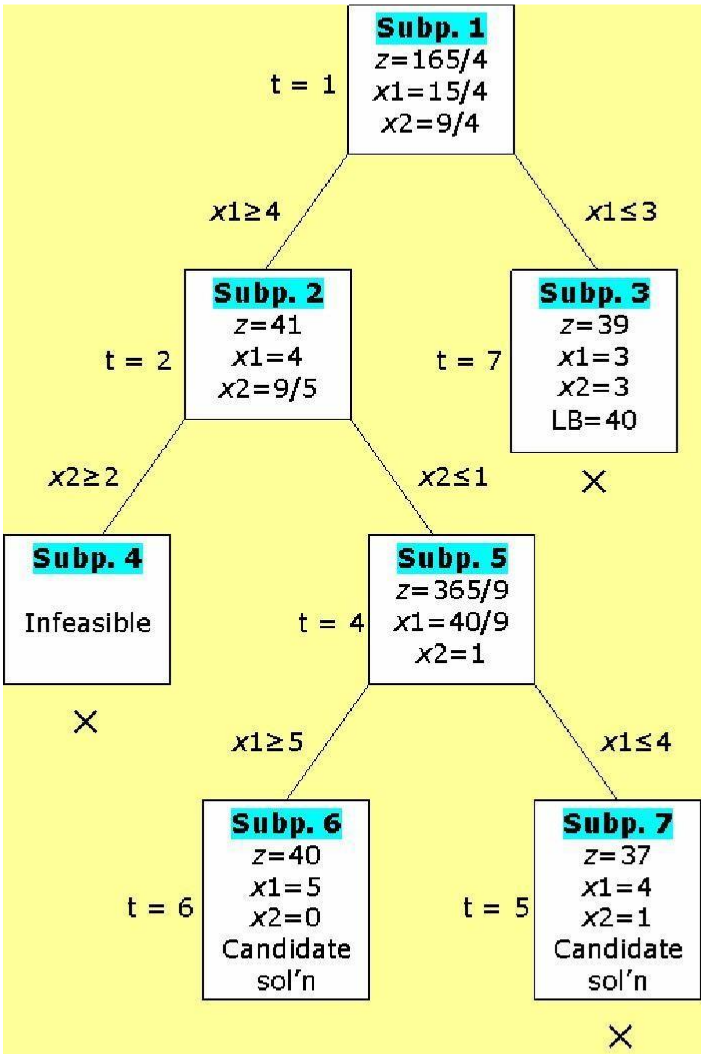
Subp.3 is the only remaining unsolved problem.

The optimal solution to subp.3 is

$z = 39$, $x_1 = 3$, $x_2 = 3$

Subp.3 cannot yield a *z*-value exceeding the current LB, so it cannot yield the optimal solution to the IP.

Final B&B Tree



Optimal Sol'n

Thus, the optimal solution to the IP

$z = 40$, $x_1 = 5$, $x_2 = 0$

B&B for Solving Mixed IP Problems

In MIP, some variables are required to be integers and others are allowed to be either integer or nonintegers.

To solve a MIP by B&B method, modify the method by branching only on variables that are required to be integers.

For a solution to a sub-problem to be a candidate solution, it need only assign integer values to those variables that are required to be integers

Example 3. Mixed IP

(*Winston 9.4., p. 523*)

max $z = 2x_1 + x_2$

s.t. $5x_1 + 2x_2 \leq 8$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$; $x1$ integer

Answer

We solve the LR (subp.1) of the problem

[replace "$x_1 \geq 0$ and integer" by "$x_1 \geq 0$"]

Then using any LP package or utilizing simplex or graphical solution method we get

z = 11/3, $x_1 = 2/3$, $x_2 = 7/3$

Because x_2 is allowed to be fractional, we do not branch on x_2.

We branch on x_1 and create two new sub-problems:

Subp.2: LR + $x_1 \geq 1$

Subp.3: LR + $x_1 \leq 0$

We see that the optimal solution to subp.2

is $z = 7/2$, $x_1 = 1$, $x_2 = 3/2$

As only x_1 assume integer value, this solution is feasible for the original MIP

Candidate solution; LB = 7/2

The optimal solution to subp.3 is

$z = 3$, $x_1 = 0$, $x_2 = 3$

Subp.3 cannot yield a z-value exceeding the current LB, so it cannot yield the optimal solution to the MIP.

Optimal Sol'n

Thus, the optimal solution to the MIP

$z = 7/2$, $x_1 = 1$, $x_2 = 3/2$

B&B for Solving Binary IP Problems

One aspect of the B&B method greatly simplify:

Due to each variable equaling 0 or 1, branching on x_i will yield

in $x_i = 0$ and $x_i = 1$

Example 4. Binary IP

max z = $0.2\,x_1 + 0.3\,x_2 + 0.5\,x_3 + 0.1\,x_4$

s.t. $0.5\,x_1 + 1\,x_2 + 1.5\,x_3 + 0.1\,x_4 \leq 3.1$

$0.3\,x_1 + 0.8\,x_2 + 1.5\,x_3 + 0.4\,x_4 \leq 2.5$

$0.2\,x_1 + 0.2\,x_2 + 0.3\,x_3 + 0.1\,x_4 \leq 0.4$

$x_j = 0$ or $1 \quad j = 1, \ldots 4$

Answer

Replace "$x_j = 0$ or 1 (j=1,...,4)" by "$0 \leq x_j \leq 1$ (j=1,...,4)" LR of the problem

Optimal solution to the LR:

z=0.65, x_2=0.5, x_3=1, x_1=x_4=0

The variable x_2 is fractional. To resolve this we can generate two new problems:

P1: LR + x_2=0

P2: LR + x_2=1

We now have two new sub-problem to solve (*jumptracking*).

If we do this we get

P1 solution: z=0.6, x_1=0.5, x_3=1, x_2=x_4=0

P2 solution: z=0.63, x_2=1, x_3=0.67, x_1=x_4=0

Choosing sub-problem P2 (the best z–value), we branch on x_3 and

get P3 (P2 + x_3=0) sol'n: z=0.5, x_1=x_2=1, x_3=x_4=0

P4 (P2 + x_3=1) sol'n: infeasible

P3 solution is feasible for the original binary IP Candidate solution; LB = 0.5

Choosing the only remaining sub-problem P1, we branch on x_1 and get

P5 (P1 + x_1=0) sol'n: z=0.6, x_3=x_4=1, x_1=x_2=0

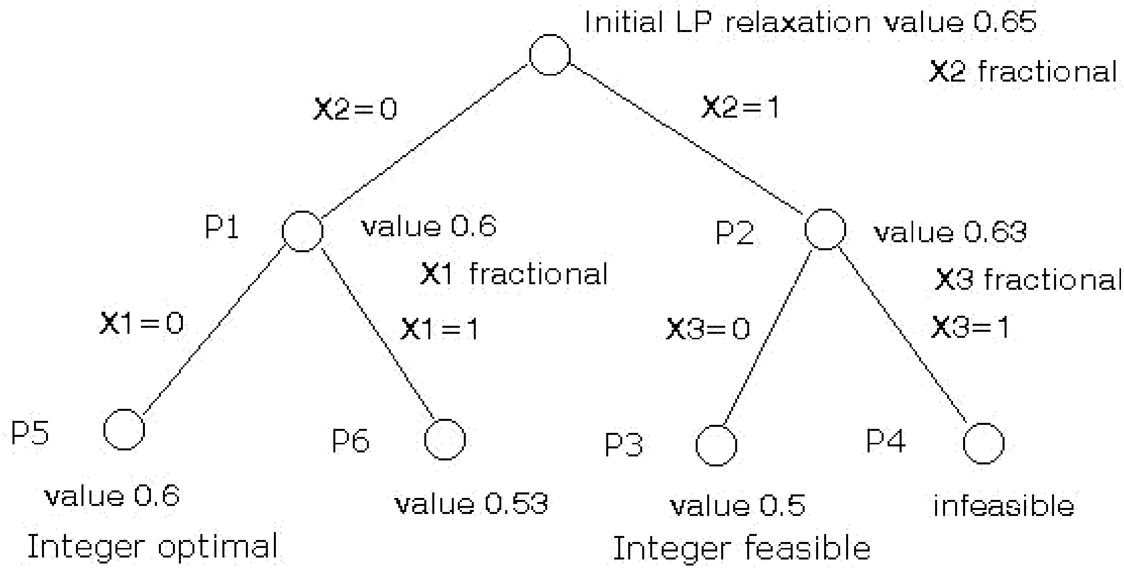
P6 (P1 + x_1=1) sol'n: z=0.53, x_1=1, x_3=0.67, x_2=x_4=0

P5 solution is feasible for the original binary IP New candidate solution; updated LB = 0.6

P6 cannot yield a z-value exceeding the current LB, so it cannot yield the optimal solution to the binary IP.

Thus, the optimal solution to the binary IP

z = 0.6, x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1



Review

Note here that B&B, like complete enumeration, also involves powers of 2 as we progress down the (binary) tree.

However also note that we did not enumerate all possible integer solutions (of which there are 16). Instead here we solved 7 LP's.

This is an important point, and indeed why tree search works at all. We do not need to examine as many LP's as there are possible solutions.

While the computational efficiency of tree search differs for different problems, it is this basic fact that enables us to solve problems that would be completely beyond us where we to try complete enumeration

B&B for Solving Knapsack Problems

Please recall that a knapsack problem is an IP, in which each variable must be equal to 0 or 1, with a single constraint:

max $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$

s.t. $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq$

b $x_i = 0$ or 1 ($i = 1, 2, \ldots, n$)

Two aspects of the B&B method greatly simplify:

- Due to each variable equaling 0 or 1, branching on x_i will yield in x_i=0 and x_i=1
- The LP relaxation may be solved by ***inspection*** instead of using any LP package or utilizing simplex or graphical solution method

Inspection

Recall that

c_i is the benefit obtained if item i is chosen

b is the total amount of an available resource

a_i is the amount of the available resource used by item i

Observe that ratio r_i (c_i/a_i) may be interpreted as the benefit item i earns for each unit of the resource used by item i.

Thus, the best items have the largest value of r and the worst items have the smallest values of r.

To solve any sub-problem resulting from a knapsack problem, compute all the ratios.

Then put the best item in the knapsack.

Then put the second best item in the knapsack.

Continue in this fashion until the best remaining item will overfill the knapsack.

Then fill the knapsack with as much of this item as possible.

Example 5. Knapsack

max z = $8x_1 + 11x_2 + 6x_3 + 4x_4$

s.t. $5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$

$x_j = 0$ or 1 $j = 1, \ldots 4$

Answer

We compute the ratios:

$r_1 = 8 / 5 = 1.6$

$r_2 = 11 / 7 = 1.57$

$r_3 = 6 / 4 = 1.5$

$r_4 = 4 / 3 = 1.33$

Using the ratios, LR solution is

$x_1 = 1, x_2 = 1, x_3 = 0.5, x_4 = 0, z =$

22 We branch on x_3 and get

P1 (LR + x_3=0) sol'n: x_3=0, x_1=x_2=1, x_4=2/3, z=21.67

P2 (LR + x_3=1) sol'n: x_3=x_1=1, x_2=5/7, x_4=0, z=21.85

Choosing sub-problem P2 (the best z–value), we branch on x_2 and get

P3 (P2 + x_2=0) sol'n: x_3=1, x_2=0, x_1=1, x_4=1, z=18

P4 (P2 + x_2=1) sol'n: x_3=x_2=1, x_1=3/5, x_4=0, z=21.8

P3 solution is feasible for the original knapsack problem Candidate solution; LB = 18

Choosing sub-problem P4, we branch on x_1 and get

P5 (P4 + x_1=0) sol'n: x_3=x_2=1, x_1=0, x_4=1, z=21

P6 (P4 + x_1=1) sol'n: Infeasible (x_3=x_2=x_1=1: LHS=16)

P5 solution is feasible for the original knapsack problem New candidate solution; updated LB = 21

The only remaining sub-problem is P1 with solution value 21.67

There is no better solution for this sub-problem than 21. But we already have a solution with value 21.

It is not useful to search for another such solution. We can fathom P1 based on this bounding argument and mark P1 as inactive.

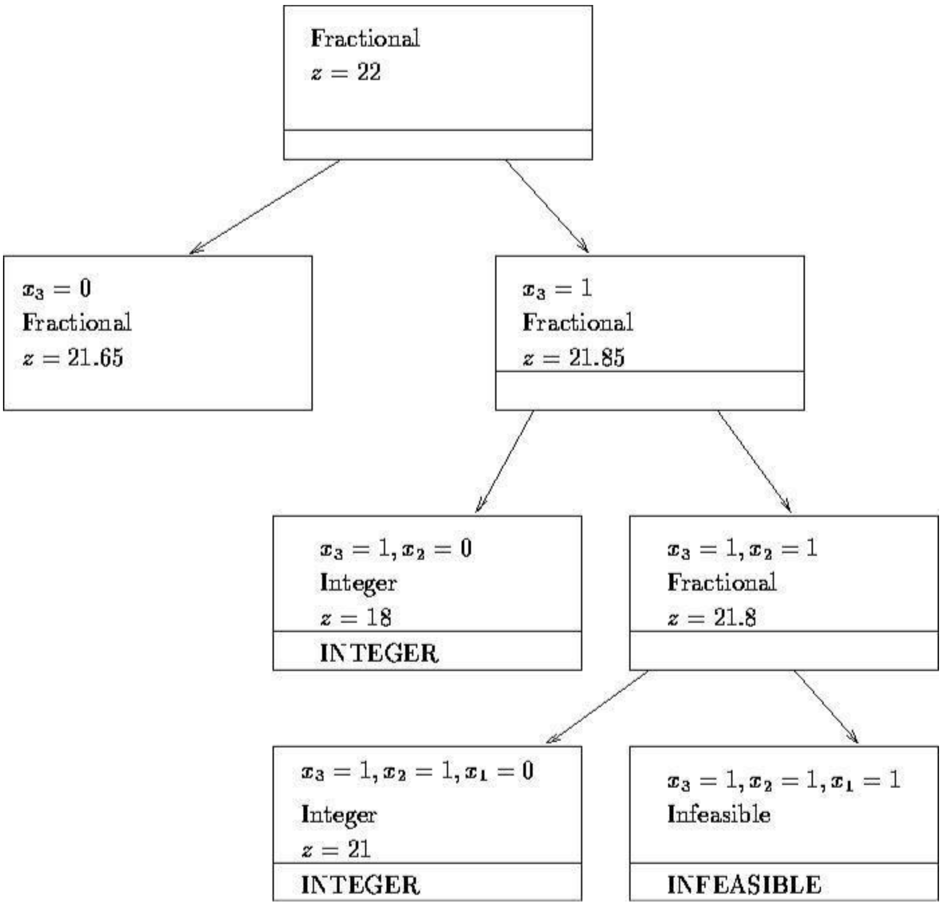
Optimal sol'n and Report

Thus, the optimal solution is

z=21, x_1=0, x_2=1, x_3=1, x_4=1

Items 2, 3, and 4 should be put in the knapsack.

In this case, the total value would be 21.



B&B for Solving Combinatorial Optimization Problems

A ***combinatorial (discrete) optimization problem*** is any optimization problem that has a finite number of feasible solutions.

A B&B approach is often an efficient way to solve them.

Examples of combinatorial optimization problems:

- Ten jobs must be processed on a single machine. It is known how long it takes to complete each job and the time at which each job must be completed (the job's due date). What ordering of the jobs minimizes the total delay of the 10 jobs?
- A salesperson must visit each of the 10 cities before returning to her/his home. What ordering of the cities minimizes the total distance the salesperson must travel before returning home? (TSP).

In each of these problems, many possible solutions must be considered.

Example 6: Machine Scheduling

Please refer to Winston 9.6. p. 528

TSP

Please recall that

We define x_{ij} as a 0-1 variable:

x_{ij} = 1 if TS goes from city *i* to city *j*;

x_{ij} = 0 otherwise

c_{ij} = distance form city *i* to city *j* (for $i \neq j$)

c_{ii} = M (a very large number relative to actual distances)

An itinerary that begins and ends at the same city and visits each city once is called a tour.

It seems reasonable that we might be able to find the answer to TSP by solving an assignment problem having a cost matrix whose *ij*th is c_{ij}.

If the optimal solution to the assignment problem yields a tour, it is the optimal solution to the TSP.

Unfortunately, the optimal solution to the assignment problem need not be a tour (may yield subtours).

If we could exclude all feasible solutions that contain subtours and then solve the assignment problem, we would obtain the optimal solution to TSP Not easy to do...

Several B&B approaches have been developed for solving TSPs.

One approach start with solving the preceding assignment problem (sub-problem 1). Because this sub-problem contains no provisions to prevent subtours, it is a relaxation of the original TSP.

Thus, if the optimal solution to the subp.1 is feasible for the TSP (no subtours), then it is also optimal for the TSP.

If it is infeasible (contain subtours), we branch on the subp.1 in a way that will prevent one of subp.1's subtours from recurring in solutions to subsequent sub-problems.

Example 7: TSP

(*Winston 9.6., p. 530*)

Joe State lives in Gary, Indiana and owns insurance agencies in Gary, Fort Wayne, Evansville, Terre Haute, and South Bend.

Each December, he visits each of his insurance agencies.

The distance between each agency:

miles	G	FW	E	TH	SB
G	0	132	217	164	58
FW	132	0	290	201	79
E	217	290	0	113	303
TH	164	201	113	0	196
SB	58	79	303	196	0

What order of visiting his agencies will minimize the total distance traveled? **Answer**

We first solve the assignment problem (subp.1) applying the Hungarian method to the cost matrix shown:

COSTS	G	FW	E	TH	SB
G	1000	132	217	164	58
FW	132	1000	290	201	79
E	217	290	1000	113	303
TH	164	201	113	1000	196
SB	58	79	303	196	1000

The optimal solution will be:

$x_{15}=x_{21}=x_{34}=x_{43}=x_{52}=1$, $z=495$

The optimal solution to subp.1 contains two subtours:

- recommends going from Gary (1) to South Bend (5), then to Fort Wayne (2), and then back to Gary (1–5–2–1).
- also suggests that if Joe is in Evansville (3), he should go to Terre Haute (4) and then to Evansville (3–4–3).

Thus, the optimal solution can not be the optimal solution to Joe's problem.

We arbitrarily choose to exclude the subtour 3-4-3.

Observe that the optimal solution to Joe's problem must have either x_{34}=0 or x_{43}=0.

Thus, we can branch on subp.1 by creating two new sub-problems.

Subp.2: Subp.1 + (x_{34}=0, or c_{34}=M)

Subp.3: Subp.1 + (x_{43}=0, or c_{43}=M)

Now arbitrarily choose subp.2 to solve.

COSTS	G	FW	E	TH	SB
G	1000	132	217	164	58
FW	132	1000	290	201	79
E	217	290	1000	1000	303
TH	164	201	113	1000	196
SB	58	79	303	196	1000

The optimal solution will be:

$x_{14}=x_{25}=x_{31}=x_{43}=x_{52}=1$, $z=652$

This solution includes the subtours 1–4–3–1 and 2–5–2.

Thus, it can not be the optimal solution to Joe's problem.

Following the LIFO approach, now branch sub-problem 2 in an effort to exclude the subtour 2-5-2. Thus we add two additional sub-problems.

Subp.4: Subp.2 + (x_{25}=0, or c_{25}=M)

Subp.5: Subp.2 + (x_{52}=0, or c_{52}=M)

By using the Hungarian method on subp.4, we obtain the optimal solution

$x_{15}=x_{24}=x_{31}=x_{43}=x_{52}=1$, $z=668$

This solution contains no subtours and yields the tour 1–5–2–4–3–1

It is a candidate solution and any node that ***cannot*** yield a z-value < 668 may be eliminated from consideration.

We next solve subp.5.

$x_{14}=x_{43}=x_{32}=x_{25}=x_{51}=1$, $z=704$

This solution also yields a tour 1–4–3–2–5–1

But z=704 is not as good as the subp.4 candidate's z=668

Thus this subp.5 may be eliminated from consideration.

Only subp.3 remains.

The optimal solution

$x_{13}=x_{25}=x_{34}=x_{41}=x_{52}=1$, $z=652$.

This solution includes the subtours 1–3–4–1 and 2–5–2.

However, it is still possible for this sub-problem to yield a solution with no subtours that beats z=668.

Next branch on sub-problem 3 creating new sub-problems.

Subp.6: Subp.3 + (x_{25}=0, or c_{25}=M)

Subp.7: Subp.3 + (x_{52}=0, or c_{52}=M)

Both of these sub-problems have a z-value that is larger than 668.

Optimal sol'n and Report

Subp.4 thus yields the optimal solution:

$x_{15}=x_{24}=x_{31}=x_{43}=x_{52}=1$, $z=668$

Joe should travel from Gary (1) to South Bend (5), from South Bend to Fort Wayne (2), from Fort Wayne to Terre Haute (4), from Terre Haute to Evansville (3), and then back to Gary.

He will travel a total distance of 668 miles.

Heuristics for TSPs

An IP formulation can be used to solve a TSP but can become unwieldy and inefficient for large TSPs.

When using B&B methods to solve TSPs with many cities, large amounts of computer time is needed.

Heuristic methods, or heuristics, can be used to quickly lead to a good (but not necessarily optimal) solution.

Two types of heuristic methods can be used to solve TSP:

1. The Nearest-Neighbor
2. The Cheapest-Insertion

The Nearest-Neighbor Heuristic

1. Begin at any city and then "visit" the nearest city.
2. Then go to the unvisited city closest to the city we have most recently visited.
3. Continue in this fashion until a tour is obtained.
4. After applying this procedure beginning at each city, take the best tour found.

Example 8. Applying the NNH to TSP

We arbitrarily choose to begin at city 1.

Of the cities 2, 3, 4, and 5, city 5 is the closest city to city 1 Generate the arc 1–5 Of the cities 2, 3, and 4, city 2 is the closest city to city 5 1–5–2

Of the cities 3 and 4, city 4 is the closest city to city 2 1–5–2–4

Joe must next visit city 3 and then return to city 1 1–5–2–4–3–1 (668 miles).

In this case, the NNH yields the optimal tour.

If we had begun at city 3, however, NNH yields the tour 3–4–1–5–2–3 (704 miles).

Thus, the NNH need not yield an optimal tour.

This procedure should be applied beginning at each city, and then the best tour found should be taken as solution.

The Cheapest-Insertion Heuristic

1. Begin at any city and find its closest neighbor.
2. Then create a subtour joining those two cities.
3. Next, replace an arc in the subtour (say, arc (*i, j*)) by the combinations of two arcs (*i*, *k*) and (*k*, *j*), where *k* is not in the current subtour that will increase the length of the subtour by the smallest (or cheapest) amount.
4. Continue with this procedure until a tour is obtained.
5. After applying this procedure beginning with each city, we take the best tour found.

Example 9. Applying the CIH to TSP

We arbitrarily choose to begin at city 1.

Of the cities 2, 3, 4, and 5, city 5 is the closest city to city 1 Generate the arc 1–5

We create a subtour (1, 5)–(5, 1)

We could replace arc (1, 5) by (1, 2)–(2, 5), (1, 3)–(3, 5), or (1, 4)–(4, 5)

We could also replace (5, 1) by (5, 2)–(2, 1), (5, 3)–(3, 1), or (5, 4)–(4, 1)

The computations used to determine which arc of (1, 5)–(5, 1) should be replaced are given in the Table:

Arc replaced	Arcs added	Added length
(1, 5)*	(1, 2)–(2, 5)	*c* 12+*c* 25–*c*
(1, 5)	(1, 3)–(3, 5)	*c* 13+*c3* 5–*c*
(1, 5)	(1, 4)–(4, 5)	*c* 14+*c* 45–*c*
(5, 1)*	(5, 2)–(2, 1)	*c* 52+*c* 21–*c*
(5, 1)	(5, 3)–(3, 1)	*c* 53+*c3* 1–*c*
(5, 1)	(5, 4)–(4, 1)	*c* 54+*c4* 1–*c*

* indicates the correct replacement: either (1, 5) or (5, 1)

We arbitrarily choose to replace arc (1, 5) by arcs (1, 2) and (2, 5) New subtour: (1, 2)–(2, 5)–(5, 1)

We then determine which arc should be replaced

Arc replaced	Arcs added	Added length
(1, 2)	(1, 3)–(3, 2)	375
(1, 2)*	(1, 4)–(4, 2)	233
(2, 5)	(2, 3)–(3, 5)	514
(2, 5)	(2, 4)–(4, 5)	318
(5, 1)	(5, 3)–(3, 1)	462
(5, 1)	(5, 4)–(4, 1)	302

We now replace arc (1, 2) by arcs (1, 4) and (4, 2) New subtour: (1, 4)–(4, 2)–(2, 5)–(5, 1)

Which arc should be replaced?

Arc replaced	Arcs added	Added length
(1, 4)*	(1, 3)–(3, 4)	166
(4, 2)	(4, 3)–(3, 2)	202
(2, 5)	(2, 3)–(3, 5)	514
(5, 1)	(5, 3)–(3, 1)	462

We now replace arc (1, 4) by arcs (1, 3) and (3, 4)

This yields the tour (1, 3)–(3, 4)–(4, 2)–(2, 5)–(5, 1)

In this case, the CIH yields the optimal tour.

But, in general, the CIH does not necessarily do so.

This procedure should be applied beginning at each city, and then the best tour found should be taken as solution.

Evaluation of Heuristics

- Performance guarantees
  Gives a worse-case bound on how far away from optimality a tour constructed by the heuristic can be
- Probabilistic analysis

A heuristic is evaluated by assuming that the location of cities follows some known probability distribution

- Empirical analysis

  Heuristics are compared to the optimal solution for a number of problems for which the optimal tour is known

UNIT - 3
REPLACEMENT

3.1. Introduction

The study of replacement is concerned with the situations that arise when some items such as equipment need replacement due to changes in their performance. This change may either be gradual or all of a sudden.

Broadly speaking, the requirement of a replacement may be in any of the following situations:

(i) An item fails and does not work at all or the item is expected to fail shortly.

(ii) An item deteriorates and need expensive maintenance.

(iii) A better design of the equipment is available.

(iv) It is economical to replace equipment in anticipation of costly failure.

In this chapter, we are interested in the first two situations. Third situation has been dealt when we studied the pay-off criteria.

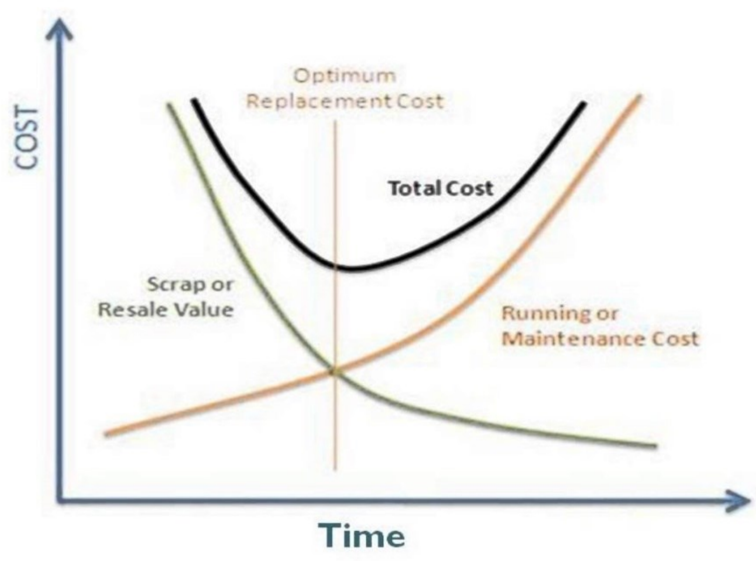
When studying the problem of replacement, we may or may not consider the time value of money.

3.2. Replacement of equipment that deteriorates gradually

Generally, the cost of maintenance and repairing of certain equipment's increases with time and ultimately the cost may become so high that it is more economical to replace theses equipment's with new ones. If the productivity of equipment decreases with time, this may also be considered as a failure. At this point a replacement is justified.

The costs associated with aging increase at an increasing rate whereas the resale value of the equipment decreases at increasing rate. The decreasing resale value results in increasing depreciation, which is the difference between the purchase price and the resale value. The depreciation of the item increases at a decreasing rate.

The optimal replacement policy for such items is to replace the equipment at a point where the total cost curve intersects the total depreciation curve



3.3. Time value of money does not change

If the value of money does not change with time, then the user of the equipment does not need to pay interest on his investments. We wish to determine the optimal time to replace the equipment.

We make use of the following notations:

C = Capital cost of the equipment
S = Scrap value of the equipment
n = Number of years that the equipment would be in use
C_m = Maintenance cost function.
ATC = Average total annual cost.

Two possibilities are there

(i) Time *t* is a continuous random variable

In this case the deterioration of the equipment is being monitored continuously. The total cost of the equipment during n years of use is given by

TC = Capital cost - Scrap value + Maintenance cost

$$= C - S + \int_0^n C_m(t)\,dt$$

$$\therefore A(n) = \frac{1}{n}TC = \frac{C-S}{n} + \frac{1}{n}\int_0^n C_m(t)\,dt$$

For minimum cost, $\frac{d}{dn}A(n) = 0$

$$\therefore -\frac{C-S}{n^2} - \frac{1}{n^2}\int_0^n C_m(t)dt + \frac{1}{n}C_m(n) = 0$$

$$\therefore C_m = \frac{C-S}{n^2} + \frac{1}{n^2}\int_0^n C_m(n)\,dt = A(n)$$

And $\frac{d^2A(n)}{dn^2} \geq 0$ at $C_m(n) = A(n)$

i.e., when the maintenance cost becomes equal to the average annual cost, the decision should be to replace the equipment.

(ii) Time *t* is a discrete random variable

In this case

$$\mathrm{A(n)} = \frac{1}{n}TC = \frac{C-S}{n} + \frac{1}{n}\sum_0^n C_m$$

A (*n*) is minimum when

$$\mathrm{A(n+1) \geq A(n) and\ A(n-1) \geq A(n)}$$

$$\text{Or, } \mathrm{A(n+1) - A(n)} \geq 0 \textit{ and } \mathrm{A(n) - A(n-1)} \leq 0$$

$$\mathrm{A(n+1)} - \mathrm{A(n)}\frac{1}{n+1}\left(C - S + \sum_{0}^{n} C_m(t)\right) + \frac{1}{n+1}C_m(n+1) - A(n)$$

$$\frac{n}{n+1}A(n) + \frac{1}{n+1}C_m(n+1) - A(n) \geq 0$$

$$\therefore A(n+1) \geq A(n)$$

Similarly

$$A(n) - A(n-1) \leq 0$$

$$\therefore C_m(n) \leq C_m(n-1)$$

Thus the optimal policy is

Replace the equipment at the end of n years if the maintenance cost in the $(n+1)^{th}$ year is more than the average total cost in the n^{th} year and the n^{th} year's maintenance cost is less than previous year's average total cost.

3.4. Present Worth Factor (Pwf):

The value of money over a period of time depends upon the nominal interest rate "r" The value of one rupee today would be equal to Rs.1 (1+r %) after one year. Or the present value of a rupee to be spent after one year is equal to Rs.1 (1 + r %)-1 at the interest rate r % per year. Similarly, the present value of a rupee to be spent after -n years is equal to (1+r)-n , and is called as -Present Worth Factor (PWF) or -Present Value Interest Factor, PVIF (r%, n) at the rate of r % per -n years. Sometimes, this is also known as -Compound Amount Factor (CAF) of one rupee spent in "n" year duration.

3.5. Class Examples

Example-1

The cost of equipment is Rs. 62,000 and its scrap value is Rs. 2,000. The life of the equipment is 8 years. The maintenance costs for each year are as given below:

Year	1	2	3	4	5	6	7	8
Maintenance Cost in Rs.	1000	2000	3500	5000	8000	11000	16000	24000

When the equipment should be replaced?

Ans:-

C = 62,000/-

Year **n**	Resale Price **S**	Maintenance Cost **C_m**	Cumulative Maintenance Cost **$\sum C_m$**	Total Cost **TC=C-S+$\sum C_m$**	Annual Total Cost **ATC = $\frac{TC}{n}$**
1	2000	1000	1000	61000	61000
2	2000	2000	3000	63000	31500
3	2000	3500	6500	65000	21666.6
4	2000	5000	11500	71500	17875
5	2000	8000	19500	79500	15900
6	**2000**	**11000**	**30500**	**90500**	**15083.3**
7	2000	16000	46500	106500	15214.2
8	24000				

$\Rightarrow$ As the avg. yearly cost is minimum for 6^{th} year the equipment should be replace after 6 year.

Example-2

A manufacturer finds from his past records that the costs per year associated with a machine with a purchase price of Rs. 50,000 are as given below:

Year	1	2	3	4	5	6	7	8
Maintenance Cost in Rs.	15000	16000	18000	21000	25000	29000	34000	40000
Scrap value In Rs.	35000	25000	17000	12000	10000	5000	4000	4000

Ans:-

C = 50,000/-

Year **n**	Resale Price **S**	Maintenance Cost C_m	Cumulative Maintenance Cost $\sum C_m$	Total Cost $TC=C-S+\sum C_m$	Annual Total Cost $ATC = \frac{TC}{n}$
1	35000	15000	15000	30000	30000
2	25000	16000	31000	58000	28000
3	17000	18000	49000	82000	27333.3
4	12000	21000	70000	108000	27000
5	**10000**	**25000**	**95000**	**135000**	**27000**
6	5000	29000	124000	169000	28166.6
7	4000	34000			
8	4000	40000			

⇒ As the avg. yearly cost is minimum for 5th year the equipment should be replace after 5 year.

Example-3

(a) Machine A cost Rs. 36,000. Annual operating costs are Rs. 800 for the first year, and then increase by Rs. 8000 every year. Determine the best age at which to replace the machine. If the optimum replacement policy if followed, what will be the yearly cost of owning and operating the machine?
(b) Machine B costs Rs. 40,000. Annual operating costs Rs. 1,600 for the first year, and then increase by Rs. 3,200 every year. You now have a machine of type A which is one year old. Should you replace it with B, if so when? Assume that both machines have no resale value.

Ans:-

(a) Machine A

C = 36,000/-

Year **n**	Resale Price **S**	Maintenance Cost C_m	Cumulative Maintenance Cost $\sum C_m$	Total Cost $TC=C-S+\sum C_m$	Annual Total Cost $ATC=\frac{TC}{n}$
1	0	800	800	36800	36800
2	0	8800	9600	45600	22800
3	**0**	**16800**	**26400**	**62400**	**20800**
4	0	24800	51200	87200	21800

(b) Machine B

C = 40,000/-

Year **n**	Resale Price **S**	Maintenance Cost C_m	Cumulative Maintenance Cost $\sum C_m$	Total Cost $TC=C-S+\sum C_m$	Annual Total Cost $ATC=\frac{TC}{n}$
1	0	1600	1600	41600	41600
2	0	4800	6400	46400	23200
3	0	8000	14400	54400	18133.3
4	0	11230	25600	65600	16400
5	**0**	**14400**	**40000**	**80000**	**16000**
6	0	17600	57600	97600	16266.6

⇒ As the avg. yearly cost is minimum for 3rd year for machine A, machine A should be replace after 3 year.

⇒ Avg. yearly cost for operating & owing the machine A is Rs. 20,800.

⇒ The avg. cost per year of operating & owing the machine B is less that of machine A. Machine A should be replaced with machine B.

N^{th} year	Cost of N^{th} year (Rs.)
2	45600-36800=8800
3	62400-45600=16800

⇒ As the cost of using machine A in 3rd year is more than avg. yearly cost of operating & owing the machine.

⇒ Machine A should be replaced machine B after 2 years. i.e. 1 year from now because of machine A is already 1 year old.

Example-4

The data on the operating costs per year and resale price of equipment A whose purchase price is Rs. 10,000 are given below:

Year	1	2	3	4	5	6	7
Maintenance Cost in Rs.	1500	1900	2300	2900	3600	4500	5500
Resale value In Rs.	5000	2500	1250	600	400	400	400

(i) What is the optimum period of replacement?

(ii) When equipment A is 2 year old equipment B which is a new model for the same usage is available. The optimum period for the replacement is 4 years with an average cost of Rs. 3,600. Should we change equipment A with that of B? If so when?

Ans:-

C = 10,000/-

Year n	Resale Price S	Maintenance Cost C_m	Cumulative Maintenance Cost $\sum C_m$	Total Cost $TC=C-S+\sum C_m$	Annual Total Cost $ATC = \frac{TC}{n}$
1	5000	1500	1500	6500	6500
2	2500	1900	3400	10900	5450
3	1250	2300	5700	14450	4816.6
4	600	2900	8600	18000	4500
5	**400**	**3600**	**12200**	**21800**	**4360**
6	400	4500	16700	26300	4383.3
7	400	5500			

⇒ As the avg. yearly cost is minimum for 5th year for machine A, machine A should be replace after 5 year.

⇒ The avg. cost per year of operating & owing the machine B is less that of machine A. Machine A should be replaced with machine B.

Nth year	Cost of Nth year (Rs.)
3	14450-10900=3550
4	18000-14450=3550
5	21800-18000=3800

⇒Machine A should be replaced machine B after 4 years. i.e. 2 year from now because of machine A is already 2 year old.

Example-5

A firm pays Rs. 10,000 for its equipment. Their operating and maintenance costs are about Rs. 2500 per year for the first two years and then go up by approximately Rs. 1,500 per year. When such equipment replaced? The discount rate is 10% per year.

Ans:-

C = 10,000/- i = 0.10

$$d = \frac{1}{1+i} = \frac{1}{1+0.1} = 0.909$$

Year n	C_m	Discount Factor d^{n-1}	Discounted Maintenance Cost $C_m * d^{n-1}$	Discounted Cumulative Maintenance Cost $\sum C_m * d^{n-1}$	TC=C-S+$\sum C_m * d^{n-1}$	$\sum d^{n-1}$	ATC = $\frac{TC}{\sum d^{n-1}}$
1	2500	1	2500	2500	12500	1	12500
2	2500	0.909	2272.5	4772.5	14772.5	1.909	7738.3
3	4000	0.826	3304	8076.5	18076.5	2.735	6609.3
4	**5500**	**0.751**	**4130.5**	**12207**	**22207**	**3.486**	**6370.3**
5	7000	0.683	4781	16988	26988	4.169	6473.4

⇒ As the avg. yearly cost is minimum for 4th year the equipment should be replace after 4 year.

Example-6

A manufacturer is offered two machines A and B. A is priced at Rs. 10,000 and running costs are estimated as Rs. 1,600 for each of the first five years, increasing by Rs. 400 per year in the sixth and subsequent years. Machine B which has a same capacity as A cost Rs. 5,000 but will have a running coats of 2,400 per year for six year, increasing by Rs. 400 per year thereafter.

If money is worth 10% per year which machine should be purchased? (Assume that the machine will eventually be sold for scrap at a negligible price)

Ans:-

Machine A

C = 10,000/- i = 0.10

$$d = \frac{1}{1+i} = \frac{1}{1+0.1} = 0.909$$

Year n	C_m	d^{n-1}	$C_m * d^{n-1}$	$\sum C_m * d^{n-1}$	TC=C-S+$\sum C_m * d^{n-1}$	$\sum d^{n-1}$	ATC = $\frac{TC}{\sum d^{n-1}}$
1	1600	1	1600	1600	11600	1	11600
2	1600	0.909	1454.4	3054.4	13054.4	1.909	6858.3
3	1600	0.826	1321.6	4376	14376	2.735	5256.3
4	1600	0.751	1201.6	5577.6	15577.6	3.486	4468.6
5	1600	0.683	1092.8	6670.4	16670.4	4.169	3999.6
6	2000	0.620	1241.2	7911.6	17911.6	4.789	3740.1
7	2400	0.564	1353.6	9265.2	19265.2	5.353	3598.9
8	2800	0.512	1435.8	10701	20701	5.866	3529.1
9	**3200**	**0.466**	**1491.2**	**12192.2**	**22192.2**	**6.332**	**3504.8**
10	3600	0.427	1525.3	13717.6	23717.6	6.755	3510.8

⇒ The avg. value of machine A is Rs. 3504.8/-

Machine B

Year n	C_m	d^{n-1}	$C_m * d^{n-1}$	$\sum C_m * d^{n-1}$	$TC = C - S + \sum C_m * d^{n-1}$	$\sum d^{n-1}$	$ATC = \frac{TC}{\sum d^{n-1}}$
1	2400	1	2400	2400	7400	1	7400
2	2400	0.909	2181.6	4581.6	9581.6	1.909	5019.1
3	2400	0.826	1982.4	6564	11564	2.735	4228.1
4	2400	0.751	1802.4	8366.4	13366.4	3.486	3834.3
5	2400	0.683	1639.2	10005.6	15005.6	4.169	3599.3
6	2400	0.620	1489.4	11495	16495	4.789	3444.3
7	2800	0.564	1579.2	13074.2	18074.2	5.353	3376.4
8	**3200**	**0.512**	**1640.9**	**14715.2**	**19715.2**	**5.866**	**3361.1**
9	3600	0.466	1677.6	16392.8	21392.8	6.332	3378.6
10	4000	0.427	1694.8	18087.6	23087.6	6.755	

⇒ The avg. value of machine B is less than machine A. So we have to purchase machine B.

Example-7

A machine which requires an initial investment of Rs. 12,000 has its salvage value at the end of the year as Rs. [7000-500(i-1)]. The operating and maintenance costs are given below:

Year	1	2	3	4	5	6	7	8	9
Maintenance Cost in Rs.	1100	1300	1700	2100	2300	2700	3100	3500	3900

Determine optimal replacement year when money increased by 12% ever year.

Ans:-

C = 12,000/- i = 0.12

$$d = \frac{1}{1+i} = \frac{1}{1+0.12} = 0.8928$$

n	d^{n-1}	S	$S^* d^{n-1}$	C_m	$C_m^* d^{n-1}$	$\sum C_m^* d^{n-1}$	$TC = C - S^* d^{n-1} + \sum C_m^* d^{n-1}$	$\sum d^{n-1}$	$ATC = \frac{TC}{\sum d^{n-1}}$
1	1	7000	6249.6	1100	1100	1100	6850	1	6850
2	0.892	6500	4796.8	1300	1160.6	2260.6	9074	1.892	4796
3	0.797	6000	4218	1700	1355	3615.7	11342.2	2.689	4218
4	0.711	5500	4002.7	2100	1494.3	5110	13610.8	3.401	4002
5	0.635	5000	3897.9	2300	1461.4	6571.4	15728.2	4.036	3897
6	**0.567**	**4500**	**3871.3**	**2700**	**1531.5**	**8103**	**17822**	**4.604**	**3871**
7	0.506	4000	3889.3	3100	1569.8	9672.8	19861.1	5.107	3889
8				3500					
9				3900					

⇒ As the avg. yearly cost is minimum for 6^{th} year the equipment should be replace after 6 year.

Example-8

The following mortality rates have been observation for certain type of light bulbs

Month	1	2	3	4	5
Percent failing by month end	10	25	50	80	100

There are 1000 bulbs in use and it costs Rs 10 to replace an individual bulb which has burnt out. If all bulbs were replaced simultaneously, it would cost Rs 2.5 per bulbs. It is proposed to replace all the bulbs at fixed interval, and individually those which fail between the intervals. What would be the best policy to adopt?

Ans:-

Month i	**Cumulative % failure up to the end of month**	**% failure during the month**	**Probability** P_i **that a new bulb shall fail during the month**
1	10	10	0.10
2	25	15	0.15
3	50	25	0.25
4	80	30	0.30
5	100	20	0.20

Month i	Bulbs failing during i^{th} month	Bulbs replaced until i^{th} month	Cost of Individual Replacement TCI	Cost of Group Replacement TCG	Total Cost TC=TCI+TCG	Average Cost per month ATC = $\frac{TC}{n}$
1	100	100	1000	2500	3500	3500
2	**160**	**260**	**2600**	**2500**	**5100**	**2550**
3	281	541	5410	2500	7910	2636.6
4	377	918	9180	2500	11680	2920
5	349	1267	12670	2500	15170	3034

$$N_0 = 1000$$

$$N_1 = N_0 \times P_1$$

$$= \frac{10}{100} \times 1000$$

$$= 100$$

$$N_2 = N_0 \times P_2 + N_1 \times P_1$$
$$= \frac{15}{100} \times 1000 + \frac{10}{100} \times 100$$
$$= 160$$
$$N_3 = N_0 \times P_3 + N_1 \times P_2 + N_2 \times P_3$$
$$= \frac{25}{100} \times 1000 + \frac{15}{100} \times 100 + \frac{10}{100} \times 160$$
$$= 281$$
$$N_4 = N_0 \times P_4 + N_1 \times P_3 + N_2 \times P_2 + N_3 \times P_1$$
$$= \frac{30}{100} \times 1000 + \frac{25}{100} \times 100 + \frac{15}{100} \times 160 + \frac{10}{100} \times 281$$
$$= 377$$
$$N_5 = N_0 \times P_5 + N_1 \times P_4 + N_2 \times P_3 + N_3 \times P_2 + N_4 \times P_1$$
$$= \frac{20}{100} \times 1000 + \frac{30}{100} \times 100 + \frac{25}{100} \times 160 + \frac{15}{100} \times 281 + \frac{10}{100} \times 377$$
$$= 349$$

Avg life = $\sum i \times P_i$

$$= 1(P_1) + 2(P_2) + 3(P_3) + 4(P_4) + 5(P_5)$$
$$= 1(0.1) + 2(0.15) + 3(0.25) + 4(0.3) + 5(0.2)$$
$$= 3.35 \text{ months}$$

No. of bulbs replaced per months = $\frac{1000}{3.35}$

$= 298$ bulbs

Cost of individual replacement = 298×10

=2980 Rs.

⇒ As cost of group replacement after every 2^{nd} month is less than cost of individual replacement.

⇒ Group replacement policy after every 2 months is better.

3.6. Lab Examples

Example -1

The maintenance cost and resale price of machine M whose purchase price is Rs. 12,000 are given as:

Year	1	2	3	4	5	6	7
Maintenance Cost in Rs.	2600	3000	3400	4000	4700	5600	6600
Resale value In Rs.	7000	4500	3250	2600	2400	2400	2400

(a) Suggest the optimal period for the replacement of the machine.

(b) When this machine is two year old, another machine N, which is a new model of machine M, is available. The optimal period for replacement of this machine N is 4 year, with an average cost of Rs. 4700. Should we change machine M with N? If so, when?

Ans:-

C =

Year **n**	Resale Price **S**	Maintenance Cost C_m	Cumulative Maintenance Cost **∑ Cm**	Total Cost **TC=C-S+∑ Cm**	Annual Total Cost $ATC = \frac{TC}{n}$

Example -2

For a machine, the following data are available

Year	0	1	2	3	4	5	6
Cost of spare in Rs.	-	200	400	700	1000	1400	1600
Salary of maintain staff In Rs.	-	1200	1200	1400	1600	2000	2600
Loss due to breakdown In Rs.	-	600	800	700	1000	1200	1600
Resale value In Rs.	12000	6000	3000	1500	800	400	400

Determine the optimum period for the replacement of the above machine.

Ans:-

C =

Year **n**	Resale Price **S**	(Spare+ Maintenance+ Breakdown) Cost **Cm**	Cumulative Maintenance Cost **$\sum$ Cm**	Total Cost **TC=C-S+$\sum$ Cm**	Annual Total Cost **ATC = $\frac{TC}{n}$**

Example -3

A piece of equipment costs Rs. 7,500 initially and requires Rs. 400 to be spent on its maintenance in the first year. The maintenance cost would increases by Rs. 500 per year in each of the subsequent years. Determine the optimal replacement for the machine when (i) future costs are not discounted, and (ii) future costs are discounted at the rate of 10% p.a.

Ans:- C =

(i) Without Discounted

Year **n**	Maintenance Cost C_m	Cumulative Maintenance Cost $\sum C_m$	Total Cost $TC = C - S + \sum C_m$	Annual Total Cost $ATC = \frac{TC}{n}$

(ii) Discounted at the rate of 10%, i = 0.1

$$d = \frac{1}{1+i} =$$

Year **n**	C_m	Discount Factor d^{n-1}	Discounted Maintenance $C_m * d^{n-1}$	Discounted Cumulative Maintenance Cost $\sum C_m * d^{n-1}$	Total Cost $TC = C - S + \sum C_m * d^{n-1}$	Cumulative Discount Factor $\sum d^{n-1}$	Annual Total Cost $ATC = \frac{TC}{\sum d^{n-1}}$

Example -4

A manufacturer has to decide between two machines M_1 and M_2, about which pertinent information is given below:

	M_1	M_2
Cost	Rs. 5000	Rs. 2500
Maintenance cost	Rs. 800 p.a. for years 1,2,…,5, increasing by Rs. 200 every year thereafter	Rs. 1,200 p.a. for years 1,2,…,6, increasing by Rs. 200 every year thereafter
Scrap value	Nil	Nil
Cost of capital	(To be used as discounted rate) 10% p.a.	

Determine optimal replacement period of M_1 and M_2. Which of the two is a better alternative?

Ans:-

$$d = \frac{1}{1+i} =$$

(i) Machine M_1,

C =

Year **n**	**C_m**	Discount Factor **d^{n-1}**	Discounted Maintenance **$C_m * d^{n-1}$**	Discounted Cumulative Maintenance Cost **$\sum C_m * d^{n-1}$**	Total Cost **TC=C-S+$\sum C_m * d^{n-1}$**	Cumulative Discount Factor **$\sum d^{n-1}$**	Annual Total Cost **ATC = $\frac{TC}{\sum d^{n-1}}$**

(ii) Machine M_2,

C =

Year **n**	C_m	Discount Factor d^{n-1}	Discounted Maintenance $C_m * d^{n-1}$	Discounted Cumulative Maintenance Cost $\sum C_m * d^{n-1}$	Total Cost $TC = C - S + \sum C_m * d^{n-1}$	Cumulative Discount Factor $\sum d^{n-1}$	Annual Total Cost $ATC = \frac{TC}{\sum d^{n-1}}$

Example -5

Good lite Company has installed 2,000 electric bulbs of a certain brand. The company follows the policy of replacing the bulbs as and when they fail. Each replacement cost Rs. 2. The probability distribution of the life of the bulbs is as given here:

Life of bulb (weeks)	1	2	3	4	5
% of bulb	0.10	0.30	0.45	0.10	0.05

Determine the cost/ week of the replacement policy in long run.

Ans:-

Example -6

A large computer installation contains 2000 components of identical nature which are subject to failure as per probability distribution that follows:

Month End	1	2	3	4	5
% Failure to date	10	25	50	80	100

Component which fail have to be replaced for efficient function of the system. If they are replaced as and when failure occurs, the cost of replacement per unit is Rs. 45. Alternatively, if all components are replaced in one lot at periodic intervals and individually replace only such failure occur between group replacements the cost of component replaced is Rs. 15.
Assess which policy of replacement would be economical.

Ans:-

Month i	**Cumulative % failure up to the end of month**	**% failure during the month**	**Probability** P_i **that a new component shall fail during the month**

Month i	Component failing during i^{th} month	Components replaced until i^{th} month	Cost of Individual Replacement TCI	Cost of Group Replacement TCG	Total Cost TC= TCI+TCG	Average Cost per month ATC = $\frac{TC}{n}$

UNIT - 4
INVENTORY

4.1. Introduction

Inventory refers to stock of goods, commodities, or other economic resources that are stored or reserved at any given period for future production or for meeting future demand.

Inventory planning is the determination of the type and quantity of inventory items that would be required at future points for maintaining production schedules. Inventory planning is generally based on information from the past and also on factors that would arise in future.

Inventory management is the function of directing the movement of goods through the entire manufacturing cycle from the requisition of raw materials to the inventory/stock of finished goods in such a manner as to meet the objectives of maximum customer service with minimum investment and efficiency.

In inventory control is primarily concerned with the inventory cost control. The objectives of inventory control are: -

1. To minimize financial investments in inventories.
2. To ensure availability of the supply of stock at all time.
3. To allow flexibility in production scheduling.
4. To ensure that the value of the material consumed is minimum.
5. Reduce surplus stock.

4.2. Types of inventories

1. Direct inventories – these include items which play a direct role in the manufacturing process and become an integral part of the finished goods, e.g. raw materials, work in progress inventories, finished goods inventories, spare parts.
2. Indirect inventories – include those items necessary for manufacturing but do not become an integral component of the finished product e.g.
   a. Lubricants
   b. Machinery/equipment
   c. Labour

Inventory decisions

1. How much of an item to order?
2. When to replenish the inventory of the item?

4.3. Factors affecting inventory

1. **Inventory or stock cost:-**

There are several:

i) Purchase/Production cost – cost of purchasing a unit of item

ii) Ordering/Acquisition/Set-up cost – costs related to acquisition of purchased items i.e. those of getting an item to a firm's store e.g. transport, loading and off-loading, inspection.

iii) Inventory carrying/ holding costs – costs associated with holding a given level of inventory e.g. warehousing, spoilage, security, pilferage, administrative, insurance, depreciation.

iv) Stock-out cost/ shortage costs – incurred due to a delay in meeting demand or inability to meet demand at all because of shortage of stock loss of future sales, cost associated with future replenishment.

2. **Order cycle** – the time period between placements of 2 successive orders.

3. **Lead time** – time between placing an order and actual replenishment of item. Also referred to as procurement time.

4. **Time horizon** – this is the period over which the inventory level will be controlled.

5. **Maximum stock** – the level beyond which stocks should not be allowed to rise.

6. **Minimum stock level/buffer stock/safety stock** – level below which stock should not be allowed to fall. It is the additional stock needed to allow for delay in delivery or for any higher than expected demand that may arise due to lead time.

7. **Reorder level** – point at which purchased order must be sent to supplier for the supply of more stock. The level of stock at which further replenishment order should be placed.

8. **Reorder quantity** – the quantity of the replacement order.

ROP (Reorder Point) = Daily Demand X Lead Time

ROP = D/T x T_L

Note that Demand is on daily basis

9. **Average stock level**

$$Average\,stock\ level = \frac{Minimum\ stock\ level + Maximum\ stock\ level}{2}$$

10. **Physical stock** – no. of items physically in stock at any given time.

11. **Stock replenishment** – rate at which items are added to the inventory.

12. **Free stock** – the physical stock plus the outstanding replenishment orders minus the unfulfilled requirements.

13. **Economic order quantity (EOQ)** – the quantity at which the cost of having stocks is minimum.

14. **Economic batch quantity (EBQ)** – quantity of stock within the enterprise. Company orders from within its own warehouses unlike in EOQ where it is ordered from elsewhere.

15. **Demand:-**

- Customer's demand, size of demand, rate of demand and pattern of demand is important
- Size of demand = no. of items demanded per period
- Can be deterministic (Static or dynamic) or probabilistic (governed by discrete or continuous probability distribution)
- The rate of demand can be variable or constant
- Pattern reflects items drawn from inventory -instantaneous (at beginning or end) or gradually at uniform rate

4.4. Inventory Costs

There are four major elements of inventory costs that should be taken for analysis, such as

(1) Item cost, Rs. C/item.
(2) Ordering cost, Rs. Co/order.
(3) Holding cost Rs. Ch/item/unit time.
(4) Shortage cost Rs. Cs/item/Unit time.

(1) Item Cost (C)

This is the cost of the item whether it is manufactured or purchased. If it is manufactured, it includes such items as direct material and labor, indirect materials and labor and overhead expenses. When the item is purchased, the item cost is the purchase price of 1 unit. Let it be denoted by Rs. C per item.

(2) Purchasing or Setup or Acquisition or Ordering Cost (C_o)

Administrative and clerical costs are involved in processing a purchase order, expediting, follow up etc., It includes transportation costs also. When a unit is manufactured, the unit set up cost includes the cost of labor and materials used in the set up and set up testing and training costs. This is denoted by Rs. C_o per set up or per order.

(3) Inventory holding cost (C_h)

If the item is held in stock, the cost involved is the item carrying or holding cost. Some of the costs included in the unit holding cost are

(1) Taxes on inventories,
(2) Insurance costs for inflammable and explosive items,
(3) Obsolescence,
(4) Deterioration of quality, theft, spillage and damage to times,
(5) Cost of maintaining inventory records.

This cost is denoted by Rs. C_h/item/unit time. The unit of time may be days, months, weeks or years.

(4) Shortage Cost (C_s)

The shortage cost is due to the delay in satisfying demand (due to wrong planning); but the demand is eventually satisfied after a period of time. Shortage cost is not considered as the opportunity cost or cost of lost sales. The unit shortage cost includes such items as,

(1) Overtime requirements due to shortage,
(2) Clerical and administrative expenses.
(3) Cost of expediting.
(4) Loss of goodwill of customers due to delay.
(5) Special handling or packaging costs.
(6) Lost production time.
This cost is denoted by Rs. C_s per item per unit time of shortage.

4.5. THE BASIC DETERMINISTIC INVENTORY MODELS

1. EOQ Model with Uniform Demand
2. EOQ Model with Different rates of Demands in different cycles
3. EOQ Model with Shortages (backorders) allowed
4. EOQ Model with Uniform Replenishment

Notations used:-

Q = number of units per order

Q^* = economic order quantity or optimal no. of units per order to minimize total cost

D = annual demand requirement (units per year)

C = cost of 1 unit of item

C_0 = ordering (preparation or set-up) cost of each order

$C_h = C_c$ = holding or carrying cost per unit per period of time

T = length of time between two successive orders

N = no. of orders or manufacturing runs per year

TC = Total Inventory cost

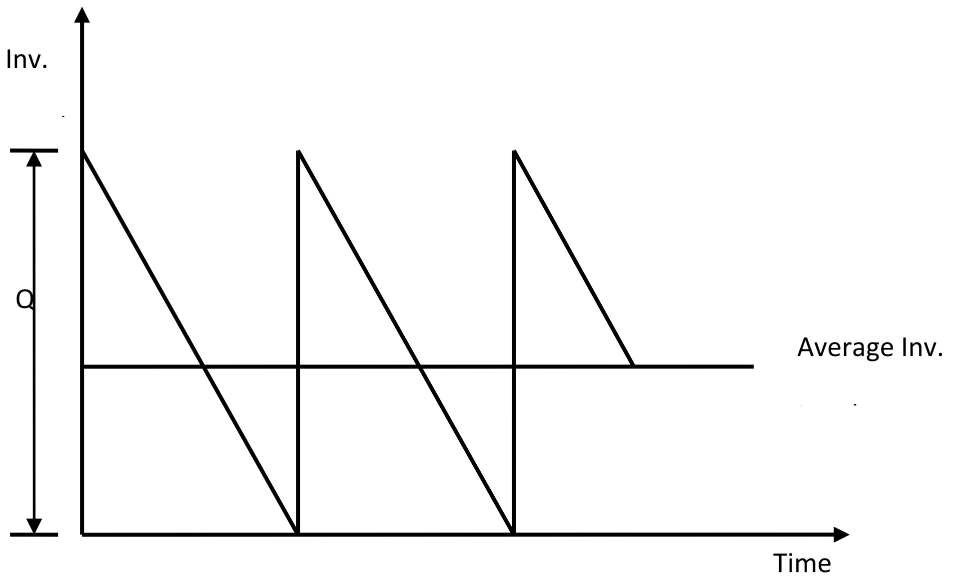
The optimal order quantity (EOQ) is at a point where the ordering cost = holding cost

Model 1- EOQ Model with Uniform Demand

Policy: Whenever the inventory level is 0, order Q items

Objective: Choose a Q that will minimize total Inventory Cost

The behavior of inventory at hand with respect to time is illustrated below:



This is the ordering quantity which minimizes the balance of cost between inventory holding cost and ordering costs.

It is based on the following assumptions:

1. A known constant stock holding cost.
2. A known constant ordering cost.
3. The rate of demand is known (is deterministic).
4. A known constant price per unit.
5. Inventory replenishment is done instantaneously.
6. No stock-out is allowed.
7. Quantity discounts are not allowed – purchase price is constant.
8. Lead time is known and fixed.

1. **Annual ordering cost**

$$Annual\ ordering\ \cos t = (no.\ of\ orders\ placed\ per\ year) \times (ordering\ \cos t\ per\ order)$$

$$= \left(\frac{Annual\ Demand}{no.\ of\ units\ in\ each\ order}\right) \times (order\ \cos t\ per\ order)$$

$$= \frac{D}{Q} \times C_o \cdots\cdots\cdots\cdots\cdots(1)$$

2. **Annual holing (or carrying) cost**

$$Annual\ ordering\ cost = (Average\ inventory\ level) \times (carrying\ cost\ per\ order)$$

$$= \frac{Q}{2} \times C_h \cdots\cdots\cdots\cdots\cdots (2)$$

3. **Equating (1) and (2) above**

Since the minimum TC occurs at the point where the ordering cost and the inventory carrying costs are equal, we equate the 2 equations above.

$$\frac{D}{Q} \times C_o = \frac{Q}{2} \times C_h$$

Solve for Q

$$2DC_o = Q^2 C_h$$

$$Q^2 = \frac{2DC_o}{C_h}$$

$$Q^* = \sqrt{\frac{2DC_o}{C_h}}$$

$$EOQ = \sqrt{\frac{2DC_o}{C_h}}$$

Note:

1. Inventory holding or carrying costs are often expressed as annual percentage(s) of the unit cost or price.
   C_o or C_h as % of unit cost or price

   I = annual inventory carrying charge (cost) as 1% of price

   C_h = IC where C is the unit price of inventory item

$$EOQ = Q^* = \sqrt{\frac{2DC_o}{C_h}}$$

2. Total cost is sum of annual C_h and annual ordering cost.

$$TC = \frac{D}{Q}.C_o + \frac{Q}{2}.C_h$$

Put value of Q* in TC,

$$TVC = \sqrt{2DC_oC_h}$$

Example -1

A supplier is required to deliver 20000 tons of raw materials in one year to a large manufacturing organization. The supplier maintains his go-down to store the material received from various resources. He finds that cost of inventory holding is 30 paisa per ton per month. His cost for ordering the material is Rs. 400. One of the conditions of the supplier contract from the manufacturing organization is that the contract will be terminated in the event of supply not being maintained as a schedule. Determine (1) in what lot size is the supplier should produce the material for minimum total associated cost of inventory? (2) At what time interval should he procure the material? It may be assume that replacement of inventory is instantaneous.

Ans:-

Given Date
D = 20000 tons
T = 12 months
C_h = Rs. 0.30 per tons per months
C_o = Rs. 400

(1) Economic order quantity

$$EOQ = \sqrt{\frac{2DC_o}{C_h}}$$

$$= \sqrt{\frac{2(12000)(400)}{0.30 \times 12}}$$

$$Q^* = 2108\, tons$$

(2) Tine interval

$$t_{co} = \frac{Q^*}{D}$$

$$= \frac{2108}{20000}$$

$$= 1.26 \text{ month}$$

Example -2

In the above example, if there is (i) 10 per cent increase in holding cost or (ii) 10 percent increase in ordering cost, in each case determine the optimal lot size and corresponding minimum total expected cost of inventory. Comment the result.

Ans:-

(i) $C_h' = 1.1\ C_h$

$$EOQ = \sqrt{\frac{2DC_o}{C_h'}}$$

$$= \sqrt{\frac{2(12000)(400)}{1.1 \times 0.30 \times 12}}$$

$$Q^* = 2010\ tons$$

$$TAC = \sqrt{2C_h' C_o D}$$

$$= \sqrt{2 \times 1.1 \times 0.30 \times 400 \times 12000}$$

$$= 7960 \text{ Rs.}$$

(ii) $C_o' = 1.1\ C_o$

$$EOQ = \sqrt{\frac{2DC_o'}{C_h}}$$

$$= \sqrt{\frac{2(12000)(400)(1.1)}{0.30 \times 12}}$$

$$Q^* = 2211\ tons$$

$$TAC = \sqrt{2C_h C_o' D}$$

$$= \sqrt{2 \times 0.30 \times 1.1 \times 400 \times 12000}$$

$$= 7960 \text{ Rs.}$$

Example -3

A certain item costs Rs. 250 per ton. The monthly requirement is 5 tons and each time the stock is replenished, there is an order cost of Rs. 120. The cost of carrying inventory has been estimated at 10% of the value of the stock per year. What is the optimal order quantity? If lead time is 3 months, determine the re order point. At what intervals the order should be placed?

Ans:-

Given Data

C = Rs. 250 per ton

C_o = Rs. 120

$C_h = 250 \times 0.1$ = Rs. 25 per ton per year

$D = 5 \times 12 = 60$ tons

$T_L = 3$ months

$$EOQ = \sqrt{\frac{2DC_o}{C_h}}$$

$$= \sqrt{\frac{2(60)(120)}{25}}$$

$$Q^* = 24\ tons$$

$$t_{co} = \frac{Q^*}{D}$$

$$= \frac{24}{60}$$

$$= 0.4 \text{ year or } 4.8 \text{ months}$$

$$Q_R = \frac{D}{T} \times T_L$$

$$= \frac{60}{12} \times 3$$

$$= 15 \text{ tons}$$

Example 4:

A manufacturer has to supply his customers with 1200 units of his product per annum. The inventory carrying cost amounts to ₹ 1.2 per unit. The set-up cost per run is ₹ 160. Find:

i) EOQ
ii) Minimum average yearly cost
iii) Optimum no of orders per year
iv) The optimum time between orders (optimum period of supply per optimum order)

Ans:-

i) Economic order quantity

$$EOQ = \sqrt{\frac{2DC_o}{C_h}}$$

$$= \sqrt{\frac{2(1200)(160)}{1.2}}$$

$$= 565.69\ or\ 566\ units$$

ii) Minimum average yearly cost

$$TC = \frac{D}{Q}.C_o + \frac{Q}{2}.C_h$$

$$TC(Q^*) = \frac{DC_o}{Q^*} + \frac{Q^* C_h}{2}$$

$$= \frac{1200(160)}{566} + \frac{566(1.2)}{2}$$

$$= 339.22 + 339.6$$

$$= \text{Rs } 678.82 \text{ } or \text{ } Rs \text{ } 679$$

iii) Optimum no. of orders per year (N^*)

$$N^* = \frac{Demand}{EOQ}$$

$$= \frac{1200}{566}$$

$$= 2.1 \text{ } orders \Rightarrow 3 \text{ } orders$$

iv) Optimum time between orders

$$T^* = \frac{no. of \text{ } working \text{ } days \text{ } in \text{ } a \text{ } year}{N^*}$$

$$= \frac{365}{3}$$

$$= 122$$

Example 5:

The annual demand per item is 6400 units. The unit cost is ₹ 12 and the inventory carrying charges 25% per annum. If the cost of procurement is ₹ 300 determine:

i) EOQ
ii) No. of orders per year
iii) Time between 2 consecutive orders
iv) Optimum cost

Ans:-

i) EOQ

$$EOQ = \sqrt{\frac{2DC_o}{C_h}}$$

$$= \sqrt{\frac{2(6400)(300)}{(0.25)(12)}}$$

$$= 1131 \text{ } units$$

ii) N^*

$$N^* = \frac{Demand}{EOQ}$$
$$= \frac{6400}{1131}$$
$$= 5.65\ orders \Rightarrow 6\ orders$$

iii) Time between 2 consecutive orders

$$T^* = \frac{no.\ of\ working\ days\ in\ a\ year}{N^*}$$
$$= \frac{365}{5.65}$$
$$= 64.60$$

OR

$$T^* = \frac{EOQ}{Demand} \times 12\ months$$
$$= \frac{1131}{6400} \times 12\ months$$
$$= 2\ months\ 4\ days$$

iv) Optimal cost

$$optimal\ \text{cost} = (unit\ \text{cost} \times demand) + \sqrt{2DC_oC_h}$$
$$= (12 \times 6400) + \sqrt{2(6400)(300)(0.25 \times 12)}$$
$$= 80194.11$$

Model 2- EOQ Model with Different Rates of Demand

⇒ Assumptions of this model are same as those of model 1 except

⇒ Demand rate is different in different cycles. The total demand D is specified as demand during time horizon T

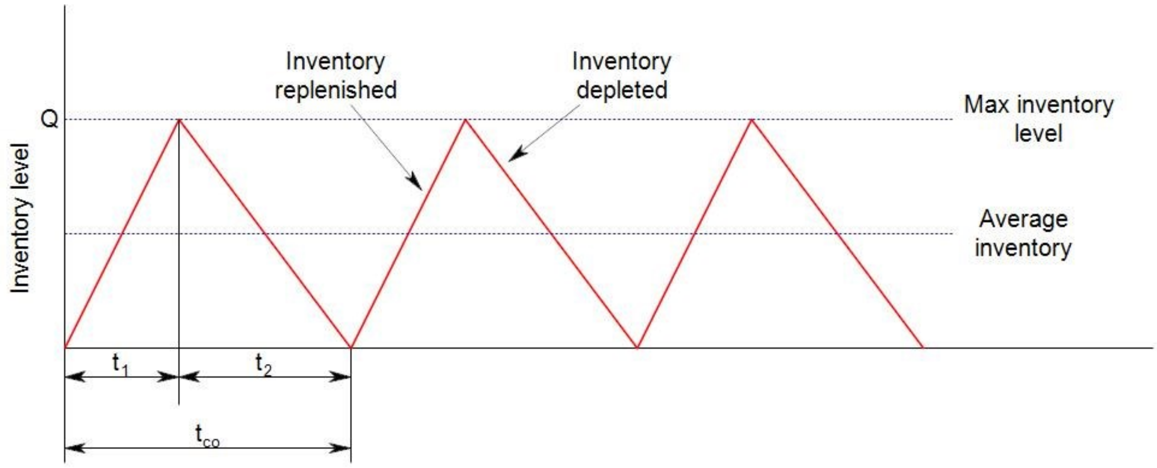
$$Holding\ Cost = \frac{Q}{2} \times \left(\frac{d-r}{d}\right)$$

$$Order / set\ \text{up cost} = \frac{D \times C_o}{Q}$$

$$EOQ = Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{d}{d-r}}$$

$$TVC = \sqrt{2rC_oC_h(\frac{d-r}{d})}$$

$$t_{co} = \frac{Q^*}{r}$$



Example -6

A manufacturing company needs 4000 units of material every month. The delivery system from the supplier is so scheduled that once delivery commences the materials is received at the rate of 6000 units per month. The cost of processing purchase order is Rs. 600 and the inventory carrying cost is 30 paisa per unit per month. Determine the optimal lot size and interval at which the order is to be placed. What is maximum inventory during a cycle?

Ans:-

Given Data
C_o = Rs. 600
C_h = Rs. 0.30 per unit per month
d = 6000 units per months
r = 4000 units per months

Optimal lot size

$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{d}{d-r}}$$

$$= \sqrt{\frac{2 \times 4000 \times 600}{0.30}} \times \sqrt{\frac{6000}{6000-4000}}$$

$$= 6928 \text{ units}$$

Interval time

$$t_{co} = \frac{Q^*}{r}$$

$$= \frac{6928}{4000}$$

$$= 1.732 \text{ months}$$

Maximum inventory Q_{max}

$$Q_{max} = Q^* - rt_1$$

$$\text{Where, } t_1 = \frac{Q^*}{d}$$

$$= \frac{6928}{6000}$$

$$= 1.154 \text{ months}$$

$$Q_{max} = 6928 - 4000 \times 1.154$$

$$= 2309.33 \text{ units}$$

Example -7

The demand for a certain item is 150 units per week. No shortages are to be permitted. Holding cost is 5 paisa per unit per week. Demand can be met either by manufacturing or purchasing. With each source the data are as follows:

	Manufacture	Purchase
Item cost Rs./ Unit	10.50	12
Set up/ Ordering cost Rs. / Order or set up	90	20
Replenishment rate units / week	260	Infinite
Lead time in weeks	4	10

Determine (a) the minimum cost procurement source and its economic advantage over its alternative resource, (b) E.O.Q. or E.B.Q. as per the source selected, (c) the minimum procurement level (Re-order point)

Ans:-

For manufacture

Given Data

C_o = Rs. 90

C_h = Rs. 0.05 per unit per month

d = 260 units per weeks

r = 150 units per weeks

T_L = 4 week
C = Rs. 10.50 per unit

$$TVC = \sqrt{2rC_oC_h(\frac{d-r}{d})} + C \times r$$

$$= \sqrt{2 \times 150 \times 90 \times 0.05 \times (\frac{260-150}{260})} + 10.50 \times 150$$

$$= 1598.9 \text{ Rs. per week}$$

For purchase

Given Data
C_o = Rs. 20
C_h = Rs. 0.05 per unit per month
r = 150 units per weeks
T_L = 4 week
C = Rs. 12 per unit

$$TVC = \sqrt{2rC_oC_h} + C \times r$$

$$= \sqrt{2 \times 150 \times 20 \times 0.05} + 12 \times 150$$

$$= 1817.32 \text{ Rs. per week}$$

(a) Minimum TC is 1598.9 per week for manufacture

(b) E.B.Q

$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{d}{d-r}}$$

$$= \sqrt{\frac{2 \times 90 \times 150}{0.05}} \times \sqrt{\frac{260}{260-150}}$$

$$= 1129.76 \text{ units}$$

(c) Re-order point

$$Q_R = r \times T_L$$

$$= 150 \times 4$$

$$= 600 \text{ units}$$

Model 3- EOQ Model with Shortages (backorders) allowed

⇒ Assumptions of this model are same as those of model 1 except Shortages is allowed.

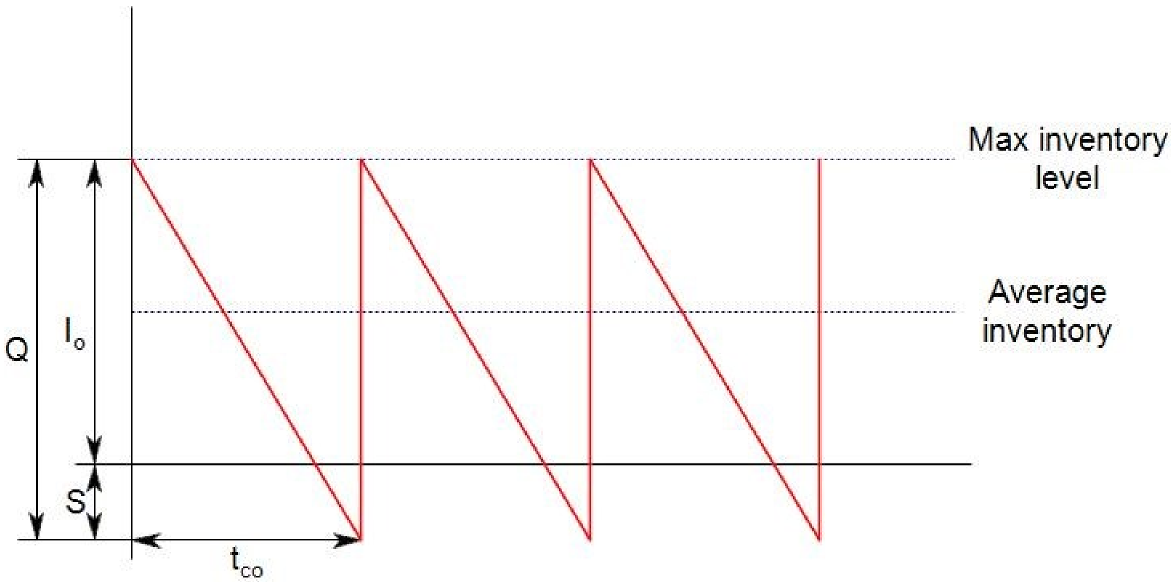
$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{C_h + C_s}{C_s}}$$

$$TVC = \sqrt{2rC_oC_h(\frac{C_s}{C_h + C_s})}$$

$$t_{co} = \frac{Q^*}{r}$$

Initial stock I_o

$$I_o = Q^* \times (\frac{C_s}{C_h + C_s})$$



Example -8

A tractor manufacturing company has entered in to a contract with M/s Auto Diesel for delivering 30 engines per day. M/s Auto Diesel has committed that for every day's delay in delivery; there will be penalty of delayed supply at the rate of Rs. 100 per engine per day. M/s Auto Diesel has the inventory holding cost of Rs. 600 per engine per month. Assume replenishment of engines as instantaneous and ordering cost as Rs. 15000. What should be initial inventory level and what should be ordering quantity for minimum associated cost of inventory? At what interval procurement should be made?

Ans:-

Given Data

C_o = Rs. 15000

C_h = Rs. 600 per engine per month

C_s = 100*30 =3000 per engine per month

r = 30*30 = 900 engine per month

$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{C_h + C_s}{C_s}}$$

$$= \sqrt{\frac{2 \times 900 \times 15000}{600}} \times \sqrt{\frac{600 + 3000}{3000}}$$

$$= 233 \text{ engines}$$

$$I_o = Q^* \times (\frac{C_s}{C_h + C_s})$$

$$= 233 \times (\frac{3000}{600 + 3000})$$

$$= 195 \text{ engines}$$

$$t_{co} = \frac{Q^*}{r}$$

$$= \frac{233}{30}$$

$$= 7.76 \text{ days}$$

Example -9

In above example, find out optimum order quantity if shortage is not permitted. Compare this with the value of obtained in above example and comment on the result.

Ans:-

$$Q^* = \sqrt{\frac{2rC_o}{C_h}}$$

$$= \sqrt{\frac{2 \times 900 \times 15000}{600}}$$

$$= 212 \text{ engines}$$

$\Rightarrow$ If shortage is not permitted, EOQ is reducing 233 engines per order to 212 engines per order.

Model 4- EQQ Model with Uniform Replenishment

⇒ Assumptions of this model are same as those of model 1 except Demand is variable and Shortages is allowed.

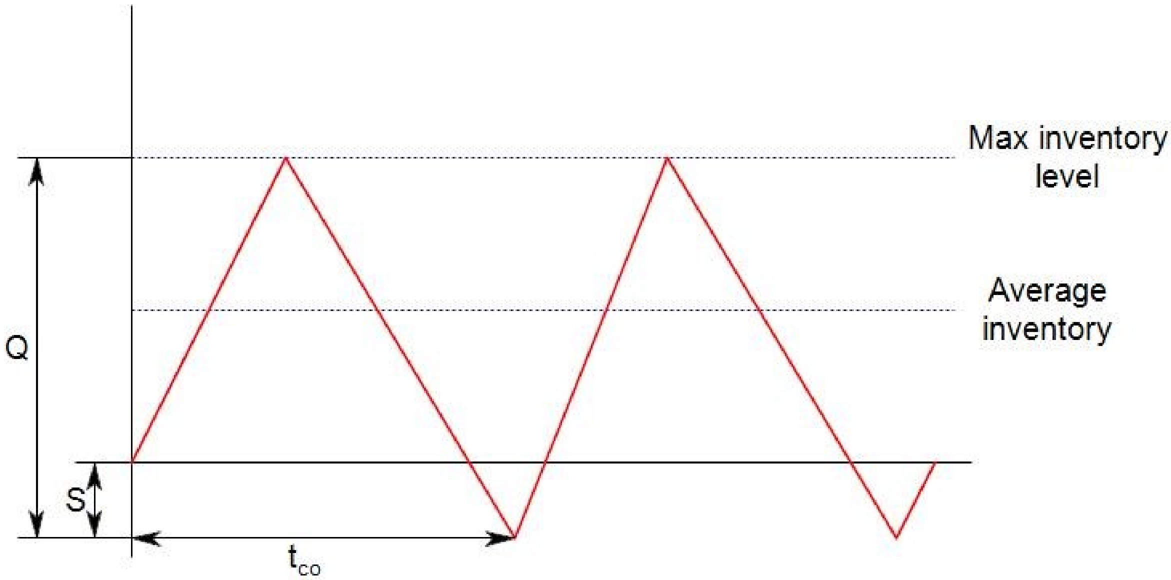
$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{d}{d-r}} \times \sqrt{\frac{C_h + C_s}{C_s}}$$

$$TVC = \sqrt{2rC_oC_h(\frac{d-r}{d})(\frac{C_s}{C_h + C_s})}$$

$$t_{co} = \frac{Q^*}{r}$$

Initial stock I_o

$$I_o = Q^* \times (\frac{C_s}{C_h + C_s}) \times (\frac{d-r}{d})$$



Example -10

The demand for an item in a company is 18000 units per year and the company can produce the item at a rate of 3000 units per month. The set up cost is Rs. 500 per set up and the annual inventory holding cost is estimated at 20 percent of the investment in average inventory. The cost of one unit short is Rs. 20 per year. Determine, (i) Optimal production batch quantity, (ii) Optimum cycle time and production time, (iii) Maximum inventory level in the cycle, (iv) Maximum shortage permitted and (v) Total associated cost per year. The cost of the items is Rs. 20 per unit.

Ans:-

Given Data

C_o = Rs. 500

C_h = 0.20*20 = Rs. 4 per unit per year
C_s = Rs. 20 per unit per year
r = 18000 unit per year
d = 3000*12 = 36000 unit per year

$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{d}{d-r}} \times \sqrt{\frac{C_h + C_s}{C_s}}$$

$$= \sqrt{\frac{2 \times 18000 \times 500}{4}} \times \sqrt{\frac{36000}{36000 - 18000}} \times \sqrt{\frac{4+20}{20}}$$

$$= 3286 \text{ units}$$

$$t_{co} = \frac{Q^*}{r}$$

$$= \frac{3286}{18000}$$

$$= 0.18255 \text{ year}$$

= 2.19 months

$$t_{po} = \frac{Q^*}{d}$$

$$= \frac{3286}{36000}$$

$$= 0.091277 \text{ year}$$

= 1.09 months

$$I_o = Q^* \times (\frac{C_s}{C_h + C_s}) \times (\frac{d-r}{d})$$

$$= 3286 \times (\frac{20}{4+20}) \times (\frac{36000 - 18000}{36000})$$

$$= 1369 \text{ units}$$

$$S = Q^* \times (\frac{C_h}{C_h + C_s}) \times (\frac{d-r}{d})$$

$$= 3286 \times (\frac{4}{4+20}) \times (\frac{36000 - 18000}{36000})$$

$$= 273.88 \text{ units}$$

$$TC = \sqrt{2rC_oC_h(\frac{d-r}{d})(\frac{C_s}{C_h+C_s})} + C \times r$$

$$TC = \sqrt{2 \times 18000 \times 500 \times 4 \times (\frac{36000-18000}{36000})(\frac{20}{4+20})} + 20 \times 18000$$

$$= 365477.22$$

Model 5- EOQ Model with Quantity Discounts

Quantity discounts occur in numerous situations where suppliers provide an incentive for large order quantities by offering a lower purchase cost when items are ordered in larger lots or quantities. In this section we show how the EOQ model can be used when quantity discounts are available.

EOQ without discounts

$$Q^* = \sqrt{\frac{2rC_o}{C_h}}$$

- EOQ with discounts

$$Q_o^* = \frac{(C_d \times r + Q^* \times i \times C)}{i(C - C_d)}$$

- Max Net Saving with discounts

$$X_{max} = \frac{(C_d \times r + Q^* \times i \times C)^2}{2 \times i \times r(C - C_d)} - C_o$$

Example -11

A wholesale dealer in bearings purchases 30000 bearings annually at intervals and order size suitable to him. The price is Rs. 150 per bearing. The manufacturing company offers the dealer a discount of Rs. 7 per bearing for the order size larger than earlier. The reorder cost is Rs. 40 and the inventory carrying cost amounts to 20 percent of the investment in purchase price. Decide the optimum order size for special discount offer purchase and the maximum benefit he can derive from this order.

Ans:-

Given Data

C_o = Rs. 40

C_h = 0.20*150 = Rs. 30 per unit per year

r = 30000 unit per year

C = Rs 150 per unit

C_d = Rs 7 per unit

i = 0.20

$$Q^* = \sqrt{\frac{2rC_o}{C_h}}$$

$$= \sqrt{\frac{2 \times 30000 \times 40}{30}}$$

$$= 283 \text{ units}$$

$$Q_o^* = \frac{(C_d \times \text{r} + \text{Q}^* \times \text{i} \times \text{C})}{i(\text{C} - \text{C}_d)}$$

$$= \frac{(7 \times 30000 + 283 \times 0.2 \times 150)}{0.2(150 - 7)}$$

$$= 7640 \text{ units}$$

$$X_{\max} = \frac{(\text{C}_d \times \text{r} + \text{Q}^* \times \text{i} \times \text{C})^2}{2 \times i \times r(\text{C} - \text{C}_d)} - C_o$$

$$= \frac{(7 \times 30000 + 283 \times 0.2 \times 150)^2}{2 \times 0.2 \times 30000(150 - 7)} - 40$$

$$= 27779$$

Model 6- Probabilistic Inventory Models

The inventory models that we have discussed thus far have been based on the assumption that the demand rate is constant and deterministic throughout the year. We developed minimum-cost order quantity and reorder-point policies based on this assumption. In situations where the demand rate is not deterministic, models have been developed that treat demand as probability distribution. In this section we consider a single-period inventory model with probability demand.

The single-period inventory model refers to inventory situations in which one order is placed for the product; at the end of the period, the product has either sold out, or there is a surplus of unsold items that will be sold for a salvage value. The single-period inventory model is applicable in situations involving seasonal or perishable items that cannot be carried in inventory and sold in future periods. Seasonal clothing (such as bathing suits and winter coats) is typically handled in a single-period manner. In these situations, a buyer places one preseason order for each item and then experiences a stock out or hold a clearance sale on the surplus stock at the end of the season. No items are carried in inventory and sold the following year. Newspapers are another example of a product that is ordered one time and is either sold or not sold during the single period. While newspapers are ordered daily, they cannot be carried in inventory and sold in later periods. Thus, newspaper orders may be treated as a sequence of single-period models; that is, each day or period is separate, and a single-period inventory decision must be made each period (day). Since we order only once for the period, the only inventory decision we must make is how much of the product to order at the start of the period. Because newspaper sales are an excellent example of a single-period situation, the single-period inventory problem is sometimes referred to as the **newsboy problem.**

- Optimum stock level

$$P_r = \frac{C_s}{C_s + C_h}$$

Example -12

A large industrial campus has decided to have its diesel generator system for street lighting, security illumination and round the clock process systems. The generator needs a tailor made for each other control unit which cost Rs. 18000 per number when ordered with the total equipment of diesel generator. A decision needs to be taken whether additional numbers of this unit should be ordered along with equipment, and if so, how many units should be ordered? These control units, though tropicalized and considered quite reliable, are known to have failed from time to time and history of failures of similar equipment give the following probability of failure.

No. of units having failed and hence No. of spare Required	0	1	2	3	4	5	6
Probability	0.6	0.2	0.1	0.05	0.03	0.02	0

It is found that if the control unit fails, the entire generator system comes to a grinding halt. When control unit fails and a spare unit is not available it is estimated that the cost rush order procurement, including the associated cost of the downtime is Rs. 50000 per unit. Considering that any investment in inventory is the cost of inventory, decide how many spare units should be ordered along with the original order. Determine total associated cost for each no. of spare unit.

Ans:-

⇒ Let us consider the elementary approach as the population of demand varies only from 0 to 5.

⇒ I = 0,

$$TAC = C_s \times 1 \times P_1 + C_s \times 2 \times P_2 + C_s \times 3 \times P_3 + C_s \times 4 \times P_4 + C_s \times 5 \times P_5$$

$$= 50000(0.2 + 2 \times 0.1 + 3 \times 0.05 + 4 \times 0.03 + 5 \times 0.02)$$

$$= 38500$$

⇒ I = 1,

$$TAC = C_h \times 1 \times P_0 + C_s \times 0 \times P_1 + C_s \times 1 \times P_2 + C_s \times 2 \times P_3 + C_s \times 3 \times P_4 + C_s \times 4 \times P_5$$

$$= 18000 \times 0.6 + 50000 \times 0.1 + 50000 \times 2 \times 0.05 + 50000 \times 3 \times 0.03 + 50000 \times 4 \times 0.02$$

$$= 29300$$

$\Rightarrow$ I = 2,

$$TAC = C_h \times 2 \times P_0 + C_h \times 1 \times P_1 + C_s \times 0 \times P_2 + C_s \times 1 \times P_3 + C_s \times 2 \times P_4 + C_s \times 3 \times P_5$$

$$= 18000 \times 2 \times 0.6 + 18000 \times 0.2 + 50000 \times 1 \times 0.05 + 50000 \times 2 \times 0.03 + 50000 \times 3 \times 0.02$$

$$= 33700$$

$\Rightarrow$ I = 3,

$$TAC = C_h \times 3 \times P_0 + C_h \times 2 \times P_1 + C_h \times 1 \times P_2 + C_s \times 0 \times P_3 + C_s \times 1 \times P_4 + C_s \times 2 \times P_5$$

$$= 18000 \times 3 \times 0.6 + 18000 \times 2 \times 0.2 + 18000 \times 1 \times 0.05 + 50000 \times 1 \times 0.03 + 50000 \times 2 \times 0.02$$

$$= 44900$$

$\Rightarrow$ I = 4,

$$TAC = C_h \times 4 \times P_0 + C_h \times 3 \times P_1 + C_h \times 2 \times P_2 + C_h \times 1 \times P_3 + C_s \times 0 \times P_4 + C_s \times 1 \times P_5$$

$$= 18000 \times 4 \times 0.6 + 18000 \times 3 \times 0.2 + 18000 \times 2 \times 0.05 + 18000 \times 1 \times 0.03 + 50000 \times 1 \times 0.02$$

$$= 59500$$

$\Rightarrow$ I = 5,

$$TAC = C_h \times 5 \times P_0 + C_h \times 4 \times P_1 + C_h \times 3 \times P_2 + C_h \times 2 \times P_3 + C_h \times 1 \times P_4 + C_s \times 0 \times P_5$$

$$= 18000 \times 5 \times 0.6 + 18000 \times 4 \times 0.2 + 18000 \times 3 \times 0.05 + 18000 \times 2 \times 0.03 + 18000 \times 1 \times 0.02$$

$$= 76140$$

Cumulative Probability Table

No. of units	$\sum_{0}^{s} P_r$
0	0.6
1	**0.8**
2	0.9
3	0.95
4	0.98
5	1.00
6	1.00

$$P_r = \frac{C_s}{C_s + C_h}$$

$$= \frac{50000}{50000 + 18000}$$

$$= 0.7352$$

- ⇒ As I = 1, total associated cost is minimum, 1 spare units should be ordered along with the original order.
- ⇒ P_r is between 0.6 and 0.8

  $0.6 < 0.7352 < 0.8$
- ⇒ Optimum stock level is 1.

Example -13

In the above problem as regular purchase price of control unit is almost one third of the estimated rush order associated cost of one unit. The management decides to buy two spare units with the first order. Having decided that, the management would like to know for what range of actual values of shortage cost, the decision is justified.

Ans:-

⇒ For I = 2 to be optimum

$$P_{r1} < \frac{C_s}{C_s + C_h} < P_{r2}$$

$$\Rightarrow P_{r1} < \frac{C_s}{C_s + C_h}$$

$$0.8 < \frac{C_s}{C_s + 6000}$$

$$24000 < C_s$$

$$\Rightarrow \frac{C_s}{C_s + C_h} < P_{r2}$$

$$\frac{C_s}{C_s + 6000} < 0.9$$

$$C_s < 54000$$

⇒ Value of C_s is between 24000 and 54000

Example -14

Probabilistic demand of sweets in a large chain of sweet marts is rectangular between 1000 kg and 1400 kg. Profit per kg of fresh sweet sold is Rs. 14.70. If sweet is not sold fresh, next day it can be sold at a loss of Rs. 2.30 per kg. Determine the optimum stock to have fresh sweet on hand every day.

Ans:-

Given Data

C_o = Rs. 14.70 per kg

C_h = Rs 2.30 per kg
Range = 1400-1000 = 400

$$f(\mathrm{r}) = \frac{1}{Range} = \frac{1}{400}$$

$$\int_{1000}^{I_0} f(\mathrm{r})\,\mathrm{dr} = \frac{C_s}{C_s + C_h}$$

$$\int_{1000}^{I_0} \frac{1}{400}\,\mathrm{dr} = \frac{14.70}{14.70 + 2.30}$$

$$\frac{1}{400}(\mathrm{I}_o - 1000) = \frac{14.70}{17}$$

$$I_o = 1.346 \text{ kg}$$

Example -15

A newspaper boy buys daily papers from vendor and gets commission of 4 paisa for each paper sold. As he is always demanding large number in a lot, he has agreed to pay 3 paisa per each copy returned unsold. He has the past experience of the demand (its probability) as under.
23 (0.01), 24 (0.03), 25 (0.06), 26(0.10), 27(0.20), 28(0.25), 29(0.15), 30(0.10), 31(0.05), 32(0.05)
How many papers should he lift from vendor for minimum associated cost?

Ans:-

Given Data
C_o = 4 paisa per paper
C_h = 3 paisa per paper

Cumulative Probability Table

No. of units	$\sum_{0}^{s} P_r$
23	0.01
24	0.04
25	0.10
26	0.20
27	0.40
28	**0.65**
29	0.80
30	0.90
31	0.95
32	1.00

$$P_r = \frac{C_s}{C_s + C_h}$$
$$= \frac{4}{4+3}$$
$$= 0.571$$

⇒ P_r is between 0.6 and 0.8

$0.4 < 0.571 < 0.65$

⇒ Optimum stock level is 28 newspapers.

4.6. ABC analysis

ABC analysis is an inventory categorization method which consists in dividing items into three categories (A, B, C):

- A being the most valuable items,
- C being the least valuable ones.

This method aims to draw managers' attention on the critical few (A-items) not on the trivial many (C-items)

The ABC approach states that a company should rate items from A to C, basing its ratings on the following rules:

- **A-items** are goods which annual consumption value is the highest; the top 70-80% of the annual consumption value of the company typically accounts for only 10-20% of total inventory items.
- **B-items** are the interclass items, with a medium consumption value; those 15-25% of annual consumption value typically accounts for 30% of total inventory items.
- **C-items** are, on the contrary, items with the lowest consumption value; the lower 5% of the annual consumption value typically accounts for 50% of total inventory items.

	Percentage of items	**Percentage value of annual usage**	
Class A items	About 20%	About 80%	Close day to day control
Class B items	About 30%	About 15%	Regular review
Class C items	About 50%	About 5%	Infrequent review

4.7. Lab Example

Example -1

M/s T.V. assembly, one man T.V. assembler – entrepreneur, needs 10000 of tubes per year. The cost of one procurement is Rs. 80. The holding cost per tube is Rs. 3 per year. The rush purchase of tubes, if not in stock, amounts to equivalent shortage cost of Rs. 6 per tube per year. If stock ordered is delivered all instantaneously, determine how much he should order, at what interval and what will then be the total associated cost of inventory?

Ans:-

Given Data

C_o = Rs. 80

C_h = 3/12 per tube per month

C_s = 6/12 per tube per month

r = 10000/12 = 833.33 tube per month

$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{C_h + C_s}{C_s}}$$

$$= \sqrt{\frac{2 \times 80 \times 10000 \times 12}{12 \times 3}} \times \sqrt{\frac{6}{6+3}}$$

$$= 894.42 \text{ tubes}$$

$$t_{co} = \frac{Q^*}{r}$$

$$= \frac{894.42}{833.33}$$

$$= 1.07 \text{ months}$$

$$TVC = \sqrt{2rC_oC_h(\frac{C_s}{C_h + C_s})}$$

$$= \sqrt{2 \times 833.33 \times 80 \times \frac{3}{12}(\frac{6}{3+6})}$$

$$= 149 \text{ Rs. per month}$$

Example -2

In above example, M/s T.V. assembly seeks to reduce holding cost to Rs. 2.4 per tube per annum, and with patronized supplier manages to reduce procurement cost to Rs. 60 per order. What percentage reduction in penalty cost for shortage should negotiates that his total associated cost of inventory is reduced by 50 percent?

Ans:-

Given Data

C_{o1} = Rs. 60

C_{h1} = 2.4/12 per tube per month

TVC = 149/2 = 74.5 Rs.

$$TVC = \sqrt{2rC_{o1}C_{h1}(\frac{C_{s1}}{C_{h1}+C_{s1}})}$$

$$74.5 = \sqrt{2\times 833.33\times 60\times \frac{2.4}{12}(\frac{C_{s1}}{2.4+C_{s1}})}$$

$$5550.2 = 2\times 833.33\times 60\times \frac{2.4}{12}(\frac{C_{s1}}{2.4+C_{s1}})$$

$$0.2775 = \frac{C_{s1}}{2.4+C_{s1}}$$

$$0.666 = 0.7225C_{s1}$$

$$C_{s1} = 0.9217 \text{ Rs per tube per annum}$$

$$\text{Percentage reduction in penalty cost} = \frac{6-0.9217}{6}\times 100$$

$$= 84.63\%$$

Example -3

A manufacturer requires 15000 units of a part annually for an assembly operation. He can produce this part at the rate of 100 units per day. The set up cost for each production run is Rs. 50. To hold one unit of this part in inventory costs Rs. 5 per year. Shortage cost is Rs. 15 per unit per year. Cost of the part is Rs. 20 per unit. Assuming 250 working days per year, what will be the optimum manufacturing quantity? What will be the time between two production runs? What will be the total annual cost of the inventory system?

Ans:-

Given Data

C = 20 per unit

C_o = Rs. 50

C_h = Rs. 5 per unit per year

C_s = Rs. 15 per unit per year

r = 15000 unit per year
d = 100*250 = 25000 unit per year

$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{d}{d-r}} \times \sqrt{\frac{C_h + C_s}{C_s}}$$

$$= \sqrt{\frac{2 \times 15000 \times 50}{5}} \times \sqrt{\frac{25000}{25000 - 15000}} \times \sqrt{\frac{5+15}{15}}$$

$$= 1000 \text{ units}$$

$$t_{co} = \frac{Q^*}{r}$$

$$= \frac{1000}{15000} \times 250 \text{ working days}$$

$$= 17 \text{ working days}$$

$$TC = \sqrt{2rC_oC_h(\frac{d-r}{d})(\frac{C_s}{C_h + C_s})} + C \times r$$

$$TC = \sqrt{2 \times 15000 \times 50 \times 5 \times (\frac{25000 - 15000}{25000})(\frac{15}{5+15})} + 20 \times 15000$$

$$= 301500 \text{ per year}$$

Example -4

A large scale truck fleet operator has to supply truck at the rate of 30 every day. If on account of prolonged repairs and maintenance he is not able to supply the trucks, he has to incur the cost of short supply, loss of profit at the rate of Rs. 100 per day per truck. On the other hand, if he has road worthy trucks in excess of requirements he has to incur holding cost of Rs. 20 per day per truck. Every time he orders the lot of truck from repairs department, he incurs the cost of Rs. 40 per order. How many trucks should be ordered from repair department at a time, and at what interval should he put the order? What is the total associated cost of inventory then?

Ans:-

Given Data
C_o = Rs. 40
C_h = Rs. 20 per truck per day
C_s = 100 per truck per day
r = 30 truck per day

$$Q^* = \sqrt{\frac{2rC_o}{C_h}} \times \sqrt{\frac{C_h + C_s}{C_s}}$$

$$= \sqrt{\frac{2 \times 40 \times 30}{20}} \times \sqrt{\frac{20 + 100}{100}}$$

$$= 12 \text{ trucks}$$

$$t_{co} = \frac{Q^*}{r}$$

$$= \frac{12}{30}$$

$$= 0.40 \text{ days}$$

$$TVC = \sqrt{2rC_oC_h(\frac{C_s}{C_h + C_s})}$$

$$= \sqrt{2 \times 30 \times 40 \times 20(\frac{100}{20 + 100})}$$

$$= 200 \text{ Rs per day}$$

Example -5

Farm equipment manufacture undertakes to have a transshipment delivery of 40 trailers every day in a huge construction plant. His short supply results in the loss of Rs. 40 per unit per day. In case he has more trailers on hand then required he has to incur the cost of Rs. 5 p trailer per day. If he makes it a policy to receive the delivery at the fixed interval of 1 month, how much he should order and what should be his stock at the beginning of the month?

Ans:-

Given Data
C_h = Rs. 5 per unit per day
C_s = 40 per unit per day
r = 40 units per day
t_{co} = 30 days

$$I_o = Q^* \times (\frac{C_s}{C_h + C_s})$$

But,

$$t_{co} = \frac{Q^*}{r}$$

$$\Rightarrow Q^* = t_{co} \times r$$

$$= 40 \times 30$$

$$= 1200 \text{ units}$$

$$I_o = 1200 \times \left(\frac{40}{5+40}\right)$$

$$= 1067 \text{ units}$$

Example -6

A newspaper stall sells feature magazine at a sale commission of Rs. 2 per copy. The sale is a probabilistic rectangular distribution between 500 to 600 copies. If the vendor has decided to book 580 copies what price reduction he must be thinking to offer for the sale of old issues?

Ans:-

Given Data
C_s = Rs. 2 per copy
I_o = Rs 580 copies
Range = 600-500 = 100

$$f(\text{r}) = \frac{1}{Range} = \frac{1}{100}$$

$$\int_{500}^{580} f(\text{r})\,\text{dr} = \frac{C_s}{C_s + C_h}$$

$$\int_{500}^{580} \frac{1}{100}\,\text{dr} = \frac{2}{2 + C_h}$$

$$\frac{1}{100}(580 - 500) = \frac{2}{2 + C_h}$$

$$2 + C_h = 2.5$$

$$C_h = 0.5 \text{ Rs per copy}$$

UNIT 5

DYNAMIC PROGRAMMING

Outline

- The simulations are carried out by the following steps:

  – Determine the input characteristics.

  – Construct a simulation table.

  – For each repetition I, generate a value for each input, evaluate the function, and calculate the value of the response Y_I.

- Simulation examples will be given in queuing, inventory, reliability and network analysis.

Simulation of Queuing Systems

- A queuing system is described by its calling population, nature of arrivals, service mechanism, system capacity and the queuing discipline.
- In a single-channel queue:

  – The calling population is infinite.

  – Arrivals for service occur one at a time in a random fashion. Once they join the waiting line they are eventually served.

- Arrivals and services are defined by the distribution of the time between arrivals and service times.
- Key concepts:

  – The system state is the number of units in the system and the status of the server (busy or idle).

  – An event is a set of circumstances that causes an instantaneous change in the system state, E.G., arrival and departure events.

  – The simulation clock is used to track simulated time.

  Simulation of Queuing Systems

- Event list: to help determine what happens next:
  – Tracks the future times at which diff erent types of events occur.

– Events usually occur at random times.

- The randomness needed to imitate real life is made possible through the use of random (pseudo-random) numbers (more on this later).

Simulation of Queuing Systems

- Single-channel queue illustration:

– Assume that the times between arrivals were generated by rolling a die 5 times and recording the up face, then input generated is:

Simulation of Queuing Systems

- Assume the only possible service times are 1, 2, 3 and 4 time units and they are equally likely to occur, with input generated as:

Customer	Service Time	Customer	Service Time
1	2	4	2
2	1	5	1
3	3	6	4

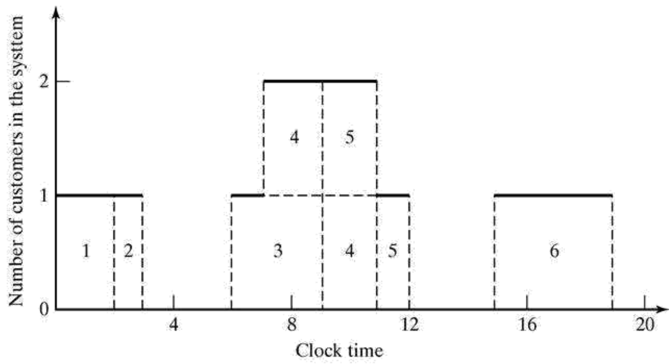
- Resulting simulation table emphasizing clock times:

Customer Number	Arrival Time (clock)	Time Service Begins (Clock)	Service Time (Duration)	Time Service Ends (clock)
1	0	0	2	2
2	2	2	1	3
3	6	6	3	9
4	7	9	2	11
5	9	11	1	12
6	15	15	4	19

Simulation of Queuing Systems

• Another presentation method, by chronological ordering of events:

Event Type	Customer Number	Clock Time
Arrival	1	0
Departure	1	2
Arrival	2	2
Departure	2	3
Arrival	3	6
Arrival	4	7
Departure	3	9
Arrival	5	9
Departure	4	11
Departure	5	12
Arrival	6	15
Departure	6	19



Simulation of Queuing Systems

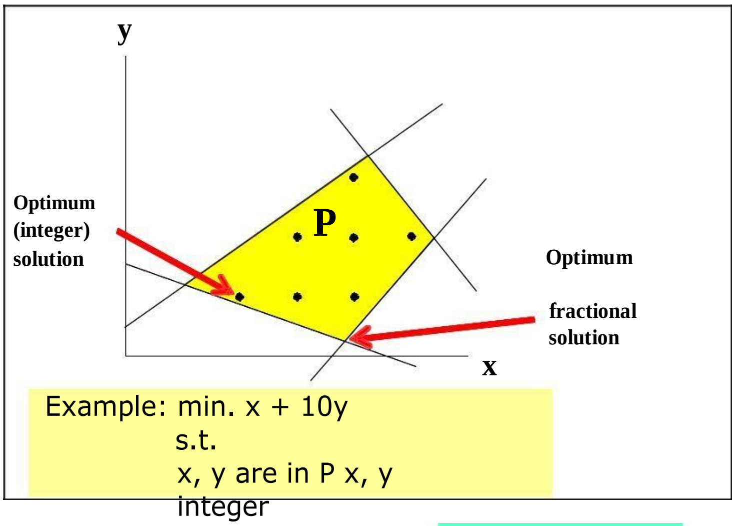
- Grocery store example with only one checkout counter:

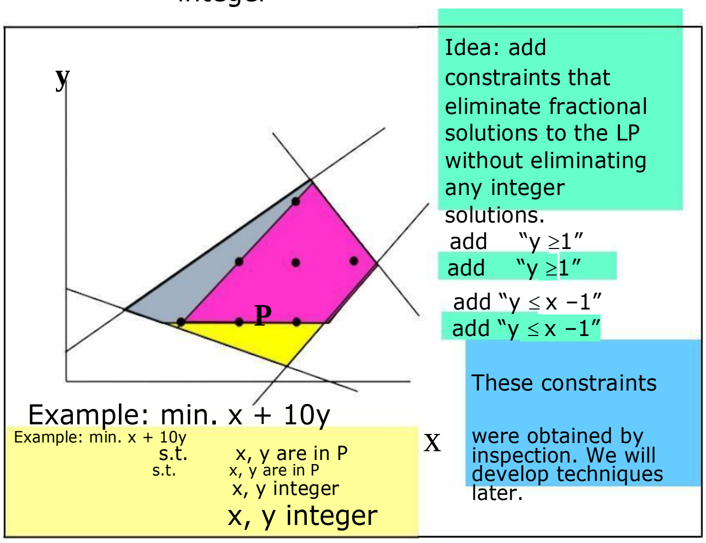
– Customers arrive at random times from 1 to 8 minutes apart, with equal probability of occurrence:

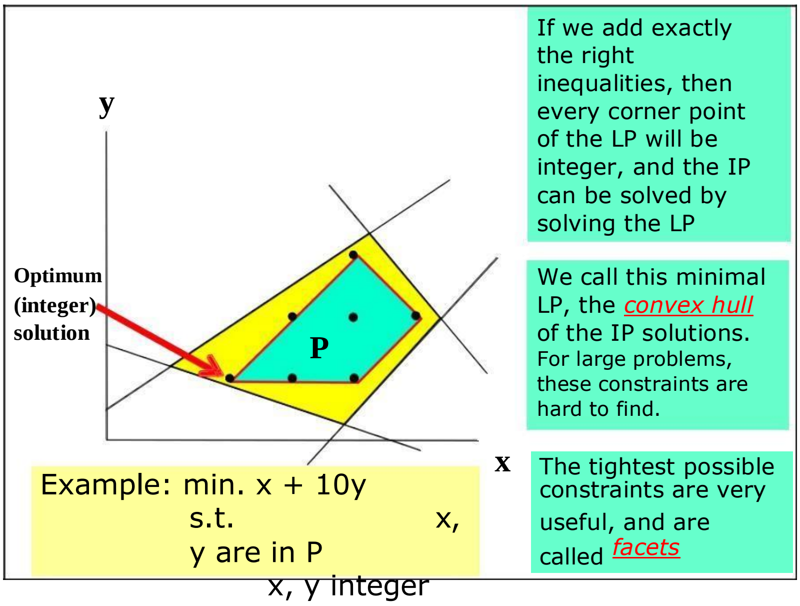
Time between Arrivals (minutes)	Probability	Cumulative Probability	Random Digit Assignment
1	0.125	0.125	001-125
2	0.125	0.250	126-250
3	0.125	0.375	251-375
4	0.125	0.500	376-500
5	0.125	0.625	501-625
6	0.125	0.750	626-750
7	0.125	0.875	751-875
8	0.125	1.000	876-000

Cutting Planes

A linear inequality is a ***valid inequality*** for a given IP problem if it holds for all integer feasible solutions to the model. Relaxations can often be strengthened dramatically by including valid inequalities that are not needed by a correct discrete model. To strengthen a relaxation, a valid inequality must cut off (render infeasible) some feasible solutions to current LR that are not feasible in the IP model.







This need to cut off noninteger relaxation solutions is why valid inequalities are sometimes called ***cutting planes***.

Cut Classification

- General purpose

  A fractional extreme point can always be separated (LP-based approach, that works for IP)

  - Disjunctive cuts
  - Gomory cutting planes
- Problem specific

  Derived from problem structure, generally facets. (Capital Budgeting (Knapsack), Set Packing...)

Cutting Plane Algorithm (Gomory cut)

Find the optimal tableau for the IP's LR.

If all variables in the optimal solution assume integer values, we have found an optimal solution! Otherwise proceed to next step

Pick a constraint in the optimal tableau whose RHS has the fractional part closest to ½.

For the constraint identified, put all of the integer parts on the left side (round down), and all the fractional parts on the right

Generate the cut as:

"RHS of the modified constraint" ≤ 0

Use the dual simplex to find the optimal solution to the LR, with the cut as an additional constraint.

- If all variables assume integer values in the optimal solution, we have found an optimal solution to the IP.
- Otherwise, pick the constraint with the most fractional right-hand side and use it to generate another cut, which is added to the tableau.

We continue this process until we obtain a solution in which all variables are integers. This will be an optimal solution to the IP.

Dual Simplex Method

Please recall that at dual simplex:

- We choose the most negative RHS.
- BV of this pivot row leaves the basis.
- For the variables that have a negative coefficient in the pivot row, we calculate the ratios (coefficient in R0 / coefficient in pivot row).
- Variable with the smallest ratio (absolute value) enters basis.

Example 10. Telfa Co.

(*Winston 9.8., p. 546*)

max z = $8x_1 + 5x_2$

s.t. $x_1 + x_2 \leq 6$

$9x_1 + 5x_2 \leq 45$

$x_1, x_2 \geq 0$ and integer

Answer

If we ignore integrality, we get the following optimal tableau:

z	*x* 1	*x* 2	*s* 1	*s* 2	*RHS*
1	0	0	1.25	0.75	41.25
0	0	1	2.25	-0.25	2.25
0	1	0	-1.25	0.25	3.75

Let's choose the constraint whose RHS has the fractional part closest to ½ (Arbitrarily choose the second constraint):

$x_1 - 1.25 s_1 + 0.25 s_2 = 3.75$

We can manipulate this to put all of the integer parts on the left side (round down), and all the fractional parts on the right to get:

$x_1 - 2 s_1 + 0 s_2 - 3 = 0.75 - 0.75 s_1 - 0.25 s_2$

Now, note that the LHS consists only of integers, so the right hand side must add up to an integer. It consists of some positive fraction minus a series of positive values. Therefore, the right hand side cannot be a positive value. Therefore, we have derived the following constraint:

$0.75 - 0.75 s_1 - 0.25 s_2 \leq 0$

This constraint is satisfied by every feasible integer solution to our original problem. But, in our current solution, s_1 and s_2 both equal 0, which is infeasible to the above constraint. This means the above constraint is a cut, called the ***Gomory cut*** after its discoverer.

We can now add this constraint to the linear program and be guaranteed to find a different solution, one that might be integer.

z	*x* 1	*x* 2	**s 1**	*s* 2	*s* 3	RHS
1	0	0	1.25	0.75	0	41.25
0	0	1	2.25	-0.25	0	2.25
0	1	0	-1.25	0.25	0	3.75
0	0	0	-0.75	-0.25	1	**-0.75**

The dual simplex ratio test indicates that s_1 should enter the basis instead of s_3.

The optimal solution is an IP solution:

$z = 40$, $x_1 = 5$, $x_2 = 0$

Example 11. Supplementary Problem

min $z = 6\,x_1 + 8\,x_2$

s.t. $3\,x_1 + x_2 \geq 4$

$x_1 + 2\,x_2 \geq 4$

$x_1, x_2 \geq 0$ and integer

Answer

Optimal tableau for LR

z	*x* 1	*x* 2	*e* 1	*e* 2	*RHS*
1	0	0	-0.80	-3.60	17.60
0	1	0	-0.40	0.20	0.80
0	0	1	0.20	-0.60	1.60

Choose the second constraint

$x_2 + 0.2\,e_1 - 0.6\,e_2 = 1.6$

Manipulate this:

$x_2 - e_2 - 1 = 0.6 - 0.2\,e_1 - 0.4\,e_2$

Cut:

$0.6 - 0.2\,e_1 - 0.4\,e_2 \leq 0$

New LP tableau

z	*x* 1	*x* 2	***e* 1**	*e* 2	*s* 3	RHS
1	0	0	-0.8	-3.6	0	17.6
0	1	0	-0.4	0.2	0	0.8
0	0	1	0.2	-0.6	0	1.6
0	0	0	-0.2	-0.4	1	**-0.6**

The dual simplex ratio test indicates that e_1 should enter the basis instead of s_3.

The optimal solution is an IP solution:

$z = 20$, $x_1 = 2$, $x_2 = 1$

SHORT & LONG ANSWERS

UNIT - I

1. Write the scope of Operations research
2. Write the Applications of Operations Research
3. Define Operations research
4. Write the assumptions in Linear programming
5. What is simulation and what is the need of simulation
6. Define scope of Operations research
7. Write the advantages of simulation

UNIT – II

1. Write the different Types in Transportation problem
2. Define the following i)Alternative optimum solution ii)unbounded solution iii)Slack variable
3. Write the formula for EOQ for the purchase model without shortages
4. Write algorithm for Least Cost Cell method
5. What is surplus variable
6. Write algorithm for Northwest corner method
7. What is artificial variable
8. Write algorithm for LCM method

UNIT – III

1. Discuss the practical application of assignment problem
2. Discuss the steps of Hungarian method
3. Write any three applications of Bellman's principle of optimality

UNIT – IV

1. What is dynamic programming.
2. What is Kendall Notation .Give the classification of queuing system based on Kendall Notation
3. What is difference between balanced and unbalanced problems in the Assignment problems
4. What is Kendall Notation .Give the classification of queuing system based on Kendall Notation
5. What is dominance property
6. Distinguish between breakdown maintenance and and preventive maintenan
7. Define the following i) balking ii) Reneging
8. Write the applications of Travelling salesman problem

UNIT - V

1. Define inventory (2M)
2. Find the value of the game

2. 6	3. 9
4. 8	5. 4

3. Define dynamic programming
4. Find the value of the game
5. Define strategy
6. Find the value of the game

1	-1	3	-1	5
-2	2	-2	4	-2

UNIT - I

1. a) Let us consider a company making single product. The estimated demand for the product for the next four months are 1000,800,1200,900 respectively. The company has a regular time capacity of 800 per month and an overtime capacity of 200 per month. The cost of regular time production is Rs.20 per unit and the cost of overtime production is Rs.25 per unit. The company can carry inventory to the next month and the holding cost is Rs.3/unit/month the demand has to be met every month. Formulate a linear programming problem for the above situation.
b) What are applications of OR

2. Solve the following LPP by Big-M penalty method Minimize $Z = 5X_1 + 3X_2$
S.T $2X_1 + 4X_2 = 12$, $2X_1 + 2X_2 = 10$, $5X_1 + 2X_2 = 10$ and $X_1, X_2 \leq 0$

2.a) Solve the following LP problem using graphical method

Maximize $Z == -x_1+2x_2$
Subjected to $x_1-x_2 \leq -1$
$-0.5x1-x2 \leq 2$ $x_1,x_2 \geq 0$
b) Explain the advantages of OR

3 a.) Explain what is meant by degeneracy in LPP? How can this be solved?

b.) Solve the following LP problem by two phase method. Maximize $Z = 5x_1+3x_2$
subjected to $3x_1+2x_2 \geq 3$

$x_1+4x_2 \geq 4$

$x_1+x_2 \leq 5$

$x_1+x_2 \geq 0$

4. Solve the following LPP problem by Two phase method Max $Z=2x_1+3x_2+5x_3$

S.T

$3x_1+10x_2+5x_3$

≤ 15 $33x_1-$

$10x_2+9x_3\leq 33$

$x_1+2x_2+3x_3\geq 4$

$x_1, x_2, x_3\geq 0$

5.a) Define the LPP. Give an example

b) Solve the following LPP using graphical method and verify by Simplex method Maximize $Z=10x_1+8x_2$ $x_1\leq 300$

$x_2\leq 500$ and $x_1, x_2, \geq 0$

6. a) A firm produces three types of biscuits A,B,C it packs them in arrangement of two sizes 1 and 11. The size 1 contains 20 biscuits of type A, 50 of type B and 10 of type C. the size 11 contains 10 biscuits of type A, 80 of type B and 60 of type C. A buyer intends to buy at least 120 biscuits of type A, 740 of type B and 240 of type C. Determine the least number of packets he should buy. Write the dual LP problem and interrupt your answer

b) Solve the following LPP using graphical method and verify by Simplex method Maximize $Z=10x_1+8x_2$ $x_1\leq 300$

$x_2\leq 500$ and $x_1, x_2, \geq 0$

7. a) Explain what is meant by degeneracy in LPP? How can this be solved? b)Solve the following LP problem by graphically Maximize $Z=2x_1+x_2$

S.T $x_1+2x_2\leq 10$, $x_1+x_2\leq 6$,$x_1-x_2\leq 2$,$x_1-2x_2\leq 1$ $x_1,x_2\geq 0$

UNIT - II

1. A company has factories at F_1, F_2 and F_3 that supply products to ware houses at W_1, W_2 and W_3 .The weekly capacities of the factories are 200,160 and 90 units. The weekly warehouse requirements are 180,120 and150/units respectively. The unit shipping costs in rupees are as follows find the optimal solution

	W1	W2	W3	supply
F1	16	20	12	200
F2	14	8	18	160
F3	26	24	16	90
Demand	180	120	150	

2. Different machines can do any of the five required jobs with different profits ring from each assignment as shown in adjusting table. Find out maximum profit possible through optimal assignment

Jobs	Machines				
	A	B	C	D	E
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

3. a) Solve the following assignment problem to minimize the total time of the

Operator

4. Jobs					
5. Operator	6. 1	7. 2	8. 3	9. 4	10. 5
11. 1	12. 6	13. 2	14. 5	15. 2	16. 6
17. 2	18. 2	19. 5	20. 8	21. 7	22. 7
23. 3	24. 7	25. 8	26. 6	27. 9	28. 8
29. 4	30. 6	31. 2	32. 3	33. 4	34. 5
35. 5	36. 9	37. 3	38. 8	39. 9	40. 7
41. 6	42. 4	43. 7	44. 4	45. 6	46. 8

b) Write the Mathematical representation of an assignment model?

4. a). Briefly explain about the assignment problems in OR and applications of assignment in OR?

   b) What do you understand by degeneracy in a transportation problem?

5. a) Give the mathematical formulation of Transportation problem

b) Use Vogel's approximate method to obtain an initial basic feasible solution of a transportation problem and find the optimal solution

Warehouse / Factory	W	X	Y	Z	Supply
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	

6. Six jobs go first on machine A, then on machine B and last on a machine C. The order of completion of jobs have no significance. The following table gives machine time for the six jobs and the three machines. Find the sequence of jobs that minimizes elapsed time to complete the jobs.

Jobs	Processing Time		
	Machine A	Machine B	Machine C
1	8	3	8
2	3	4	7
3	7	5	6
4	2	2	9
5	5	1	10
6	1	6	9

a) What do you understand by degeneracy in a transportation problem?
b) Obtain initial solution in the following transportation problem by using VAM and LCM

Source	D1	D2	D3	D4	D5	Availability
S1	5	3	8	6	6	1100
S2	4	5	7	6	7	900
S3	8	4	4	6	6	700
Requirement	800	400	500	400	600	

7. Different machines can do any of the five required jobs with different profits resulting from each assignment as shown in the adjusting table. Find out maximum profit possible through optimal assignment.

Jobs	Machines				
	A	B	C	D	E
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

8. a) State the assignment problem mathematically.

b) For the assignment table, find the assignment of salesmen to districts that will result in maximum sales

Districts / Sales people	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

9. a) What do you understand by degeneracy in a transportation problem?

10. b)A company has three plants at locations A.B.C which supply to Warehouse located at D,E,F,G and H. Monthly plant capacities are 800,500, and 900 respectively. Monthly warehouse requirements are 400,500,400 and 800units.Unit Transportation cost in rupees is

	D	E	F	G	H
A	5	8	6	6	3
B	4	7	7	6	5
C	8	4	6	6	4

11. Determine the optimum distribution for the company in order to minimize total transportation cost by NWCR

UNIT –III

1. Solve the following sequence problem given optimal solution when passing is not allowed

	Jobs				
Operator	1	2	3	4	5
1	6	2	5	2	6
2	2	5	8	7	7
3	7	8	6	9	8
4	6	2	3	4	5
5	9	3	8	9	7
6	4	7	4	6	8

2. Machine A costs of Rs:80,000. Annually operating cost are Rs:2,000 for the first years and they increase by Rs:15,000 every years (for example in the fourth year the operating cost are Rs:47,000) .Determine the least age at which to replace the machine. If the optional replacement policy is followed (a)What will be the average yearly cost of operating and owing the machine (Assume that the reset value of the machine is zero when replaced, and that future costs are not discounted

b) Another machine B cost Rs:1,00,000.Annual operating cost for the first year is Rs:4,000 and they increase by Rs:7,000 every year .The following firm has a ma(chine of type A which is one year old. Should the firm replace it with B and if so when?

(c) Suppose the firm is just ready to replace the M/c A with another M/c of the same type, just the the firm gets an information that the M/c B will become available in a year .What should firm do?

3. A book binder has one printing press, one binding machine and manuscripts of 7 different books The time required for performing printing and binding operations for different books are shown below

Book	1	2	3	4	5	6	7
Printing time (hr)	20	90	80	20	120	15	65
Binding time(hrs)	25	60	75	30	90	35	50

Decide the optimum sequence of processing of books binder to minimize the total time required to bring out all the books

4. Six jobs are to be processed on three machines A, B, C with the order of processing jobs as BCA

Job	U	V	W	X	Y	Z
Proc,time on machine A	12	10	9	14	7	9
Proc,time on machine B	7	6	6	5	4	4
Proc,time on machine C	6	5	6	4	2	4

The suggested sequence is Y-W-Z-V-U-X. Find out the elapsed time for the sequence suggested. Is it optimal? If it is not optimal, then find out the optimal sequence and the minimum total elapsed time associated with it.

5. The data collected in running a Machine the cost of which is Rs: 60,000 are

Resale value	1	2	3	4	5
Resale value(R	42,000	30,000	20,400	14,400	9,650
Cost of Spares (	4,000	4,270	4,880	5,700	6,800
Cost of Labor	14,000	16,000	18,000	21,000	25,00

Find the time when the machine should be replaced?

7. Find the most economic batch quantity of a product on machine if the production rate of the item on the machine is 300 pieces per day and the demand is uniform at the rate of 150 pieces/day. The set up Cost is Rs.300 per batch and the cost of holding one item in inventory is Rs.0.81/per day. How will the batch quantity vary if the machine production n rate was infinite?

8. A salesman has to visit five cities A,B,C,D,E. The intercity distances are tabulated below

9.	10. A	11. B	12. C	13. D	14. E
15. A	16. -	17. 12	18. 24	19. 25	20. 15
21. B	22. 6	23. -	24. 16	25. 18	26. 7
27. C	28. 10	29. 11	30. -	31. 18	32. 12
33. D	34. 14	35. 17	36. 22	37. -	38. 16
39. E	40. 12	41. 13	42. 23	43. 25	44. -

Find the shortest route covering all the cities.

9 .a) Explain the terminology of sequencing techniques in operations research?

b) Solve the following sequence problem, given an optimal solution when passing is not allowed

Machines	Jobs				
	A	B	C	D	E
M1	11	13	9	16	17
M2	4	3	5	2	6
M3	6	7	5	8	4
M4	15	8	13	9	11

9. a) State Group of replacement policy

b) The following failure rates have been observed for a certain type of light bulbs

10. End of week	11. Probability of failure date
12. 1	13. 0.05
14. 2	15. 0.13
16. 3	17. 0.25
18. 4	19. 0.43
20. 5	21. 0.68
22. 6	23. 0.88
24. 7	25. 0.96
26. 8	27. 1.00

The cost of replacing an individual failed bulb is Rs.1.25.the decision is made to replace all bulbs simultaneously at fixed intervals and also to replace individual bulbs as they fall in service. If the cost of group replacement is 30 paise per bulb, what is the best interval between group replacements? At what group replacement price per bulb would a policy of strictly individual replacement become preferable to the adopted policy?

9. a) A firm is considering the replacement of a machine, whose cost price is Rs.12,200 and its shop value is Rs.200. From experience the running (maintenance and operating) costs are found to be as follows.

Year	1	2	3	4	5	6	7	8
Ruuning cost	200	500	800	1200	1800	2500	3200	4000

10. a) When should the machine be replaced?

b) Explain two person zero sum game and n person game?

UNIT –IV

1. Obtain the optimal strategies for both pensions and the value of the game for two persons zero sum game whose payoff matrix is as follows

		player-B	
		B1	B2
Player-A	A1	1	-3
	A2	3	5
	A3	-1	6
	A4	4	1
	A5	2	2
	A6	-5	0

2. A) The Production department of a company required 3600 Kg, if raw material for manufacturing a particular item for a year. It has been estimated that the cost of placing an order is Rs. 36 and the cost of carrying inventory is 25% for the investment in the inventories, the price is Rs. 10/Kg. help the purchase manager to determine and ordering policy for raw material, determine optimal lot size.
   i. Define group replacement policy.

   b) A computer contains 10000 resistors. When any resistor fails, it is replaced the cost of replacing a resistor individually is Rs.1 only. If all the resistors are replaced at the same time, cost per resistor would be reduced to 35 paisa. The % of surviving resistors say S(t) at the end of month t and the P(t) the probability of failure during

The month t is.

t	0	1	2	3	4	5	6
S(t)	100	97	90	70	30	15	0
P(t)	-	0.03	0.07	0.2	0.4	0.15	0.15

What is the optimal replacement policy?

3. a)Explain the terms
   i) Maxmin criteria and Minimax criteria ii) Strategies: Pure and mixed strategies.

   b) Solve the following game graphically

	Player B		
Player A	B_1	B_2	B_3
A_1	1	3	11
A_2	8	5	2

4. a)Explain the terms i)Rectangular games ii) type of Strategies

b) Solve the following game graphically where pay off matrix for player A has been prepared

1	5	-7	4	2
2	4	9	-3	1

5. A dealer supplies you the following information with regards to a product that he deals in annual demand =10,000 units, ordering cost Rs.10/order. Price Rs.20/unit. Inventory carrying cost is 20% of the value of inventory per year. The dealer is considering the possibility of allowing some back orders to occur. He has estimated that the annual cost of back ordering will be 25% of the value of inventory

a. What should be the optimum no of units he should buy in 1 lot?
b. What qty of the product should be allowed to be back ordered
c. What would be the max qty of inventory at any time of year

Would you recommend to allow backordering? If so what would be the annual cost saving by adopting the policy of back ordering.

6. .a) Purchase manager places order each time for a lot of 500 no of particular item from the available data the following results are obtained, inventory carrying 40%, ordering cost order Rs.600, cost per unit Rs.50 annual demand 1000 find out the loser to the organization due to his policy

b) What are inventory models? Enumerate various types of inventory models and describe them briefly

7. a)What are characteristics of a game?

b) Reduce the following Game by dominance and the fid the game value

PlayerA					
		I	II	III	IV
	I	3	2	4	0
	II	3	4	2	4
	III	4	2	4	0
	IV	0	4	0	8

8. The demand for a purchased item 1000 units per month and shortages are allowed. If the unit cost is Rs. 1.50 per unit, the cost of making one purchase is Rs.600,the holding cost for one unit is Rs.2 per year and one shortage isRs.10 per year. Determine

a. The optimum purchase quantity ii)The number of orders per year iii)The optimal total yearly cost

9. a) Obtain the optimal strategies for both pensions and the value of the game for two persons zero sum game whose payoff matrix is as follows.

		player-B	
		B1	B2
Player-A	A1	1	-3
	A2	3	5
	A3	-1	6
	A4	4	1
	A5	2	2
	A6	-5	0

b) Explain pay of matrix and types of strategy in game theory?

UNIT - V

1. Customers arrive at box office windows being manned by a single individual according to a poisson input process with a mean rate of 20/hr. the time required to same a customer has an exponential distribution with a mean of 90 sec. Find the average waiting time of customers. Also determine the average number of customers in the system and average queue length.

2. a) What is simulation ? Discuss applications of simulation?

   b) Minimize $z = y_1^2 + y_2^2 + y_3^2$ subjected to y1+y2+y3 =10 and y1, y2, y3≥0 solve using Bellman's principle
3. Customers arrive at box office windows being manned by a single individual according to a poisson input process with a mean rate of 20/hr. the time required to save a customer has an exponential distribution with a mean of 90 sec. Find the average waiting time of the customers. Also determine the average number of customers in the system and average queue length

4. a) State and explain the Bellman"s principle of optimality.

b) Solve the LPP by dynamic programming approach Maximize z = 4x1 + 14x2

such that 2x 1 + 7x2 ≤
21 7x1 + 2x2 ≤ 21 x1,
x2 ≥ 0

5. a) Explain how the queues are classified and give their notations

   c) In a bank, cheques are cashed at a single "teller" counter. Customers arrive at the counter in a Poisson manner at an average rate of 30 customers/hr. The teller takes on an average 1.5 minutes to cash a cheque. The service time has been shown to be exponentially distributed.

   i) Calculate the percentage of time the teller is busy

   ii) Calculate the average time a customer is expected to wait.

6. Use dynamic programming to solve the following LPP
   Max $z= 3x_1+5x_2$

   Subjected to $x_1 \leq 4$.

   $X_2 \leq 6$,

   $3x_1+2x_2 \leq 18$,

   $x1,x2 \geq 0$

7. A bakery keeps stock of a popular brand of cake. Previous experience show the daily demand pattern for the item with associated probabilities as given

Daily damand (number)	0	10	20	30	40	50
Probability	0.01	0.20	0.15	0.50	0.12	0.02

use the following sequence of random numbers to simulate the demand for next 10 days Random numbers: 25,39,65,76,12,05,73,89,19,49 Also estimate the daily average demand for the cakes on the basis of the

8. Solve using dynamic programming

   Max $z= 50x_1+100x_2$

   S.T $2x_1+3x_2 \leq 48$

   $x_1+3x_2 \leq 42$

9. a) Define simulation why simulation uses. Give one application area when this technique is used in practice

   b) Use dynamic programming to solve the following LPP Max $z= 3x_1+5x_2$
   Subjected to $x_1 \leq 4$.
   $X_2 \leq 6$,
   $3x_1+2x_2 \leq 18$,
   $x1,x2 \geq 0$

10. a)What are the applications of the dynamic programming?.Explain Bellman's principle of optimality.

b)Using dynamic programming approach solve the below problem

Maximize $z = 8x_1 + 7x_2$

S.T $2x_1 + x_2 \le 8, 5x_2 + 2x_2 \le 15, x_1, x_2 \ge 0$